A FAST FOUR-PARAMETER SINE WAVE ESTIMATION METHOD[1] (ANNUAL RESEARCH REPORT)

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I. Introduction

This work is an annual research report, based on the parts of paper [5]. The author would like to suggest reading paper [5] instead of this lecture.

Accurate estimation of sine wave parameters is an important field of signal processing. The recommended method by the IEEE standard for ADC testing [1] is the four parameter fitting algorithm, using the time domain samples. However, this method could be very time consuming for high number of samples. In this paper an alternative, frequency domain method is presented with the capability to provide the sine parameters much faster, without significant loss of precision.

II. Estimation of sine parameters

A Implementation

To speed up the fitting algorithm, the frequency domain samples are used instead of the time domain samples. To compress the information in the frequency domain, the four-term Blackman-Harris window is used which has side lobes under -91 dB [2]. The time domain expression of the window is [2]:

$$w(k) = a_0 + a_1 \cdot \cos\left(\frac{2\pi k}{N}\right) + a_2 \cdot \cos\left(\frac{4\pi k}{N}\right) + a_3 \cdot \cos\left(\frac{6\pi k}{N}\right) \qquad (1)$$

where

$$a_0 = 0.35875, a_1 = -0.48829, a_2 = 0.14128, a_3 = -0.01168. \qquad (2)$$

If we want to use the formula of a windowed sine wave in the frequency domain, first we need the expression of the nonwindowed *N* length sine (which takes leakage into account):

$$X_{sin}(k) = X_{sin}^{-}(k) + X_{sin}^{+}(k) \qquad (3)$$

$$X_{sin}^{-}(k) = e^{-j\pi(k-f)\frac{N-1}{N}} \cdot \frac{A+jB}{2} \cdot \frac{\sin\big(\pi(k-f)\big)}{\sin\left(\pi(k-f)\frac{1}{N}\right)} \qquad (4)$$

$$X_{sin}^{+}(k) = e^{-j\pi(k+f)\frac{N-1}{N}} \cdot \frac{A-jB}{2} \cdot \frac{\sin\big(\pi(k+f)\big)}{\sin\left(\pi(k+f)\frac{1}{N}\right)} \qquad (5)$$

where

$$A = R \cdot \cos(\varphi)\,, B = R \cdot \sin(\varphi) \qquad (6)$$

and the expression of the sine wave in the time domain is

$$x(t) = R \cdot cos(2\pi f_x t + \varphi) \qquad (7)$$

where R, f_x, ϕ are the amplitude, frequency and initial phase of the sine wave, respectively. In the sine fitting procedure the following formulas are used to avoid large arguments in the input of the trigonometric functions (and as a consequence of large arguments, roundoff errors):

$$f = \lfloor f \rfloor + \langle f \rangle, f_i = \lfloor f \rfloor, f_f = \langle f \rangle \qquad (8)$$

$$X_{sin}^{-} = e^{j\pi\left(f_f + \frac{k-f}{N}\right)} \cdot \frac{A+jB}{2} \cdot \frac{\sin(-f_f\pi)}{\sin\left(\frac{k-f}{N}\pi\right)} \qquad (9)$$

$$X_{sin}^{+} = e^{j\pi\left(-f_f + \frac{k+f}{N}\right)} \cdot \frac{A-jB}{2} \cdot \frac{\sin(f_f\pi)}{\sin\left(\frac{k+f}{N}\pi\right)} \qquad (10)$$

[1] This one year research report is based on the parts of [5].

where f_i and $\lfloor f \rfloor$ is the integer part and f_f and $\langle f \rangle$ is the fractional part of f, which is the number of periods the sine wave contained.

