ANALYSIS OF THE SIGN-ERROR FXLMS ALGORITHM

György OROSZ
Advisors: Gábor PÉCELI and László SUJBERT

I. Introduction

Sign-error algorithms are mainly used due to their simple realization and low computational demand [1, 2]. These algorithms have extensive literature, however, the sign-error adaptive controller algorithm hasn't been investigated yet. This paper introduces the analysis of the Sign-Error Filtered-x Least-Mean Square (SE-FxLMS) adaptive controller algorithm. The advantage of the sign-error adaptive controller, above the reduced computational complexity, is that it makes possible *data compression* in the control loop *in a very simple way*. The signal compressing feature of sign-error algorithms is demonstrated in [4] that introduces a wireless active noise control system where a sign-error resonator based adaptive controller is used for noise control. In spite of the sign-error algorithm introduced in [4], the SE-FxLMS can also be applied in the case of general stochastic reference and control signals, and it preserves the capability of signal compression.

The paper is structured as follows. Section II provides a brief description of the SE-FxLMS algorithm. In Section III, it is shown that the mean-absolute error of the algorithm is bounded for any value of the convergence parameter. The characteristic properties of the sign-error FxLMS algorithm are demonstrated with simulation in Section IV.

II. Introduction to the SE-FxLMS Algorithm

FxLMS [3] is one of the most well-known adaptive controller algorithms. It has simple structure and ensures high degree of freedom due to the large number of free parameters so it is capable of the controlling of systems with complicated transfer function (e.g. acoustic and complex mechanical systems) [5].

The block diagram of the sign-error variant of the FxLMS algorithm can be seen in Fig. 1. In the figure, $S(z)$ denotes the plant to be controlled. The input of $S(z)$ i.e. the output of the controller is u_n and the output of $S(z)$ is denoted by y'_n. y_n and e_n denote the desired value of y'_n and the error signal respectively. n_n denotes the noise. x_n is the so-called reference signal that is correlated with y_n.

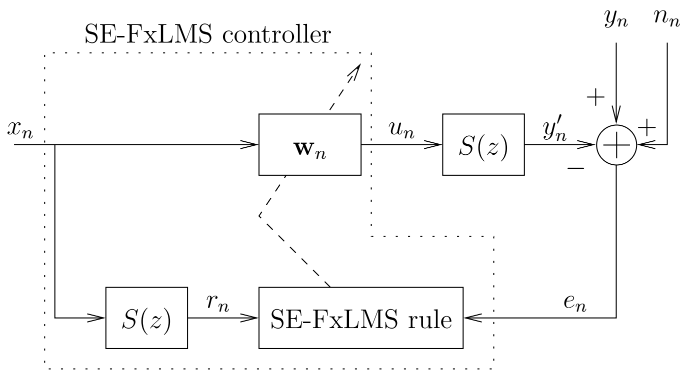

Figure 1: Block diagram of the SE-FxLMS algorithm

A typical application of the FxLMS algorithm is the active noise control (ANC) [5]. In the ANC systems, y_n is the noise to be suppressed, $S(z)$ is an acoustic system and the control signal u_n is the so-called anti-noise that is radiated by a loudspeaker—that is the actuator. The superposition of the noise and the anti-noise is the residual noise i.e. the error signal e_n that is sensed by a microphone. u_n should be inverted in order to achieve subtraction at the microphone as shown in Fig. 1. In a practical application, it can easily be ensured that x_n is correlated with y_n.

The control signal u_n is produced by filtering x_n with the adaptive transversal filter $\mathbf{w}_n$:

$$u_n = \sum_{i=0}^{N-1} w_{i,n} x_{n-i} = \mathbf{x}_n^{\mathrm{T}} \mathbf{w}_n, \quad (1)$$

where $\mathbf{w}_n = [w_{0,n} \dots w_{N-1,n}]^\mathrm{T}$ and $\mathbf{x}_n = [x_n \dots x_{n-N+1}]^\mathrm{T}$. In the case of the sign-error algorithms, the weights of $\mathbf{w}_n$ are updated by the steepest descent method so that the absolute value of the error i.e. $|e_n|$ is minimized [2]. The SE-FxLMS algorithm can be derived from the "conventional" FxLMS by replacing the error with its sign in the updating term of $\mathbf{w}_n$. Since the updating rule of the FxLMS is $\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e_n \mathbf{r}_n$ [3] so the SE-FxLMS algorithm is as follows:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu \,\mathrm{sign}(e_n)\mathbf{r}_n, \tag{2}$$

