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Doctoral School of Business and Management
Budapest University of Technology and Economics

Marcell Dülk

# Methodological improvements in financial analyses

Doctoral thesis

Advisor: György Andor, PhD

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# I. INTRODUCTION

A fundamental question in finance is how much an asset is worth. By “asset” a claim for funds is meant, such as a stock of a firm, which entitles its holder to monetary earnings resulting from operation of the firm. Over the past century, the notion of “present value” has been settled as the principal measure of worth, which is defined as the discounted sum of all future funds expected from the asset as of the time of valuation, including, of course, any funds available instantly upon acquisition of the asset. Instead of the term “fund,” the term “cash flow” is mostly used (and will be used hereinafter also), referring to the fundamentals of firm valuation. Discounting is the procedure which derives the value of a cash flow expected at some future point in time as of the time of valuation, i.e., at present. Technically, this is performed by using a discount rate accounting for the timing and the risk of the cash flow. Since value is derived by discounting future expected cash flows to present, the approach and the rules of such analyses are referred to as the discounted cash flow (DCF) framework.

Two main fields of finance are investments and corporate finance, and present value is a key concept in both. Contemporary corporate finance is concerned with maximization of stockholders’ wealth. Thus, managers of firms should acquire “valuable” assets, that is, assets which increase the value of stocks. The related decision-making process is called capital budgeting. Increasing the value of stocks requires identification of assets that are worth more than the price they can currently be purchased for, and it is the present value which is used to determine the worth of the assets. The difference between the present value and the current price is referred to as the net present value. Generally speaking, the field of investments is concerned with the behavior of asset prices in capital markets, which prices are obviously connected to the values of capital assets. And again, asset values are determined by their present values. Since our thesis is dedicated to aspects of present value computation, it is not “limited to” corporate finance; nevertheless, capital budgeting is where our findings could principally be applied, as the present value is rather merely an input or a reference point to the field of investments.

The typical textbook computation of the present value requires the following main inputs: 1) definition of the length of a time period (which is mostly a year) referred to as the “interest period;” 2) determination of the expected economic life of the asset under valuation, expressed in the same time unit as the interest period; 3) estimation of the aggregate expected cash flows of the asset for each of the periods during the expected life, assuming that they occur at the ends of the periods; 4) estimation of the discount rate for the interest period, which is identical across the periods. In this textbook approach, present value calculation is technically simple, however, simplicity, as usually, comes at a cost. This textbook approach may contain the following errors, for example: estimation errors regarding the expected cash flows and the discount rate, negligence of the fact that cash flows need not occur exclusively at the ends of the periods, and negligence of the uncertainty of the asset's economic life, provided that, of course, the asset's economic life is stochastic. Elimination of these errors inevitably calls for a toolbox of more complicated mathematical methods, making the computation of present value technically more involved. Therefore, it is particularly worth to explore the characteristics of these errors and to assess whether or not, or under what circumstances, these errors are so severe that the sacrifice of some simplicity is desirable.

Our research addresses three of the above-mentioned sources of error. First, we relax the assumption that all cash flows within an interest period occur at the end of the period. Second, we account for uncertainty related to the economic life of assets. Third, we strive to provide a more precise estimate of the discount rate applicable to energy efficiency projects related to space heating and cooling. We note that we tackle these three problem areas separately; a thorough study of the aggregate effects of errors from various sources is left for future research.

As mentioned above, textbooks typically suggest calculating the present value of an asset by aggregating all cash flows within a given interest period to the end of that period, and then discounting them to the present. This is called "end-of-period" convention. Cash flows of an asset, in reality, may obviously occur throughout the interest period, so it is essential to assess the possible errors an analyst may make by approximating the present value by the end-of-period convention. In addition, our research was motivated by finding a relatively simple, easy-to-use formula which the end-of-period present value could be adjusted with to mitigate such errors. We developed a novel approximation formula called "harmonic" convention, which is based on the harmonic mean of the beginning-of-period and the end-of-period present values. Analogous to the end-of-period convention, the beginning-of-period convention aggregates all cash flows within a given interest period to the beginning of that period. We use the relative error (in fact, its absolute value) as the measure of error, defined as the approximative present value divided by the correct, actual present value of the asset, less one. We show that the harmonic convention minimizes the maximum possible error for the general case when the underlying cash flow pattern of the asset is not considered. Accordingly, our first thesis is the following:

**Thesis 1 (Andor and Dülk, 2013): The harmonic mean of the end-of-period and the beginning-of-period present values minimizes the maximum possible error when approximating the present value of intraperiod cash flows. This harmonic convention is defined mathematically as**

$$P_H = P_E \frac{1+i}{1+i/2}$$

where $P_H$ and $P_E$ are the harmonic and the end-of-period present values, respectively; and $i$ is the discrete discount rate for the given interest period.

We extend the evaluation of errors to specific cash flow pattern assumptions, and compare the performance of the harmonic convention with alternative conventions given in the relevant literature. These alternative conventions can all be stated as adjustments to the end-of-period convention, and are the beginning-of-period and mid-period conventions, in particular. The mid-period convention, as its name suggests, aggregates all cash flows within a given interest period exactly to the mid-point of the period. The cash flow pattern assumptions we examine apply to a single period and are defined by a single parameter; in particular, cash flows are described by triangular, PERT, and intermittently uniform (termed "semester estimate") distributions. These distributions provide a reasonable coverage of realistic cases. We find that the harmonic convention, even though not being the best choice for all the scenarios, has acceptable errors in the majority of cases, and thus can be a good tool for practitioners. Thus, our second thesis is:

**Thesis 2 (Andor and Dülk, 2013): For realistic cash flow patterns the approximation error of the harmonic convention is usually small (less than 5%) and smaller than that of other conventions, especially when the majority of cash flows fall into the second half of the period.**

Our second research endeavor is the thorough mathematical analysis of present value calculation when the cessation point of an asset's cash flow pattern is uncertain. Textbooks typically suggest computing present value according only to the expected life of an asset, without further consideration of the probability distribution of the asset life. This is, in effect, equivalent to treating the asset life as deterministic. In reality, the life of, e.g., equipment is stochastic (cf. reliability engineering); thus, such uncertainties should be involved in financial analyses. To do this, the expected present value should be calculated to base investment decisions on, instead of the present value according to the expected life. We examine both the case of continuous exponential cash flow pattern combined with exponentially distributed life, and their discrete equivalent of geometric gradient series pattern with geometrically distributed life. We present in detail the mathematical framework for obtaining closed form solutions for the present values with cessation point uncertainty in general, and the formulas particularly for the two cases just highlighted. We evaluate possible errors, by using the same relative error measure (in fact, its absolute value) as previously, defined as the present value according to the conventional approach divided by the correct, expected present value, less one. We generally find that if the discount rate is equal to the growth rate, the error is always zero. If the growth rate exceeds the discount rate, then the error can reach the theoretical maximum of 100%. However, if the growth rate is less than the discount rate, which can be considered to be the usual case, the error cannot exceed 30%. This is actually the value of a local maximum of the error function, which local maximum exists for every expected life. The main difference between the continuous and the discrete case is that the value of this local maximum is invariant regarding the expected life and it is (approximately) 30% in the continuous case, but varies with expected life in the discrete case, being smaller for shorter expected lives but being at least 12.5%. We formulate the following general rule of thumb: for a given discount rate – growth rate combination, the longer the expected life, or, alternatively, for a given expected life, the larger the difference in absolute value between the two rates, the larger is the error in magnitude. In particular, we find that even a small percentage point difference between the discount rate and the growth rate can lead to considerable errors. For typical expected lives of 10 to 20 periods, a rate difference of 2% to 1% gives a non-negligible error of approximately 10%. These results also call attention to the importance of precision in discount rate and growth rate estimation. Our third thesis concludes these:

**<u>Thesis 3 (Andor and Dülk, 2014*):</u> In typical cases, neglecting the uncertainty of asset life may often lead to considerable error (above 10%) in the present value but may be regarded as tolerable for a rough estimation as the error cannot exceed 30%.**

Finally, the third part of our research is somewhat distinct from the technicalities of the previous two. We aim at estimating the cost of capital, the relevant risk in particular, of energy efficiency projects related to space heating and cooling. We conduct the analysis in the Capital Asset Pricing Model (CAPM) framework, and strive to estimate empirically the beta parameter of such projects, which parameter expresses the relevant risk of an asset in the CAPM. We do the estimation from historical energy price and weather data, as these two factors determine fundamentally the risk of energy-saving projects. This follows from the cash flows of savings in energy bills taking up the vast majority of the project's (risky) cash flows, and the savings flows being the product of the unit price of energy and the amount of consumption, which latter is assumed to depend exclusively on the weather. Our research is motivated by providing a more precise assessment of project risk than those provided by the industry betas, which serve obviously just as rough estimates. We face the apparent problem of lack of relevant historical stock price data on energy efficiency projects, from which beta could be directly estimated. Therefore, we take the indirect approach of estimating from historical data on the factors affecting the project's cash flows, i.e., energy price and weather, which are available. We test the hypothesis whether the project beta is statistically significantly different from zero. Since beta is determined by the correlation of returns on the asset with returns on the market portfolio in a multiplicative way, the beta is zero if this correlation is zero. Because asset returns are not observed, as noted above, we compute every pair-wise correlation between the cash flow risk factors and market portfolio returns. We argue that if none of these pair-wise correlations is different from zero, then the correlation of asset returns with market portfolio returns should not be different from zero either. In our case of energy efficiency projects, this means testing the null hypothesis of zero correlation in the following three instances: 1) energy unit price measure and market portfolio returns; 2) weather measure and market portfolio returns; 3) energy unit price measure and weather measure. We follow the approach, found also in related studies, of treating these factors as investments, and calculate historical percentage changes on them as their measure. Thus, the energy unit price uncertainty is proxied by historical percentage changes in the unit price of energy, and weather uncertainty is proxied by historical percentage changes in heating degree-days. We examine natural gas and electricity as two energy carriers on which a project might save on, in several European countries, both for households and businesses. None of the cases is found to have correlation significantly different from zero; therefore, the hypothesis that energy efficiency projects have a zero beta and consequently their cost of capital is the risk-free rate, cannot be rejected. We remark that our results are constrained in the sense that we do not quantify the beta, which would obviously require a solid theoretical framework linking the evolution of cash flows and stock prices. An attempt regarding this, which also challenges the results of previous related studies and accords with our results reported here, is made in a recent yet unpublished draft of ours (Andor and Dülk, 2012). Our fourth thesis summarizes the above:

**<u>Thesis 4 (Dülk, 2012a, 2012b):</u> The hypothesis that the cost of capital is the risk-free rate for space heating or cooling energy efficiency projects saving on electricity or natural gas cannot be rejected.**

The rest of our discussion is divided into three main chapters corresponding to the above-summarized three fields of research. First, we present the harmonic convention in more detail as an improvement over the end-of-period convention. Second, we provide a more thorough overview of our results regarding uncertainty as to asset life. Third, the cost of capital estimation for energy efficiency projects related to space heating and cooling close the discussion.

# II. HARMONIC MEAN AS AN APPROXIMATION FOR DISCOUNTING INTRAPERIOD CASH FLOWS[1]

## II.1. Introduction

Economic analysis today is most often conducted in the discounted cash flow (DCF) framework, where the present value of future funds is derived as the indicator of an asset's worth. The present value of a single amount is determined by its timing and the discount rate for a given interest period (Figure 1.1) and is expressed by the formula in (1.1) (proven by, e.g., Fleischer, 1986):

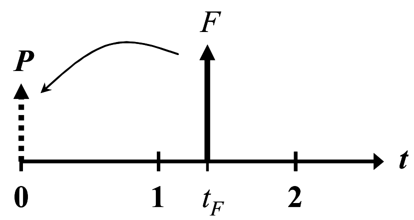


**Figure 1.1**: Discounting an intraperiod cash flow.

$$P = F(1+i)^{-t_F} \tag{1.1}$$

where $P$ is the present value of $F$, $F$ is a cash flow, $i$ is the effective interest rate (discount rate) for the interest period, and $t_F$ is the time of occurrence of cash flow $F$, expressed in the time unit of the interest period (i.e., if the interest period is one year, then $t_F$ is expressed in years).

Here, we consider only the case of discrete compounding (as shown in (1.1)) and not that of continuous compounding. The use of the former is more widespread, and the discount rates for the two approaches can be easily derived from one another as follows:

$$i = e^r - 1 \text{, or equivalently } r = \ln(1+i) \tag{1.2}$$

where $r$ is the continuous discount rate and $e$ is the base of the natural logarithm.

Following from (1.1), the present value of an asset with multiple cash flows (Figure 1.2) is derived as (1.3):

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[1] The work reported in this main chapter has been developed in the framework of the project "Talent care and cultivation in the scientific workshops of BME." This project is supported by the grant TÁMOP - 4.2.2.B-10/1--2010-0009.

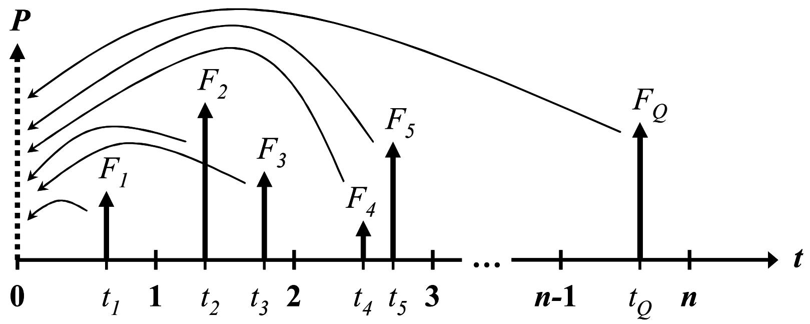



**Figure 1.2**: Discounting multiple cash flows of an asset.

$$P = \sum_{q=1}^{Q} F_q (1+i)^{-t_q} \tag{1.3}$$

where $Q$ is the total number of cash flows of the asset, $q$ is the index of cash flows and their timing, and $n$ is the index of interest periods in Figure 1.2. (We remark that we start the indexing from 1 because it refers to the number of the cash flow and not its timing; thus, an initial cash flow at time 0 [in some texts noted as $F_0$ or $A_0$] would correspond to $F_1$ in (1.3).)

Formula (1.3) naturally assumes that cash flows are discretely distributed in time (i.e., they are received and disbursed in discrete quantities), which is typically the case in the real world. Employing the procedure suggested by (1.3) is quite cumbersome, especially when there are a large number of cash flows. Therefore, it is reasonable to use approximations that may not be completely accurate but that are simpler and easier-to-use.

One group of approximations considers a continuous cash flow pattern (i.e., cash flow is a continuous function of time; see Figure 1.3) and computes the present value by integration (see, e.g., Park and Sharp-Bette, 1990), as shown in (1.4):

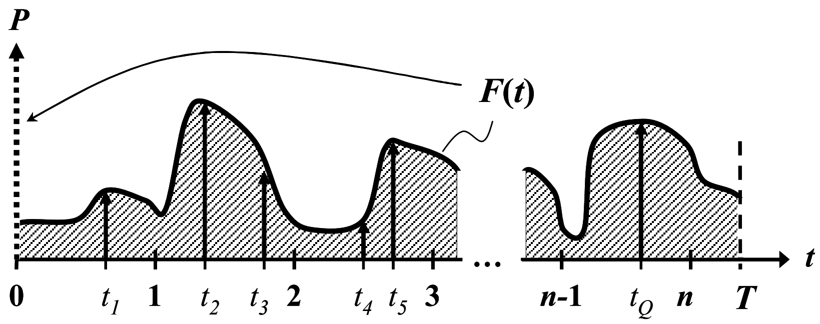



**Figure 1.3**: Discounting a continuous cash flow stream.

$$P = \int_0^T F(t)(1+i)^{-t}\, dt \tag{1.4}$$

where $F(t)$ is the cash flow function, and $T$ is the life of the asset in the same time unit as the interest period.

If we address cash flow patterns that can be approximately described by continuous functions for which closed form present value solutions exist, then this approach is handy; however, in most cases, this approach still yields a mathematical formula that is too complicated. See Almond and Remer (1979) and Remer et al. (1984) for a variety of these cash flow patterns and their discounting. Here we note, as well, that the issue of discounting a continuous cash flow stream is equivalent to discounting a single cash flow with timing uncertainty that exhibits the same pattern. More precisely, let the probability density function $p(t)$ of the timing of the single cash flow $F$ be the following:

$$p(t) \equiv \frac{F(t)}{F}, \text{ where } F \equiv \int_{-\infty}^{+\infty} F(t)dt, \text{ thus } \int_{-\infty}^{+\infty} p(t)dt = 1, \text{ and}$$

$$P = \int_{-\infty}^{+\infty} Fp(t)(1+i)^{-t}\,dt = \int_{-\infty}^{+\infty} F(t)(1+i)^{-t}\,dt \tag{1.5}$$

Another approach used by some approximations simplifies the present value calculation by aggregating all cash flows within an interest period to a specific time point in that period and then applies (1.3). The most notable and widely used of them is the so-called end-of-period convention, which assumes that all cash flows in a given interest period occur at the end of the period. This method is also fundamental in engineering economy textbooks (e.g., Eschenbach, 2011; Fleischer, 1994; Hartman, 2007; Park, 2011; Thuesen and Fabrycky, 2001) and can be calculated by the following formula:

$$P_E = \sum_{n=1}^{N} A_n (1+i)^{-n} \tag{1.6}$$

where index $E$ of $P$ refers to "end-of-period", $N$ is the total number of interest periods during the asset's life, and $A_n$ is the total amount of cash flows in interest period $n$. (We intentionally start the indexing from 1, whereas it is generally started from 0 in the related textbooks when calculating net present value [NPV]. The problem with using 0 rather than 1 is that it adds an essentially "not-to-be-discounted" cash flow $A_0$ to the discounted ones, which would distort our relative error calculations of discounting conventions. Additionally, the interpretation of the timing of $A_0$ can be problematic because "now" can be the end of period 0 or the beginning of period 1. Because of these issues, we omit $A_0$ and instead simply focus on present value, by working with (1.6).)

The end-of-period convention is indeed simple, but the problem of its strict timing assumption has long been recognized (de la Mare, 1975; Luneski, 1967).

Another notable method that builds similarly on timing adjustment is the mid-period convention (indexed by *M*), in which all cash flows are aggregated to the midpoint of their interest periods. This method uses the following formula:

$$P_M = \sum_{n=1}^{N} A_n (1+i)^{-n+\frac{1}{2}} \text{, which is actually } P_M = P_E \sqrt{1+i} \qquad (1.7)$$

As (1.7) shows, the mid-period present value is easily derived from the end-of-period present value (and vice versa).