If the measured signal contains a dc component, the mathematical model has to be modified:

$$x(t) = R \cdot \cos\left(\frac{2\pi f n}{N} + \varphi\right) + C \tag{11}$$

where C is the level of the dc component and n=0,1,…,N-1. This means that an X_{dc} part has to be added to the X_{sin} part in the frequency domain to represent the signal correctly:

$$X(k) = DFT\left\{R \cdot \cos\left(\frac{2\pi f n}{N} + \varphi\right) + C\right\} \tag{12}$$

$$X(k) = X_{dc}(k) + X_{sin}(k) \tag{13}$$

where

$$X_{dc}(k) = C \cdot N, if\ k = 0 \pm N \pm 2 \cdot N \pm 3 \cdot N \pm \cdots \tag{14}$$

and

$$X_{dc}(k) = 0\ otherwise \tag{15}$$

If the signal is windowed with samples of the four term Blackman-Harris window, the multiplication in the time domain becomes a convolution in the frequency domain:

$$X_{BH}(k) = DFT\{x(t) \cdot w(t)\} = \underline{a}^T \cdot \underline{y} \tag{16}$$

$$\underline{a} = \left[\frac{a_3}{2}\ \frac{a_2}{2}\ \frac{a_1}{2}\ a_0\ \frac{a_1}{2}\ \frac{a_2}{2}\ \frac{a_3}{2}\right]^T \tag{17}$$

$$\underline{y}^T = [X(k-3), X(k-2), X(k-1), X(k), X(k+1), X(k+2), X(k+3)]. \tag{18}$$

Above expression is linear in the amplitude, initial phase (more precisely in the terms A and B which are related to R and ϕ), and the dc component. However, it is nonlinear in the frequency. This means that during the fitting an iterative numerical method has to be used. For this purpose, the Newton-Gauss method was chosen because it has two advantageous properties [3]:

- The second order derivatives are neglected in the Hesse-matrix,
- The positive semi-definiteness of the Hesse-matrix is ensured.

The Newton-Gauss method was performed on the following cost-function:

$$K = \frac{1}{2}\underline{e}^T\underline{e} \tag{19}$$

where e is the vector of residuals. The definition of the Newton-Gauss step is

$$\Delta\underline{p} = (\boldsymbol{J}^T \boldsymbol{C}^{-1}\boldsymbol{J})\boldsymbol{J}^T\boldsymbol{\Sigma}^{-1}\underline{e} \tag{20}$$

where $\underline{p}$, $\boldsymbol{J}$, $\boldsymbol{\Sigma}$ are the vector of parameters (A, B, C, f), the so-called Jacobian matrix and the covariance matrix, respectively. The Jacobian matrix has a size N×4 and consists of the derivatives with respect to the parameters A, B, C and f.

$$\boldsymbol{J} = \left[\frac{\partial X_{BH}}{\partial A}, \frac{\partial X_{BH}}{\partial B}, \frac{\partial X_{BH}}{\partial C}, \frac{\partial X_{BH}}{\partial f}\right] \tag{21}$$

This is calculated using the derivatives of the expression

$$X(k) = DFT\left\{R \cdot \cos\left(\frac{2\pi f t}{N} + \varphi\right) + C\right\} \tag{22}$$

$$\frac{\partial X(k)}{\partial A} = \frac{\partial X_{sin}^-(k)}{\partial A} + \frac{\partial X_{sin}^+(k)}{\partial A} \tag{23}$$

$$\frac{\partial X(k)}{\partial B} = \frac{\partial X_{sin}^-(k)}{\partial B} + \frac{\partial X_{sin}^+(k)}{\partial B} \tag{24}$$

$$\frac{\partial X(k)}{\partial f} = \frac{\partial X_{sin}^-(k)}{\partial f} + \frac{\partial X_{sin}^+(k)}{\partial f} \tag{25}$$

$$\frac{\partial X(k)}{\partial C} = \frac{\partial X_{dc}(k)}{\partial C} \tag{26}$$

$$\frac{\partial X_{sin}^-(k)}{\partial A} = e^{j\pi\left(f_f + \frac{k-f}{N}\right)} \cdot \frac{1}{2} \cdot \frac{\sin(-f_f\pi)}{\sin\left(\frac{k-f}{N}\pi\right)} \tag{27}$$

$$\frac{\partial X_{sin}^+(k)}{\partial A} = e^{j\pi\left(-f_f + \frac{k+f}{N}\right)} \cdot \frac{1}{2} \cdot \frac{\sin(f_f\pi)}{\sin\left(\frac{k+f}{N}\pi\right)} \tag{28}$$

