The Cumulative Idle Time in an $nD/D/1$ queue

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Abstract

An $nD/D/1$ queueing model means that n independent periodic sources are served by a single server and the packets have the same size. These models have received close attention as general queueing models in telecommunications. Both discrete models, where it is permitted to transmit packets only at fixed time instants, and also continuous models, where the time of transmission is not restricted, can be applied in the modelling. This paper provides the exact distribution of the cumulative idle time duration in such queuing systems and also proposes accurate approximation formulae for large models. The results of this paper are of practical significance, because the proposed formulae allow one to improve some existing algorithms through using the proposed formulae instead of available approximations of the distribution of the cumulative idle time.

Keywords: nD/D/1 queue, cumulative idle time, approximation

1 Introduction

This work was originally inspired by the analysis of a particular part of the Universal Mobile Telecommunications System (UMTS). A mobile user gets access to UMTS through the UMTS Terrestrial Radio Network (UTRAN). A base station of UTRAN terminates the air interface and forwards the traffic to the Radio Network Controller (RNC). Many papers focus on the Iub interface which is the interface between the RNC and base station. The radio protocols which are used to convey the data result in deterministic interarrival times between packets transferred through the Iub interface. The deterministic interarrival times motivate the application of the $nD/D/1$ queuing models in the analysis of the Iub interface.

The $nD/D/1$ queuing models are intensively used in telecommunications problems. In [3], authors describe a simulation model suitable for dimensioning the transport link of Iub interface. In this simulation model, the considered performance measure is the queueing delay of packets. In [4], authors propose an admission control mechanism which guarantees low packet delay on the Iub interface. In both papers, the authors have used the $nD/D/1$ queueing models.

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Other papers investigate a multiplexer which uses non-preemptive priority scheduling fed by periodic sources. In [11], the authors perform a delay analysis of the high priority sources. They assume that the buffer of the low priority sources is saturated so that at least one packet always exists in the buffer. This is the worst-case scenario for the high priority packet delay. They provide an exact and closed form of the delay distribution of high priority packets. In [2], the authors study different segmentation methods of low priority packets and they provide an approximation formula for the delay distribution of a low priority packet at the restricted case of having a single low priority packet in the system. They also show that the delay distribution of a low priority packet can be determined using the cumulative idle time distribution of the high priority queue.

The $nD/D/1$ queueing models have also received close attention as general queueing models. In [5], the distribution of the typical work and the maximum work are studied both for finite number of sources and in the limit of large number of sources. In [6], authors provide an exact formula for the queue length distribution in the case where all sources have the same period and they also provide tight upper and lower bounds on this distribution when the periods are different. In [7], the distribution of idle time duration is given in closed form and in [10], the distribution of the busy time duration is also given. In [8], the Ballot Theorems are applied which are the basis of the calculation of the queue length distribution. This distribution is given for systems where the time axis is slotted as well as for systems working in continuous time. The typical waiting time distribution is discussed as well. Finally, a detailed review of the work on $nD/D/1$ models is given in [9]. The cumulative idle time is a more complex measure than the idle time duration or the queue length distribution. It provides a more detailed system description and its detailed analysis has not been considered in the literature yet.

In this paper exact formulae and accurate approximations for the cumulative idle time distribution both in discrete-time (slotted) models and in continuous-time models are added to the results above. The application of the proposed formulae can increase the accuracy of the admission control mechanism proposed in [4], and the application of exact formulae allows an exact analysis of different segmentation methods studied in [2].

The rest of the paper is structured as follows. The next section describes the model and introduces the model parameters. Section 3 contains two theorems which provide the cumulative idle time distribution of discrete and continuous models. These are presented in Section 3.1 and 3.2, respectively. The asymptotic behavior of the cumulative idle time distribution is studied in Section 4. Section 5 proposes approximations for faster calculation of the cumulative idle time distribution and Section 6 concludes the paper.

2 Model description

This section describes the behavior of the $nD/D/1$ queuing models. We study discrete-time models, referred to as *discrete models*, which operate with time-slots and continuous-time models, referred to as *continuous models*, where there is no restriction to the time of transmission.

A work-conserving server serves N sources. Every source generates packets periodically with the deterministic interarrival time T. The i-th source generates one packet at $\ldots, \tau_i -$

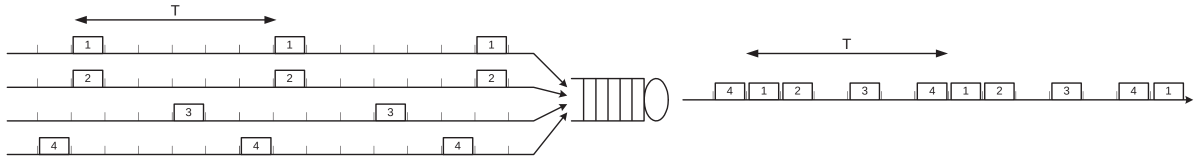

Figure 1: A discrete model with 4 sources

$T, \tau_i, \tau_i + T, \ldots$, where the random offset τ_i is uniformly distributed between 0 and T. The deterministic service time of packets is the time unit. The sources are independent of each other (i.e. $\tau_1, \tau_2, \ldots, \tau_N$ are independent) and the buffer is large enough to avoid packet loss. Figure 1 shows an example model.

In the continuous models there is no restriction to the start of transmitting a packet but in the discrete models the transmissions can be started only at the beginnings of slots. The size of the slots is one time unit.

In both cases we consider additionally that the system is stable which means that the queue empties in each period. In other words the number of packets arrived in a time interval of duration T is less than T.

The performance measure considered is $Idle(t_1, t_2)$ which gives the cumulative length of the time intervals in (t_1, t_2) during which no data is transmitted.

The $Idle(t_1, t_2)$ has some special properties. The $Idle(t_1, t_2)$ is stationary i.e.

$$\forall t_0 > 0 \quad Idle(t_1, t_2) \stackrel{d}{=} Idle(t_1 + t_0, t_2 + t_0).^1 \tag{1}$$

Additionally, if the starting point (t_1) is fixed this quantity is periodic except for its trend i.e.

$$Idle(t_1, t_2 + T) \stackrel{d}{=} Idle(t_1, t_2) + T - N. \tag{2}$$

The subject of analysis is the distribution of the measure $Idle_t = Idle(0, t)$, referred to as the distribution of cumulative idle time duration or cumulative idle time distribution, where $t \in (0, T)$ because it determines the two-variable $Idle$ function considering (1) and (2). Principally, the beginning of an arbitrary time interval with duration t is referred to as 0 and the end of the interval as t. In the discrete models, T and t are measured in time slots.

[1] The notation $\stackrel{d}{=}$ means equality in distribution i.e. for all fixed t_0, t_1, t_2 the distribution function of $Idle(t_1, t_2)$ is equal to the distribution function of $Idle(t_1 + t_0, t_2 + t_0)$.

