M Ű E G Y E T E M 1 7 8 2

Budapest University of Technology and Economics
Faculty of Mechanical Engineering
Department of Machine- and Product Design

# A MATHEMATICAL MODEL AND SIMULATION FOR ANALYSING THE PRODUCTION GEOMETRY OF SPATIAL GEARS

Ph.D. thesis

Author:
**Groma István**
groma.istvan@gt3.bme.hu

Supervisor:
**Dr. Bercsey Tibor**
bercsey.tibor@gt3.bme.hu
professor of engineering science

Budapest
2010

# 1 Introduction

The development of industry requires more accurate, more exact and more efficient power transmission techniques. In order to create efficiently meshing gears, the better understanding of the properties of meshing teeth and improvement in satisfying various design assumption are mandatory. Former gearing studies tend to neglect distinct inaccuracies of certain manufacturing process, and so omit or simplify geometrical and mechanical errors of the gear during manufacturing and loading.

As a result of unavoidable manufacturing limitations and inaccuracies the sheer geometry of the manufactured piece is only an approximation of the ideal design or production geometry. Hence the goal of my approach is to work out a general mathematical model which is applicable for handling manufacturing errors, utilises modern CAD[1] tools. It relies on former approaches in the field of theory of gearing.

The new approach is applicable for predicting and analysing specific manufacturing processes with recognized geometrical and kinematic features. In addition, in terms of mensurable errors it allows to figure out the probable quality of products produced with a given manufacturing process or to determine the necessary precision of technology in order to achieve a particular accuracy of manufactured gears.

The methodology proposed is based on the results and ideas of mechanical engineering, theory of gearing, probability and statistical theories, information theory, computer science and the mixture of these.

# 2 Background

The fundamental idea of modelling tooth surfaces with conjugate surfaces was introduced by Olivier [Théodore Olivier, 1842] and was improved by Gohman who personally developed the first generalized and analytical theory of gearing for spatial gears [H. I. Gohman, 1886].

His theory simplified the calculation of contact lines on conjugate surfaces which were influenced by well-known methods of differential geometry. On the other hand, however, Gohman's approach was still complex and far from being applicable in practical engineering. This inspired Litvin and others to elaborate the "kinematic method of determination" [Faydor L. Litvin and Alfonso F
The main idea of the kinematic method reads as follows: the relative displacement of the conjugate surface pair can be formalized with kinematic (the motion theory of solid bodies) equation.

Tajnafői's theory called "derivation of the structures of mechanisms" exceeds the approach of spatial conjugate surfaces and assumes the more general proposal of derivation of structures [Bercsey Tibor, 1977, Tajnafői József, 1991]. The theory uses the term "source surface" for the working surface of the tool (representing either a surface or an edge) and "derivative surface" for the enveloping surface of the cutaway volume along the path of the tool. The source surface is treated as a solid, non-wearing, stationary and frictionless theoretical surface of the

[1] The abbreviation of Computer-Aided Design.

manufacturing tool. The derivative surface arises as a result of the shape of the tool and the relative displacements between the tool and the piece providing the formal system of static and dynamic action quantities.

The generic description of worm-gear tooth surfaces was worked out by Dudás Illés and Balajti Zsuzsanna. Their abstract approach unifies the different tooth surfaces on a cylindrical or a conical worm gear [Balajti Zsuzsanna, 2007, Dudás Illés, 1988]. The generic description is driven from the kinematic method. The authors use the terms "direct problem" when an appropriate tool is determined for a given helical surface, and the "indirect problem" for finding the resulting surface of an imaginary manufacturing process including a tool with specific shape. Both directions of evolution are important in everyday engineering practice.

Former publication studying tooth-profile or shape errors originating from inaccurate action quantities in general is literally non-existing. The novel approach is intended to fill this gap and to improve the model of derivative production geometries too.