$$\frac{\partial X_{sin}^{-}(k)}{\partial B} = e^{j\pi\left(f_f + \frac{k-f}{N}\right)} \cdot \frac{j}{2} \cdot \frac{\sin(-f_f\pi)}{\sin\left(\frac{k-f}{N}\pi\right)} \tag{29}$$

$$\frac{\partial X_{sin}^{+}(k)}{\partial B} = e^{j\pi\left(-f_f + \frac{k+f}{N}\right)} \cdot \frac{-j}{2} \cdot \frac{\sin(f_f\pi)}{\sin\left(\frac{k+f}{N}\pi\right)} \tag{30}$$

$$\frac{\partial X_{sine}^{-}(k)}{\partial f} = j\pi\frac{N-1}{N} \cdot X_{sin}^{-}(k) + e^{j\pi\left(f_f + \frac{k-f}{N}\right)} \cdot \frac{A+jB}{2} \cdot \ldots$$
$$\cdot \frac{-\pi\cdot\cos(f_f\pi)\cdot\sin\left(\frac{k-f}{N}\pi\right) + \frac{\pi}{N}\cos\left(\frac{k-f}{N}\pi\right).\sin(-f_f\pi)}{\frac{1-\cos\left(\frac{2\pi(k-f)}{N}\right)}{2}} \tag{31}$$

$$\frac{\partial X_{sin}^{+}(k)}{\partial f} = -j\pi\frac{N-1}{N} \cdot X_{sin}^{+}(k) + e^{j\pi\left(-f_f + \frac{k+f}{N}\right)} \cdot \frac{A-jB}{2} \cdot \ldots$$
$$\cdot \frac{\pi\cdot\cos(f_f\pi)\cdot\sin\left(\frac{k+f}{N}\pi\right) - \frac{\pi}{N}\cos\left(\frac{k+f}{N}\pi\right).\sin(f_f\pi)}{\frac{1-\cos\left(\frac{2\pi(k+f)}{N}\right)}{2}} \tag{32}$$

$$\frac{\partial X_{dc}(k)}{\partial C} = N \text{ for } k = 0 \pm N \pm 2\cdot N \pm 3\cdot N \pm \cdots \tag{33}$$

$$\frac{\partial X_{dc}(k)}{\partial C} = 0 \text{ otherwise} \tag{34}$$

Using the above expressions the derivatives of X_{BH} can be calculated as

$$\frac{\partial X_{BH}(k)}{\partial A} = \underline{a}^T \cdot \frac{\partial \underline{y}^T}{\partial A}, \frac{\partial X_{BH}(k)}{\partial B} = \underline{a}^T \cdot \frac{\partial \underline{y}^T}{\partial B} \tag{35}$$

$$\frac{\partial X_{BH}(k)}{\partial f} = \underline{a}^T \cdot \frac{\partial \underline{y}^T}{\partial f}, \frac{\partial X_{BH}(k)}{\partial C} = \underline{a}^T \cdot \frac{\partial \underline{y}^T}{\partial C} \tag{36}$$

Since the model (16) is nonlinear in parameter f, it is important to estimate accurately its value to ensure the convergence of the Newton-Gauss method. For this purpose, interpolated FFT was used where the initial guess for the frequency was calculated using the analytic solution for the rectangular windows (for more details see [4]).

Using the above described method, the fit can be done using only the points of the Blackman-Harris windows around the sine frequency and the dc frequency, because the significant part of the information about the sine is compressed into these points. During the fitting algorithm 35 points are used (12 points represent the dc part and 23 the sine). This way only the time consumption of the FFT depends on the number of samples, the computational burden of the fit does not. This is an important difference between this and the original fitting algorithm which uses the time domain samples, and results in a significant increase of speed compared to the original method.

B Properties of the estimator

Since the distribution of the noise in the time domain is unknown (a measured sine may contain Gaussian noise, harmonic distortion, quantization noise, etc.), as an approximation we can use the Central Limit Theorem, and assume that the noise in the frequency domain is more or less Gaussian since on every frequency it is the linear combination of a noise samples which has unknown distribution. Then the properties of the least-squares estimator will be approximately the following:

- the estimator is asymptotically normally distributed,
- the estimator is asymptotically unbiased (the mean value of the estimation error is 0),
- the covariance matrix of the estimator asymptotically reaches the Cramer-Rao lower bound.