A third method, not frequently used but a logical alternative, is the beginning-of-period convention (indexed by *B*), which is also related to the end-of-period present value:

$$P_B = \sum_{n=1}^{N} A_n (1+i)^{-n+1} \text{, which is actually } P_B = P_E (1+i) \qquad (1.8)$$

As seen in (1.6)-(1.8), these approximations can be written in the next general form, where the approximative present value is obtained via an adjustment to the most widely used end-of-period present value:

$$P_{approx} = P_E k(i) \text{, according to which } k_E(i) = 1,\ k_M(i) = \sqrt{1+i} \text{, and } k_B(i) = 1+i \qquad (1.9)$$

where $P_{approx}$ is the approximative present value of the given convention, and $k(i)$ is an adjustment function of the discount rate $i$.

We introduce a new adjustment function that is superior to all of the above mentioned conventions in many ways. We refer to this as the "harmonic" approximation (indexed by *H*) because it actually yields the harmonic mean of the end-of-period and beginning-of-period present values:

$$P_H = \frac{2}{1/P_E + 1/P_B} \text{, from which } P_H = P_E \frac{1+i}{1+i/2} \text{, and } k_H(i) = \frac{1+i}{1+i/2} \qquad (1.10)$$

The remainder of this main chapter is organized as follows. First, we identify the relevant range of discount rates for realistic investment analyses. We then deduce that the harmonic approximation is the best adjustment of the end-of-period present value to limit possible approximation errors regardless of the actual cash flow patterns or timings. Next, we develop nomograms (two-dimensional diagrams) for specific cash flow pattern (or probability density function) styles to show the possible calculation errors in the case of identical patterns over the asset's life. We then discuss the issues of varying patterns across periods and provide possible solutions. Finally, a real world-based example is provided as an illustration.

## II.2. Relevant range of discount rates

The approximation error is obviously a function of the discount rate *i*. Therefore, it is worthwhile to explore discount rates that may occur in everyday calculations, thus quantifying the upper limits of possible errors. To determine the highest possible discount rate, we turn to the Capital Asset Pricing Model (CAPM), which is widely used for deriving the cost of capital using the following formula (see, e.g., Brealey and Myers, 1996):

$$i = R_f + \beta E(R_p) \tag{1.11}$$

where $R_f$ is the risk-free rate, $\beta$ is the risk parameter of the given asset, and $E(R_p)$ is the expected market risk premium.

It may well be assumed that $R_f \approx 2\%$ and $E(R_p) \approx 6\%$ annually in real terms (see, e.g., Damodaran, 2002; Ibbotson, 2010). Thus, the relevant range of $i$ is assigned by the relevant range of $\beta$. To identify what values $\beta$ might take on, we use Damodaran's online financial dataset as a cited, publicly accessible source on such financial information. (Damodaran's data is available at *http://pages.stern.nyu.edu/~adamodar/New_Home_Page/data.html*. We used the data titled "Levered and Unlevered Betas by Industry", "Total Beta by Industry Sector", and "Risk Premiums for Other Markets".) Provided that investors hold efficient portfolios, a common assumption in financial analysis, we can consider the unlevered betas (corrected for cash), of which the highest is 2.42. This translates to roughly a 17% maximum discount rate. However, investors are often not well-diversified, violating the efficient portfolio assumption. In such cases, the so-called total betas can be used in the CAPM-equation to reflect the total risk of the investment. Total betas are obviously higher than "normal" betas because the investors are not taking advantage of diversification. According to Damodaran's dataset, the highest total beta is 4.09, yielding a maximum possible discount rate of approximately 27%. As Damodaran (2003) discusses, the discount rate is often further increased by a country risk premium for specific (e.g., political) risks associated with the country of the investment. Among such estimates in his dataset, 12.75% is the highest. This raises the upper limit to a rate of approximately 40% in real terms. If the analysis is performed in nominal terms, then inflation is also added. However, because extreme cases of inflation may even double the real rate, we decided to cap our range of relevance at 50%, which implies a top inflation rate of approximately 10%, a relatively elevated rate. Related literature, such as Lawrence (2009), cites court cases when discount rates above 20% were used and also notes that many venture capitalists demand a return even greater than 30%. Horvath (1995) examines discount rates up to 50%, justifying this upper limit with the expectations of venture capital firms. Although discount rates exceeding 30% could be rare in practice, we evaluate approximation errors up to 50% to achieve a more complete illustration of error characteristics. The condition of $i \geq 0$ is also set.

## II.3. Harmonic approximation and maximum possible errors

We apply the relative error as the measure of error, as defined in the study by Lohmann and Oakford (1984):

$$\varepsilon = \frac{P_{approx}}{P_{actual}} - 1 \text{, which is, in our discussion, equivalent to } \varepsilon = \frac{P_E k(i)}{P_{actual}} - 1 \qquad (1.12)$$

where $\varepsilon$ is the relative error of an approximation, $P_{approx}$ is the approximative present value according to the approximation, and $P_{actual}$ is the actual present value of the given cash flow pattern. A positive error means overstatement, and a negative error means understatement of the actual present value.

We aim to develop a formula that minimizes the maximum possible error in the case of any cash flow pattern in order to provide a general method of adjustment. An analyst, then, need not bother with the estimation of the cash flow pattern. This generality may prove useful in cases such as when the timing of cash flows, and therefore their pattern, is uncertain or if the analyst is an "outsider," valuing a company from its financial statements without access to information about the pattern.

To calculate the maximum possible error, extreme cases must be identified first. Cash flow estimates are typically made for interest periods (e.g., years), and the adjustment is the same in each period (as follows from (1.9)). Therefore, the focus can be narrowed to a single period, but the findings are equally valid for multiple periods. The exact present value of cash flows within an interest period must fall between the end-of-period and beginning-of-period present values. These are the two extremes, then, when all cash flows occur either at the end or at the beginning of a given period. Thus, the maximum possible error of any approximation is

$$\varepsilon_{\max} = \max\left\{\left|\frac{P_{approx}}{P_E} - 1\right|, \left|\frac{P_{approx}}{P_B} - 1\right|\right\} \tag{1.13}$$

That is, the larger of the two in absolute value, measured against the extremes of present values. We take the absolute value of errors because we are interested in the magnitude of errors, which fundamentally determines the "quality" of an approximation. We do not attribute more importance to either over- or underestimation; only the magnitude matters.

It can be seen in (1.13) that, taking $P_E$ and $P_B$ fixed, the two error terms change in opposite directions as $P_{approx}$ changes. In other words, if an approximation yields a present value that is closer to $P_E$, then its absolute error compared to $P_E$ is smaller, while its absolute error compared to $P_B$ is larger (and vice versa). Therefore, the maximum possible error is minimal when the two terms are equal:

$$\left|\frac{P_E k(i)}{P_E} - 1\right| = \left|\frac{P_E k(i)}{P_E(1+i)} - 1\right|, \text{ that is } \frac{P_E k(i)}{P_E} - 1 = -\frac{P_E k(i)}{P_E(1+i)} + 1, \text{ from which}$$

$$k(i) = \frac{1+i}{1+i/2} = k_H(i) \tag{1.14}$$

The adjustment function in (1.14) is the one we have called harmonic approximation (see (1.10)).

Following from (1.13), the maximum possible error of the harmonic approximation is

$$\varepsilon_{H,\max} = \max\left\{\left|\frac{P_E k_H(i)}{P_E} - 1\right|, \left|\frac{P_E k_H(i)}{P_E(1+i)} - 1\right|\right\} = \max\left\{\left|\frac{i}{2+i}\right|, \left|\frac{-i}{2+i}\right|\right\} = \frac{i}{2+i} \tag{1.15}$$

The same properties of the end-of-period, mid-period, and beginning-of-period conventions are (cf. Lohmann and Oakford, 1984):

$$\varepsilon_{E,\max} = \max\left\{0, \left|\frac{-i}{1+i}\right|\right\} = \frac{i}{1+i} \tag{1.16}$$

$$\varepsilon_{M,\max} = \max\left\{\left|\sqrt{1+i}-1\right|, \left|\frac{1}{\sqrt{1+i}}-1\right|\right\} = \sqrt{1+i}-1 \quad (1.17)$$

$$\varepsilon_{B,\max} = \max\{|i|, 0\} = i \quad (1.18)$$

Figure 1.4 exhibits the maximum possible errors as functions of the discount rate.

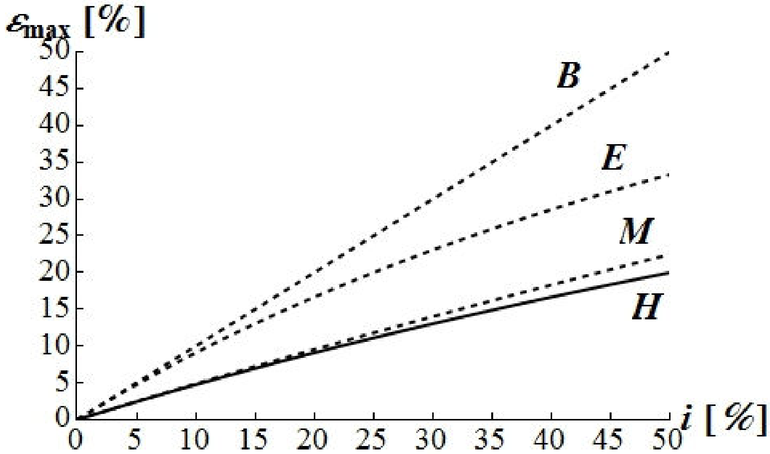



**Figure 1.4**: Comparison of maximum possible errors of the beginning-of-period (*B*), end-of-period (*E*), mid-period (*M*), and harmonic (*H*) approximations.

As Figure 1.4 shows, the order of the maximum possible errors is $\varepsilon_{H,\max} < \varepsilon_{M,\max} < \varepsilon_{E,\max} < \varepsilon_{B,\max}$ for any positive discount rate. The harmonic approximation is, in fact, the one that yields the smallest maximum possible error among all such possible approximations. Recalling the relevant range of discount rates and substituting $i = 50\%$, the maximum possible errors are 50%, 33.33%, 22.47%, and 20.00% for the beginning-of-period, end-of-period, mid-period, and the harmonic conventions, respectively. This result indicates that it is quite worthwhile to adjust the end-of-period present value with the harmonic approximation; however, it is only slightly better than the mid-period, even at the extreme.

## II.4. Evaluation for selected single-parameter cash flow patterns

The evaluation in the previous section was performed for the general case, i.e., when the analyst does not consider the actual cash flow pattern, and it assessed the maximum possible errors in (1.13). It was argued that, with respect to the maximum possible error, there is no difference between multi-period and single period cases. That is, for a given convention, the maximum possible error in any single period is the same as for the asset as a whole.

This section evaluates the relative errors defined in (1.12) for several actual cash flow patterns (or, similarly, for possible timings of one cash flow). Similar analyses are reported by, for example, Fleischer et al. (1998), Lawrence (2009), and Lohmann and Oakford (1984). We note that the findings for the single and multiple period cases (for a given convention), with respect to relative error, are the same if and only if the patterns are the same in each period during the life of the asset. Then, the relative error in any single period is the same as for the asset as a whole. If the patterns vary, however, the relative error may differ across periods. Then, the relative error regarding the asset as a whole is a function of the relative errors, the cash flow amounts, and the order of the patterns.

The discussion that follows is divided into three subsections. First, we introduce the pattern styles that we use in the analysis. Second, we perform the evaluation under the assumption that the periods have identical patterns. This assumption might appear to be restrictive, but it is a typical case in real life that the same, e.g., seasonal, pattern recurs year after year. Third, we discuss the issue of varying patterns and provide possible solutions.

### *II.4.1. Patterns used in the analysis*

We examine three cash flow pattern styles that describe cash flows within one interest period and are defined by a single parameter. Practitioners may find these pattern styles useful for incorporating their knowledge about the actual cash flow pattern of an asset under evaluation. Two of these styles are based on the PERT distribution and the triangular distribution, while we refer to the third style as “semester estimates” (see Figure 1.5 for their illustration).

The PERT and triangular distributions are popular in project analysis (see, e.g., Rees, 2008) because they are defined only by three parameters: lower bound, upper bound, and likeliest value. We employ their probability density functions as the cash flow function within a given interest period by setting the lower bound to the beginning of the period and the upper bound to the end of the period. Thus, the only parameter an analyst needs to estimate is the mode of the probability density function (denoted by $c$), i.e., the time point when the cash flow function peaks. In the case of the semester estimates, the analyst judges what proportions of a period’s cash flow are expected to occur in the first and second half of the period. The cash flow pattern within the semester is assumed to be continuous uniform. For better comparability with the PERT and triangular styles, we assign $c$ to the estimate for the second half of the period. The first half, then, has the $1–c$ proportion of cash flows. We note that although we use continuous patterns, they are reasonable alternatives to frequent discrete cash flows within an interest period, e.g., monthly cash flows within a year. We believe that these pattern styles, by their adjustable parameter, can cover a wide range of useful patterns.

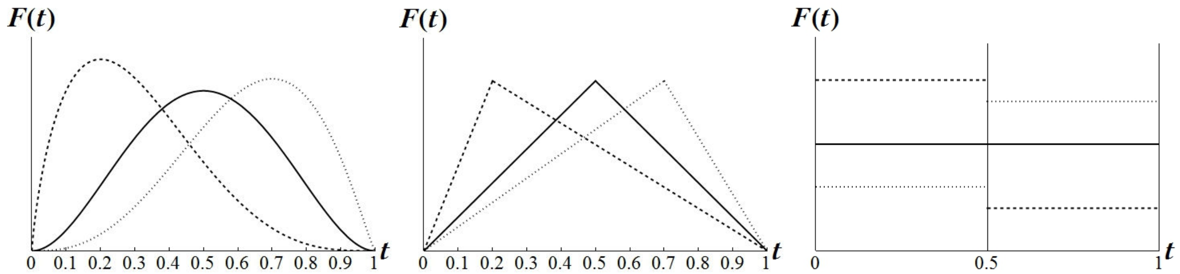



**Figure 1.5**: The PERT, triangular, and semester patterns (from left to right), for $c$ = 0.2 (dashed), 0.5 (solid), and 0.7 (dotted).

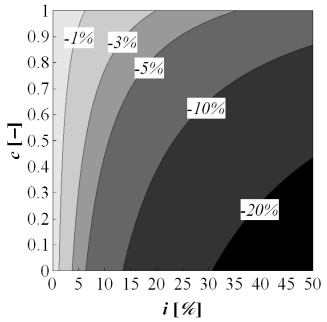
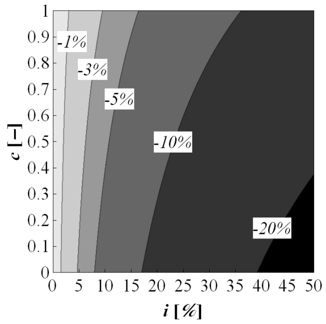
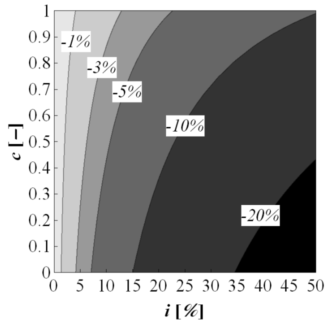
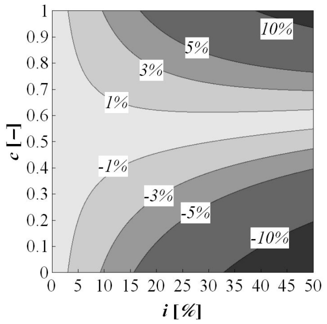
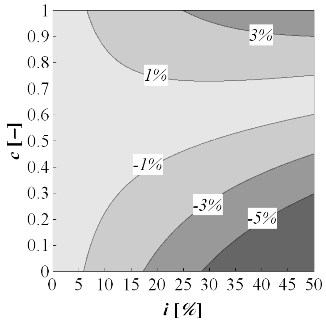
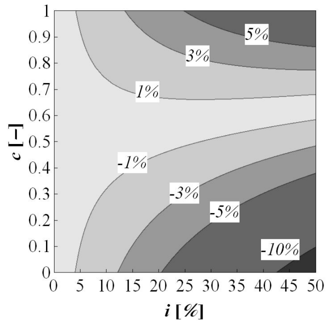
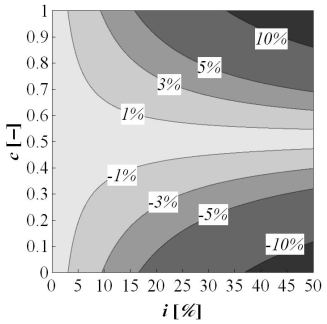
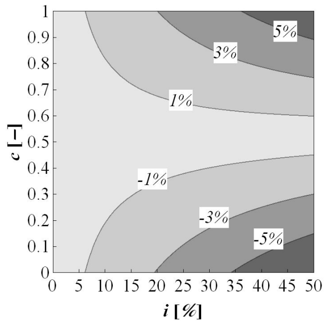
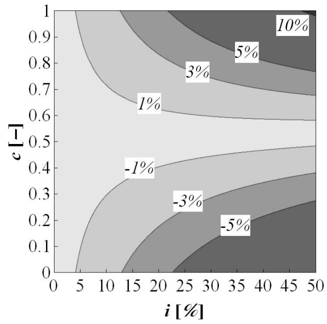
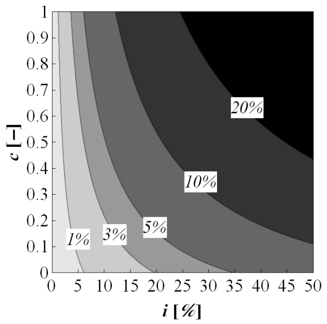
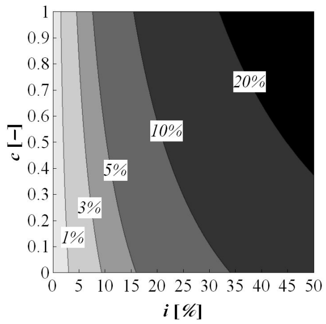
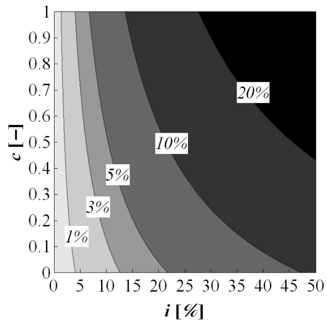
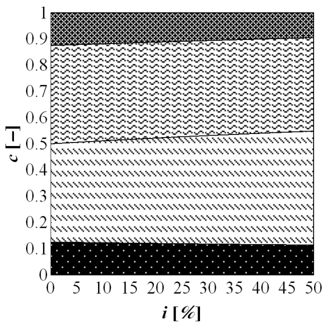
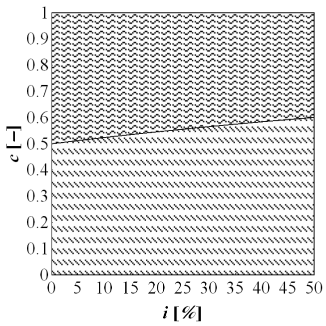
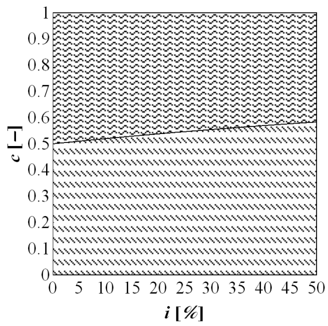
### *II.4.2. Case of identical patterns*

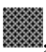
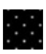
Table 1.1 evaluates the approximation errors of the harmonic, mid-period, end-of-period, and beginning-of-period conventions for each of the three pattern styles described above, assuming that the pattern (both its style and $c$ parameter) is the same in each period during the life of the asset. Because for the given convention, the relative errors are then the same in any single period, for convenience, we perform the evaluation with period boundaries set to 0 and 1, but the conclusions are the same for any arbitrary $n$ and $n$+1 period boundaries. This is logical, following from making the same adjustment in each period (see (1.9)). We further remark that, considering one year as the length of the period, $t$ = 0 does not necessarily equate to January 1 because the starting point of a project, for example, can be arbitrarily assigned.