where $\mathrm{sign}(\cdot)$ denotes the sign function, μ is the positive convergence parameter and $\mathbf{r}_n = [r_n \dots r_{n-N+1}]^\mathrm{T}$ is the vector of the filtered reference signal:

$$r_n = \sum_{k=0}^{F-1} s_k x_{n-k} \leftrightarrow \mathbf{r}_n = \sum_{k=0}^{F-1} s_k \mathbf{x}_{n-k}, \tag{3}$$

where $\{s_k, k = 0 \dots F-1\}$ is the impulse response of $S(z)$.

Note that (2) uses only the sign of the error signal, which makes the computation simple since the multiplication by a sign function means only the manipulation of the sign of the multiplicand. Another advantage of the algorithm emerges when in the control system, the central controller—that implements the SE-FxLMS algorithm—and the sensor are separated and they are connected over a low bandwidth communication channel. If the sensor transmits only the sign of the error signal, significant reduction in the amount of data can be achieved [4]. The error signal can be measured directly by the sensor—e.g. in the ANC systems the residual noise—or it can be calculated if the desired signal y_n is known by the sensor that measures y'_n.

III. Upper Bound of the Mean-Absolute Error

As (2) shows, the parameter $\mathbf{w}_n$ is updated irrespectively of the magnitude of the error signal even in the tight region of the optimum where the error is small. This causes the fluctuation of the parameters around the optimum so the error signal can't be set to zero even in optimal case. This residual error is one of the most commonly investigated property of sign-error algorithms [2] and it is generally characterized by the mean-absolute error (MAE) that will be derived for the SE-FxLMS in this section.

According to (2), one can obtain a general updating rule for $\mathbf{w}_n$:

$$\mathbf{w}_n = \mathbf{w}_{n-k} + \sum_{q=1}^{k} \mu \,\mathrm{sign}(e_{n-q})\mathbf{r}_{n-q}, \quad k \geq 1. \tag{4}$$

In order to obtain a formula for e_n, first, y'_n should be calculated:

$$y'_n = \sum_{k=0}^{F-1} s_k u_{n-k} = \sum_{k=0}^{F-1} s_k \mathbf{x}_{n-k}^\mathrm{T} \mathbf{w}_{n-k} = s_0 \mathbf{x}_n^\mathrm{T} \mathbf{w}_n + \sum_{k=1}^{F-1} s_k \mathbf{x}_{n-k}^\mathrm{T} \left[\mathbf{w}_n - \sum_{q=1}^{k} \mu \,\mathrm{sign}(e_{n-q})\mathbf{r}_{n-q} \right], \tag{5}$$

where (1) and (4) are used. Due to (3), (5) can be rewritten as follows:

$$y'_n = \mathbf{r}_n^\mathrm{T} \mathbf{w}_n - \sum_{k=1}^{F-1} s_k \mathbf{x}_{n-k}^\mathrm{T} \sum_{q=1}^{k} \mu \,\mathrm{sign}(e_{n-q})\mathbf{r}_{n-q} = \mathbf{r}_n^\mathrm{T} \mathbf{w}_n - h_n, \tag{6}$$

$$h_n = \mu \sum_{k=1}^{F-1} \sum_{q=1}^{k} \mathrm{sign}(e_{n-q}) s_k \mathbf{x}_{n-k}^\mathrm{T} \mathbf{r}_{n-q}. \tag{7}$$

Let $\mathbf{w}_{opt}$ denote the optimal value of $\mathbf{w}_n$—that makes the absolute error minimal—so the error of the parameter vector is $\widetilde{\mathbf{w}}_n = \mathbf{w}_n - \mathbf{w}_{opt}$. It is assumed that y_n is of the form:

$$y_n = \mathbf{r}_n^{\mathrm{T}} \mathbf{w}_{opt} + \nu_n, \tag{8}$$

where ν_n is the component of y_n that cannot be tracked even in optimal case. In the following, ν_n will be included into the noise and a resultant noise $\varepsilon_n = n_n + \nu_n$ will be used. According to the definition of y_n, the error signal can be written as follows:

$$e_n = y_n - y_n' + n_n = \mathbf{r}_n^{\mathrm{T}} \mathbf{w}_{opt} - \mathbf{r}_n^{\mathrm{T}} \mathbf{w}_n + h_n + \varepsilon_n = -\mathbf{r}_n^{\mathrm{T}} \widetilde{\mathbf{w}}_n + h_n + \varepsilon_n. \tag{9}$$