III. Comparison of the new and the original algorithm

A Precision and accuracy

In this section the estimation errors of the algorithm are compared from the point of view of the mean value and variance. In these simulations a sine disturbed by Gaussian noise (zero mean, *LSB*/3 standard deviation) was quantized with a nonideal 10 bit ADC. The INL and DNL characteristics of the quantizer can be seen in Fig. 1. 10000 records were created where the amplitude, frequency, dc component and initial phase were random variables. The parameters of each record were estimated with both sine fitting procedures, and then the statistical properties were compared. Table I. shows the results where μ_t is the mean value of the original estimator (which uses time domain samples), μ_f is the mean value of the new estimator (which uses frequency domain samples), σ_t is the standard deviation of the original estimator and σ_f, is the standard deviation of the new estimator.

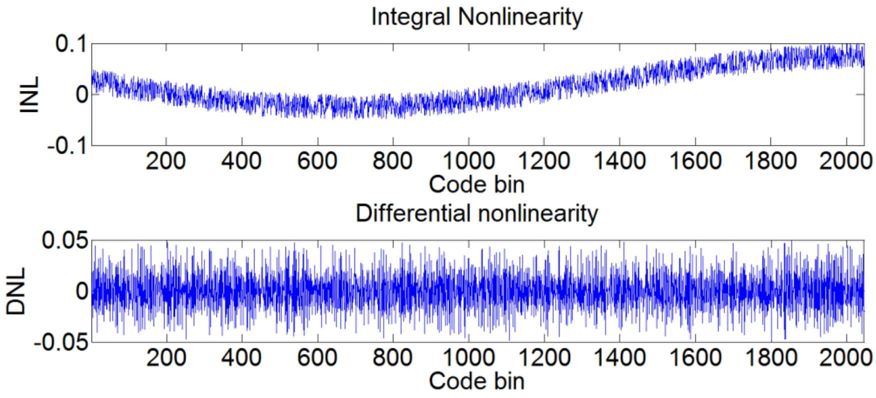

Figure 1: INL and DNL of 10 bit nonideal quantizer

$\hat{A}$		$\hat{B}$		$\hat{C}$		$\hat{f}$	
μ_t/μ_f	σ_t/σ_f	μ_t/μ_f	σ_t/σ_f	μ_t/μ_f	σ_t/σ_f	μ_t/μ_f	σ_t/σ_f
0,788	0,996	0,914	0,988	1,000	0.993	0,377	0,949

Table I: Comparison of statistical properties for each parameter

B Comparison of speed

Table II. shows the different computational times of the algorithms for different record lengths where t_t is the time requirement of the original method and t_f is of the new method (both in seconds). The time consumption of the FFT in the new method is included in these results.

2^{14} samples		2^{16} samples		2^{18} samples		2^{20} samples		2^{22} samples	
t_t	t_f	t_t	t_f	t_t	t_f	t_t	t_f	t_t	t_f
0.11	0.09	0.45	0.10	1.98	0.15	7.73	0.31	14.95	0.64

Table II: Comparison of computational burden

IV. Conclusion

In this lecture an alternative fitting method was presented. It was shown that it is able to provide the sine parameters without significant loss in the precision but much faster than the original algorithm.

References

[1] "IEEE Standard for Terminology and Test Methods for Analog-to-Digital Converters," IEEE Std 1241-2010 (Revision of IEEE Std 1241-2000) , vol., no., pp.1-139, Jan. 14 2011.

[2] Harris, F.J.; , "On the use of windows for harmonic analysis with the discrete Fourier transform," Proceedings of the IEEE , vol.66, no.1, pp. 51- 83, Jan. 1978.

[3] Adriaan van den Bos (2007), “Parameter Estimation for Scientists and Engineers”, Wiley-Interscience.

[4] Bilau, T. Z.; Megyeri, T.; Sárhegyi, A.; Márkus, J; Kollár, I.; , “Four-parameter fitting of sine wave testing result: iteration and convergence,” Computer Standards and Interfaces, vol. 26, pp. 51-56, 2004.

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