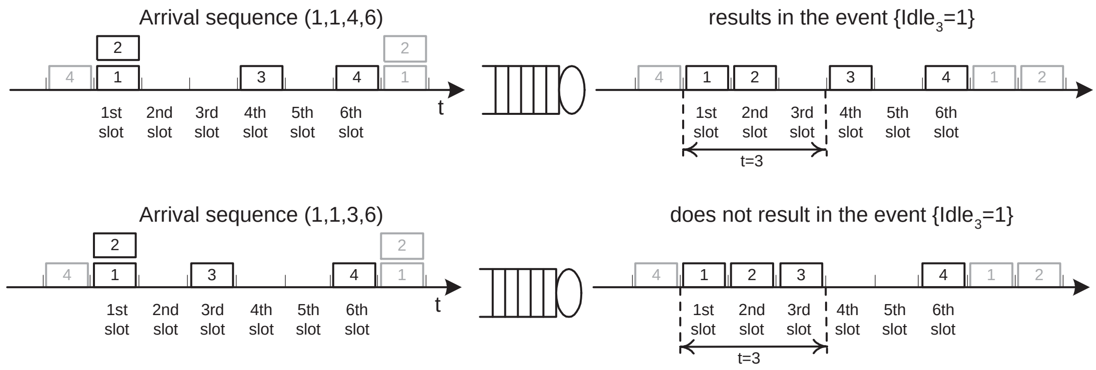

Figure 2: Classification of arrival sequences; model parameters: $T = 6$, $N = 4$ and $t = 3$

3 Analysis

This section presents the algorithm which provides the distribution of $Idle_t$. The development is based purely on probabilistic arguments. Before presenting the idea of the proposed algorithm we introduce the arrival sequences.

The arrival sequence is the sequence of arrival times of the packets arrived in the interval $[0, T]$. Due to the periodicity in the packet arrivals, each source generates exactly one packet in the interval $[0, T]$. Thus an arrival sequence contains N elements. The i-th element of an arrival sequence is the arrival time of the source-i packet that arrived in the interval $[0, T]$. Basically, an arrival sequence contains the random offsets of the N sources and therefore completely determines the arrival times of all packets. For example in the model with parameters $T = 6$ and $N = 4$, $(1, 1, 4, 6)$ is a possible arrival sequence (see the upper part of Figure 2) and means that the packets of the source-1 fall into slots $\ldots, 1, 7, 13, \ldots$, the packets of the source-2 fall into slots $\ldots, 1, 7, 13, \ldots$, the packets of the source-3 fall into slots $\ldots, 4, 10, 16, \ldots$ and the packets of the source-4 fall into slots $\ldots, 6, 12, 18, \ldots$ Assuming the discrete models, the set of all possible arrival sequences is $\mathcal{S} = \{1, 2 \ldots, T\}^N$ considering T slots and N sources and the set of all possible arrival sequences is $(0, T)^N$ in the continuous models. Each arrival sequence has the same probability.

3.1 The discrete models

Consider the discrete models. The idea behind the proposed algorithm is as follows. The arrival sequences are divided into two sets: $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. The set $\mathcal{S}_1$ contains the arrival sequences which result in the event $\{Idle_t = k\}$. Figure 2 shows two arrival sequences and their service patterns which are the packet sequence at the output of the queue[2]. The first arrival sequence $(1, 1, 4, 6)$ results 2 packets in the interval $[1..3]$ therefore if we consider the event $\{Idle_3 = 1\}$ then this arrival sequence is element of $\mathcal{S}_1$. The second arrival sequence $(1, 1, 3, 6)$ results 3

[2]We always consider only an interval of size T and we draw larger interval only for ease of understanding.

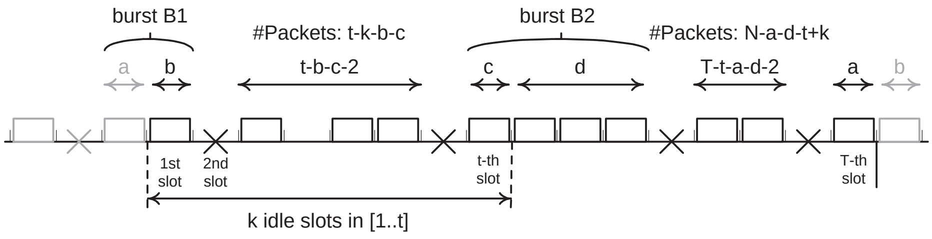

Figure 3: The general structure of the service patterns which result in the event $\{Idle_t = k\}$ in the discrete models

packets in the interval [1..3] therefore if we consider the event $\{Idle_3 = 1\}$ then this arrival sequence is element of $\mathcal{S}_2$. The probability of $\{Idle_t = k\}$ is the measure of the set $\mathcal{S}_1$.

Here, we determine the probability of the event $\{Idle_t = k\}$ in a discrete model. This event will be built up from its subsets. We consider here only a regular case where at least 2 idle slots are in the interval $[1, t]$ and $[t+1, T]$. Other cases will be considered in Theorem 1. In order to decompose the event the whole $\mathcal{S}$ is partitioned into sets indexed by four non-negative integers. The set $\mathcal{S}_{a,b,c,d}$ contains the arrival sequences which result in a burst (B1) of size $a+b$ and time 0 is in this burst, a burst (B2) of size $c+d$ and time t is in this burst, and finally the interval $[1, t]$ contains b packets of the burst B1 and c packets of the burst B2. Figure 3 shows this partitioning. The sets $\mathcal{S}_{a,b,c,d}$ are disjoint and $\bigcup \mathcal{S}_{a,b,c,d} = \mathcal{S}$, therefore the probability of the event $\{Idle_t = k\}$ can be calculated as follows:

$$\Pr\{Idle_t = k\} = \Pr\{\mathcal{S}_1\} = \sum_{\forall a,b,c,d} \Pr\{\mathcal{S}_1 \cap \mathcal{S}_{a,b,c,d}\}. \tag{3}$$

Figure 3 presents the general structure of the service patterns which result in the event $\{Idle_t = k\}$ and using this general structure we rewrite (3) as follows:

$$\Pr\{Idle_t = k\} = \sum_{\forall a,b,c,d} \Pr\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\} \tag{4}$$

and the events are defined as follows:

$$\begin{aligned}
\mathcal{A}_1 &= \{a+b \text{ packets arrived in the interval } [T-a-1, T+b]\}, \\
\mathcal{A}_2 &= \{c+d \text{ packets arrived in the interval } [t-c+1, t+d-1]\}, \\
\mathcal{A}_3 &= \{t-k-b-c \text{ packets arrived in the interval } [b+2, t-c-1]\}, \\
\mathcal{A}_4 &= \{N-a-d-t+k \text{ packets arrived in the interval } [t+d+2, T-a-1]\}, \\
\mathcal{B}_1 &= \{\text{slot } b+1 \text{ is empty}\}, \qquad \mathcal{B}_2 = \{\text{slot } t-c \text{ is empty}\}, \\
\mathcal{B}_3 &= \{\text{slot } t+d+1 \text{ is empty}\}, \qquad \mathcal{B}_4 = \{\text{slot } T-a \text{ is empty}\}.
\end{aligned}$$

Note that the events above depend on the actual values of N, T, t, a, b, c and d, but to avoid notational complexity we do not indicate this dependency. Using the definition of conditional probability, we rewrite (4) as follows:

$$\Pr\{Idle_t = k\} = \sum_{\forall a,b,c,d} \Pr\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} \Pr\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 | \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}. \quad (5)$$

Using the identity provided in Lemma 4 in the Appendix we rewrite (5) as follows:

$$\Pr\{Idle_t = k\} = \quad (6)$$
$$\sum_{\forall a,b,c,d} \Pr\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} \Pr\{\mathcal{B}_1 | \mathcal{A}_1, \mathcal{B}_4\} \Pr\{\mathcal{B}_2 | \mathcal{A}_2, \mathcal{B}_1\} \Pr\{\mathcal{B}_3 | \mathcal{A}_3, \mathcal{B}_2\} \Pr\{\mathcal{B}_4 | \mathcal{A}_4, \mathcal{B}_3\}.$$

The value of $\Pr\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$ is calculated as follows:

$$\Pr\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} = \binom{N}{a+b} \binom{N-a-b}{c+d} \binom{N-a-b-c-d}{t-k-b-c} \times \quad (7)$$
$$\left(\frac{a+b}{T}\right)^{a+b} \left(\frac{c+d}{T}\right)^{c+d} \left(\frac{t-b-c-2}{T}\right)^{t-k-b-c} \left(\frac{T-t-a-d-2}{T}\right)^{N-a-d-t+k}.$$

The main difficulty in the calculation of $\Pr\{Idle_t = k\}$ is the evaluation of the conditional probabilities in (6). The following lemma gives formulae for these probabilities.