## 3 Materials and methods

In my approach the original **kinematic method** used for gear design and analysis was extended with a **probability space** where design variables of the manufacturing process are represented with random variables. This approach utilises not only fundamental geometrical concepts but the basic ideas of theory of gearing and was firmly influenced by the theory of **derivation of the structures of mechanisms**. In addition, the modern knowledge of **information theory** also had important contribution to my research and to achieving results.

In case of an **indirect problem** the working surface of the tool can be formalized with the discipline of differential geometry [Faydor L. Litvin, 1971, Dudás Illés, 1988]. The relative displacements occurring during a manufacturing process can be modelled as **continuous transformations**. Furthermore, most cases fall into the more precise group of transformations called **homogeneous linear transformations**. Thus, it is possible to formalize them as 4-dimensional square matrices. The measurable properties of relative displacements and working surfaces, called **static and dynamic action quantities**, should be treated as design variables while their actual values having great influence on the derivative surface[2] and geometry.

According to the kinematic method the derivative surface is the **conjugate surface** of the source surface[3] over its path. The derivative surface can be implicitly formalized with one or two coordinates and design variables of the manufacturing process or more generally with action quantities:

$$\vec{\mathbf{H}}(u_1, u_2;\ \underbrace{p_1, \ldots, p_l}_{\text{stat. a. q.}};\underbrace{p_{l+1}(u_1), \ldots, p_n(u_1)}_{\text{dyn. a. q.}}).$$

It is reasonable to assume that all design variables either have some measuring error or some

[2] The manufactured flank of the gear.

[3] The working surface of the tool.

random noise during evolution. Sometimes both are present. Hence a stochastic, time-varying error function (or signal) was added to each design variable. So that, the momentary amplitudes of the error functions would be Gaussian-distributed providing a **Gauss noise** on the variables. Additionally, it was concluded that momentary amplitudes are independent from each other[4] making the error functions behave as a **white noise**. In order to use these variables with the kinematic method, all design variables should be continuous over time which is in consonance with physical observation. These considerations led to the idea of $\texttt{RN}\left[.\right](t)$ formalism[5] providing a solid tool for the different inaccurate design variables:

$$\begin{aligned} &\texttt{RN}\left[s \pm a \circlearrowright f\right](t) := S_3\left(t; \hat{P}_0, \ldots, \hat{P}_{\lfloor fT \rfloor}\right), \\ &\forall i : \hat{P}_i \cong \Phi\left(x; \mu := s, \sigma := \tfrac{a}{3}\right),\ t \in [0, T]. \end{aligned}$$

Table 1: The $\texttt{RN}\left[.\right](t)$ formalism

| | |
|---:|---|
| $a$ | The maximal amplitude of the design variable. |
| $f$ | The sampling rate of the design variable. |
| $\hat{P}_i$ | Independent random variables representing an instantaneous value of the design variable in ascending order. |
| $s$ | The numerical base value of the design variable. |
| $S_3\left(t; \ldots\right)$ | Natural cubic spline interpolation on a set of data points using independent variable $t$. |
| $T$ | The duration of the manufacturing process. |
| $\mu$ | The expected value of a Gaussian random variable. |
| $\sigma$ | The scale of a Gaussian random variable. |
| $\Phi\left(x; \mu, \sigma\right)$ | The probability density function of Gaussian-distribution. |

The total duration of the manufacturing process can be slitted into smaller distinct intervals according to the sampling rate. An independent **Gaussian random variable** is assigned to each time interval. These random variables are isolated by their related discrete intervals so, an arbitrary **interpolation methods** could be used to achieve an continuous function over the random variables.

Among the numerous interpolation methods it is reasonable to apply the **natural cubic spline interpolation** since this interpolations does not oscillate at interpolating points[6], and by the same token this type of interpolation can be handled easily with formal calculus [Szirmay-Kalos László et al., 2000, Horváth Imre and Juhász Imre, 1996].

The introduction of the probability space transforms the classical production geometry into a random production geometry which resembles sorely the statistical properties of a sample of manufactured gears. Hence, this way it becomes possible to analyse the geometrical properties of an inadequately manufactured piece prior actual manufacturing. The spatial point of the random production geometry will be referred to as **stochastic points** due to their unpredictable nature.