Since (2) and (9) are formally similar to the equations that describe simple sign-error LMS (SE-LMS) algorithm which was investigated in [1] hence, those results can be applied for the SE-FxLMS algorithm as well—SE-LMS is a special case of SE-FxLMS with $S(z) = 1$. The following equations describe the SE-LMS algorithm and they were used for the derivation of the MAE of the SE-LMS [1]:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu \operatorname{sign}(e_n) \mathbf{x}_n \quad \text{and} \quad e_n = -\mathbf{x}_n^{\mathrm{T}} \widetilde{\mathbf{w}}_n + \varepsilon_n, \tag{10}$$

and the MAE—that is denoted by E^a—of the SE-LMS is:

$$E^a = \sum_{k=1}^{n} E\left\{|e_k|\right\} \le \frac{\widetilde{\mathbf{w}}_1}{2\mu n} + \frac{1}{2}\mu N R_{xx}(0) + E\{|\varepsilon_n|\}, \tag{11}$$

where $E\{|\varepsilon_n|\}$ denotes the expected value of the absolute value of the noise and $R_{xx}(0) = E\left\{x_n^2\right\}$ is the autocorrelation function of x_n in the origin, i.e. the variance of x_n if $E\left\{x_n\right\} = 0$.

Comparing (10) with (2) and (9), it can be noted that in the SE-FxLMS, $\mathbf{r}_n$ is used instead of $\mathbf{x}_n$ so in the calculation of the MAE of the SE-FxLMS, $R_{rr}(0)$ should be used instead of $R_{xx}(0)$.

On the other hand, the effect of the dynamic system appears in the error as an additive term—see h_n in (9)—hence, it can be handled as the part of the noise. Since the noise increases the MAE (11) by its mean-absolute value i.e. by $E\left\{|\varepsilon_n|\right\}$ thus the dynamic system increases the MAE by $E\left\{|h_n|\right\}$ i.e. by the expected value of $|h_n|$.

An upper bound of $E\left\{|h_n|\right\}$ can be calculated according to (7) using that $|a+b| \le |a| + |b|$ and $E\left\{|\mathbf{x}_{n-k}^{\mathrm{T}} \mathbf{r}_{n-q}|\right\} \le E\left\{\sum_{i=0}^{N-1} |x_{n-k-i}||r_{n-q-i}|\right\} = N R_{|x||r|}(k-q)$ where $R_{|x||r|}(k-q)$ is the cross-correlation of $|x_n|$ and $|r_n|$ if it is assumed that they are stationer. Finally, one obtains:

$$E\left\{|h_n|\right\} \le \mu \sum_{k=1}^{F-1} \sum_{q=1}^{k} |s_k| N R_{|x||r|}(k-q) = \mu\eta. \tag{12}$$

Using (11), (12) and the above discussion, the MAE of the SE-FxLMS can be calculated:

$$E^a \le \frac{\widetilde{\mathbf{w}}_1}{2\mu n} + \frac{1}{2}\mu N R_{rr}(0) + E\left\{|h_n|\right\} + E\left\{|\varepsilon_n|\right\} \le \frac{\widetilde{\mathbf{w}}_1}{2\mu n} + \frac{1}{2}\mu N R_{rr}(0) + \mu\eta + E\left\{|\varepsilon_n|\right\}. \tag{13}$$

(13) can be partitioned into three groups:

- $\frac{1}{2\mu n}\widetilde{\mathbf{w}}_1$ is the effect of the transient and it vanishes if $n \to \infty$. Since it is inversely proportional to μ, the smaller the μ is, the longer the transient phase is.
- $\mu\left[\frac{1}{2}N R_{rr}(0) + \eta\right]$ is the effect of the constant step size, i.e. the parameters are modified irrespectively of the magnitude of the error. This term is proportional to μ so the MAE can be decreased by decreasing μ.
- $E\left\{|\varepsilon_n|\right\}$ is the effect of the noise, it isn't influenced by the convergence parameter.