Definition 1 *Let $G(L, n)$ denote the probability that the n packets arrive into a time interval with duration L slots randomly in such way that all of them are served inside the interval assuming that the system was empty at the beginning of the interval. Denote $g(L, n)$ the number of the associated arrival sequences, i.e. $g(L, n) = L^n\, G(L, n)$.*

Definition 2 *Let $F(n)$ denote the probability that the n packets arrive into a time interval with duration n slots randomly in such way that all of them stay together in a burst provided that the first one arrives into the first slot of the interval assuming that the system was empty at the beginning of the interval. Denote $f(n)$ the the number of the associated arrival arrival sequences, i.e. $f(n) = n^n\, F(n)$.*

Lemma 1 *The quantities $G(L, n)$, $g(L, n)$, $F(n)$ and $f(n)$ quantities are given as follows:*

$$G(L, n) = \frac{(L-n+1)(L+1)^{n-1}}{L^n}, \quad g(L, n) = (L-n+1)(L+1)^{n-1},$$

$$F(n) = G(n, n)\frac{(n+1)^{n-1}}{n^n}, \quad f(n) = (n+1)^{n-1}.$$

The proof is given in the Appendix.

Theorem 1 summarizes the calculation of $\Pr\{Idle_t = k\}$ for all possible cases (there is no idle slot in the interval $[1, t]$, there is only one idle slot in the interval $[1, t]$, etc.).

Theorem 1 *The distribution of $Idle_t$ is given by the following equations, where a, b, c and d are non-negative integers:*

(A) If $T-N \geq 4$ *and* $2 \leq k \leq T-N-2$*:*

$$\Pr\{Idle_t = k\} = \frac{1}{T^N} \sum_{\substack{a+d \leq N-t+k \\ b+c \leq t-k}} \binom{N}{a+b} \binom{N-a-b}{c+d} \binom{N-a-b-c-d}{N-a-d-t+k} \times$$

$$f(a+b)\ f(c+d)\ g(T-t-a-d-2, N-a-d-t+k)\ g(t-b-c-2, t-k-b-c),$$

(B) If $T-N \geq 3$*:*

$$\Pr\{Idle_t = T-N-1\} = \frac{1}{T^N} \sum_{\substack{a \leq T-t-1,\ b \leq t-2 \\ b+c \leq N-T+t+1}} \binom{N}{a+b} \binom{N-a-b}{c+T-t-a-1} \times$$

$$f(a+b)\ f(c+T-t-a-1)\ g(t-b-c-2, N-T+t+1-b-c),$$

(C) If $T-N \geq 3$*:*

$$\Pr\{Idle_t = 1\} = \frac{1}{T^N} \sum_{\substack{a \leq T-t-2,\ b \leq t-1 \\ a+d \leq N-t+1}} \binom{N}{a+b} \binom{N-a-b}{t-b-1+d} \times$$

$$f(a+b)\ f(t-b-1+d)\ g(T-t-a-d-2, N-t+1-a-d),$$

(D) If $T-N \geq 2$*:*

$$\Pr\{Idle_t = T-N\} = \frac{1}{T^N} \sum_{\substack{a \geq T-t \\ b \leq t-2}} \binom{N}{a+b} f(a+b)\ g(T-a-b-2, N-a-b),$$

(E) If $T-N \geq 2$ *and* $N > t$*:*

$$\Pr\{Idle_t = 0\} = \frac{1}{T^N} \sum_{\substack{a \leq T-t-2 \\ b \geq t}} \binom{N}{a+b} f(a+b)\ g(T-a-b-2, N-a-b),$$

(F) If $T-N = 2$*:*

$$\Pr\{Idle_t = 1\} = \frac{1}{T^N} \sum_{\substack{a \leq \min(N, T-t-1) \\ b \leq \min(N-a, t-1)}} \binom{N}{a+b} f(a+b)\ f(N-a-b),$$

(G) If $T-N = 1$*:*

$$\Pr\{Idle_t = 1\} = \frac{t}{T^N}\ f(T-1),$$

(H) If $T-N = 1$*:*

$$\Pr\{Idle_t = 0\} = \frac{T-t}{T^N}\ f(T-1).$$

The proof is given in the Appendix.

3.2 The continuous models

In the continuous models the cumulative idle time is a mixed random variable. Figure 6 in the Appendix shows an example and the discrete part of the mixed distribution causes jumps in the distribution function. The discrete part of the distribution is caused by arrival sequences which produce an empty system at time 0 and at time t. Figure 4 shows the general structure of the service pattern which result in the event $\{Idle_t = t - k\}$ (discrete part) and the calculation the probability of this event is based on the following identity:

$$\Pr\{Idle_t = t - k\} = \Pr\{k \text{ packets arrived and served in } (0,t), \;\; N-k \text{ packets arrived and served in } (t,T)\}.$$

The case (C) in Theorem 2 summarizes the calculation of $\Pr\{Idle_t = t - k\}$.

The continuous part of the mixed distribution is caused by arrival sequences which produce a non-empty system at time 0 or at time t. Figure 5 shows the general structure of the service patterns which result in τ cumulative idle time in the interval $(0,t)$ and the figure also introduces the applied notations. There are two ways of obtaining τ cumulative idle time in the interval $(0,t)$: (i) $z_1 + z_2 \leq 1$ and $z_1 + z_2 = \{t - \tau\}$ and (ii) $z_1 + z_2 > 1$ and $z_1 + z_2 - 1 = \{t - \tau\}$; where $\{x\}$ denotes the remainder of x, i.e. $\{x\} = x - \lfloor x \rfloor$. The evaluation of case (i) starts from the following identity:

$$\Pr\{\tau \leq Idle_t \leq \tau + \Delta\} = \tag{8}$$
$$\sum_{\forall a,b,c,d} \int_{z_1=0}^{\{t-\tau\}} \Pr\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 | \; z_1, \; \{t-\tau\}-z_1 \leq z_2 \leq \{t-\tau\}-z_1 + \Delta\} \times$$
$$\Pr\{z_1, \; \{t-\tau\}-z_1 \leq z_2 \leq \{t-\tau\}-z_1+\Delta\} dz_1 + o(\Delta)$$

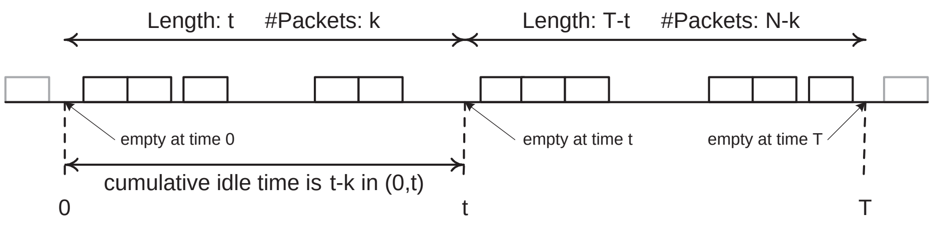

Figure 4: The general structure of the service patterns which result in the event $\{Idle_t = t - k\}$