The errors are computed as defined in (1.12), where $P_{approx}$ is the approximative present value according to the given method (e.g., $P_H$), and $P_{actual}$ is the precise present value of the given pattern (e.g., PERT) calculated according to the integral in (1.4). This obviously captures the deviations of approximations only from the present value of the given single-parameter pattern and not the difference attributable to using the specific pattern style. The latter deviation is itself another approximation because the real pattern may not be described precisely by the given pattern; however, we do not go into detail about such errors.

The first four rows in Table 1.1 show the absolute values of relative errors individually, by method. We also indicate the sign of the errors on the contours. The last row depicts in what regions of parameters a particular method is the best choice over the others, based on the absolute values of relative errors.

**Table 1.1**: Evaluation of approximations for selected single-parameter cash flow pattern styles.

| | PERT | Triangular | Semester estimates |
|---|---|---|---|
| End-of-period |  |  |  |
| Harmonic |  |  |  |
| Mid-period |  |  |  |
| Beginning-of-period |  |  |  |
| Regions of preference |  |  |  |

Textures , , , and  refer to the end-of-period, harmonic, mid-period, and beginning-of-period approximations, respectively; on nomograms, darker colors indicate more severe errors.

As Table 1.1 shows, the harmonic and the mid-period conventions are separated by approximately the $c$ = 0.5 line, which is steeper for the triangular and semester estimate patterns, in favor of the mid-period. The region plots also demonstrate that if the PERT style is used, the end-of-period and the beginning-of-period conventions prove to be the best choice in cases when $c$ is more than approximately 0.9 and less than approximately 0.1, respectively. If either the triangular or semester style is used, these conventions are outperformed by the mid-period and the harmonic formulas for any combination of $c$ and $i$. From the region plots, it can be generally concluded that if the cash flow peak falls into the second half of the period, then the harmonic approximation is preferred; if it falls into the first half, then the mid-period is best.

The individual error characteristics show that the end-of-period convention always understates the actual present value, and the beginning-of-period always overstates it. The harmonic and mid-period conventions can both overstate and understate the actual present value, depending on which half of the period, approximately, the actual value falls into. We can observe that even for smaller discount rates (approximately 7%), the end-of-period and the beginning-period conventions' relative errors can be above 5% magnitude, which may be regarded as the limit of negligibility. Their error magnitudes are above 10%, which can be considered severe, for a large number of possible combinations. However, the errors of the harmonic and the mid-period formulas hardly ever exceed the 10% level. These larger errors occur only when the peak and the discount rate are both in the extreme. Similar to their maximum possible errors, the difference in accuracy between these two methods is slight. The individual error plots also show that the harmonic method is typically better when the cash flow peak is in the second half of the period, whereas the mid-period is better when the peak is in the first half. Regarding over- and understatement, the harmonic and mid-period conventions typically have the same sign for any $c - i$ combination.

The nomograms in Table 1.1 make adjustments based on the errors possible. They show the sign and magnitude of the relative error so that, for any given pattern and discount rate combination, the analyst may derive the actual present value by correcting for the error by rearranging (1.12):

$$P_{actual} = \frac{P_{approx}}{1+\varepsilon} \tag{1.19}$$

This approach actually makes the adjustment of the end-of-period present value unnecessary. With the help of the end-of-period nomogram, the precise present value of any pattern (of the given style) and discount rate combination can be calculated. However, a small error may still be present due to the reading of the nomograms as they exhibit discrete error levels. In this respect, the adjustment formulas may still be desirable if their nomograms can be read more accurately for the case under evaluation.

In conclusion, we can establish that for small discount rates, i.e., below 7%, it does not matter which approximation is used, or, alternatively, whether the end-of-period present value is adjusted or not. However, the use of the harmonic is suggested because it has the smallest maximum possible error. For rates above 7%, if most of the cash flows are expected to occur in the first half of the period, then the mid-period is recommended; however, if they are expected to occur in the second half, then the harmonic approximation should be preferred. With the help of the nomograms, the approximative present values can also be corrected to arrive at the close-to-actual present value.

### *II.4.3. Case of varying patterns*

Of course, the cash flow patterns of an asset are not necessarily the same in each period during the life of the asset. That is, the style of the patterns or the pattern parameters (e.g., the $c$ parameter) and, consequently, the relative error of a given approximation may differ across periods. In such cases, both the cash flow amounts and the order of the patterns affect the relative error for the asset as a whole. In other words, it matters how much cash flow is described by a given pattern and when that pattern occurs. Because of these additional relevant parameters, convenient two-dimensional diagrams as in Table 1.1 cannot be generated.

We still, however, may rely on some of the previous findings in the case of varying patterns. First and most generally, we can say that without considering the actual cash flow patterns, the harmonic approximation can be used because it has the smallest maximum possible error. Second, if we address only pattern styles evaluated in Table 1.1 and the discount rate is under 7%, then it does not matter which approximation is used because the relative error for the asset as a whole is not greater than approximately 5%; however, the harmonic approximation is again recommended due to its smallest maximum possible error. Third, periods identical in some respect (e.g., the relative error is the same or the suggested best approximation method is the same for each) can be grouped and error-corrected or adjusted separately by groups according to Table 1.1. The present values of the groups are then added up. This approach, of course, makes sense only if there are few "outlying" periods, i.e., the majority of periods are the same.

Finally, we propose another approach that applies to a single period but can be more useful in a multi-period analysis. As noted above, patterns may vary across periods, and they need not be of the styles evaluated in Table 1.1. However, the differing pattern may be approximated by the one-period semester estimate pattern style, similar to the approximation of an integral by rectangles or a probability density function by histograms (see Figure 1.6).

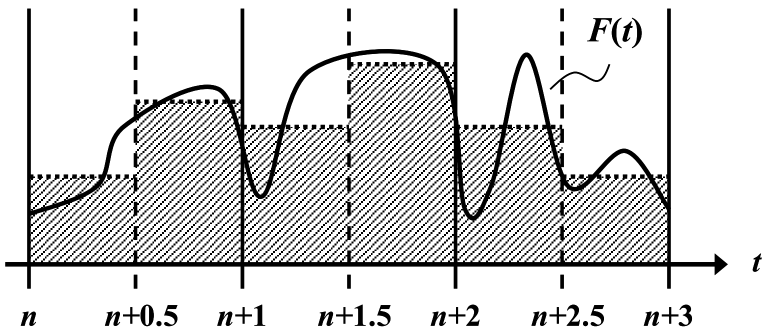



**Figure 1.6**: Approximating a cash flow pattern by one-period semester estimates over multiple periods.

We note that the PERT and triangular styles can also be approximated this way. Practitioners may find this approach useful because it requires that they only refer to the semester estimate nomograms. However, it is important to note that this substitution with semester estimates is itself an additional approximation. Because our purpose is to evaluate the errors attributable to the various discounting conventions, we note this possible application of semester estimates but do not detail the errors introduced by this approximation procedure.

## II.5. Illustrative example

To illustrate the use of the above suggested techniques and the significance of possible errors by neglecting the intraperiod trait of cash flows, we present below a realistic capital budgeting example.

Consider a manufacturing firm that plans to install photovoltaic cells on the rooftop of its factory to save on electricity costs. The photovoltaic cell project is planned to span 20 years ($N = 20$), at the end of which there will be no salvage value of the project.

The annual cash flow from energy savings is expected to be 100 ($A = 100$) in the first year and to grow at 6% annually ($g = 6\%$, to account for rising electricity prices and/or consumption of the company), both in nominal terms. The project is undertaken in a country where the most intense sunshine is expected on June 30, while there are very few sunny days in January and February. The analyst employs PERT distribution to describe the intensity of sunshine and, consequently, the cash flow pattern of the savings during a year. The project is assumed to start on February 1 (thus, $t = 0$ is February 1). The time, then, between February 1 and June 30 (which is the day when the sunshine is expected to be the most intensive) is 5 months, that is, 5/12 = 0.42 fraction of a year. Hence, the *c* parameter of the PERT-style yearly cash flow pattern is 0.42 (see Figure 1.7).

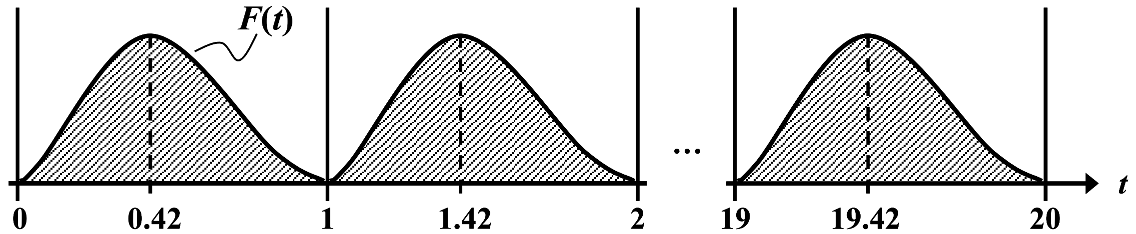



**Figure 1.7**: Cash flow pattern of the energy savings over the project's life.

The cells need one overhaul during the planning horizon (after a given number of operating hours) that is estimated to cost 200 in nominal terms. Because the overhaul depends on operating hours, the timing of this cash flow is uncertain: it is assumed to occur sometime between the middle of year 9 and 11 and is described by a uniform probability distribution (see Figure 1.8).

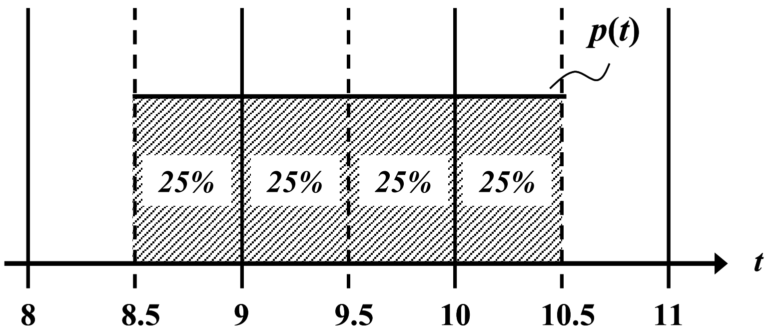



**Figure 1.8**: Probability density function of the overhaul cost.

This means nominal expected cash outflows of 25% x 200 = 50 both in year 9 and 11 and 50% x 200 = 100 in year 10. Additionally, the probability density function over these years is actually a series of one-period semester estimates, with parameters $c = 1$, 0.5, and 0 in year 9, 10, and 11, respectively.

The price of the cells including installation costs is 500 and is paid upfront (on February 1 in the first year). An annual nominal cost of capital (discount rate) of 25% is assumed.

If the analyst employs the end-of-period convention without considering the actual cash flow pattern of the saving and the timing distribution of the overhaul, the following calculation is made (based on (1.6), employing the geometric gradient series formula for the savings):

$$P_E = \frac{A}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^N\right) - \frac{50}{(1+i)^9} - \frac{100}{(1+i)^{10}} - \frac{50}{(1+i)^{11}} = 507 - 7 - 11 - 4 = 485,$$

from which the net present value (NPV) of the project is 485 – 500 = –15; therefore, based on the end-of-period present value, the project would be rejected. The maximum possible error the analyst may make by relying on this calculation is, from (1.16) as well as from Figure 1.4, the following:

$$\varepsilon_{E,\max} = \frac{i}{1+i} = \frac{0.25}{1+0.25} = 0.2 = 20\%;$$

that is, we may obtain the true present value – not the NPV! – wrong by 20%, at most. We note that the initial cash flow does not conform to the end-of-period framework because it is treated as a separate beginning-of-period cash flow in the regular NPV calculation. Therefore, for consistency, it must be excluded from the error measurement.

If the knowledge about patterns and timings is considered, then the analyst can use the nomograms in Table 1.1 and look up the errors related to the end-of-period present values. However, in this case, we face the issue of varying patterns over the project's life (i.e., PERT-style savings and semester-style overhaul costs). Therefore, a single error correction cannot be determined. Because the errors can be larger than 10%, according to the nomograms, it is worthwhile to make corrections, which can be performed separately for the cash flow groups. According to the end-of-period nomograms in Table 1.1, the relative error at $i$ = 25% for the savings (PERT, $c$ = 0.42), year 9 overhaul (semester, $c$ = 1), year 10 overhaul (semester, $c$ = 0.5), and year 11 overhaul (semester, $c$ = 0) is about $\varepsilon$ = –12%, –6%, –11%, and –15%, respectively. Then, correcting for the errors according to (1.19) yields the following:

$$P_{actual} = \frac{P_{savings}}{1-\varepsilon_{savings}} + \sum_{n=9}^{11} \frac{P_{overhaul,n}}{1-\varepsilon_{overhaul,n}} \approx \frac{507}{1-0.12} - \frac{7}{1-0.06} - \frac{11}{1-0.11} - \frac{4}{1-0.15} \approx 552\,,$$

from which the NPV is 552 – 500 = 52, which is now positive, versus the NPV of –15 arrived at via the end-of-period convention. For reference, the precise actual present value of the project calculated by the integration formula (1.4) is 550, from which the deviation of the correction based on nomogram-reading is minute. Substituting 550 for $P_{actual}$ and 485 for $P_{approx}$ into (1.12) shows that the simple end-of-period convention understates the true present value of the project by approximately 12%, which is significant. The results also show that relying on the end-of-period convention can lead to improper rejection of projects. In the calculation, we corrected all terms for errors, but it most likely would have been sufficient to adjust only the savings because they make up the majority of cash flows. This presumption relates to our earlier comment that the effort of separate adjustments may not be rewarded by a comparable increase in accuracy.

If the analyst again does not consider the actual patterns and timings but uses the harmonic formula in (1.10), the following computation is made:

$$P_H = P_E \frac{1+i}{1+i/2} = 485 \times \frac{1+0.25}{1+0.125} = 539\,,$$

from which the NPV is 539 – 500 = 39, which leads to the right decision, although somewhat smaller than the precise actual NPV of 550 – 500 = 50. By the application of the harmonic approximation, the maximum possible error is reduced to the following, as shown by (1.15) and Figure 1.4:

$$\varepsilon_{H,\max} = \frac{i}{2+i} = \frac{0.25}{2+0.25} = 0.11 = 11\%\,,$$

which is significantly lower, at almost half of the end-of-period maximum possible error of 20%. Thus, we can be sure that the approximation error of the harmonic formula does not exceed 11%, regardless of the actual patterns or timings.

If the knowledge about patterns and timings is considered, the nomograms in Table 1.1 show that the harmonic present values are much closer to the actual present value than the end-of-period present values. The nomograms of the harmonic approximation in Table 1.1 show that for $i$ = 25%, the error of the savings (PERT, $c$ = 0.42) is around –2%, and the errors of the overhaul costs (semester estimates with $c$ = 0, 0.5, and 1) hardly exceed the 5%

magnitude. Thus, the harmonic present values can be considered close enough to the actual present value that no further adjustment is necessary. Indeed, substituting 550 for $P_{actual}$ and 539 for $P_{approx}$ into (1.12) shows a 2% understatement, which is, not surprisingly, essentially the same as the error related to the savings.

As the best approximation for the given project, the "Regions of preference" nomograms in Table 1.1 suggest either the mid-period or the harmonic. The mid-period convention is better for the savings flows, which account for most of the project's cash flows, but computing the mid-period present value of the project yields the following:

$$P_M = P_E\sqrt{1+i} = 485 \times \sqrt{1+0.25} = 542\,,$$

which differs less than 1% from the harmonic present value of 539 and understates the precise actual present value of 550 by approximately 1.5%. This value is not much less than the 2% understatement of the harmonic formula.

## II.6. Conclusion

There is a new formula for adjusting the end-of-period present value to minimize the maximum possible computation error in cases of intraperiod cash flows. This so-called harmonic approximation is based on the harmonic mean of the end-of-period and the beginning-of-period present values. This method is recommended if the practitioner has very limited knowledge about the actual cash flow patterns or the probability density functions of the cash flow timings. The harmonic approximation is also a good alternative convention with acceptable error for numerous realistic cash flow assumptions, especially when cash flows tend to occur in the second half of the period. Routine application of the standard end-of-period convention can lead to considerable errors, even for relatively low discount rates. Extensive graphical illustrations are presented to map the error characteristics.

# III. PRESENT VALUE UNDER UNCERTAIN LIFE: AN EVALUATION OF RELATIVE ERROR

## III.1. Introduction

In the discounted cash flow (DCF) framework, the worth of a capital asset (henceforth simply asset) is computed by discounting its expected future cash flows at an appropriate rate (discount rate) that expresses the cost of capital. As described in related textbooks (e.g., especially Gönen, 1990; Park and Sharp-Bette, 1990; Remer et al., 1984; and also Eschenbach, 2011; Fleischer, 1994; Hartman, 2007; Park, 2011; Thuesen and Fabrycky, 2001), the computation can typically take one of two forms: discounting discrete cash flows (i.e., cash flows occurring at equally spaced time periods, typically years) at a discrete discount rate (i.e., the effective rate for one period) or discounting a continuous cash flow stream (i.e., cash flow is a continuous function of time) at a continuous discount rate (i.e., the rate assuming an infinitesimal period of capitalization). Mathematically, the former approach is defined by (2.1), and the latter is defined by (2.2). We assume, as usual, an identical discount rate across periods.

$$P_d = \sum_{n=1}^{N} F_n (1+i)^{-n} \tag{2.1}$$

$$P_c = \int_0^T F(t) e^{-rt} dt \tag{2.2}$$

where $P$ is the present value; the indices $d$ and $c$ refer to the given approach, that is, "discrete" and "continuous," respectively; $n$ is the index of periods in the discrete case; $N$ (an integer) is the life of the asset expressed in periods; $F_n$ is cash flow at the end of period $n$; $T$ (a real number) is the life of the asset expressed in periods; $t$ is time; $F(t)$ is the continuous cash flow function; $i$ is a discrete discount rate; $r$ is a continuous discount rate; and $e$ is the base of the natural logarithm.