As (13) shows, the SE-FxLMS preserves the characteristic property of the sign-error algorithms that in ideal case, the steady state error can be set to an arbitrary small value, however, at the expense of the settling time [1, 2].

IV. Simulation results

The properties of the sign-error FxLMS algorithm were also investigated with simulation and it was compared with the original FxLMS algorithm as well. The plant in the simulation was a second order system that has relatively high dynamics:

$$S(z) = 2\frac{z^2 + 0.6627z + 0.6214}{z^2 - 0.3373z + 0.81} \tag{14}$$

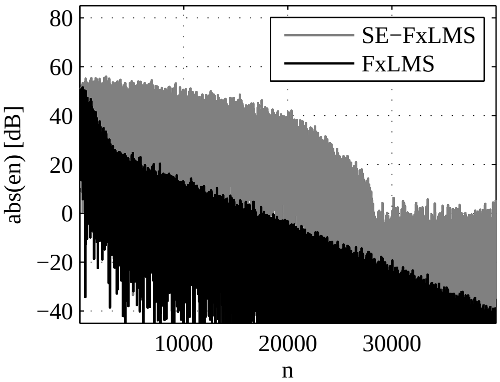

Figure 2: Transients of the SE-FxLMS and the FxLMS algorithms

In Fig. 2, the transients of the SE-FxLMS (gray line) and FxLMS (black line) algorithms can be seen. The parameters of the algorithms were: $\mu = 10^{-3}$ for SE-FxLMS and $\mu = 10^{-4}$ for FxLMS, $N = 10$ and the reference signal x_n was Gaussian random process with the variance of $R_{xx}(0) = 1 \rightarrow R_{rr}(0) = 15.6$. Comparing the transients of the algorithms, one can observe the most characteristic property of the SE-FxLMS algorithm: in spite of the FxLMS that ensures exponentially decreasing error, in the case of the SE-FxLMS, the average absolute error doesn't decrease below a certain level that is determined by the convergence parameter.

The degradation of the residual error in the case of the SE-FxLMS is the result of the fact that the magnitude of the error is neglected during the adaptation. However, this simplification of the algorithm makes also possible the implementation of the algorithm in systems with limited resources [4].

The steady state MAE ($\mathrm{MAE_{ss}}$) of the sign-error algorithms is generally an important design parameter that can be estimated according to (13) when $n \rightarrow \infty$. In the figure, the SE-FxLMS is in steady state for $n > 28000$. In the simulation, the $\mathrm{MAE_{ss}}$ was 0.303 and the estimated $\mathrm{MAE_{ss}}$ was 3.44, which shows the validity of (13). If the $\mathrm{MAE_{ss}}$ were calculated according to (11), i.e. neither the filtering of the reference signal nor the effect of the plant $S(z)$ were taken into account, the $\mathrm{MAE_{ss}}$ would be 0.005 so the dynamic system in the feedback loop of the sign-error algorithm has significant effect.

V. Conclusions

In this paper, the SE-FxLMS algorithm was introduced and its analysis was presented. It was shown that the bound of the MAE of the SE-FxLMS algorithm can be given for any value of the convergence parameter. Both the analytical and simulation results show that the dynamic plant in the feedback loop of the sign-error algorithm can significantly influence the MAE.

In the future, the extension of the results on multiple-input multiple-output case can be expected.

References

[1] A. Gersho, "Adaptive Filtering with Binary Reinforcement," *IEEE Transactions on Information Theory,* Vol. IT-30, No. 2, Mar. 1984., pp. 191–199.

[2] O. Macchi, "Advances in Adaptive Filtering," in *Digital Communications,* Amsterdam, The Netherlands: North-Holland, 1986., pp. 41–57.

[3] B. Widrow, *Adaptive inverse control*, Prentice-Hall, Inc. 1996, pp. 160–208.

[4] Gy. Orosz, L. Sujbert, G. Péceli, "Spectral Observer with Reduced Information Demand," *Proc. of the IEEE International Instrumentation and Measurement Technology Conference*, Victoria, Canada, May 12-15. 2008., pp. 2155–2160.

[5] M.O. Tokhi, S.M. Veres, *Active Sound and Vibration Control: theory and applications*, Bath, UK, The Institution of Electrical Engineers, 2002.