Minimally $1/2$-tough series-parallel graphs

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Abstract: Let t be a positive real number. A graph is called *t-tough* if the removal of any vertex set S that disconnects the graph leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough if the toughness of the graph is t, and the deletion of any edge from the graph decreases the toughness. Series–parallel graphs are graphs with two distinguished vertices called terminals, formed recursively by two simple composition operations, series and parallel joins. They can be used to model series and parallel electric circuits.

We characterize the minimally t-tough series-parallel graphs for $t = 1/2$. This will show that most of the series-parallel graphs with toughness $1/2$ are not minimally $1/2$-tough. It is interesting that, on the other hand, if $1/2 < t < 1$, then most of the t-tough series-parallel graphs are minimally t-tough.

Keywords: toughness, minimally tough graphs, series–parallel graph

1 Introduction

All graphs considered in this paper are undirected but may contain loops and parallel edges. Let $c(G)$ denote the number of components, and $\kappa(G)$ the connectivity number of the graph G. For a connected graph G, a vertex set $S \subseteq V(G)$ is called a *cutset* if $c(G-S) > 1$. The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. In the present paper, we only use this notation when $E_1 \cap E_2 = \emptyset$ and $|V_1 \cap V_2| \leq 2$. For and edge uv where $u, v \in V_1$ the union of G_1 and uv is $G_1 \cup uv = (V_1, E_1 \cup uv)$.

The notion of toughness was introduced by Chvátal in [2].

Definition 1 *Let t be a real number. A graph G is called* t-tough *if $|S| \geq t \cdot c(G-S)$ for any set $S \subseteq V(G)$ with $c(G-S) > 1$. The* toughness *of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$. We say that a cutset $S \subseteq V(G)$ is a* tough set *if $|S| = \tau(G) \cdot c(G-S)$.*

Note that a graph is disconnected if and only if its toughness is 0.

This notion is one of the widely used measures in network reliability, it is mentioned in various applications for different types of networks. So, it is a natural question to investigate the following concept of minimally t-tough graphs, introduced by Broersma in [9].

Definition 2 *Let t be a real number. A graph G is said to be* minimally t-tough *if $\tau(G) = t$ and $\tau(G-e) < t$ for every $e \in E(G)$.*

It follows directly from the definition that every t-tough noncomplete graph is $2t$-connected, implying $\kappa(G) \geq 2\tau(G)$ for noncomplete graphs. Therefore, the minimum degree of any t-tough noncomplete graph is at least $\lceil 2t \rceil$.

The following conjecture is motivated by a theorem of Mader [5], which states that every minimally k-connected graph has a vertex of degree k.

Conjecture 3 (Kriesell [4]) *Every minimally 1-tough graph has a vertex of degree 2.*

This conjecture can be naturally generalized.

Conjecture 4 (Generalized Kriesell's Conjecture [7]) *Every minimally t-tough graph has a vertex of degree $\lceil 2t \rceil$.*

Recently, Conjecture 4 was disproved. In [11] Zheng and Sun constructed minimally $\left(1 + \frac{1}{2k-1}\right)$-tough 4-regular graphs for every $k \geq 2$. Since $\left\lceil 2\left(1 + \frac{1}{2k-1}\right)\right\rceil < 3$, the Conjecture 4 is not true for all values of t, but it still might be true for many other values of t or for all irregular graphs [11].

On the other hand, Conjecture 4 was verified for several graph classes and various values of t. In [7] Conjecture 4 was proved for minimally t-tough, split graphs when for all t, minimally t-tough, chordal graphs when $t \leq 1$, and minimally t-tough, claw-free graphs when $t \leq 1$. Also, in [3] Ma et al. showed that for minimally $\frac{3}{2}$-tough, claw-free graphs the conjecture is true.

This paper focuses on series–parallel graphs, which are important in graph theory and computer science due to their structured and hierarchical properties. This inherent organization simplifies their analysis compared to general graphs. Moreover, series–parallel graphs are particularly valuable in applications such as network design, circuit layout, and algorithm development, as their structure allows many computationally intensive algorithms to run more efficiently.