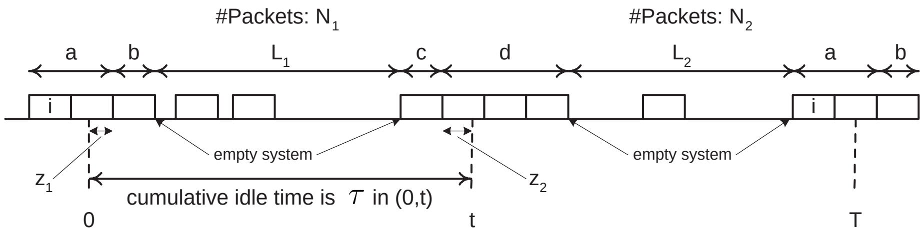

Figure 5: The general structure of the service patterns which result in τ cumulative idle time in the interval $(0, t)$

and the events above are defined as follows:

$$\begin{aligned}
\mathcal{C}_1 &= \{N_1 = \lfloor t-\tau \rfloor - b - c \text{ packets arrived in the interval } L_1 = (z_1+b, t-c-z_2)\},\\
\mathcal{C}_2 &= \{N_2 = N - \lfloor t-\tau \rfloor - a - d \text{ packets arrived in the interval } L_2 = (t+d-z_2, T-a+z_1)\},\\
\mathcal{C}_3 &= \{a+b \text{ packets arrived in the interval } (T-a+z_1, T+z_1+b)\},\\
\mathcal{C}_4 &= \{c+d \text{ packets arrived in the interval } (t-c-z_2, t+d-z_2)\},\\
\mathcal{D}_1 &= \{\text{system is empty at time } z_1+b\}, \qquad \mathcal{D}_2 = \{\text{system is empty at time } t-c-z_2\},\\
\mathcal{D}_3 &= \{\text{system is empty at time } t+d-z_2\}, \quad \mathcal{D}_4 = \{\text{system is empty at time } T-a+z_1\}.
\end{aligned}$$

Note that the events above depend on the actual values of N, T, t, a, b, c, d, z_1 and z_2, but to avoid notational complexity we do not indicate this dependency. Because the event $\{z_1 = x\}$ is equivalent to the event {packet-i arrives at $T - a + z_1$} (see Figure 5) and the packet arrivals are distributed uniformly, we rewrite (8) as follows:

$$\Pr\{\tau \le Idle_t \le \tau + \Delta\} = \\ \frac{\Delta}{T}\frac{\{t-\tau\}}{T}\sum_{\forall a,b,c,d} \Pr\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 |\ z_1 + z_2 = \{t-\tau\}\} + o(\Delta).$$

Now we can determine the density function of $Idle_t$ as follows:

$$\begin{aligned}
f(\tau) &= \lim_{\Delta\to 0} \frac{\Pr\{\tau \le Idle_t \le \tau + \Delta\}}{\Delta}\\
&= \frac{\{t-\tau\}}{T^2}\sum_{\forall a,b,c,d} \Pr\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 |\ z_1 + z_2 = \{t-\tau\}\}.
\end{aligned}$$

The further derivation of $f(\tau)$ follows a similar way as we have done in the case of discrete models. The evaluation of cases (ii) is similar to the evaluation of case (i). Finally, Theorem 2 summarizes the calculation of the cumulative idle time distribution for all possible cases.

Before presenting Theorem 2, we introduce two definitions and we also provide a lemma which will be used in the theorem.

Definition 3 *Let $P(L, n)$ denote the probability that the n packets arrive into the time interval with duration L randomly in such way that all of them are served inside the interval assuming that the system was empty at the beginning of the interval. Let $p(L, n) = L^n\ P(L, n)$.*

Definition 4 *Let $R(n)$ denote the probability that the n packets arrive into the time interval with duration n randomly in such way that all of them stay together in a burst provided that the first packet arrives at the beginning of the interval assuming that the system was empty at the beginning of the interval. Let $r(n) = n^n \; R(n)$.*

Lemma 2 *The quantities $p(L, n)$ and $r(n)$ are given as follows:*

$$P(L,n) = \frac{L-n}{L}, \quad p(L,n) = (L-n)L^{n-1}, \qquad R(n) = \frac{1}{n}, \quad r(n) = n^{n-1}.$$

The proof is given in the Appendix.

Theorem 2 summarizes the calculation of cumulative idle time distribution for all possible cases.

Theorem 2 *The distribution of $Idle_t$ has discrete and continuous parts. Denote $f(\tau) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Pr\{\tau \le Idle_t \le \tau + \Delta\}$ the density function of the $Idle_t$, where the limit exists. The distribution of $Idle_t$ is given by the following equations, where a, b, c and d are non-negative integers:*

(A) If $N \ge \lceil t \rceil$, then

$$\begin{aligned}
\Pr\{Idle_t = 0\} &= \frac{1}{T^N} \sum_{a \ge \lceil t \rceil} \binom{N}{a} (a-t) r(a) \; p(T-a, N-a) \\
&= \frac{1}{T^N} \sum_{a \ge \lceil t \rceil} \binom{N}{a} (a-t) a^{a-1} (T-N)(T-a)^{N-a-1}
\end{aligned}$$

else $\Pr\{Idle_t = 0\} = 0$.

(B) If $N \ge \lceil T - t \rceil$, then

$$\begin{aligned}
\Pr\{Idle_t = T-N\} &= \frac{1}{T^N} \sum_{a \ge \lceil T-t \rceil} \binom{N}{a} (a-T+t) r(a) \; p(T-a, N-a) \\
&= \frac{1}{T^N} \sum_{a \ge \lceil T-t \rceil} \binom{N}{a} (a-T+t) a^{a-1} (T-N)(T-a)^{N-a-1}
\end{aligned}$$

else $\Pr\{Idle_t = T-N\} = 0$.

(C) If $\max(0, N - \lfloor T-t \rfloor) \le k \le \min(N, \lfloor t \rfloor)$ and $k \in \mathbb{N}$, then

$$\begin{aligned}
\Pr\{Idle_t = t-k\} &= \frac{1}{T^N} \binom{N}{k} \; p(t,k) \; p(T-t, N-k) \\
&= \binom{N}{k} \left(\frac{t}{T}\right)^k \left(1 - \frac{t}{T}\right)^{N-k} \left(1 - \frac{k}{t}\right) \left(1 - \frac{N-k}{T-t}\right)
\end{aligned}$$

else $\Pr\{Idle_t = t-k\} = 0$.

(D) If $\tau \leq \min(t, T-N)$, $\tau \geq \max(0, t-N)$ *and expect the cases above the* $f(\tau)$ *exists and*

$$f(\tau) = f_A(\tau) + f_B(\tau)$$

where

$$f_A(\tau) = \frac{1}{T^N} \sum_{\forall a,b,c,d} \binom{N}{a+b} \binom{N-a-b}{c+d} \binom{N-a-b-c-d}{N_1^A} \times \\ r(a+b)\ r(c+d)\ p(L_1^A, N_1^A)\ p(L_2^A, N_2^A)\ \{t-\tau\}$$

$$f_B(\tau) = \frac{1}{T^N} \sum_{\forall a,b,c,d} \binom{N}{a+b} \binom{N-a-b}{c+d} \binom{N-a-b-c-d}{N_1^B} \times \\ r(a+b)\ r(c+d)\ p(L_1^B, N_1^B)\ p(L_2^B, N_2^B)\ (1 - \{t-\tau\})$$

and

$N_1^A = \lfloor t-\tau \rfloor - b - c$, $\quad N_1^B = \lfloor t-\tau \rfloor - b - c - 1$,

$N_2^A = N - N_1^A - a - b - c - d$, $\quad N_2^B = N - N_1^B - a - b - c - d$,

$L_1^A = t - b - c - \{t-\tau\}$, $\quad L_1^B = t - b - c - \{t-\tau\}$,

$L_2^A = T - t - a - d + \{t-\tau\}$, $\quad L_2^B = T - t - a - d + \{t-\tau\}$,

where $\{x\}$ *denotes the remainder of* x, *i.e.* $\{x\} = x - \lfloor x \rfloor$.