The discrete and continuous discount rates are related to one another as

$$i = e^r - 1 \text{, or equivalently } r = \ln(1+i) \tag{2.3}$$

We note that we use the term "asset," generally defined as any series of cash flows; therefore, we consider only present value. In this general interpretation, cash flows of a capital budgeting project, for example, are not distinguished as usually done (e.g., initial investment, net return or revenues/disbursements, salvage value – see, e.g., in Tufekci and Young, 1987; Zinn et al., 1977), and consequently net present value is – as it is conceptually – just a specific present value. For convenience, we start the indexing in (2.1) from 1, although our findings are not affected by starting from 0, as is done in some textbooks.

In (2.1) and (2.2), essentially all variables, such as the cash flows, discount rates and asset lives may be uncertain; that is, they may be random variables. Therefore, valuation and investment decisions should be based on the *expected* present value (see, e.g., Park and Sharp-Bette, 1990; Tufekci and Young, 1987 and references therein). Although textbooks usually recognize the stochastic nature of these variables, for computational simplicity, they substitute the expected values of the random variables into the present value formula instead of taking the expectation of the whole expression. This is an obvious source of computational error, leading to biased present value and, possibly, to incorrect investment decision.

Here, we are concerned with error attributable to the uncertainty of the economic life of an asset, $N$ or $T$ (henceforth usually simply "life") – that is, the moment in time when the series of cash flows terminates. For a more tractable analysis, we assume non-stochastic discount rates and that cash flows are stochastically independent of asset life (for a numerical method of dealing with correlation between cash flows and life in discrete time, see Van Horne, 1972). Based on this latter assumption, for notational parsimony, we omit the expectations operator for cash flows, so that $F$ itself will henceforth denote expected cash flow. Under our assumptions, the usual but incorrect textbook computations (henceforth referred to as "conventional") are as follows (e.g., Park and Sharp-Bette, 1990; Remer et al., 1984):

$$\hat{P}_d = \sum_{n=1}^{E(N)} F_n (1+i)^{-n} \tag{2.4}$$

$$\hat{P}_c = \int_0^{E(T)} F(t) e^{-rt} dt \tag{2.5}$$

where $E(.)$ is the expectations operator, and a hat indicates that a present value is approximate.

In contrast, the correct approach is to calculate $E(P)$, on which investment decisions should be based. The difference between $E(P)$ and $\hat{P}$ clearly depends on the cash flow pattern and the distribution of asset life. We examine the geometric gradient series pattern and geometrically distributed asset life for the discrete case, and the exponential pattern and exponentially distributed life for the continuous case. In other words, these patterns are growing annuities, which are generally used in practice because their present values can be given in nice closed forms. We assume that growth rates are non-stochastic. The geometric and exponential distributions are also each other's discrete versus continuous counterparts and are perhaps also the most frequently used distributions in reliability engineering, in modeling failure rates (e.g., Grosh, 1989; Ireson et al., 1996). These distributions are the only "memoryless" distributions; that is, the life of an item does not depend on how long the item has survived (e.g., Grosh, 1989; Ireson et al., 1996). They describe completely random failures – the middle (and longest) section of the bathtub curve in reliability engineering (e.g., Kövesi et al., 2012). Thus, we can also say that incorporating life uncertainty into present value calculations is an explicit consideration of underlying technical characteristics, and it is desirable to reconcile technology and economics (Andor, 1996).

Our contributions are threefold. First, under the above-described pattern, distribution, and parameter assumptions, we evaluate the relative error of computing present value based on expected life instead of the expected present value. This error measure is defined as

$$\varepsilon = \frac{\hat{P}}{E(P)} - 1 \tag{2.6}$$

where $\varepsilon$ denotes the relative error. In this formulation, a positive error means that the conventional approach overstates the correct present value, and a negative error implies understatement. (Note that these points apply to the magnitude of the present values; thus, in the case of negative present values, the reverse is true if we look at the actual values. To avoid ambiguity, we will be concerned with the magnitude, i.e., the absolute value, of present values.)

We are the first to look at the relative error, which we deem a better descriptor of computational error than the absolute error, i.e., $\hat{P} - E(P)$, which was examined by Chen and Manes (1986). Relative error is dimensionless and is expressed as a percentage deviation, whereas absolute error is expressed in dollars. As we demonstrate, in using relative error, we are able to establish limiting conditions on errors that are unobtainable in working with absolute error. For example, we establish that the relative error cannot exceed approximately 30%, provided that the discount rate exceeds the growth rate. Such limits are not obtainable using absolute error. Additionally, we present convenient graphical illustrations of the behavior of relative error, which, again, cannot be done with absolute error, which depends on actual dollar amounts.

Second, we explicitly extend the analysis to the negative domain of growth rates and discount rates. Consideration of negative growth rates is quite plausible, but the inclusion of negative discount rates might seem less obvious and require some justification. A negative discount rate corresponds to an asset with negative expected return – such investment opportunities can be considered as “hedges” or “insurance” against asset pricing factors. In the terminology of the most widely used model for estimation of the cost of capital, the Capital Asset Pricing Model (CAPM), this means assets that have such a large negative beta that the risk premium, in absolute value, exceeds the risk-free rate, yielding a negative discount rate. In fact, empirical evidence has recently shown that this may be the case with, e.g., energy efficiency and renewable energy projects (e.g., Awerbuch, 1993, 1995, 2003; Awerbuch and Deehan, 1995; Bolinger et al. 2006; Kahn and Stoft 1993; Metcalf, 1994; Andor and Dülk, 2012), as it is argued that energy prices exhibit contra-cyclical behavior (e.g., Awerbuch, 2000; Awerbuch and Sauter, 2006; Khan and Fiorino, 1992; Papapetrou, 2001; Sadorsky, 1999). We consider the theoretical range of both the growth and discount rates in the –1 to 1 open interval.

Third, we provide a thorough comparison of the discrete and the continuous case. Such a comparative study, to the best of our knowledge, has not yet been published. Although the two cases bear many similarities, there are essential differences.

Our findings can be summarized as follows for both the discrete and the continuous case, as they are found to bear a close resemblance to one another. If the discount rate exceeds the growth rate, then the conventional approach always overstates the expected present value, with an error that can be at most approximately 30%. However, for any given expected life, there exist several growth rate and discount rate combinations for which such a maximum is attained. In the discrete case, as the main point of distinction with the continuous case, the value of this maximum depends on the expected life and is lower for shorter lives but cannot be lower than approximately 13%. If the discount rate is equal to the growth rate, then the error is always zero; thus, in this case, the conventional approach is perfectly accurate. If the discount rate is less than the growth rate, the conventional approach always understates the expected present value, with an error that can reach the theoretical limit of 100% in magnitude, for any expected life. We highlight that, in this case, the error is very sensitive to the difference between the discount rate and the growth rate. As a rule of thumb, we can establish that for a given discount rate and growth rate combination, a longer expected life – or, conversely, for a given expected life, a larger difference in absolute value between the two rates – results in a larger error. In particular, we find that for realistic cases of 10 to 20 years of expected life and a 2% to 1% difference between rates, the error may exceed 10%, which can be considered significant. These results also call attention to the importance of precision in growth rate and discount rate estimation.

The remainder of this main chapter is structured as follows. The next two main sections discuss the continuous case and the discrete case separately. The continuous case is taken first because its results are simpler and more tractable. These main sections are also divided into two subsections: mathematical derivations are followed by evaluations of errors. A discussion section consolidates the results also in graphical form, and provides a simple illustrative numerical example. A final section concludes.

## III.2. Continuous case

### *III.2.1. Derivation of present value formulas*

Recall from the introduction that we assume a continuous exponential cash flow pattern. Thus, the evolution of cash flows is mathematically represented as

$$F(t) = Ce^{jt} \tag{2.7}$$

where $j$ is a non-stochastic and time-invariant continuous growth rate, and $C$ is a constant.

The present value of such a pattern, following from (2.2), is computed as (e.g., Park and Sharp-Bette, 1990; Remer et al., 1984)

$$P = \int_0^T Ce^{-(r-j)t}dt = \frac{C}{r-j}\left(1 - e^{-(r-j)T}\right) \tag{2.8}$$

(We have dropped the $c$ index, as we focus solely on the continuous case in this section.)

Recall that the life of the asset, $T$, is a random variable. Thus, under the conventional approach, the present value is calculated as

$$\hat{P} = \frac{C}{r-j}\left(1 - e^{-(r-j)E(T)}\right) \tag{2.9}$$

The correct calculation, by contrast, is

$$E(P) = E\left(\frac{C}{r-j}\left(1 - e^{-(r-j)T}\right)\right) \tag{2.10}$$

Notice that the present value in (2.8) is a function, in other words, a transform of the random variable $T$. Assuming a positive $C$, it can be established from the second (partial) derivatives with respect to $T$ that the present value function in (2.8) is strictly convex if and only if $r < j$, and it is strictly concave if and only if $r > j$. Now, based on Jensen's inequality, the direction of the bias between (2.9) and (2.10) can be established. That is, $\hat{P} < E(P)$ if and only if $r < j$; then, the conventional approach provides an understated present value. Conversely, $\hat{P} > E(P)$ if and only if $r > j$; then, the conventional approach provides an overstated present value. (In the case of negative $C$, i.e., negative present values, just the opposite is true.) Note that equality occurs when $r = j$ (for then the second [partial] derivative is zero, so that the function in (2.8) is both convex and concave). In this case, the exponent in (2.8) is zero and the integral becomes

$$P = \int_0^T C dt = CT \tag{2.11}$$

Because $C$ is a constant, it is apparent that $CE(T) = E(CT)$, that is, $\hat{P} = E(P)$. Thus, the conventional approach is perfectly accurate if and only if $r = j$.

It is important to note that the findings thus far are independent of the actual probability distribution of $T$.

If we rewrite the expected present value in (2.10) as

$$E(P) = \frac{C}{r-j}\left(1 - E\left(e^{-(r-j)T}\right)\right) \tag{2.12}$$

we notice that the expectation term is, in fact, the moment-generating function of $T$. Thus, knowing the distribution of $T$, and provided that its moment-generating function exists, the exact formula for the expected present value can be given. Alternatively, a general approximation based on a Taylor series expansion (at $T = E(T)$) can be used, usually by retaining only first and second order terms, as follows (e.g., Chen and Manes, 1986; Young and Contreras, 1975):

$$E\left(e^{-(r-j)T}\right) \approx e^{-(r-j)E(T)} + \frac{(r-j)^2}{2} e^{-(r-j)E(T)} V(T) \tag{2.13}$$

where $V(T)$ is the variance of $T$. Note that (2.13) can also be interpreted as an adjustment to the conventional approach, $\hat{P}$ (cf. Young and Contreras, 1975), namely

$$E(P) \approx \hat{P} - \frac{C(r-j)}{2} e^{-(r-j)E(T)} V(T) \tag{2.14}$$

We note another possible formulation for computing the expected present value, $E(P)$. Building on the definition of expected value, we seek the solution of

$$\int_{-\infty}^{+\infty} P f_P(P) dP = E(P) \tag{2.15}$$

where $f_P(P)$ is the probability density function (pdf) of the present value $P$.

Because $P$ is a transform of the random variable $T$, (2.15) can be rewritten by substituting (2.8) as follows (see, e.g., Bean, 2001):

$$\int_{-\infty}^{+\infty} P f_T(T) dT = \int_{-\infty}^{+\infty} \frac{C}{r-j}\left(1 - e^{-(r-j)T}\right) f_T(T) dT = \frac{C}{r-j}\left(1 - \int_{-\infty}^{+\infty} e^{-(r-j)T} f_T(T) dT\right) = E(P) \tag{2.16}$$

where $f_T(T)$ is the pdf of asset life $T$. Notice that the integral expression on the left is actually the moment-generating function; thus, this formula is equivalent to (2.12). The parameters in (2.16) may, of course, be required to meet some restrictive conditions in order to achieve convergence of the integral, that is, for the moment-generating function, and thus the expected present value, to exist.

We further note that the technique of the Laplace transform can also be used to compute the expected present value. For a discussion of this approach see, e.g., Buck and Hill (1971, 1975), Grubbström (1967).

Now, recalling our assumption that asset life is exponentially distributed, we substitute the moment-generating function of the exponential distribution into (2.12) to obtain the closed form solution for the expected present value (see, e.g., Chen and Manes, 1986; Zinn et al., 1977):

$$E(P) = \frac{C}{r-j}\left(1 - \frac{1}{1 - \frac{-(r-j)}{\lambda}}\right) = \frac{C}{r-j}\left(\frac{\theta(r-j)}{1+\theta(r-j)}\right) \tag{2.17}$$

where $\lambda$ is the parameter of the exponential distribution, and $\theta$ denotes the mean of the exponential distribution, that is, the expected life of the asset (in other words, $E(T) = \theta$), which is related to the distribution parameter via $\theta = 1/\lambda$.

Verifying the convergence criteria from (2.16), we find that, assuming positive expected asset life, $\theta(r - j) > -1$ must hold in order for $E(P)$ to exist; otherwise, $E(P) = \infty$. (If $\theta$ is zero, then $E(P)$ is also zero, and the relative error is undetermined due to division by zero.)

### *III.2.2. Evaluation of relative error*

Dividing (2.9) by (2.17) and subtracting 1, the relative error is

$$\varepsilon = \frac{\frac{C}{r-j}\left(1 - e^{-\theta(r-j)}\right)}{\frac{C}{r-j}\left(\frac{\theta(r-j)}{1+\theta(r-j)}\right)} - 1 = \left(1 - e^{-\theta(r-j)}\right)\left(1 + \frac{1}{\theta(r-j)}\right) - 1 \tag{2.18}$$

Notice that the relative error is independent of the cash flow parameter $C$. It should also be noticed that (2.18) is, in fact, a function of a single variable: substituting $x = \theta(r - j)$, the error can be expressed as

$$\varepsilon = \left(1 - e^{-x}\right)\left(1 + \frac{1}{x}\right) - 1 \tag{2.19}$$

It follows from the previous discussion that if $x \leq -1$, then, because $E(P)$ equals infinity, the error is –100%, and if $x = 0$, which occurs when $r = j$, the error is zero. ($x = 0$ occurs also in the theoretical case where $\theta = 0$, but then we have a degenerate distribution, and the error is undetermined due to division by zero, see earlier. We assume a positive expected life.) Figure 2.1 shows the plot of the absolute value of the error function in (2.19). We take the absolute value of the error to express the magnitude. The issue of sign has been discussed above in reference to Jensen's inequality.

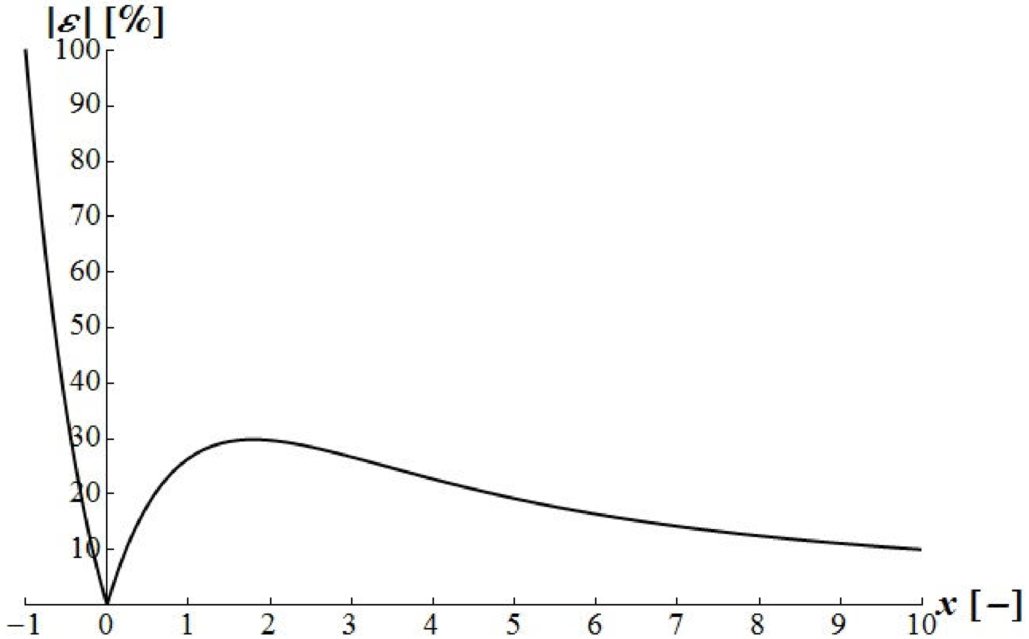



**Figure 2.1**: Function of the absolute value of the relative error in variable $x = \theta(r - j)$.

As Figure 2.1 shows, the relative error has a local maximum. Differentiating (2.19) with respect to $x$ and solving for zero produces only one real root, at $x \approx 1.79$, where $\varepsilon \approx 29.84\%$. Because we take the absolute value of the error, the global maximum is 100%. It can be seen in Figure 2.1 that in the negative domain, the error is very sensitive to $x$. We can also establish that if $x$ converges to infinity, then the error converges to zero. However, if we examine the composition of the variable $x$, the interpretations of the limiting cases become more subtle. First, consider what is probably the least unrealistic of the limiting cases, namely, that the expected life is infinity. This limiting case is meaningful if and only if $r > j$, which is the convergence criterion of the present value integrals when $\theta$ is infinity, and if this criterion holds, the error is indeed zero. Otherwise, neither the conventional nor the correct present value exist (i.e., both are infinity). Plausibly, uncertainty does not matter in the case of an asset that is expected to be infinitely lived (provided, of course, that its present value exists). Convergence of either the discount rate or the growth rate to either positive or negative infinity yield meaningless results. If the discount rate is infinitely large or the growth rate converges to negative infinity, both the conventional and the correct present values are zero; thus, the relative error, which is a ratio, is not meaningful. If the discount rate converges to negative infinity or the growth rate is infinitely large, neither the conventional nor the correct present value exists. (Thus, convergence of $x$ to negative infinity is not meaningful.)

Now, further examining the composition of the variable $x$ and knowing that, as $\theta$ is positive, $x$ is positive if and only if $r > j$, and $x$ is negative if and only if $r < j$, the results can be interpreted as follows. If the discount rate exceeds the growth rate, then the error cannot exceed 30%, but this maximum can be attained for every expected life $\theta$, as there exist several (in fact, infinitely many) $r - j$ combinations for which the extremum condition is satisfied. This is because the parametric equation $x = \theta(r - j)$ by itself is an underdetermined system of equations and thus has infinitely many solutions. In other words, the real number $x$ can be written as the product of two real numbers in infinitely many ways. If the discount rate is smaller than the growth rate, then the upper limit of the error is 100%, which again, for the reason just mentioned, can be attained for every expected life. Figure 2.1 clearly shows that, in this case, the error is very sensitive to the difference between the discount rate and the growth rate. When the two rates are equal, the error is zero, as discussed earlier.