There are several ways to define series–parallel graphs. The following definition basically follows the one used in [8].

Definition 5 (Series–Parallel Graph) *A graph $G(s,t)$ is a* series–parallel graph *(sp-graph for short) with terminals s and t, if either G consists of one edge connecting s and t, or G is derived from two or more series–parallel graphs by one of the following two operations.*

- *Series join: Given k series–parallel graphs $G_1(s_1,t_1), G_2(s_2,t_2), \ldots, G_k(s_k,t_k)$, form a new graph $G(s,t)$ by identifying the vertices t_i and s_{i+1} for all $1 \leq i \leq k-1$ and setting $s = s_1$ and $t = t_k$.*
- *Parallel join: Given k series–parallel graphs $G_1(s_1,t_1), G_2(s_2,t_2), \ldots, G_k(s_k,t_k)$, construct a new graph $G(s,t)$ by identifying the vertices $s_1, s_2, \ldots, s_k$ as a single vertex s, and similarly identifying the vertices $t_1, t_2, \ldots, t_k$ as a single vertex t.*

In either operation, instead of joining k graphs in one step, we could make $k-1$ joins each time joining precisely 2 graphs.

Note that the ordering $G_1(s_1,t_1), G_2(s_2,t_2), \ldots, G_k(s_k,t_k)$ matters for the series join, while it does not matter for the parallel join.

The recursive definition of the series–parallel graph above naturally gives a rooted, leveled, vertex–ordered tree, called a *series–parallel tree* (sp-tree for short), for each series–parallel graph $G(s,t)$. To avoid confusion, we refer to the vertices of this tree as *nodes*, while the word *vertex* will always refer to a vertex of the series–parallel graph.

Each leaf in an sp-tree corresponds to an edge of $G(s,t)$, and each non-leaf node in an sp-tree corresponds to either a series or a parallel join. We call such nodes *normal, series*, or *parallel*, respectively.

The parent of a normal node can be either a series or a parallel node. The parent of a series or a parallel node could be a series or a parallel node. However, it is easy to see that the operations can always be chosen so that the parent of a series node is always a parallel node, and the parent of a parallel node is always a series node. We can also assume that the children of series and parallel nodes are ordered from left to right in the same order as the corresponding operations were performed. We denote this unique series–parallel tree of G by T_G.

2 Preliminary results

It is easy to see that the toughness of any sp-graph is at most 1, therefore there is no minimally t-tough graph with toughness more than 1. It is also easy to see that minimally t-tough graphs cannot contain loops and parallel edges.

In [6] we characterized all minimally t-tough graphs with $\frac{1}{2} < t \leq 1$.

Theorem 6 ([6]) *If G is a minimally 1-tough sp-graph, then G is a cycle.*

Before stating our other results, we first introduce some necessary definitions.

The edges e, whose parent in the sp-tree (i.e. its first join) is a parallel join, are special. Note that in this case, the siblings of e can not be edges themselves since we assume that there are no parallel edges. So, all siblings of e must be series nodes. As we will see, certain edges exhibit distinct properties, motivating the definition of two special types of edges.

Definition 7 *An edge e is a* leap-edge *if its parent is a parallel node, which is the root of the sp-tree, and it has only one sibling (which is a series node). An edge is a* jump-edge *if its parent is a parallel node that is not the root or the parent is the root, but there are at least two series node siblings (i.e. it is not a leap-edge).*

Notice that if two edges are parallel, then both of them are either leap-edges or jump-edges.

Theorem 8 ([6]) *An sp-graph is minimally t-tough with $\frac{1}{2} < t < 1$ if and only if there are no jump-edges.*

3 Main result

The main result of the present paper is the characterization of minimally $\frac{1}{2}$-tough graphs.