[4]The error function has equal power within domain interval.
[5]The abbreviation of Random Noise.
[6]Runge's phenomenon.

$$\begin{array}{ccccc} & & \vec{\mathbf{H}}\left(u_1, u_2; p_1, \ldots, p_n\left(u_1\right)\right) & & \\ & \swarrow & & \searrow & \\ \mathtt{RN}\left[s_1 \pm a_1 \circlearrowleft f_1\right](t), & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & & , \mathtt{RN}\left[s_n \pm a_n \circlearrowleft f_n\right](t) \\ \swarrow & & & & \searrow \\ \underbrace{\hat{P}_{1,0}, \ldots, \hat{P}_{1,\lfloor f_1 T \rfloor},} & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & & , \hat{P}_{n,0}, \ldots, \hat{P}_{n,\lfloor f_n T \rfloor} \end{array}$$

$$\underbrace{\hat{P}_{1,0}, \ldots, \hat{P}_{1,\lfloor f_1 T \rfloor}, \quad \ldots\ldots\ldots \quad , \hat{P}_{n,0}, \ldots, \hat{P}_{n,\lfloor f_n T \rfloor}}_{N}$$

Figure 1: The idea of substituting inaccurate design variables into the implicit equation of the manufactured tooth flank.

The representation of input error of the manufacturing process is nothing but the specific random variables[7] shown on the bottom layer of Figure 1. If these input random variables have a certain distribution, the adequate random production geometry considered to be determined.

Due to its complexity it is difficult to apply the random production geometry model directly. Since, the whole random production geometry can be interpreted as a continuous set of potential production geometries with a related probability distribution, the random production geometry could be examined through a **statistical approach**. In order to analyse the geometrical properties of the derivative surface the random production geometry should be substituted with a random sample of proper size. The elements of the sample represent deterministic derivative surfaces or, more precisely, explicit geometries of particular flanks on gears and it can be treated with **CAD surface modelling**.

The proper size of the sample should be calculated by adapting a well-known statistical equation for Gaussian random variables established in the ASTM E122 standard:

$$Sample\ size \geq \prod_i \frac{\Phi_{0,1}^{-1}\left(1-\frac{\alpha}{2}\right)^2 \sigma_i^2}{ME^2},$$

where **significance level** denoted by the Greek letter $\alpha$ reads as a percentage and had to be chosen to suite the desired confidence. $ME$ stands for the **margin of error** – an overall limit of the distance between elements and the mean of the sample – which specifies the size of the confidence interval.

The validity and efficiency of the random production geometry model was tested by comparing measured data with computer simulation results [Waldemar Steinhilper and Bernd Sauer, 2006, Erney György, 1983, W. Höfler, 1967]. The object of the verification process was a sample of involute spur gears (40 piece with the following characteristic properties: $z_1 = 9; m = 5\text{mm}; \alpha_0 = 20°; \beta = 0°; x_1 = +0,07; b = 44,8\text{mm}$) manufactured with a **Niles ZSTZ 315 C1 grinding machine**. Their data was compared with the corresponding results of a computer simulation designed to study the imaginary random production geometry of a similar spur gear.

[7] A Gaussian random variable can be clearly defined with two parameters: mean and variance.

The input of the simulation was the approximated accuracy of the machine settings and the goal was to produce a random set of surface models with a practical **Monte Carlo algorithm** developed for this particular purpose (see Figure 2).