It is also worthwhile to more closely examine the monotonicity of the error function depicted in Figure 2.1. In the negative domain of $x$, the function is strictly decreasing (more precisely, on the –1 to 0 open interval because below –1, the error is identically 100%; see earlier), and in the positive domain, it is strictly increasing up to the local maximum at $x \approx 2$, beyond which the function is strictly decreasing. As $x$ is the product of two variables, if we take one of them as fixed, $x$ changes in magnitude in the same direction as the other variable changes in magnitude. That is, for a given rate difference, $r - j$, $x$ increases/decreases in absolute value as the expected life, $\theta$ (which is assumed to be always positive), increases/decreases. Similarly, for a given expected life, $x$ increases/decreases in absolute value as the rate difference increases/decreases in absolute value. We can argue that, for realistic cases, the rate difference may rarely exceed 10% in absolute value (e.g., a 10% discount rate paired with a 0% growth rate) and the planning horizon (i.e., the expected life) may rarely exceed 20 years. These practical constraints correspond to a limit of $x = 2$ in the positive domain, approximately the point of the local maximum. In fact, this implies that the local error maximum is unlikely to be attained in practice. Under these tentative assumptions, the relevant range of $x$ can be confined to –1 to 2. (The range of –2 to –1 can be neglected, as the error is already at the theoretical maximum of 100% in that region.) Thus, recalling our monotonicity observations, the following rule of thumb can be stated: for a given rate difference, the larger the expected life – or, alternatively, for a given expected life, the larger the rate difference in absolute value – the larger is the magnitude of the error. A caveat must be added, however, for the sake of precision. Because the absolute value of the error function is not symmetric around $x = 0$ (i.e., in the negative domain of $x$, it is “steeper”), it may occur that, given an expected life, a negative rate difference has a larger error magnitude than what a more positive rate difference (i.e., the absolute value of which is larger than the absolute value of the negative difference) has. In other words, for a given expected life, a decrease/increase in the rate difference to the extent that it changes sign may result in an increase/decrease in the magnitude of the error even though the absolute value of the rate difference decreases/increases. Therefore, we make the clarification that by "a larger rate difference in absolute value (or magnitude)" we mean henceforth that a positive (negative) rate difference is made more positive (negative); thus, a change of sign is ruled out. (We note that this change of sign issue does not occur when the rate difference is given because the expected life is assumed to be always positive. Thus, that kind of formulation of our rule of thumb requires no such caveat.)

Figure 2.2 plots error functions for specific expected lives, to better illustrate the above points (regarding, for example, local error maximum attainable for every expected life and monotonicity under realistic constraints).

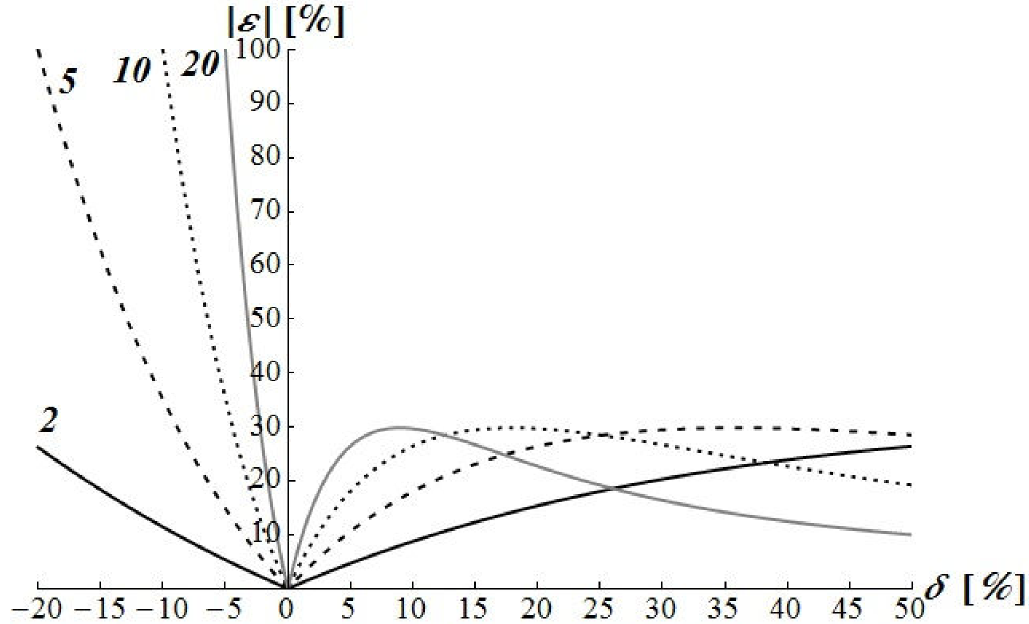



**Figure 2.2**: Absolute value of the relative error as a function of the rate difference $\delta = r - j$ for expected lives of $\theta = 2$ (solid), 5 (dashed), 10 (dotted), and 20 (gray).

As a side-observation, we note that similar conclusions can be drawn using the Taylor series approximation formula in (2.13). Knowing that the variance of an exponentially distributed random variable is the square of its expected value, we can approximate the error function (omitting here tedious algebraic manipulations) as

$$\varepsilon \approx \frac{x^2}{2e^x - 2 - x^2} \tag{2.20}$$

which behaves similarly to the exact function in (2.19) and also has a single local maximum in the positive domain, at $x \approx 1.59$, with an approximate value of the local error maximum of 48%. These results can be seen as close to those obtained via the precise analysis.

Finally, we identify the advantage of working with the relative error, or more precisely, why the local maximum cannot be obtained using the absolute error. The absolute error can be written as (following Chen and Manes, 1986)

$$\varphi = \hat{P} - E(P) = \frac{C}{r-j}\left(1 - e^{-\theta(r-j)}\right) - \frac{C}{r-j}\left(\frac{\theta(r-j)}{1+\theta(r-j)}\right) = C\frac{1-\left(1+\theta(r-j)\right)e^{-\theta(r-j)}}{\left(1+\theta(r-j)\right)(r-j)} \quad (2.21)$$

It is obvious that (2.21) cannot be rewritten as a function of a single variable; in fact, it is a function of at least three variables because the cash flow parameter cannot be eliminated. The system of equations in which the partial derivatives are all zero cannot be solved; therefore, the necessary conditions for the existence of a local extremum are not fulfilled. This is plausible because several absolute errors exist for which the same relative error is attained, and thus, although the relative error is maximal, the absolute error can be arbitrarily large.

## III.3. Discrete case

### *III.3.1. Derivation of present value formulas*

We recall from the introduction that, in the discrete case, we assume a geometric gradient series cash flow pattern. Thus, the evolution of cash flows is defined mathematically as

$$F_n = F_1(1+g)^{n-1} \quad (2.22)$$

where $g$ is a non-stochastic and time-invariant discrete growth rate.

We also note an interesting similarity between the continuous and discrete growth patterns used in our analysis: the "period-wise" integrals of the exponential cash flow function in (2.7), that is, the series of integrals over successive periods, produces a geometric series (i.e., also an exponentially growing series, but in discrete time). To show this, we first compute $F_n$ as a period-wise integral of (2.7) as

$$F_n = \int_{n-1}^{n} Ce^{jt}\,dt = e^{j(n-1)}\frac{C}{j}\left(e^{j} - 1\right) \quad (2.23)$$

From this, we can evaluate the discrete growth rate (for any $n$) as

$$1+g=\frac{F_n}{F_{n-1}}=\frac{e^{j(n-1)}}{e^{j(n-2)}}=e^j \qquad (2.24)$$

which completes the demonstration. Notice that, not surprisingly, we arrived at the same relationship between $g$ and $j$ as that between $i$ and $r$ in (2.3).

Returning to (2.22), the present value of $F_n$, as of time zero, is computed as

$$P_n=F_n(1+i)^{-n}=\frac{F_1}{1+g}\left(\frac{1+g}{1+i}\right)^n \qquad (2.25)$$

where $P_n$ denotes the present value of $F_n$.

Thus, $P_n$ represents a geometric series, and using the well-known formula for the sum of a geometric series, the present value of the series of cash flows (shown previously in (2.1)) can be given as (e.g., Park and Sharp-Bette, 1990; Remer et al., 1984)

$$P=\frac{F_1}{1+i}\frac{\left(\frac{1+g}{1+i}\right)^N-1}{\frac{1+g}{1+i}-1}=\frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^N\right) \qquad (2.26)$$

(We have omitted the $d$ index, as it is clear that we are discussing the discrete case in this section.)

Recalling that the life of the asset, $N$, is a random variable, under the conventional approach, the present value is calculated as

$$\hat{P}=\frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^{E(N)}\right) \qquad (2.27)$$

The correct calculation, by contrast, is

$$E(P)=E\left(\frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^N\right)\right) \qquad (2.28)$$

Note that the present value in (2.26) is a function, i.e., a transform of the random variable $N$. As both $g$ and $i$ are assumed to be greater than –1, the quotient $(1 + g)/(1 + i)$ must be positive. Thus, assuming a positive $F_1$, it can be established from the second (partial) derivatives with respect to $N$ (assuming for the moment that $N$ is a continuous variable) that the present value function in (2.26) is strictly convex if and only if $i < g$ and strictly concave if and only if $i > g$. Now, based on Jensen's inequality, the direction of the bias between (2.27) and (2.28) can be established. That is, $\hat{P} < E(P)$ if and only if $i < g$; then, the conventional approach provides an understated present value. Conversely, $\hat{P} > E(P)$ if and only if $i > g$; then, the conventional approach provides an overstated present value. (In the case of negative $F_1$, i.e., negative present values, just the opposite is true.) Equality occurs when $i = g$, although then the second (partial) derivative and the closed form present value formula in (2.26) are both undetermined due to division by zero. This is actually the result of the violation of the convergence criterion for the sum of the geometric series, namely, that the ratio of successive terms must be less than 1 (in absolute value). Recalling the present value formula for the individual terms of the series in (2.25), we obtain in this case, also by substituting $i$ for $g$ (see, e.g., Park and Sharp-Bette, 1990; Remer et al., 1984)

$$P_n = \frac{F_1}{1+i} \; \forall n \geq 1, \text{ from which } P = N\frac{F_1}{1+i} \tag{2.29}$$

Because both $F_1$ and $i$ are non-stochastic, it is apparent that $\frac{F_1}{1+i}E(N) = E\left(N\frac{F_1}{1+i}\right)$, that is, $\hat{P} = E(P)$. Thus, the conventional approach is perfectly accurate if and only if $i = g$.

It is important to note that the findings thus far are the same as those that have been established in the continuous case and are again independent of the actual probability distribution of asset life.

If we rewrite the expected present value in (2.28) as

$$E(P) = \frac{F_1}{i-g}\left(1 - E\left(\left(\frac{1+g}{1+i}\right)^N\right)\right) \tag{2.30}$$

we see that the expectation term is actually the probability-generating function of $N$. Thus, knowing the distribution of $N$, and provided that the power series representation corresponding to its distribution converges, the exact formula for the expected present value can be given.

Alternatively, we can again retreat to an approximation based on a Taylor series. Assuming for the moment that $N$ is a continuous variable and expanding the expectation term about $N = E(N)$ and retaining only first and second order terms, we have

$$E\left(\left(\frac{1+g}{1+i}\right)^{N}\right)\approx\left(\frac{1+g}{1+i}\right)^{E(N)}+\left(\frac{1+g}{1+i}\right)^{E(N)}\frac{\ln^{2}\left(\frac{1+g}{1+i}\right)}{2}V(N) \tag{2.31}$$

where $V(N)$ is the variance of $N$. Notice that (2.31) can, again, be interpreted as an adjustment to the conventional approach, $\hat{P}$ (similar to (2.14) in the continuous case). Specifically,

$$E(P)\approx\hat{P}-\frac{F_{1}}{i-g}\left(\frac{1+g}{1+i}\right)^{E(N)}\frac{\ln^{2}\left(\frac{1+g}{1+i}\right)}{2}V(N) \tag{2.32}$$

Similarly to the continuous case, an alternative formulation for computing the expected present value, $E(P)$, can be stated. Building on the definition of expected value for a discrete random variable, we seek the solution of

$$\sum_{n=0}^{\infty}p_{P}(n)P\big|_{N=n}=E(P) \tag{2.33}$$

where $p_P(n)$ is the probability mass function (pmf) of the present value $P$.

Again, because $P$ is a transform of the random variable $N$, (2.33) can be rewritten, by substituting (2.26) into it and exploiting the fact that the sum of the infinite series of a pmf equals 1 by definition, as follows (see, e.g., Bean, 2001):

$$\sum_{n=0}^{\infty}p_{N}(n)\frac{F_{1}}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^{n}\right)=\frac{F_{1}}{i-g}\left(1-\sum_{n=0}^{\infty}p_{N}(n)\left(\frac{1+g}{1+i}\right)^{n}\right)=E(P) \tag{2.34}$$

where $p_N(n)$ is the pmf of life $N$. The parameters in (2.34) may, of course, be required to meet some restrictive conditions in order to achieve convergence of the sum of the infinite series, that is, for the expected present value to exist. Notice that this power series is, in fact, the probability-generating function of $N$, so we have arrived at a formulation equivalent to (2.30).

We further note that the technique of the Z-transform (which can be regarded as the discrete-time equivalent of the Laplace transform) can also be used to compute the expected present value. For a discussion of this approach see, e.g., Buck and Hill (1974), Tanchoco and Buck (1977).

Recalling our assumption that asset life is geometrically distributed and substituting the corresponding probability-generating function into (2.30), we find the closed form solution for the expected present value (see, e.g., Gerchak and Åstebro, 2000):

$$E(P)=\frac{F_1}{i-g}\left(1-\frac{\alpha\frac{1+g}{1+i}}{1-(1-\alpha)\frac{1+g}{1+i}}\right)=\frac{F_1}{i-g}\left(\frac{1}{1+\frac{1+g}{\eta(i-g)}}\right) \tag{2.35}$$

where $\alpha$ is the parameter of the geometric distribution, and $\eta$ denotes the mean of the geometric distribution, that is, the expected life of the asset (in other words $E(N) = \eta$), which is related to the distribution parameter via $\eta = 1/\alpha$. There are actually two versions of the geometric distribution – we use the one for the domain of positive integers; that is, asset life cannot be zero (see Gerchak and Åstebro, 2000).

Verifying the convergence criteria from (2.34), we find, exploiting the fact that expected asset life must be positive, that $\left|\frac{1+g}{1+i}\right|<\frac{\eta}{\eta-1}$ must hold in order for $E(P)$ to exist; otherwise, $E(P) = \infty$. Because both $g$ and $i$ are assumed to be greater than –1, the absolute value sign can be omitted, as the ratio is always positive. Note also that $\eta$ cannot be zero, as asset life is assumed to be at least 1 period.

### *III.3.2. Evaluation of relative error*

Dividing (2.27) by (2.35) and subtracting 1, we obtain the relative error, after considerable algebraic manipulation:

$$\varepsilon=\frac{\frac{F_1}{i-g}\left(1-\left(\frac{1+g}{1+i}\right)^{\eta}\right)}{\frac{F_1}{i-g}\left(\frac{1}{1+\frac{1+g}{\eta(i-g)}}\right)}-1=\left(1-\left(\frac{1+g}{1+i}\right)^{\eta}\right)\left(1+\frac{1}{\eta\left(\frac{1+i}{1+g}-1\right)}\right)-1 \tag{2.36}$$

Notice that the relative error, similar to that of the continuous case, is independent of the cash flow parameter $F_1$. More important, however, is the fact that, in contrast to the continuous case in (2.18), the error function in (2.36) cannot be rewritten as function of a single variable. This is clear from the fact that $\eta$ is an exponent in one term and a multiplier in another. Due to this limitation, the analysis of the discrete case is more involved than that of the continuous case. However, the above formulation helps us recognize that (2.36) can be rewritten as a function of two variables, if we substitute into the expression $y = (1 + g)/(1 + i)$. Then,

$$\varepsilon = \left(1 - y^{\eta}\right)\left(1 + \frac{1}{\eta\left(\frac{1}{y} - 1\right)}\right) - 1 \tag{2.37}$$

It follows from the previous discussion that if $y \geq \eta/(\eta - 1)$, then, because $E(P)$ equals infinity, the error is –100%; and if $y = 1$, which occurs when $i = g$, the error is zero. Note also that for $\eta = 1$, the error is also always zero (in this case, we have a degenerate distribution similar to $\theta = 0$ in the continuous case).

Although (2.37) is a function of two variables, in essence, it is very similar to the single-variable error function (2.19) in the continuous case. To see this, substitute $z = \eta(1/y - 1)$ and apply a first order Taylor series approximation to show that

$$e^{-z} = \left(e^{\frac{1}{y}-1}\right)^{-\eta} \approx \left(1 + \frac{1}{y} - 1\right)^{-\eta} = y^{\eta},$$

from which (2.37) can be rewritten as

$$\varepsilon \approx \left(1 - e^{-z}\right)\left(1 + \frac{1}{z}\right) - 1 \tag{2.38}$$

which has the same form in $z$ as (2.19) has in $x$. Thus, we can expect results in the discrete case to be similar to those we found in the continuous case.

Figure 2.3 shows the plot of the absolute value of the error function of two variables, given in (2.37), for specific values of $\eta$. Again, we take the absolute value of the error to express the magnitude. The issue of sign has been discussed above, in reference to Jensen's inequality.

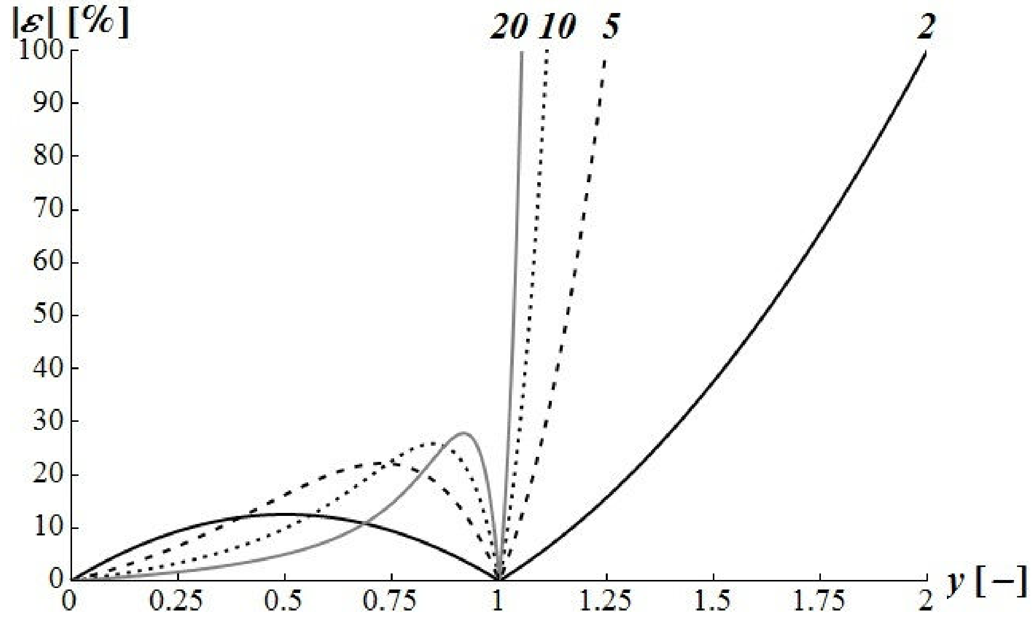



**Figure 2.3**: Function of the absolute value of the relative error in variable $y = (1 + g)/(1 + i)$ for expected lives of $\eta = 2$ (solid), 5 (dashed), 10 (dotted), and 20 (gray).