4 Asymptotic behavior of the cumulative idle time distribution

First, we study what the relation is between a discrete model and a continuous model if these models have the same parameters. In other words, the question is how the cumulative idle time distributions differ from each others in these models. Next, we examine how the distribution of the idle time duration behaves as the size of the model is increased, or more specifically, the limiting distribution of the cumulative idle time will be determined. The size of a model is measured by the number of sources (N), and T/N and t/N remain constant throughout the determination of limiting distribution.

Consider a discrete and a continuous model with the same parameters (N, T, t). The following lemma states that the difference between the two cumulative idle time durations is always less or equal than one time unit.

Lemma 3 *Denote by* $Idle_t^d$ *the cumulative idle time duration random variable of a discrete model with parameters* (N, T, t) *and denote by* $Idle_t^c$ *the same variable of a continuous model with the same parameters. Then*

$$|Idle_t^d - Idle_t^c| \leq 1 \qquad \forall t \in [0, T].$$

The proof is given in the Appendix.

The following theorem states that the limiting distribution of the cumulative idle time duration has normal distribution.

Theorem 3 *The distribution of the cumulative idle time duration is a normal distribution with mean $t(T-N)/T$ and variance $tN(T-t)/T^2$ if T/N and t/N remain constant when $N \to \infty$.*

The proof is given in the Appendix.

5 Approximation of the cumulative idle time distribution

This section proposes approximations which enable a fast and accurate calculation of the cumulative idle time distribution. In general, the approximations are based on the Theorem 3. In case of discrete models, the cumulative idle time distribution is approximated by a truncated and discretized normal distribution. In case of continuous models, the cumulative idle time distribution is approximated in two steps: first we determine the discrete part of the distribution using Theorem 2, and then we approximate the continuous part by a truncated normal distribution. The proposed approximations are as follows:

Approximation 1 *Consider the discrete models. The cumulative idle time distribution, $F_d(x) = \Pr\{Idle_t \leq x\}$, can be approximated by the following distribution:*

$$\widetilde{F}_d(x) = \begin{cases} 0, & x < 0 \\ 1, & x > t \text{ or } x > T - N \\ \Phi(\lfloor x \rfloor + 0.5;\ \mu, \sigma^2), & \text{otherwise.} \end{cases} \tag{9}$$

where $\mu = t(T-N)/T$, $\sigma^2 = Nt(T-t)/T^2$ and $\Phi(x;\ \mu, \sigma^2)$ denotes the normal distribution function with mean μ and variance σ^2.

Approximation 2 *Consider the continuous models. The cumulative idle time distribution, $F_c(x) = \Pr\{Idle_t \leq x\}$, can be approximated by the following distribution:*

$$\widetilde{F}_c(x) = F_A(x) + pF_B(x) \tag{10}$$

where $F_A(x)$ is the discrete part of the distribution and $F_B(x)$ is the continuous part of the distribution. Because Theorem 2 enables fast calculation of $F_A(x)$ we use the exact $F_A(x)$ in the approximation. The $F_B(x)$ is approximated by a normal distribution where the parameters of the distribution and the weight parameter (p) are set such a way that the mean value and the variance of $\widetilde{F}_c(x)$ will be $\mu = t(T-N)/T$ and $\sigma^2 = Nt(T-t)/T^2$, respectively.

Figure 6 and Figure 7 in the Appendix show the distribution of the cumulative idle time for a small model ($N = 10$) and for a moderate size model ($N = 50$). Considering discrete and a continuous models with the same parameters, the difference between the two models is smaller when the model is larger and for infinitely large models this difference is vanishing. This means that the discrete and the continuous models converge to each other.

In general, the calculation of the cumulative idle time distribution in a discrete model is easier, i.e. requires fewer computations than the analysis of a continuous model. However using Lemma 3 an upper and a lower bound can be constructed for the cumulative idle time distribution in a continuous model. Figure 8 and Figure 9 in the Appendix show these bounds in the example models. The examples show that the difference between the upper and lower bounds is smaller when the model is larger. This means that for large continuous models we can give tight upper and lower bounds.

To investigate the accuracy of Approximation 1 and Approximation 2 we introduce the following error measures:

$$\varepsilon_{abs} = \sup_x |\widetilde{F}(x) - F(x)| \qquad \varepsilon_2 = \int (\widetilde{F}(x) - F(x))^2 dx$$

where $F(x)$ and $\widetilde{F}(x)$ denote the exact and the approximated distribution functions, respectively. The value of ε_{abs} represents the largest difference between the distribution functions and the value of ε_2 gives the "average" difference between the distribution functions.

Figure 10 and Figure 11 in the Appendix show how the accuracy depends on the model parameters and how the accuracy of the approximation is getting better if the number of sources increase from 20 to 100. We can observe that if N/T, which is the utilization of the system, is smaller than 0.6 the approximation also provides accurate results for small number of sources ($N = 20$). Figure 12 in the Appendix show how "fast" the error of the approximation decreases as the number of sources increase. We can observe that the error measure ε_2 converges faster to zero than the absolute error. Finally, Figure 13–Figure 16 in the Appendix present the exact and the approximated distribution functions for some particular cases.

Consequently, the proposed approximations give accurate results for also moderate size of models while the evaluation of the approximation does not require significant computational effort. For smaller models where the approximation can be inaccurate, the evaluation of exact formulae does not require significant computational effort. The exact formulae and the approximations together provide a powerful tool for the analysis of an $nD/D/1$ queue.

6 Conclusion

The cumulative idle time distribution of an $nD/D/1$ model has been determined in closed form both in discrete time (slotted) models and in continuous time models. The asymptotic behavior of the cumulative idle time distribution has been investigated and a normal distribution has been found as the limit of the cumulative idle time distribution. Based on this convergence, we have also proposed approximation formulae for large models. The exact formulae and the approximations together provide a powerful tool for analysis the cumulative idle time distribution of an $nD/D/1$ queue.

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A Lemma

Lemma 4 *Consider the events $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_4$ defined in Section 3. The following identity is valid:*

$$\Pr\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 | \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} = \Pr\{\mathcal{B}_1|\mathcal{A}_1, \mathcal{B}_4\} \Pr\{\mathcal{B}_2|\mathcal{A}_2, \mathcal{B}_1\} \Pr\{\mathcal{B}_3|\mathcal{A}_3, \mathcal{B}_2\} \Pr\{\mathcal{B}_4|\mathcal{A}_4, \mathcal{B}_3\}.$$

Proof:

We take into account arrival sequences where the event $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4$ occurs, i.e. $x_1 = a+b$ packets arrived in the interval $I_1 = [T-a-1, \ldots, T+b]$ and $x_2 = c+d$ packets arrived in the interval $I_2 = [t-c+1, \ldots, t+d-1]$ and $x_3 = t-k-b-c$ packets arrived in the interval $I_3 = [b+2, \ldots, t-c-1]$ and $x_4 = N-a-d-t+k$ packets arrived in the interval $I_4 = [t+d+2, \ldots, T-a-1]$.