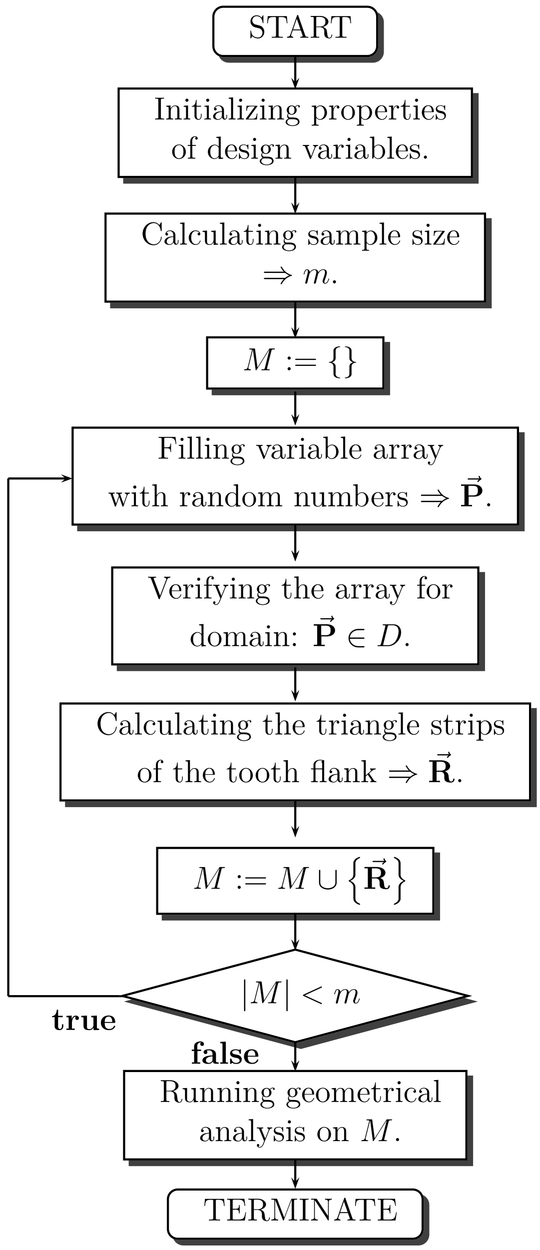


Figure 2: The flowchart of the random production geometry analysis process

Calculations (representing measurements) were carried out on the output of the simulation parallel with industrial audit trails of spur gear manufacturing process. The simulated and the measured data were compared by means of **statistical mean** and **standard deviation**. If the relative errors of mean and variance were under 15% the results are graded as well fitting while a relative error above 50% empathises a poor fit. A relative error in the order of 100% implicates that the simulation is inadequate for modelling the manufacturing process. The results given in Table 2 confirm the capabilities of the random production geometry model introduced in the study.

Table 2: Comparison of measured data and simulation results (mean ± variance)

| Measurement | Measured value | Computed value | Relative error |
|---|---|---|---|
| Working radial run-out ($F_{rr}$) | $10,6{\pm}3,7\mu$m | $10,5{\pm}1,2\mu$m | 1±68% |
| Tooth alignment error ($F_{\beta r}$) | $6,9{\pm}4,8\mu$m | $6,6{\pm}4,3\mu$m | 4±10% |
| Span measurement variation ($F_{vWr}$) | $10,2{\pm}2,9\mu$m | $13,3{\pm}3,0\mu$m | 30± 3% |
| Tooth profile error ($f_{fr}$) | $4,8{\pm}3,7\mu$m | $4,5{\pm}1,2\mu$m | 6±68% |
| Base pitch error ($f_{pbr}$) | $7,1{\pm}2,8\mu$m | $6,4{\pm}2,9\mu$m | 10± 4% |

Besides modelling the flank of spur gears the random production geometry model can be applied for far more complex cases like **helical tooth surfaces** on worm-gears. Thus to obtain a random production geometry for helical surfaces the variables of the general equation of helical surfaces should be substituted with random variables. The helical surface is actually the conjugate surface of the manufacturing tool. So, in order to find the explicit equation for helical surfaces the **vectorial equation of contact in 3-dimensional space** should be solved. This equation cannot be solved direly for a random production geometry, but by evoking the idea of random sampling one can solve the equation for each element of the sample.