As Figure 2.3 shows, for every $\eta$, for $y < 1$, the relative error has a local maximum, the value of which is conditional on $\eta$. Because we take the absolute value of the error, the global maximum is 100% for every $\eta$. In the domain where $y > 1$, the error is very sensitive to $y$. These observations closely accord with those for the continuous case, with the difference that in the discrete case the shape of the error function, more precisely, the value of the local maximum, varies (slightly) with $\eta$. The main characteristics, however, are the same – that is, for a given expected life, the discrete and continuous errors behave identically. This is confirmed when we examine more closely the composition of the variable $y$. If $y < 1$, which occurs if and only if $i > g$, there exists a local error maximum. The error is very sensitive to values of $y > 1$, which occur if and only if $i < g$. The error is zero when $y = 1$, which occurs if and only if $i = g$. (Our assumptions do not allow $y$ to equal 0.)

The main difference between the discrete and the continuous case, which can be observed in Figure 2.3, is that in the discrete case, the value of the local error maximum is not the same for every expected life, as was the case in the continuous case. For example, for an expected life of 5 periods, the value of the local error maximum is approximately 22%, in contrast to the approximately 30% attainable in the continuous case. Figure 2.3 also shows that (and this can be verified by examining the partial derivatives) the location and the value of the local maximum are both increasing in $\eta$. That is, they are lowest for $\eta = 2$, when the value of the local error maximum is 12.5%, obtained at $y = 0.5$, and highest as $\eta$ approaches infinity. For example, for $\eta$ = 1,000, the value of the local error maximum is 29.8%, which is the same maximum value we found in the continuous case and is obtained at $y$ = 0.998. It is intuitive that the largest possible value of the local error maximum in the discrete case should be the same as that in the continuous case because, from the "view of infinity," the length of the discrete periods is infinitesimal, that is, the discrete case looks to be continuous. Therefore, we can establish that the error in the discrete case cannot exceed 30% if $i > g$. We also note that this theoretical extreme is approached quite rapidly, e.g., for $\eta$ = 20, the value of the local error maximum is 27.8%.

Additionally, it is important to recognize that because both $i$ and $g$ are typically small (i.e., close to zero), the ratio $y$ behaves quite analogously to the rate difference $i - g$. (This can be verified, e.g., via a first-order Taylor series approximation of the ratio in two variables.) That is, the larger the difference in magnitude, the further the ratio is away from 1, where the error is zero. Thus, the results so far accord with those established in the continuous case. That is, for a longer expected life, the rate difference must be smaller to attain the local error maximum. Thus, as $\eta$ increases, the point of the local maximum moves closer to $y$ = 1 (see in Figure 2.3).

In summary, we can draw conclusions very similar to those we drew in the continuous case. If the discount rate exceeds the growth rate, then the error cannot exceed 30%, but a maximum of at least 12.5% can be attained for every expected life $\eta$ (except for 1), as there exist several $i - g$ combinations for which the extremum condition is satisfied (in fact, infinitely many because both $i$ and $g$ can be real numbers). This is because the parametric equation $y = (1 + g)/(1 + i)$ by itself is an underdetermined system of equations and thus has infinitely many solutions. In other words, the real number $y$ can be written as the ratio of two real numbers in infinitely many ways. If the discount rate is smaller than the growth rate, then the upper limit of the error is 100%, which can be attained for every expected life. Figure 2.3 clearly shows that, in this case, the error is very sensitive to the ratio of the rates. When the two rates are equal, the error is zero, as discussed earlier.

It is also worthwhile to examine some limiting cases of the error function. If the expected life converges to infinity, the error is meaningful if and only if $i > g$, which becomes the convergence criterion of the series composing the present values, and if this criterion holds, the error is indeed zero. Otherwise, neither the conventional nor the correct present value exists (i.e., they are both infinity). This is the same observation we made in the continuous case, and again it reflects that uncertainty does not matter in case of an asset that is expected to be infinitely lived (provided, of course, that its present value exists). Convergence of either the discount rate or the growth rate to either positive or negative infinity yield meaningless results in the discrete case as well. If the discount rate converges to positive or negative infinity, all terms in the series are zero (see (2.25)), so both the conventional and the correct present values are also zero; thus, the relative error, which is a ratio, is not meaningful. If the growth rate converges to positive or negative infinity, all terms in the series are infinitely large, so neither the conventional nor the correct present value exists.

Finally, the same rule of thumb that we formulated for the continuous case can be formulated for the discrete case. Let us now recall the practical boundaries introduced in the evaluation of the continuous case. First, the rate difference may rarely exceed 10% in magnitude. Adding that the individual rates themselves are unlikely to exceed 20% in magnitude, this translates into a minimal $y$ of approximately 0.9. Second, the expected life may rarely exceed 20 years – for $\eta = 20$, the local maximum is obtained at $y \approx 0.9$, and for smaller values of $\eta$, it is obtained at smaller values of $y$ (see the previous discussion). Thus, we can establish that, under these practical constraints, the local maximum is unlikely to be attained, and thus it is sufficient to consider only the two monotonic ranges above and below $y = 1$. It can also be observed in Figure 2.3 (and verified by examining the partial derivative) that in this range, the error is increasing in $\eta$. In conclusion, for a given rate difference, the larger the expected life – or, alternatively, for a given expected life, the larger the magnitude of the rate difference – the larger is the magnitude of the error. This is the same rule as the one formulated above for the continuous case. The same caveat made in the continuous case applies here as well, that is, because the absolute value of the error function is not symmetric around $y = 1$, we clarify that "larger" as pertains to the magnitude of the rate difference means that the rate difference is "more positive (negative)".

Due to its complexity and limited relevance, we omit the discussion of the similarity of the results obtainable by using the Taylor series approximation formula in (2.31) compared to the precise analysis. Additionally, discussion of the difference between absolute and relative errors is omitted for the same reason.

## III.4. Discussion of results

As we have shown, the error characteristics in the continuous and the discrete case are quite similar. It might be argued that a discount rate higher than the growth rate is the typical case in real life. If so, then it is somewhat reassuring to know that, given any rate difference and asset life combination, the error cannot exceed 30%. However, it is discomforting that, at least theoretically, a non-negligible error maximum may occur for any expected life (or any rate difference). Generally, a long (short) expected life associated with a small (large) rate difference may give rise to such an extreme situation. As was argued, however, analysts may rarely encounter this error maximum in practice. If the growth rate exceeds the discount rate, the picture is much more dire. Then, the error has no limit except the theoretical maximum of 100% and, in addition, is more sensitive to the rate difference. The case of negative discount rates is particularly relevant here: then, even for very small, near-zero growth rates, the error may be severe, and for this to occur, the discount rate need not be very negative, as we show below. Perhaps the best way to demonstrate possible computational errors attributable to using expected asset life, rather than computing the expected present value, is via a graphical illustration. Figure 2.4 shows nomograms (two-dimensional diagrams) that exhibit the severity of errors for various discount rate and growth rate combinations for expected lives of 2, 5, 10, and 20 periods, in both the continuous and discrete cases. For a more tractable demonstration, we cut off rates above 20% in magnitude.

As the nomograms show, the continuous and discrete cases are very similar, as expected. It is alarming to observe that the error may easily exceed 10%, which can be considered significant, and that for that to happen, the rate difference need only be a few percentage points. For example, in the continuous case, for an expected life of 10 periods, rate differences of 2% and –2% produce errors of 8.8% and –11.4%, respectively. For an expected life of 20 periods, rate differences of 1% and –1% produce the same error percentages. As follows from our previous findings and as seen in Figure 2.4, as the expected life increases, the band of rate combinations associated with errors of less than 10% shrinks. That is, a given rate difference (typically) implies a more severe error, the longer is the expected life (as stated in our rule of thumb). These results call attention to the importance of precision in growth rate and discount rate estimation, as even a small percentage point deviation can have serious consequences. That is, for imprecise rate estimates the error may be found to be negligible, so the analyst would wrongly think that it is unnecessary to bother with computing the expected present value and would simply use the expected life. Whereas, the error may be much larger if the correct rates were used, and the computation of the expected present value would be clearly preferred. The nomograms presented here help assess such possible errors.

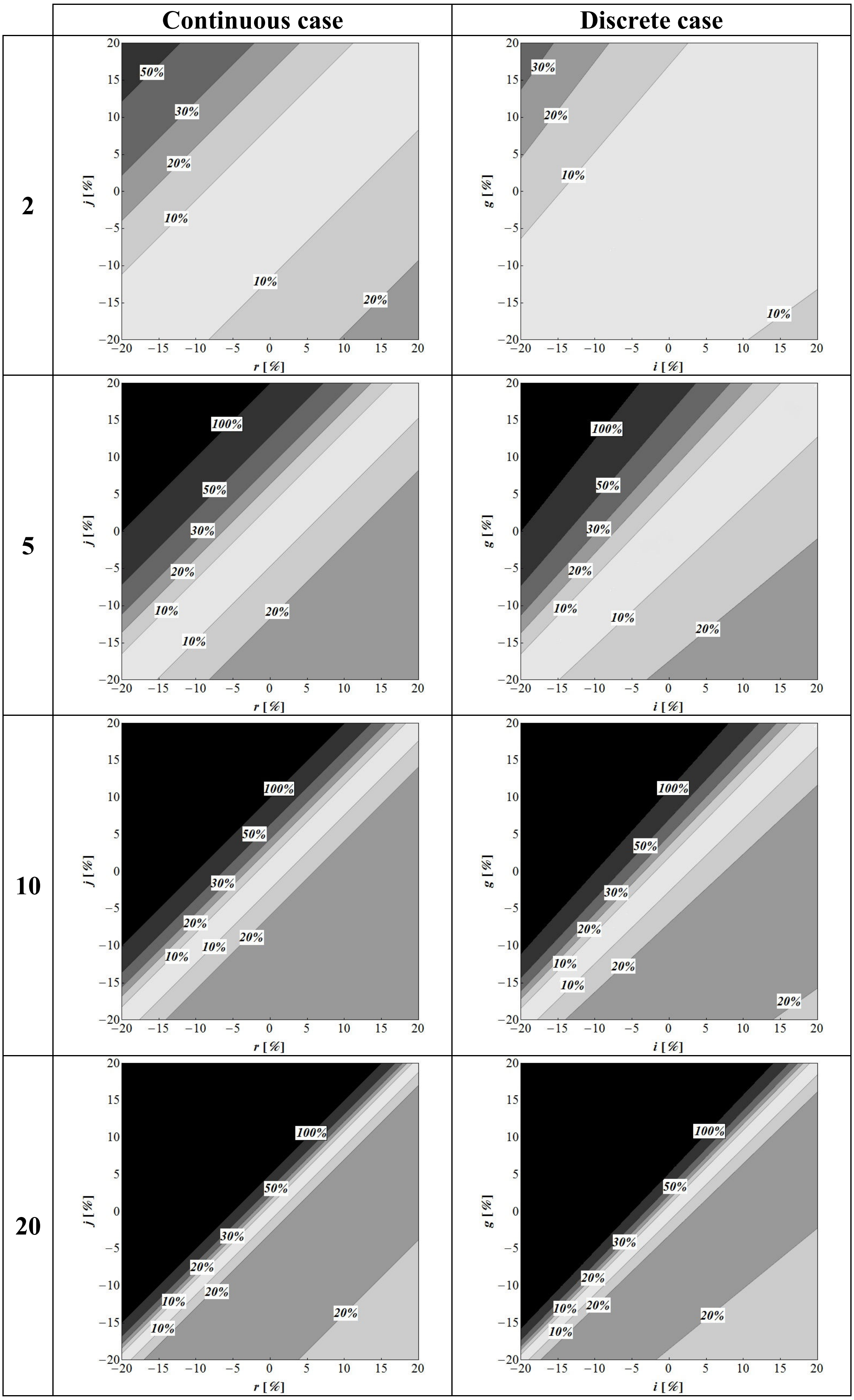



**Figure 2.4**: Comparative evaluation of errors for various expected lives and discount rate – growth rate combinations; darker colors indicate more severe errors.

Finally, we provide a brief illustrative example to present the error characteristics numerically. Consider an energy efficiency project, specifically, an installation of photovoltaic cells on the rooftop of a building, where the returns consist of a single cash flow stream resulting from savings on energy costs. (Other relevant cash flows, e.g., operation and maintenance costs, can usually be neglected for such projects, so consideration of the savings flows alone is a good working approximation.) Such a project can be fairly described as a growing annuity of cash flows, accounting for possibly rising fuel prices. The life of photovoltaic cells may be approximated by a memoryless probability distribution, such as the exponential or geometric distribution (e.g., Mishra and Joshi, 1996; Zini et al., 2011). We assume in this example a reasonable 20-year expected life for the project. As discussed in the introduction, several papers have argued for the counter-cyclical nature of energy prices and found empirically that related projects may have a negative beta in the CAPM. A negative beta may imply a negative discount rate, depending on one's assumptions about the market parameters (i.e., the risk-free rate and the expected market risk premium). Because some of these papers (e.g., Andor and Dülk, 2012; Bolinger et al., 2006; Metcalf, 1994) observe that the possibility of a positive beta generally cannot be ruled out for energy-saving projects (i.e., they cannot reject the null hypothesis of zero beta, cf. Dülk, 2012), and because Andor and Dülk (2012) conclude that the risk-free rate is generally a fair approximation for the cost of capital for such projects, we consider a narrow range of discount rates around zero. The growth rate is examined in a similarly narrow range, as it is not expected to be large for an extended period of time. For reference, some negative growth rates are also included. We consider one year as the interest period. We note that the intraperiod pattern of cash flows may not be precisely described as an exponential growth in the continuous case, or as an end-of-period lump sum in the discrete case, but the error attributable to this issue is negligible for the small discount rates we use (see Andor and Dülk, 2013 for a thorough discussion of this topic). Table 2.1 summarizes the errors for the various rate combinations.

As Table 2.1 shows, the errors are not negligible and may even be severe for the realistic scenarios examined. This is not surprising in light of our earlier results, as here we are dealing with a long life combined with a small rate difference. Perhaps the most realistic scenario is the 2% risk-free rate as the discount rate and a zero growth rate (if we are thinking in real terms), for which the error is approximately 15%, which is significant. Therefore, computation of the expected present value is desirable in the case of such an energy efficiency project.

**Table 2.1**: Relative errors (rounded to integers) for various discount rate and growth rate combinations for the illustrative project with an expected life of 20 years. A positive error means overstatement, and a negative error means understatement, of the correct present value.

| **Continuous case** | | | | | | | |
|---|---|---|---|---|---|---|---|
| ***j* \\\\ *r*** | **-6%** | **-4%** | **-2%** | **0%** | **2%** | **4%** | **6%** |
| **-5%** | -11% | 9% | 20% | 26% | 29% | 30% | 29% |
| **-3%** | -45% | -11% | 9% | 20% | 26% | 29% | 30% |
| **-1%** | -100% | -45% | -11% | 9% | 20% | 26% | 29% |
| **0%** | -100% | -69% | -26% | 0% | 15% | 24% | 28% |
| **1%** | -100% | -100% | -45% | -11% | 9% | 20% | 26% |
| **3%** | -100% | -100% | -100% | -45% | -11% | 9% | 20% |
| **5%** | -100% | -100% | -100% | -100% | -45% | -11% | 9% |
| **Discrete case** | | | | | | | |
| ***g* \\\\ *i*** | **-6%** | **-4%** | **-2%** | **0%** | **2%** | **4%** | **6%** |
| **-5%** | -12% | 9% | 20% | 25% | 27% | 28% | 27% |
| **-3%** | -46% | -11% | 8% | 19% | 25% | 27% | 28% |
| **-1%** | -100% | -45% | -11% | 8% | 19% | 25% | 27% |
| **0%** | -100% | -68% | -25% | 0% | 14% | 22% | 26% |
| **1%** | -100% | -98% | -43% | -11% | 8% | 19% | 25% |
| **3%** | -100% | -100% | -95% | -42% | -11% | 8% | 19% |
| **5%** | -100% | -100% | -100% | -92% | -41% | -10% | 8% |

As a complement to practical recommendations, we note that for the valuation of companies the issue of expected present value is probably not relevant, as such valuations typically assume an infinite life, along with the condition that the discount rate exceeds the growth rate, and, recalling our related findings, in this case the error is zero.

## III.5. Conclusions

We have shown that if the economic life of an asset is uncertain, the expected present value should be calculated instead of simply substituting the expected life of the asset into the present value formula. The computational error attributable to the latter approach depends on the cash flow pattern of the asset, the cost of capital, and the probability distribution of the asset life. We have evaluated relative errors for the case of a continuous exponential cash flow pattern and exponentially distributed life, as well as for their discrete equivalents of a geometric gradient cash flow series and geometrically distributed life, for various costs of capital. By using the relative error we have gained insights that previous studies have not achieved because they were working with absolute, rather than relative, error. Most notably, we found a local error maximum if the discount rate exceeds the growth rate and showed that this maximum can be attained for any expected life. We are the first to present a detailed comparison of the discrete and continuous cases and to demonstrate that very similar conclusions can be drawn in the two cases. The most essential difference concerns the local maximum, the value of which is found to vary with the expected life in the discrete case, while it is invariant in the continuous case. More concretely, we found that if the discount rate exceeds the growth rate, then the error cannot be larger than 30% in either the discrete or the continuous case, and the correct present value is overstated. This value of the local error maximum can be attained for every expected life in the continuous case, but it can be attained only for relatively long (approximately 20 periods or longer) expected lives in the discrete case. In the discrete case, the value of the local error maximum is smaller for a shorter expected life and is found to be at least 12.5%, except for an expected life of one year, when the error is zero. The error is also always zero if the discount rate is exactly equal to the growth rate. We found that the error may easily become severe if the growth rate exceeds the discount rate and may even reach the theoretical maximum of 100%, both in the discrete and continuous case. The correct present value is then understated. Setting practical constraints on the possible range of rates and expected lives, we have formulated the following general rule of thumb: for a given difference between the discount rate and the growth rate, a larger expected life – or, alternatively, for a given expected life, a larger rate difference in magnitude – results in a larger magnitude of the error. Our findings call attention to the need for precision in discount rate and growth rate estimation, as, for realistic cases, even a single percentage point rate difference may result in a non-negligible error. An empirically supported illustrative example was presented to highlight possible errors, including cases with negative discount and/or growth rates. Analyses similar to that presented here could be conducted for other cash flow patterns and probability distributions.