The event $\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \cap \mathcal{B}_4$ means that the system is empty between the intervals I_1, I_2, I_3 and I_4, i.e. slot $b+1$ is empty and slot $t-c$ is empty and slot $t+d+1$ is empty and slot $T-a$ is empty. Assuming that the event $\mathcal{A}$ occurs, the event $\mathcal{B}$ means the following: the x_1 packets arrived in the interval I_1 are served within the interval I_1 and the x_2 packets arrived in the interval I_2 are served within the interval I_2 and the x_3 packets arrived in the interval I_3 are served within the interval I_3 and the x_4 packets arrived in the interval I_4 are served within the interval I_4.

Denote by $\alpha(x,l)$ the number of possible combination of the x packet arrivals in an interval of length l. Using elementary calculation we get $\alpha(x,l) = l^x$. Denote $\beta(x,l)$ the number of possible combination of the x packet arrivals in an interval of length l such way all these packets served in the considered interval. For example $\alpha(2,2) = 4$ and $\beta(2,2) = 3$, because the following arrivals are possible: $(\tau_1 = 0, \tau_2 = 0), (\tau_1 = 0, \tau_2 = 1), (\tau_1 = 1, \tau_2 = 0), (\tau_1 = 1, \tau_2 = 1)$ and only the arrival sequence $(\tau_1 = 1, \tau_2 = 1)$ can not be served in the considered interval.

The probability of event $\mathcal{B}$ assuming that the event $\mathcal{A}$ occurs can be expressed using the function $\alpha(x,l)$ and $\beta(x,l)$ as follows:

$$\Pr\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 | \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} = \frac{\beta(x_1,l_1)}{\alpha(x_1,l_1)} \frac{\beta(x_2,l_2)}{\alpha(x_2,l_2)} \frac{\beta(x_3,l_3)}{\alpha(x_3,l_3)} \frac{\beta(x_4,l_4)}{\alpha(x_4,l_4)}$$

where l_k is the length of the interval I_k, $k = 1,2,3,4$. The $\frac{\beta(x_1,l_1)}{\alpha(x_1,l_1)}$ is equal to $\Pr\{\mathcal{B}_1|\mathcal{A}_1, \mathcal{B}_4\}$ and $\frac{\beta(x_2,l_2)}{\alpha(x_2,l_3)}$ is equal to $\Pr\{\mathcal{B}_2|\mathcal{A}_2, \mathcal{B}_1\}$ and $\frac{\beta(x_3,l_3)}{\alpha(x_3,l_3)}$ is equal to $\Pr\{\mathcal{B}_3|\mathcal{A}_3, \mathcal{B}_2\}$ and $\frac{\beta(x_4,l_4)}{\alpha(x_4,l_4)}$ is equal to $\Pr\{\mathcal{B}_4|\mathcal{A}_4, \mathcal{B}_3\}$, therefore the lemma is proved. □

B Proofs

Proof of Lemma 1 In the discrete models, the Benes formula [8] is used for the queue length distribution with $q = 1$.

$$g(L,n) = L^n \cdot \Pr(Q(L,n) = 1) = L^n \left[1 - \sum_{r=1}^{n-1} \binom{n}{r+1} \left(\frac{r}{L}\right)^{r+1} \left(1 - \frac{r}{L}\right)^{n-r-1} \frac{L-n+1}{L-r}\right], \tag{11}$$

where $Q(L,n)$ is the queue length in the periodic system with length L and with n users. The substitution of $q=1$ comes from the difference between the models. In [8] the departures take place at the beginning of slots arrivals during of slots and the queue length is considered at the end of the slots. In our model the packet arrived into a slot can be served in the same slot because the ceiling of the uniform random arrival time, i.e. the smallest integer greater than or equal to it, is taken. It means that $Q_{t-1}=1$ is needed for an idle slot at t.

The closed form equivalent of the formula (11) is

$$g(L,n) = (L-n+1)(L+1)^{n-1}.$$

□

Proof of Theorem 1 Based on the model parameters (N,T,t) the theorem considers the following cases: (A) there are at least two idle slots in $[1,\dots,t]$ as well as in $[t+1,\dots,T]$, (B) there are at least two idle slots in $[1,\dots,t]$ and exactly one in $[t+1,\dots,T]$, (C) there are exactly one idle slot in $[1,\dots,t]$ and at least two ones in $[t+1,\dots,T]$, (D) there are at least two idle slot in $[1,\dots,t]$ and there is none in $[t+1,\dots,T]$, (E) there is no idle slot in $[1,\dots,t]$ and there are at least two ones in $[t+1,\dots,T]$, (F) there is exactly one idle slot in $[1,\dots,t]$ and there is one in $[t+1,\dots,T]$, (G) there is exactly one idle slot in $[1,\dots,t]$ and there is none in $[t+1,\dots,T]$ and (H) there is no idle slot in $[1,\dots,t]$ and there is exactly one in $[t+1,\dots,T]$. Applying Lemma 1 in Eq. (6) and (7) gives the way of the calculation of $\Pr\{Idle_t = k\}$ in case (A). The remaining cases can be calculated using the same argument that was used in case (A). □

Proof of Lemma 2

$$p(L,n) = \Pr\left(t_1^* \le L-1, t_2^* \le L-2, \dots, t_n^* \le L-n\right)$$

where t_i^* denotes the sorted packet arrivals. If we use the packet arrivals instead of the sorted packet arrivals, then we obtain the following identity:

$$\begin{aligned} p(L,n) &= n!\ \Pr\left(t_1 \in [0, L-n], t_2 \in [t_1, L-n+1], \dots, t_n \in [t_{n-1}, L-1]\right) \\ &= \frac{n!}{L^n} \int_0^{L-n} \int_{t_1}^{L-n+1} \dots \int_{t_{n-1}}^{L-1} dt_n \dots dt_2 dt_1 \end{aligned}$$

where $n!$ is the number of possible order of n arrivals. Introduce the following notations:

$$I_{L,n} = \int_0^{L-n} \int_{t_1}^{L-n+1} \dots \int_{t_{n-1}}^{L-1} dt_n \dots dt_2 dt_1 \qquad K_{L,n}(t_1) = \int_{t_1}^{L-n+1} \dots \int_{t_{n-1}}^{L-1} dt_n \dots dt_2.$$

Applying these notations

$$I_{L,n} = \int_0^{L-n} K_{L,n}(t) dt.$$

The following two lemmas give closed form expressions for $K_{L,n}(t)$ and $I_{L,n}$, and $r(n)$ is the limit of $\partial p(L,n)/\partial L$ at $L=n$.