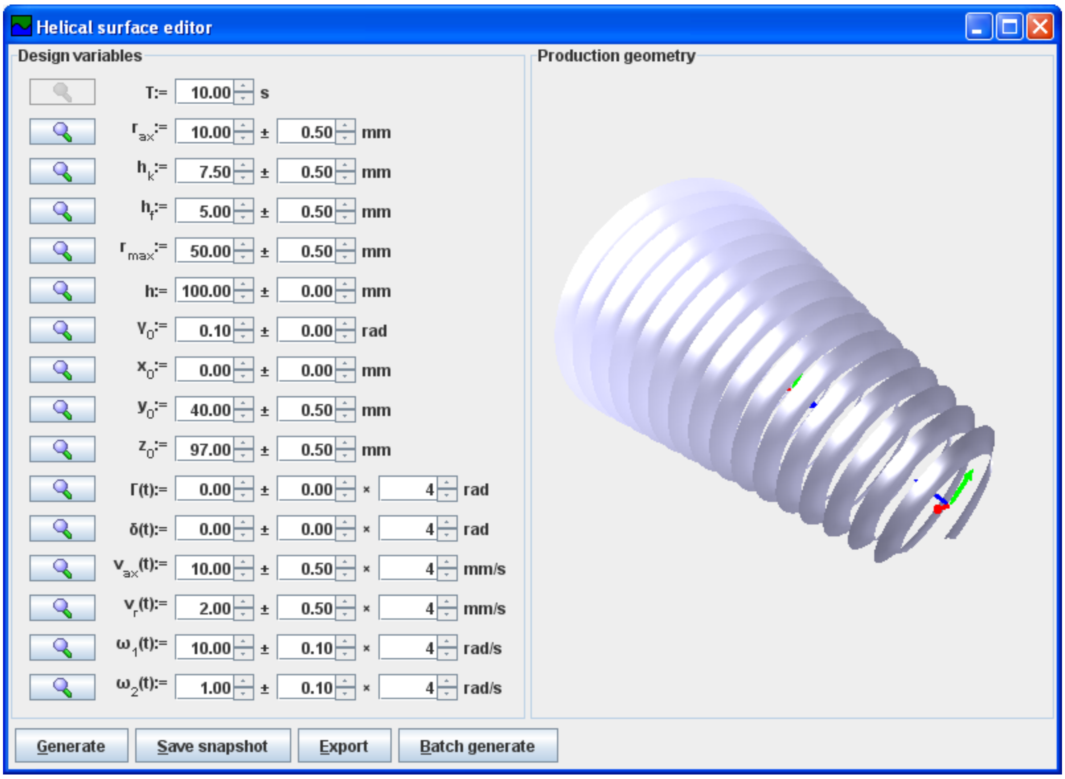


Figure 3: A screenshot showing the design variable configuration panel and the random production geometry viewer of the application

For the sake of computing the complex calculation of my approach a computer software was developed. The software is capable for analysing the helical tooth surface of worm-gears with circular profile having shape errors (see Figure 3). Besides visual representation, the software is able to predict shape and geometrical errors of the helical surface. It was studied by numerous test trials with different design variations.

By utilizing the random production geometry model one can determine the exact value of feasible shape errors for a given manufacturing process with recognized design variables and accuracy. Moreover, it is possible to find the optimal requisite accuracy of the manufacturing process in order to cover a certain product quality. This approach for modelling spatial gears provides closer relationship between design and production in gear manufacturing. The fundamental advantage of the random production geometry model is being able to analyse geometrical or even functional properties of the gear carrying some unavoidable shape errors prior actual manufacturing.

# 4 References

[Théodore Olivier, 1842] Théodore Olivier: Théorie géometrique des engrenages. Paris, 1842

[H. I. Gohman, 1886] H. I. Gohman: Theory of Gearing Generalized and Developed Analytically. Odessa, 1886

[Faydor L. Litvin and Alfonso Fuentes, 2004] Faydor L. Litvin, Alfonso Fuentes: Gear Geometry and Applied Theory. Cambridge University Press, Cambridge, 2004

[Bercsey Tibor, 1977] Bercsey Tibor: Toroidhajtások elmélete. Academic dissertation, Budapest, 1977