# IV. COST OF CAPITAL OF ENERGY EFFICIENCY PROJECTS: THE CASE OF SPACE HEATING AND COOLING[2]

## IV.1. Introduction

Energy consumption worldwide has nearly doubled since the 1970s, with fossil fuels still being the fundamental source of energy (IEA, 2010). The scarcity of natural resources and the consequences of the fossil-dominated energy mix (e.g., pollution, climate change) have placed energy efficiency at the forefront of global challenges. For instance, one of the key objectives of the EU2020 strategy is to reduce the European Union's (EU) energy consumption by 20% by 2020. Endeavors are aimed not only at business entities but households as well because the latter group has a share of about 25% in total final energy consumption in Europe (Eurostat, 2010). One particular area of possible energy saving is space heating and cooling. Space heating typically accounts for about 70% of a household's energy consumption in the EU (Market observatory for energy, 2010; Odyssee-Mure project, 2009); cooling is rather negligible accounting for less than 1% (Atanasiu and Bertoldi, 2007). However, the global air conditioning market has been expanding in the last decade and is expected to continue to expand in the coming years (BSRIA, 2008; GIA, 2010), raising the importance of cooling energy efficiency. Studies have shown considerable energy-saving potential related to space heating and cooling in both the residential and commercial sectors globally (Levine et al., 2007; Novikova and Ürge-Vorsatz, 2008). From a technological perspective, space heating and cooling energy can be saved by employing more efficient equipment, such as condensation furnaces, inverter air conditioners, and walls and windows with better thermal insulation, among others. There are, of course, other areas of possible energy saving such as lighting, electrical appliances, and so on. Here, however, we focus only on projects that reduce space heating or cooling energy consumption, which are responsible for the majority of residential energy use. We examine also whether there is any difference if a business appraises such projects, because businesses may also be interested in cutting their costs this way.

Replacing the existing technologies with more efficient ones requires capital investment; thus, financial analysis must be conducted to determine if it is worth upgrading. It is often stated that such energy efficiency projects are not profitable by themselves, and therefore, they would not be undertaken without subsidies from either the local government or

[2] The work reported in this main chapter is connected to the scientific program of the "Development of quality-oriented and harmonized R+D+I strategy and functional model at BME" project. This project is supported by the New Széchenyi Plan (Project ID: TÁMOP-4.2.1/B-09/1/KMR-2010-0002).

other organizations such as the EU (Amstalden et al., 2007). It is mainly because the potential cut in energy bills is not considered to sufficiently compensate the high price of new technologies. Capital budgeting is most often performed in a discounted cash flow (DCF) framework, in which the expected values of future cash flows are estimated and discounted to their present values by the cost of capital. Thus, the net present value (NPV) of the project is derived, and the project is considered economically favorable and acceptable if its NPV is positive.

Estimating future cash flows is relatively straightforward, but determining the appropriate cost of capital is more problematic; therefore, we focus on this issue. The cost of capital indicates how much it costs to devote funds to a particular project. It is an opportunity cost because it reflects the benefits lost from other opportunities that are foregone for the sake of the given project. The lost opportunities are considered to be those available in capital markets – more precisely, the expected returns on those investments; hence, the cost of capital is given as an interest rate. However, opportunities in capital markets differ by their riskiness. Therefore, the cost of capital must be the expected return on an investment with risk identical to that of the project. In finance, investors are typically assumed to hold a portfolio of investments. Thus, according to portfolio theory, an asset's risk relevant to the investor is lower than its risk if it is held alone. This means that the perceived, relevant risk of an energy efficiency project is lower in a portfolio than by itself (called total risk) because a portion of the total risk is eliminated in the portfolio. This elimination is attributable to the underlying correlations between the portfolio elements and the project. We argue that although space heating/cooling energy efficiency projects might have high total risk, this risk can be considered to be completely eliminated in a well-diversified portfolio; thus, an investor can see such projects as risk-free opportunities and consequently employ a risk-free discount rate. Choosing a rate according to the total risk yields a smaller NPV because the cost of capital is in the denominator and is higher for a risky project than a risk-free one. Thus, the profitability of space heating/cooling energy efficiency projects could be significantly undervalued, possibly leading to the erroneous rejection of positive NPV projects.

Estimating the risk of a project is a difficult task. It is known that risk indicates possible deviations from the expected, which, in the case of projects, indicates the possible deviations of future cash flows from their expected values. In our analysis, we start by identifying the cash flows of space heating/cooling energy efficiency projects and their sources of uncertainty. We find that such projects' risk stems fundamentally from the variability of energy bills, which are comprised of two principal elements: energy consumption (which depends on the weather) and energy prices. We examine natural gas (henceforth: gas) and electricity prices as two important energy carriers used for heating/cooling purposes. Then, we discuss what effects the weather and price uncertainties have in the Capital Asset Pricing Model (CAPM), which is a generally applied framework for cost of capital estimation. Based on data from a number of European countries, we show that the hypothesis that the appropriate discount rate is the risk-free rate for both households and businesses cannot be rejected and argue that the same conclusion is likely to be drawn for projects in other countries as well as for other types of energy efficiency projects.

## IV.2. Cash flows of space heating/cooling energy efficiency projects

An energy efficiency project typically means replacing existing technology with a better one. This implies the following cash flows, which are listed chronologically:

- Investment in the new technology (includes price of equipment and installation costs)
- Salvage value of existing technology
- Difference in operation and maintenance costs (O&M)
- Difference in taxes
- Savings in energy bills
- Salvage value of the new technology

As noted earlier, the initial investment in the new technology may be large. It might be partly offset by the salvage value of existing technology, but there is often no such value because it cannot be sold (e.g., old thermal insulation material). The difference between the O&M costs of the old and new equipment can well be regarded as negligible. Taxes are the particular elements that make the distinction between a household's and a business' perspective: Households have to pay everything in gross terms (that is, including value-added tax [VAT]), whereas for general businesses, only the net amount is relevant. However, businesses have to pay other types of taxes, e.g., corporate taxes, which are affected by the project through the depreciation of new technology and cost reduction. The way corporate taxes are calculated depends on a country's legislation, which is largely unique to that particular country. Therefore, we omit any specific calculation example and the tax effects for businesses to provide a more generally valid discussion. The savings in energy bills are probably the most crucial among the cash flows because they represent the major source of cash inflows. We note here that for households, the difference in taxes is incorporated into the energy cost savings because households pay the bills in gross terms. The salvage value of the new technology at the end of its life can also be neglected similarly to the old one or can be accounted for only in the decision about a later upgrade as the then-existing technology.

Now that the list of cash flows is assembled, we can assess the risks associated with them. The initial investment and salvage value of existing technology (if any) are paid upfront at the beginning of the project and thus do not affect profitability over the project's life-span. Difference in O&M costs is paid over time but is assumed to be negligible. Taxes may change if the tax regime changes – we argue that this does not happen often or to a significant extent, and thus, this risk factor can also be neglected. The salvage value of the new technology can also be omitted for reasons previously mentioned. The one item remaining is the energy bill, which has the greatest impact on the attractiveness of the project. As noted in the introduction, the annual savings depend on the amount of consumption on the one hand, which is a function of the weather, i.e., how cold the winter is or how hot the summer is. On the other hand, savings depend on the price of energy, which may change considerably over time.

Heating and cooling energy consumption is influenced mainly by the following factors: the difference between the desired indoor and the actual outdoor temperature, the construction of the building, and the efficiency of the equipment used. Usually, all of these parameters are constant or can at least be regarded as constant over time, except for the actual outdoor temperature, which varies continuously according to climate characteristics. This means that one source of project risk is the uncertainty related to the outdoor temperature. We note here that there are relatively easy-to-use engineering models (see, e.g., Dülk, 2010 for illustration) that apply Gaussian normal distribution to the outdoor temperature to estimate energy consumption.

Determining the price of energy is not as simple as it may seem. As noted earlier, we focus only on two groups of prices: gas and electricity. The price the end-user pays for these energy types is comprised mainly of two elements: the price of energy itself and the fee of delivering it to the end-user, i.e., distribution fees. The distribution network is, in most cases, operated as a natural monopoly; thus, such fees are regulated. This means that there is a somewhat uniform price component, which relatively seldom changes. The price of energy itself may also fall under regulation, as is often the case with households, but it may also be a market price shaped by supply and demand. The fact that there are many sorts of energy "products" traded makes the picture more complex. For example, there are forward contracts, which sell gas or electricity months or years in advance. There is obviously a different price for the same kWh sold one year or two years ahead; moreover, the price for the same product varies by exchanges. Because electricity cannot be stored, its pricing is more complicated than that of natural gas. Furthermore, energy traders, who sell the energy to end-users, purchase a multitude of different products and typically give a unique quotation according to each client's own consumption characteristics. We do not intend to go into more detail about the pricing of energy carriers, and therefore, we point out only two key difficulties: it is not unequivocal which product's price to use for analysis; and there is usually a unique price for each consumer. We believe that the best way to bridge these problems, in our case, is to determine an average "typical" price charged in the past. This means first identifying a group of consumers deemed to be representative and then collecting data on the prices they paid. Such international samples have been collected by, e.g., the International Energy Agency (IEA) and Eurostat. We argue that such prices provide a fair approximation of gas and electricity prices relevant to space heating/cooling energy efficiency projects.

Now that the factors of uncertainty have been established, we examine how they influence the project's cost of capital.

## IV.3. Cost of capital of space heating/cooling energy efficiency projects

### *IV.3.1. Theoretical background*

As outlined in the introduction, the risk of any asset is to be analyzed as part of a portfolio and not by itself – except in special cases, when investors cannot or do not want to hold portfolios. Theory holds that combining different securities into a portfolio offers risk-reduction opportunities with possibly no sacrifice of return. This is called diversification, and if the creation of portfolios is free of costs, then a rational investor does diversify. In practice, there are costs associated with diversification, e.g., transaction costs of buying securities, but they are typically small, and therefore, holding many investments is still desirable. The return on risky investments is commonly assumed to be normally distributed; thus, an investment is defined by its expected return and the standard deviation of the return, which indicates its risk. This is also referred to as mean-variance framework. A portfolio of any $n$ pieces of investment opportunities is then characterized as follows:

$$E(r_P) = \sum_{i=1}^{n} a_i E(r_i) \text{, and } \sigma(r_P) = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma^2(r_i) + 2\sum_{i,j}^{n,n} k_{i,j} a_i a_j \sigma(r_i)\sigma(r_j)}\ ,$$

$$\text{with the conditions } \sum_{i=1}^{n} a_i = 1 \text{, and } i \neq j \text{, and } i,j \neq j,i \qquad (3.1)$$

where $i$ and $j$ are indices for securities in the portfolio $P$, $a$ denotes the weight in the portfolio, $E(r)$ denotes the expected return, $\sigma(r)$ denotes the standard deviation, and $k_{i,j}$ denotes the correlation between investment $i$ and $j$.

Equation (3.1) shows that if $k_{i,j} = 1$ for any pair of $i$ and $j$, then no risk can be diversified away, and the risk of the portfolio is simply the weighted average of the standard deviations of the individual investments. However, correlations other than 1 provide incentive for diversification, and investors would combine all available investments to take full advantage of diversification. It can be shown that the smaller correlation an investment has with the other assets, the greater risk-reduction benefits it provides. The portfolios that offer the highest possible expected return for a given level of risk or, equivalently, offer a given expected return for the lowest possible risk are called efficient portfolios.

The Capital Asset Pricing Model (CAPM) was formulated on the foundations of portfolio theory with the critical assumption, among others, that investors hold an efficient portfolio; in particular, a combination of the risky market portfolio and a risk-free investment. The composition of the market portfolio is the same for every investor and contains all the available investments in the world with the same weights as they are represented in the world. The risk-free investment is, in practice, approximated mostly by U.S. treasury bills. The CAPM states that there is a linear relationship between any security's expected return and its risk (graphically referred to as the security market line [SML]):

$$E(r_i) = r_f + \beta_i \left(E(r_M) - r_f\right) \tag{3.2}$$

where $r_f$ is the risk-free rate, $E(r_M)$ is the expected return on the market portfolio $M$, and $\beta_i$ is the risk parameter of investment $i$.

The most crucial parameter of the CAPM is $\beta$, which determines the relevant risk of the investment. As mentioned earlier, in the CAPM, investors hold a combination of the risk-free investment and the market portfolio. The risk-free asset has zero correlation with the other assets and zero standard deviation, and therefore, it is not involved in diversification. Thus, in the CAPM, each investment opportunity is evaluated according to its impact on the risky market portfolio. The relevant (systematic) risk of an investment is that portion of its total risk that is not diversified in the market portfolio. This is determined by the correlation between the given asset and the market portfolio, as follows from (3.1). We note that in this theoretical framework, every asset is already part of the market portfolio, and therefore, comparing individual securities to the market is a simplification but is considered an acceptable approximation. Mathematically, the beta parameter of any investment is calculated as

$$\beta_i = k_{i,M} \frac{\sigma(r_i)}{\sigma(r_M)}, \text{ the relevant risk being } \sigma(r_i)_{relevant} = \beta_i \sigma(r_M) \qquad (3.3)$$

Because the cost of capital ($r_{alt}$) is the expected return of a capital market investment with identical risk, it is then given in the CAPM framework as

$$r_{alt} = r_f + \beta_{project}\left(E(r_M) - r_f\right) \qquad (3.4)$$

In financial practice, there is good consensus on the values of $r_f$ and $E(r_M)$, which are annually about 2% and 8%, respectively, in real terms (Andor and Tóth, 2009). Thus, practically, the beta of a project is the only parameter that needs to be determined to determine the cost of capital.

### *IV.3.2. Zero-beta hypothesis of space heating/cooling energy efficiency projects*

As (3.3) shows, $\beta$ is estimated from three parameters: the standard deviations of the return on the market portfolio and that of the project and the correlation between the two. These parameters reflect future expectations, but in practice, the betas of securities are calculated from historical data, with the assumption that the past is a good predictor for the future. Calculating the beta of public companies is a relatively simple task because there is usually a sufficiently long time series of past stock returns. There are also several indices available through which the market portfolio can be approximated. Projects, however, are not traded publicly, and therefore there are no price data available from which returns could be calculated. In financial practice, a project's beta is often estimated by taking an average of betas of public firms with risk characteristics similar to the given project, which typically means firms in industries similar to the project (Brealey and Myers, 1996; Damodaran, 2002). In our case, this suggests taking the average beta of firms in the energy sector or, more narrowly, that of electricity or gas companies. We argue that this approach yields a distorted estimate in our case because energy companies are more exposed to some risks to which our project is less exposed (e.g., from human resources, suppliers, and prices of other resources) and conversely may be less exposed to other risks that affect our project more (e.g., due to protection by regulation and governmental aid). We do not aim to exactly quantify the beta of a given space heating/cooling energy efficiency project. We argue that the beta of such projects should be taken zero, in contrast to the positive betas suggested by those of energy companies.

To test the hypothesis of a zero project beta, we focus on the correlation term in (3.3). If the correlation coefficient is zero, then the beta of the project is also zero. Correlation refers to the stochastic relationship between the return on the project and the return on the market portfolio. As noted earlier, unfortunately, there are no historical return data for our projects. However, there are data available for energy prices and weather, which represent the main sources of risk in space heating/cooling energy efficiency projects. We can state that if we find no correlation between any risk factor and the market portfolio or between any two risk factors, then the overall correlation of the project must also be zero. In our case, this requires the examination of three correlations: between gas/electricity consumption and the market portfolio, between the price of gas/electricity and the market portfolio, and between gas/electricity consumption and the price of gas/electricity.

*IV.3.2.a Energy consumption and weather equivalency assumption*

Unfortunately, there are no data publicly available regarding individual consumption in the past, from which correlations could be calculated. There are, however, detailed records on temperature, which could be used alternatively. Working with weather data instead of actual consumptions yields the same correlation results only if the two correlate perfectly with one another. This might not be the case if consumption is influenced by factors other than the weather. In fact, energy prices may have an impact on consumption, e.g., if the price of gas goes up, we may be inclined to heat less. However, it can be hypothesized that the amount of energy used for heating/cooling purposes is rather independent of prices because climatic comfort is considered a necessity, a price-inelastic good. This is supported by, e.g., the empirical study by Lee and Lee (2010), who show that the total energy and electricity demands are price-inelastic in OECD countries. Liu (2004) also confirms the inelasticities of natural gas and electricity demands, reporting that industrial consumers are more insensitive to price. Another supporting argument is that even if energy consumption is somewhat affected by energy price, it is probably dominantly more sensitive to climatic conditions. In conclusion, we make the assumption that energy consumption depends only on the weather; therefore, they correlate equivalently with any other variable. Consequently, in the next sections, we examine the relationship between energy prices, the market portfolio, and the weather.

*IV.3.2.b Correlation of weather and market portfolio*

It seems plausible that financial processes have no impact on outdoor temperature and conversely, that weather in any country has no impact on the world economy. Therefore, no correlation can be assumed without any calculation. Calculations in this respect are also unnecessary because a coefficient significantly different from zero could be accidentally measured, which would contradict common sense, as explained. Here, we note that this is also a reason for not measuring the correlation between hypothesized past energy bills and the market, although the variability of the project stems from the variability of energy bills. Calculating the correlation between energy bills and the market portfolio could provide false, misleading results because of the involvement of the weather. We remark, however, that a recent study by Yang et al. (2011) showed zero or little correlation among international weather indices and stock market indices, which supports our independence hypothesis.

*IV.3.2.c Correlation of energy prices and market portfolio*

To determine the correlation of gas and electricity prices with the market, we use the MSCI World Index in U.S. dollars as the proxy for a global market portfolio. The end-user energy prices are taken from the Eurostat database, which provides a collection of semi-annual prices (observed January 1 and July 1 every year) for various consumer groups, separately for domestic and industrial consumers. These prices are total prices, including distribution fees, the price of energy itself, and any related taxes. Data for years prior to 1991 provide prices only for the first semesters; therefore, we work with data starting from 1991. The method of data collection was changed in 2007; therefore, we use data for the 1991–2007 period to have a consistent time series. For households, we take the electricity prices of group “Dc” and gas prices of group “D3” in euros, with all taxes included. For businesses, we take the electricity prices of group “Ie” and gas prices of group “I3-1” in euros, excluding VAT. We choose these groups because they provide the largest number of observations and can be regarded as a fair approximation of an average consumer. We note here that although we previously argued for the negligibility of tax risks, we decide to include them for households because all such risks can be easily observed by simply considering gross prices. This is the reason why domestic consumer prices including all taxes are used.

For our analysis, we select those European countries that have at least 30 out of the 34 maximum possible observations. The following nine countries meet this requirement: Germany, France, United Kingdom, Italy, Belgium, Spain, Luxembourg, the Netherlands, and

Hungary. We convert prices to U.S. dollars using the average exchange rates for January and July in Eurostat. Then, we consider gas and electricity as investments and calculate the real log returns on them – an approach found in several related studies, e.g., Awerbuch and Deehan (1995), Bolinger et al. (2006), Metcalf (1994) – and on the market according to (3.5):

$$(\ln p_t - \ln p_{t-1}) - (\ln CPI_t - \ln CPI_{t-1}) = r_{t,real} \qquad (3.5)$$

where $p_t$ is the price of gas or electricity or the value of the market index for semester $t$; $CPI_t$ is the consumer price index "CPI-U, all urban consumers, U.S. city average, all items" for semester $t$, provided by the Bureau of Labor Statistics, U.S. Department of Labor; and $r_{t,real}$ is the real log return in semester $t$. We note that the market index and CPI data are freely available as of the ends of months, whereas energy prices are available as of the beginnings of the given months. Therefore, as the closest approximation, we pair January 1 energy prices with the previous year's end-of-December market index and CPI, and July 1 prices with the same year's end-of-June market index and CPI. In the cases of missing data, we use simple listwise deletion.