Lemma 5 *Closed-form expression for $K_{L,n}(t)$:*

$$K_{L,n}(t) = \frac{1}{(n-1)!}\,(L-t)^{n-2}\,(L-t-n+1)\,.$$

Proof

$$\begin{aligned}
K_{L,n+1}(t) &= \int_t^{L-n} K_{L,n}(y)dy = \int_t^{L-n} \frac{1}{(n-1)!}\,\underbrace{(L-y)^{n-2}}_{f'(y)}\,\underbrace{(L-y-n+1)}_{g(y)}\,dy \overset{\text{partial}}{\underset{\text{integration}}{=}} \\
&= \frac{1}{(n-1)!}\left[\frac{-\,(L-y)^{n-1}}{n-1}\,(L-y-n+1) - \int \frac{(L-y)^{n-1}}{n-1}dy\right]_{y=t}^{L-n} \\
&= \frac{1}{(n-1)!}\left[\frac{-\,(L-y)^{n-1}}{n-1}\,(L-y-n+1) + \frac{1}{n-1}\frac{1}{n}\,(L-y)^{n}\right]_{y=t}^{L-n} \\
&= \frac{1}{n!}\,(L-t)^{n-1}\,(L-t-n)\,.
\end{aligned}$$

□

Lemma 6 *Closed-form expression for $I_{L,n}$:*

$$I_{L,n} = \frac{1}{n!}L^{n-1}(L-n).$$

Proof

$$\begin{aligned}
I_{L,n} &= \int_0^{L-n} K_{L,n}(t)dt = \int_0^{L-n} \frac{1}{(n-1)!}\,\underbrace{(L-t)^{n-2}}_{f'(t)}\,\underbrace{(L-t-n+1)}_{g(t)}\,dt \overset{\text{pantial}}{\underset{\text{integration}}{=}} \\
&= \frac{1}{(n-1)!}\left[\frac{-(L-t)^{n-1}}{n-1}(L-t-n+1) - \int \frac{(L-t)^{n-1}}{n-1}dt\right]_{t=0}^{L-n} \\
&= \frac{1}{(n-1)!}\left[\frac{-(L-t)^{n-1}}{n-1}(L-t-n+1) + \frac{1}{n-1}\frac{(L-t)^{n}}{n}\right]_{t=0}^{L-n} \\
&= \frac{1}{n!}L^{n-1}(L-n).
\end{aligned}$$

□

Proof of Lemma 3 Take a look at the bursts of the two systems. Some packets (arriving at $\tau_1, \ldots, \tau_m$) are served one after another in a burst in the continuous model if and only if $\forall i = 2, \ldots, m\ \tau_i \leq \tau_1 + i - 1$ where modulo-T arithmetic is used. Taking the ceil of the both sides of the inequality we get $\lceil \tau_i \rceil \leq \lceil \tau_1 \rceil + i - 1$. This means that the packages are served together in the same burst in the discrete model, too, i.e. the ”discretization” keeps the packets together in a burst.

The discretization can fuse some bursts together. We show that the starts of bursts always shift by less than one time unit. Look at one of the idle slots in the discrete model. The arrival of the next packet served after this time shifted by less than one time unit i.e. the burst starting with this packet shifted less than one time unit, too. It means that the first packet of the next burst is shifted less than one time unit by the previous burst. It can be predicated that the next burst is shifted by less than one time unit, too, because the burst ends at an integer time where the service of the next packet can be started. This is true for all bursts recursively.

We use again the fact that the first packet of a burst is shifted by less than one time unit. It means that the departure times of the packets in the same burst are shifted the same (the whole burst stays together).

The continuous bursts inside $(0, t)$ stay inside the interval but there is at most two bursts which are at the limits of the interval. These bursts are moved ahead by discretization less than one time unit. This fact gives the lemma. □

Proof of Theorem 3 Consider the continuous models. Observe that the distribution of the cumulative idle time duration can be represented using the $Arr(0, t)$ arrival function and the queue length random variables in 0 and t (denoted by Q_0 and Q_t respectively).

$$Idle_t = t - Arr(0, t) - Q_0 + Q_t.$$

The arrivals have binomial distribution

$$Arr(0, t) \stackrel{d}{=} Binom\left(N, t/T\right).$$

Using the Moivre-Laplace Theorem

$$\Pr\left\{Arr(0, t) \leq x\right\} \to \Phi\left(x, Nt/T, Nt(T-t)/T^2\right)$$

where $\Phi(x, m, \sigma^2)$ denotes the normal distribution function. Standardizing the equation we get

$$\frac{t(T-N)/T - Idle_t}{\sqrt{Nt(T-t)/T^2}} = \frac{Arr(0,t) - Nt/T}{\sqrt{Nt(T-t)/T^2}} + \frac{Q_0}{\sqrt{Nt(T-t)/T^2}} - \frac{Q_t}{\sqrt{Nt(T-t)/T^2}}.$$

The first expression in the right side converges to the standard normal distribution. The second and the last expression is estimated using the fact that the queue length variables (Q_τ) are less than or equal to the length of the burst at time τ denoted by Z_τ. Using [9] (see pages 101 − 103).

$$E(Q_\tau) \leq E(Z_\tau) = \frac{T}{T - N + 1} \to \frac{1}{1 - N/T} = const.$$

and therefore $\frac{Q_0}{\sqrt{Nt(T-t)/T^2}}$ and $\frac{Q_t}{\sqrt{Nt(T-t)/T^2}}$ converge to the deterministic 0 random variable. It means that $Idle_t$ converges to a normal distribution with mean $t(T-N)/T$ and variance $Nt(T-t)/T^2$.

Consider the discrete models. The limiting distribution of these models is the same according to Lemma 3 □

C Figures

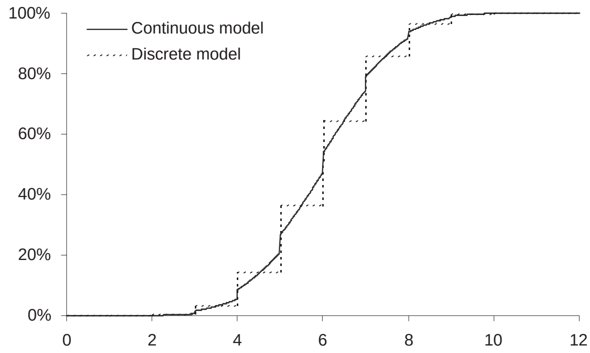

Figure 6: Distribution function of $Idle_t$; model parameters: $N = 10$, $t = 12$ and $T = 20$

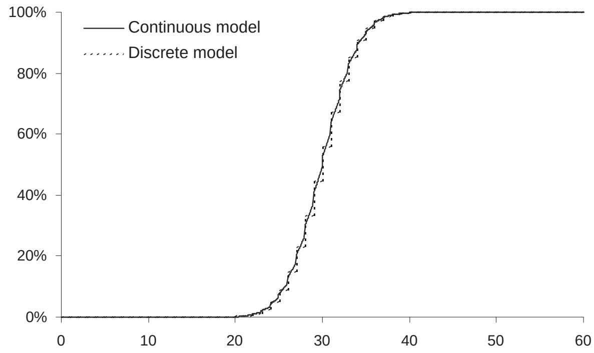

Figure 7: Distribution function of $Idle_t$; model parameters: $N = 50$, $t = 60$ and $T = 100$

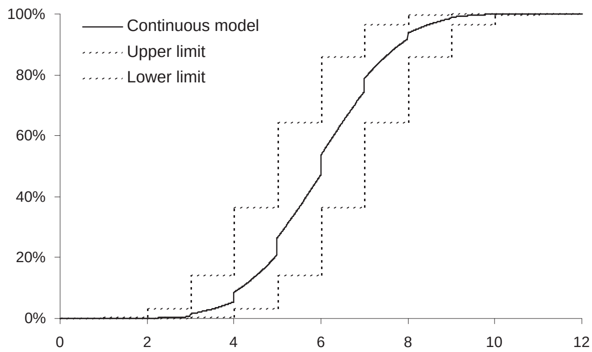

Figure 8: Bounds on the distribution of $Idle_t$ in a continuous model; model parameters: $N = 10$, $t = 12$ and $T = 20$