Next, we define a few specific *substructures* of the sp-tree. A substructure is like a subtree, but (only) the root may have other siblings in the sp-tree. A subtree is always a substructure as well. A special case is when the substructure is the whole graph.

Definition 9 *Let P_2 be the substructure that is a series join of two edges, and let R_2 denote that substructure that is a parallel join of two copies of P_2.*

Definition 10 *A* necklace graph *is a series join of edges and some R_2 subgraphs, such that the first and last subgraphs are edges. (See Figure 1.) A path is a special necklace graph.*

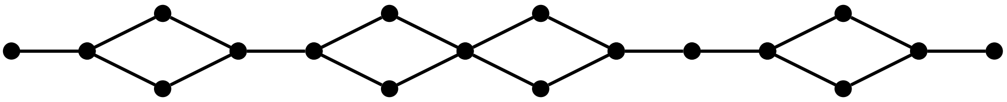

Figure 1: Necklace graph

In [6] we have proved the following theorem.

Theorem 11 ([6]) *All necklace graphs are minimally $\frac{1}{2}$-tough.*

To find the characterization, we show that it is enough to investigate sp-graphs with a simplified structure.

Definition 12 *Let $r(G)$,* the reduced graph of G, *be the graph obtained from G by repeatedly replacing an induced path of length at least 3 with a path of length 2 until there is no induced path of length at least 3.*

Lemma 13 ([6]) *G is minimally t-tough if and only if $r(G)$ is minimally t-tough.*

Now, we are ready to state the main result of the paper.

Theorem 14 *A reduced graph is minimally $\frac{1}{2}$-tough if and only if it is a necklace graph.*

Therefore, using Lemma 13 we obtain the following characterization.

Theorem 15 *G is minimally $\frac{1}{2}$-tough if and only if $r(G)$ is a necklace graph. (See Figure 2.)*

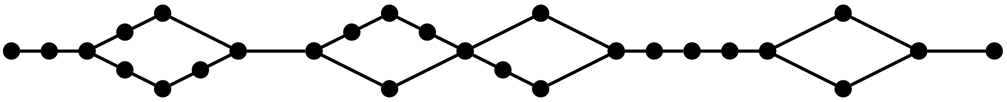

Figure 2: A minimally $\frac{1}{2}$-tough graph

This clearly implies that the Generalized Kriesell conjecture holds for minimally $\frac{1}{2}$-tough graphs.

The proof of Theorem 14 is fairly complicated and quite long. We just mention two interesting Lemmas that are crucial in the argument.

A major difficulty in determining the toughness for an arbitrary graph is that essentially any subset of the vertices might be a unique tough set. However, for sp-graphs this is not the case, we can rule out many subsets.

The main tool in this is the following lemma, which gives a useful inequality about the *mediant* of two fractions. The *mediant* of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$. Lemma 16 appeared a long time ago in [1], and an easy proof can be found in [10].

Lemma 16 (Mediant Inequality) *Let a, b, c, d be positive integers. If $\frac{a}{b} \leq \frac{c}{d}$ then*

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}.$$

Moreover, equalities hold if and only if $\frac{a}{b} = \frac{c}{d}$.

The other important tool is the following Lemma. We only use it for $k = 2$, but it might be interesting in this more general form. Note that the claim is not necessarily true if the toughness is not in the form $\frac{1}{k}$.

Lemma 17 *If $\tau(G) = \frac{1}{k}$, for some integer $k \geq 2$. Let e be an edge so that $\tau(G - e) < \tau(G)$ and S a tough set in $G - e$. Now S is tough set in G as well.*

4 Open problems

The most evident open problem is whether the Generalized Kriesell conjecture holds for minimally t-tough graphs when $t < \frac{1}{2}$. Additionally, obtaining a characterization of such graphs would be valuable.

Our methods may be applicable when $t = \frac{1}{k}$ for some integer $k \geq 3$, as Lemma 17 can be used in these cases. However, we do not yet see how to characterize these graphs.

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