Table 3.1 summarizes the correlations between the real log returns on the market portfolio and energy prices. As Table 3.1 shows, all of the coefficients are small, and none of them are significantly different from zero at the customary levels of either 0.05 or 0.10; therefore, the null hypothesis of no correlation should be accepted in all cases. It is to be added that the sample size is rather limited, which may raise concerns about the power of the test, namely, accepting the null hypothesis when it is actually false and thus making a type II error. The above correlation estimates are based on no fewer than 31 observations, except for the Netherlands business figures. Allowing for two tails at a 0.05 significance level, the test is powerful enough (with a power of 0.80) for true population correlations below -0.46 and above 0.46 for a sample size of 31. If the significance level is relaxed to 0.10, then the test is powerful for coefficients below -0.41 and above 0.41. However, our sample size is given, and we work with the largest possible sample, and therefore, there is nothing more we can do about the type II error in this respect. Further research should be conducted to determine if the current power of the test is sufficient, i.e., true correlations ranging between approximately -0.40 and 0.40 are within the tolerance limits.

**Table 3.1**: Correlation coefficients of real log returns on domestic and industrial gas and electricity prices in selected European countries measured against the real log returns on the MSCI World Index for the 1991–2007 period. Two-tailed *p*-values are provided in parentheses.

| | ***Households*** | | ***Businesses*** | |
|---|---|---|---|---|
| | *Electricity* | *Gas* | *Electricity* | *Gas* |
| Germany | 0.007<br>(0.969) | -0.010<br>(0.956) | -0.111<br>(0.547) | -0.104<br>(0.564) |
| France | 0.035<br>(0.848) | -0.129<br>(0.476) | 0.001<br>(0.995) | 0.005<br>(0.978) |
| United Kingdom | 0.169<br>(0.346) | 0.182<br>(0.311) | 0.177<br>(0.333) | -0.007<br>(0.968) |
| Italy | 0.151<br>(0.400) | 0.049<br>(0.790) | 0.027<br>(0.881) | 0.066<br>(0.718) |
| Belgium | 0.098<br>(0.587) | -0.096<br>(0.608) | -0.012<br>(0.947) | 0.009<br>(0.959) |
| Spain | 0.038<br>(0.841) | -0.001<br>(0.995) | 0.125<br>(0.503) | -0.075<br>(0.687) |
| Luxembourg | -0.025<br>(0.892) | -0.053<br>(0.778) | 0.086<br>(0.639) | -0.064<br>(0.732) |
| The Netherlands | 0.070<br>(0.702) | 0.053<br>(0.792) | 0.157<br>(0.454) | -0.059<br>(0.750) |
| Hungary | 0.019<br>(0.920) | 0.019<br>(0.920) | 0.095<br>(0.613) | -0.211<br>(0.255) |

*IV.3.2.d Correlation of energy prices and weather*

We mentioned in the introduction that air conditioning is responsible for only a negligible share in energy consumption. Thus, an increase in the demand for cooling is unlikely to have any effect on energy prices. Prices clearly have no impact on the weather; therefore, in the case of cooling, these two variables are independent, and consequently, there is no correlation between the two.

The case of heating is not as straightforward because it accounts for a vast share in consumption. Therefore, we test if any reasonable positive correlation can be found between gas or electricity prices and the coldness of the weather. We use the heating degree-days (HDD) as the proxy for the demand for heating, with a higher HDD indicating colder weather. In particular, we consider monthly data on actual heating degree-days from Eurostat (noted as ADD in the database) and sum them up for each half-year, i.e., January to June and July to December, to arrive at comparable semi-annual data for the 1991–2007 period. Because the periodicity of the weather is a year, annual changes are to be examined. Insisting on calendar years would cut our sample size in half, thus largely reducing the power of the test. To maintain the largest possible sample size while keeping the annual scope, we suggest that annual changes are recalculated semi-annually. This approach is similar in spirit to that of Erdős and Ormos (2009). We use the following formula:

$$\ln(x_t + x_{t-1}) - \ln(x_{t-2} + x_{t-3}) = r_{annual,t} \tag{3.6}$$

where $x$ is the real (CPI-corrected) price of gas or electricity in U.S. dollars or semi-annual ADD for semester $t$ and $r_{annual,t}$ is the annual change for semester $t$.

Equation (3.6) reflects the log difference of average prices, but it should be noted that division by 2 is omitted because of mathematical equivalence. We measure the correlation between the derived real log returns on annual average energy prices and the corresponding log differences of annual heating degree-days for the same group of European countries. We examine both gas and electricity prices because either can be used for heating purposes; however, their share may vary by country. The coefficients are expected to be positive if different from zero because a higher demand (i.e., higher ADD) is expected to provoke an increase in prices, ceteris paribus. Therefore, we perform a one-tailed test of significance. Missing data are again treated with listwise deletion. Table 3.2 summarizes the results.

As Table 3.2 shows, the coefficients are again not significantly different from zero, except in six cases: three for the UK, two for Italy, and one for Spain. However, in these six cases, the correlation is negative, which reflects a contradiction. Only a positive coefficient is meaningful, and therefore, we discard these results. The caveats made earlier in connection with the power of the test apply here as well: the sample size is not smaller than 27 for these estimates, except for the business figures from the Netherlands. Here, however, we use a one-tailed test, which is powerful enough (with a power of 0.80) at a 0.05 significance level for true population correlations of 0.44 and above for a sample size of 27. If the significance level is relaxed to 0.10, then the test is powerful for coefficients greater than or equal to 0.38. As shown, the test has slightly better power despite the smaller sample size due to being one-tailed. Again, the sample size is given, and therefore, we make the best possible estimate, which suggests no correlation between energy prices and the weather.

**Table 3.2**: Correlation coefficients of semi-annually calculated annual real log returns on domestic and industrial gas and electricity prices in selected European countries measured against the semi-annually calculated log differences of annual actual heating degree-days (ADD) for the 1991–2007 period. One-tailed *p*-values are provided in parentheses. *, **, and *** denote significance at the 0.10, 0.05, and 0.01 level, respectively.

| | ***Households*** | | ***Businesses*** | |
|---|---|---|---|---|
| | *Electricity* | *Gas* | *Electricity* | *Gas* |
| Germany | 0.100<br>(0.300) | 0.038<br>(0.419) | 0.097<br>(0.304) | -0.011<br>(0.477) |
| France | 0.126<br>(0.249) | 0.036<br>(0.424) | 0.118<br>(0.263) | -0.028<br>(0.441) |
| United Kingdom | -0.494***<br>(0.002) | -0.524***<br>(0.001) | -0.208<br>(0.135) | -0.448***<br>(0.007) |
| Italy | -0.122<br>(0.256) | -0.200<br>(0.144) | -0.242*<br>(0.095) | -0.296*<br>(0.056) |
| Belgium | 0.086<br>(0.322) | 0.127<br>(0.264) | 0.036<br>(0.423) | 0.006<br>(0.487) |
| Spain | -0.007<br>(0.487) | -0.097<br>(0.315) | 0.008<br>(0.485) | -0.434**<br>(0.012) |
| Luxembourg | 0.179<br>(0.172) | 0.078<br>(0.349) | 0.052<br>(0.393) | 0.096<br>(0.317) |
| The Netherlands | 0.127<br>(0.253) | 0.028<br>(0.442) | -0.029<br>(0.451) | -0.013<br>(0.476) |
| Hungary | 0.123<br>(0.262) | -0.010<br>(0.480) | 0.165<br>(0.196) | -0.052<br>(0.395) |

## IV.4. Discussion of results

Summarizing our results, we find no statistically significant correlation between gas or electricity prices, the weather, and the market portfolio. Consequently, the hypothesis that the beta of space heating/cooling energy efficiency projects is zero cannot be rejected, for both households and businesses. Substituting a zero beta into (3.4) yields the risk-free rate as the cost of capital, which is approximately 2% annually, in real terms. On the contrary, industry betas (Damodaran, 2011) suggest the following rates: 6.68% for power with a beta of 0.78, 4.94 % for electric utilities with a beta of 0.49, and 4.70% for natural gas utilities with a beta of 0.45. Using these discount rates would understate the value of space heating/cooling energy efficiency projects. For illustration, we consider discount rates and cash flows in real terms, and describe the project as a growing annuity (to account for possibly rising energy prices). That is, cash flows are expected to increase at a constant rate $g$, in real terms, year after year for $N$ years. Thus, the parameter $g$ can be interpreted as the rise in the price of energy, and the parameter $N$ is the expected life of the project. Figure 3.1 exhibits the difference in present values for various $g$ and $N$ combinations when instead of a 2% discount rate a 5% discount rate, i.e., the approximate rate suggested by industry betas, is used in the present value computation. The negative sign indicates that application of the rate suggested by industry betas provides an understated present value.

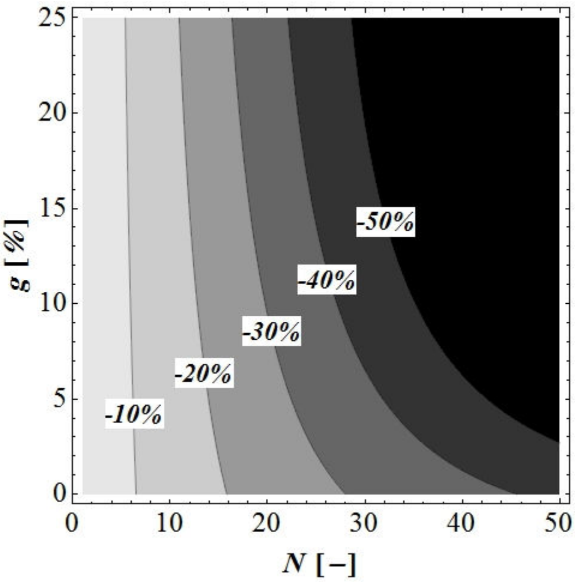



**Figure 3.1**: Understatement in present value for various growing annuities when instead of a 2% discount rate a 5% discount rate is used

As Figure 3.1 shows, neglecting any rise in the price of energy (i.e., assuming $g = 0$) the difference in present values for a typical project spanning 20 years is approximately 25%, which is large. (It can also be observed in the figure that the error is not very sensitive to the growth rate.) In view of the possible magnitudes and the sign of the error it can be established that relying on a discount rate obtained from industry betas may easily lead to wrongful rejection of otherwise profitable projects, or, similarly, may overstate the amount of (government) subsidies required to make such projects worthy of undertaking.

Next, we discuss some implications to project financing. Often, projects are financed not entirely with equity but, to some – and sometimes considerable – extent, they are financed with debt. Besides, some form of government subsidy is often available, e.g., as a lump sum or reduced interest rate. Therefore, it is worth to discuss what effects subsidized debt financing might have on project valuation. First and foremost, we make the usual assumption that debt financing by itself (i.e., without subsidies) has no effect on the value of a project. In other words, debt financing does not alter either the cash flows from the operation of the project nor the risk of these cash flows. What happens then is only that a part of the cash flows generated by business operations goes to debt holders, and equity holders receive the residual part. Without any theoretical derivations (which can be found in, e.g., Andor and Tóth, 2009), it can be seen that, in case of a low debt-to-equity ratio, debt should be risk-free because there are abundant cash flows available to meet debt service obligations. The risk of debt, at the same time, cannot be larger than the asset risk (i.e., the risk of cash flows from operation) of the project – in the unrealistic case of financing entirely with debt, all cash flows from the project would go to debt holders, and thus debt holders would bear the entire risk of the project. Pursuing the concept of dividing cash flows from business operations between debt holders and equity holders, we can state that the NPV of a project can also be divided as debt holders' NPV (denoted by *D*) and equity holders' NPV (denoted by *E*).

$$NPV_{project} = NPV_E + NPV_D \tag{3.7}$$

The NPV of the project is computed by discounting the cash flows from the operation of the project at a discount rate reflecting the relevant risk of the project's cash flows as a whole. The equity and debt NPVs are the cash flows to equity and debt discounted at the rates appropriate to their risk. It is important to add that it is equity holders who decide if a project will be undertaken or not, thus a project is deemed profitable if and only if $NPV_E > 0$.

Another cornerstone of financial analyses is that, assuming a perfect capital market, capital market transactions, and thus debt issuance, must have zero NPV. (This can be seen also by competition for debt issuance pushing interest rates down to the level where economic profit vanishes.) Thus, assuming a perfect capital market, equity holders receive the NPV of a project completely, in other words, $NPV_D = 0$ and thus $NPV_E = NPV_{project}$.

Here, we note that in the preceding analysis we estimated the asset beta of the project since cash flows were not broken down into equity- and debt-side components. Therefore, in our case, under the null hypothesis, $NPV_{project}$ is the cash flows from operation discounted at the risk-free rate, which is equivalent to the equity NPV at the same time, assuming, of course, a perfect capital market. Thus, because the cost of capital of the project is assumed to be the risk-free rate, and because the cost of debt cannot be smaller than the risk-free rate, we have the special case that the cost of debt for space heating/cooling energy efficiency projects is the risk-free rate regardless of the debt-to-equity ratio. Consequently, debt providers should charge no more than 2% interest per annum, in real terms (again, under the assumption of a perfect capital market).

Now we turn to the case when debt is not "fairly" priced, i.e., the capital market is not perfect. An example is subsidized debt with reduced interest rate, which is "underpriced" by prescribing less interest payment than its risk demands in the market. It may occur also that higher interest rate is charged than that demanded by the market for the given risk. $NPV_D$ is not equal to zero in either case: if the interest rate charged is lower (higher) than the market rate, this is a negative (positive) NPV transaction for the debt holder. In our case of energy efficiency projects we may encounter both cases. On the one hand, the government may provide debt with an interest rate lower than even the risk-free rate, which has a positive effect on the equity NPV. On the other hand, if the debt provider, e.g., a bank, does not consider these projects riskless, they may charge an interest rate higher than the risk-free rate, thereby decreasing the attractiveness of the project to equity holders. Recalling (3.7) it can be established that it may occur that the equity NPV of a project with an otherwise negative NPV is positive due to subsidized debt, or, conversely, the equity NPV of a project with an otherwise positive NPV is negative due to "overpriced" debt. In view of our results, may we have any of the two cases of debt, the "fair" cost of debt for space heating/cooling energy efficiency projects, under our null hypothesis, is the risk-free rate. Thus, $NPV_D$ can be computed by discounting the cash flows to debt at the risk-free rate. Subtracting this from $NPV_{project}$, which is the cash flows from operation discounted also at the risk-free rate, we get $NPV_E$, which should be positive for the project to be worth undertaking.

After the financing discussion, we consider the validity of our results. First, we conducted the analysis assuming that the CAPM as a model is true. This is more commonly accepted for developed countries, but Andor et al. (1999) present evidence from Hungary that even in a small country with a young capital market, the CAPM can still be valid. The findings of the authors are basically unchanged if a global or local market index is used. This implies that our results may apply even to relatively segmented investors who hold "local efficient" portfolios. Nonetheless, the zero-beta hypothesis holds only for well-diversified investors.

Another point is that we analyzed only energy efficiency projects that would be undertaken in certain European countries, which raises two important issues: that of country risk premium and the applicability of results in other countries.

It is discussed in finance textbooks that investments in foreign countries are exposed to specific (e.g., political) risks, which should be accounted for. This is preferably accomplished through the adjustment of the discount rate, increasing it in the case of riskier countries. It should be noted that any adjustment for country risk premium is relevant only if the country risk is not diversifiable, which is still a subject of debate. There is, more or less, a practical consensus towards adjustment due to the interdependence of countries in the era of globalization (Damodaran, 2003). Our results, however, need no adjustment provided that the risk factors we examined contain all the relevant sources of such risks. Then, as these risk factors have been compared directly to a global market portfolio in case of each country, the relevant risk a globally diversified investor would perceive from holding such a project has been exactly captured. In other words, the risk factors we measured are assumed to already contain all uncertainty specific to the particular country, including any country-specific risks, and their correlation with the market reflects just the relevant risk for a globally diversified investor.

Although our results are based on data from selected European countries, we argue that the same conclusions can be drawn for other countries. First, the return on the market portfolio is presumably uncorrelated with the weather in any other country. Second, heating and cooling energy consumption represent only a portion of the total national consumption, thus probably having a similarly weak impact on energy prices in other countries as well. Third, a significant proportion of end-user electricity and gas prices may be determined by regulation (e.g., distribution fees) and not by market forces. Additionally, some studies (Chen et al., 1986; Huang et al., 1996) examining the relationship between market indices and oil prices find no systematic relationship. Kilian and Vega (2011) find no convincing evidence that energy prices react to macroeconomic news in the United States. Because oil may be regarded as the most influential energy carrier, these findings are likely to apply to gas and electricity as well.

Finally, our conclusions can be extended to other types of energy efficiency projects. For example, the same results likely apply to the case of lighting, whether in a residential or public context. The amount of time during which lights are switched on in homes or in a park is independent of the world economy and is presumably uncorrelated with electricity prices because it is a price-inelastic good. The same argument can be applied to TVs, washing machines, and other appliances.

Following the arguments above, the hypothesis can be made that any energy efficiency project using gas or electricity has a zero beta in any country, and the cost of capital of the project is the risk-free rate, which needs no adjustment for country risk. Even if there is a small correlation between risk factors, the risk-free rate might still be a better approximation than industry betas. However, these are only hypotheses and require empirical testing to be confirmed.

## IV.5. Conclusions

We developed the hypothesis of zero betas for space heating and cooling energy efficiency projects. We hypothesized that although such projects may have high total risk, it is completely eliminated in the market portfolio. Thus, the cost of capital is the risk-free rate for a well-diversified investor. In support of this theory, we first identified the relevant cash flows of space heating/cooling energy efficiency projects and their sources of uncertainty. We found that two components of energy bills have the most impact on project risk: one is the weather, which determines heating/cooling energy consumption, and the other is the price of energy. Having made the assumption that energy consumption depends only on the weather, we presented arguments that zero correlation should be considered between the market portfolio and the weather, and in a number of European countries, we found no statistically significant correlation between gas or electricity prices and the market portfolio, and between gas or electricity prices and heating degree-days used as a proxy for heating energy consumption. Because none of the risk factors correlate statistically significantly with the market or one another, we established the project's beta should also be assumed zero. We found this to be the case for both households and businesses appraising such projects. We hypothesized that the same conclusion of a zero beta is likely to be drawn in other countries as well. Moreover, we presumed that our results are valid for other types of energy efficiency projects as well. The extensibility of our findings, however, requires further empirical testing to be confirmed.

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