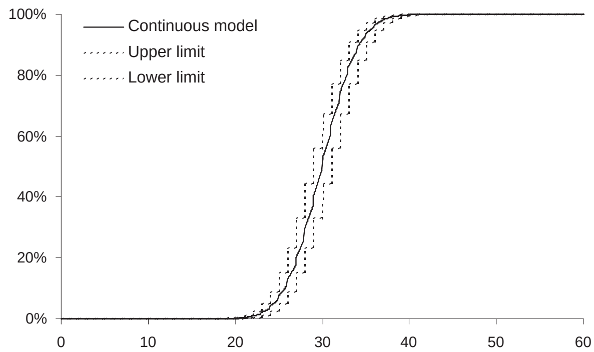

Figure 9: Bounds on the distribution of $Idle_t$ in a continuous model; model parameters: $N = 50$, $t = 60$ and $T = 100$

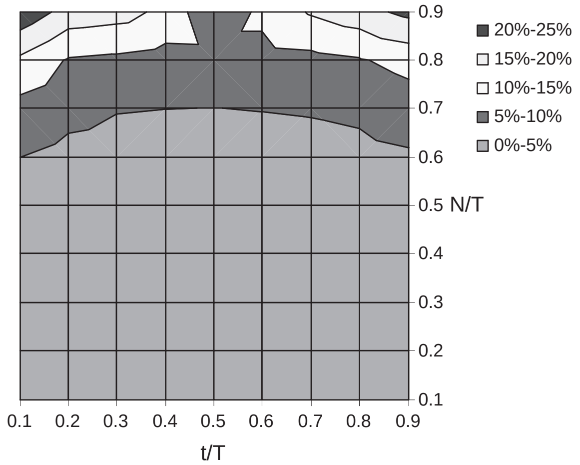

Figure 10: Absolute error of the approximation, e_{abs}; discrete model and $N = 20$

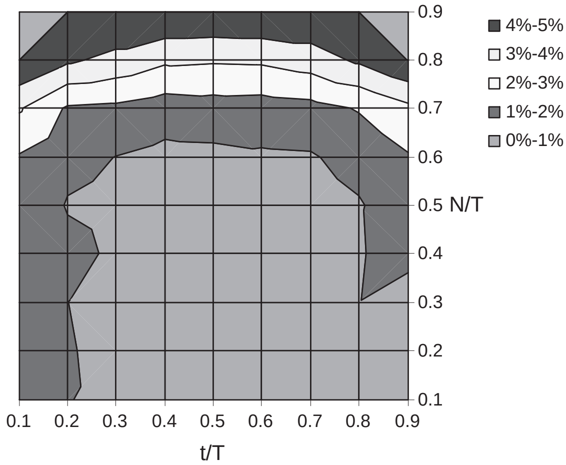

Figure 11: Absolute error of the approximation, e_{abs}; discrete model and $N = 100$

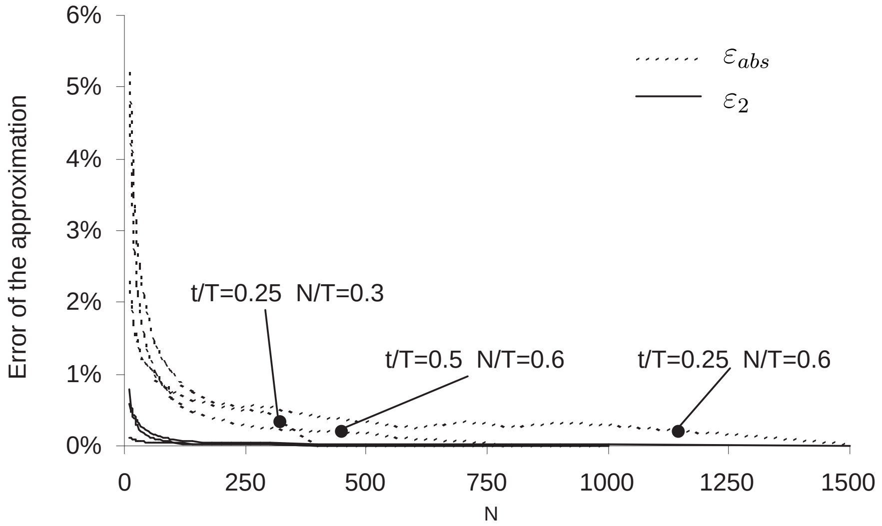

Figure 12: Error of the approximation vs. model size; discrete models

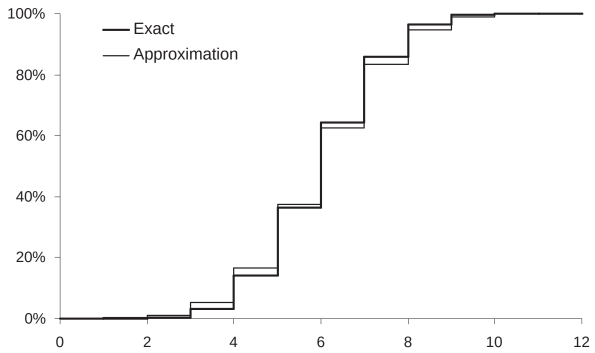

Figure 13: Approximation of the distribution function of $Idle_t$ in a discrete model; model parameters: $N = 10$, $t = 12$ and $T = 20$

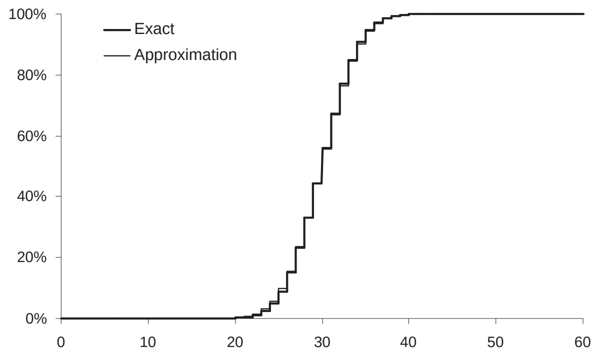

Figure 14: Approximation of the distribution function of $Idle_t$ in a discrete model; model parameters: $N = 50$, $t = 60$ and $T = 100$

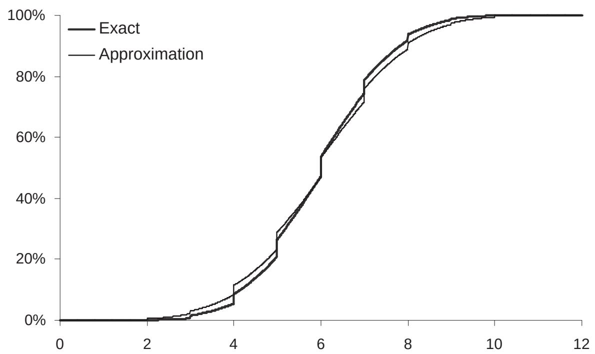

Figure 15: Approximation of the distribution function of $Idle_t$ in a continuous model; model parameters: $N = 10$, $t = 12$ and $T = 20$

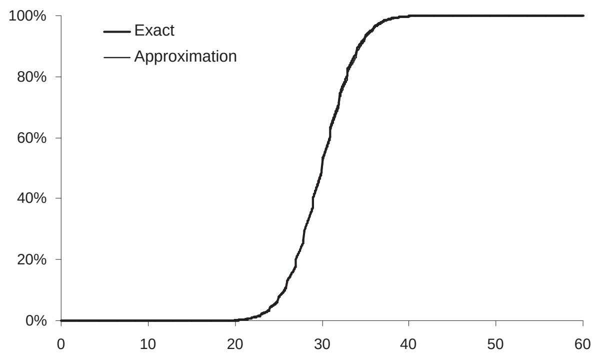

Figure 16: Approximation of the distribution function of $Idle_t$ in a continuous model; model parameters: $N = 50$, $t = 60$ and $T = 100$