[Tajnafői József, 1991] Tajnafői József: Mechanizmusok származtatáselméletének alapjai és hatása a kreatív gondolkodásra. Academic dissertation, Miskolc, 1991

[Balajti Zsuzsanna, 2007] Balajti Zsuzsanna: Kinematikai hajtópárok gyártásgeometriájának fejlesztése. Ph.D. thesis, University of Miskolc, Miskolc, 2007

[Dudás Illés, 1988] Dudás Illés: Csavarfelületek gyártásának elmélete. Academic dissertation, Miskolc, 1988

[Faydor L. Litvin, 1971] Faydor L. Litvin: A fogaskerékkapcsolás elmélete. Műszaki Könyvkiadó, Budapest, 1971

[Szirmay-Kalos László et al., 2000] Szirmay-Kalos László, Antal György, Csonka Ferenc: Háromdimenziós grafika, animáció és játékfejlesztés. ComputerBooks, Budapest, 2000

[Horváth Imre and Juhász Imre, 1996] Horváth Imre, Juhász Imre: Számítógéppel segített gépészeti tervezés. Műszaki Könyvkiadó, Budapest, 1996

[Waldemar Steinhilper and Bernd Sauer, 2006] Waldemar Steinhilper, Bernd Sauer: Konstruktionselemente des Maschinenbaus 2. Springer, Berlin, 2006

[Erney György, 1983] Erney György: Fogaskerekek. Műszaki Könyvkiadó, Budapest, 1983

[W. Höfler, 1967] W. Höfler: A fogaskerékellenőrzés új módszerei. Műszaki Könyvkiadó, Budapest, 1967

# 5 Summary of new results

**Theorem 1:** The inaccuracies of the manufacturing mechanism along with the shape errors of the manufactured piece can be expressed generally by substituting the static and dynamic action quantities of the manufacturing process with appropriate random variables. This approach is an expedient generalisation of the former model of derivation of the structures of mechanism and holds the potential to extend the formal production geometry with adequate manufacturing errors. The exact distribution of the introduced random variables should suit the stochastic features of the particular manufacturing process [2, 8, 5, 4, 13, 10, 1, 3, 7, 6, 9, 11, 12].

**Theorem 2:** If the mapping of the source surface into the derivative surface can be formalized as a continuous function of independent action quantity variables, it is feasible to interpreted this function as a transformation of an adequate multivariate random variable. The result of the transformation at given coordinates is actually a distribution of spatial points, namely a manifold of stochastic points. The continuous manifold of the coordinate variables together with the corresponding manifold of stochastic points provide a definite model for the random production geometry. The random production geometry of a particular derivative surface can be achieved by substituting random variables for the independent action quantities of global and local kinematic relations of the related manufacturing process [2, 8, 5, 4, 13, 10, 1, 3, 7, 6, 9, 11, 12].

**Theorem 3:** In order to examine the model of random production geometry, a computer software with corresponding methodology was developed on the basis of finite random sampling. To confirm the appropriateness of the approach and method a sample consisting of 40 involute spur gears ($z = 9; m = 5\text{mm}; \alpha_0 = 20°; x_1 = +0,07$) was analysed by means of shape errors. The mean value and the variance of measured manufacturing error along with the predicted errors of computer simulation (working radial run-out, tooth alignment error, span measurement variation, tooth profile error, base pitch error) generally fit well to each other. The results lack a relative error between experiment and simulation in the order of 100% or beyond [12].

**Theorem 4:** The method of derivation of spatial flanks, especially helical tooth surfaces on the grounds of kinematic method, can be refined with the inaccuracies of the manufacturing tool and process arising a certain production geometry. This combined geometry is the subject of a developed computer software operating with finite random error samples. The measurement of the shape errors on the computer representation of helical tooth surface are the matter of differential-geometrical calculations. This provides a straight way to predict the manufactured tooth accuracy on a given machine and tool setting in a statistical manner [2, 8, 5, 4, 13, 10, 1, 3, 7, 6, 9, 11, 12].

# 6 Publications related to the thesis

[1] Groma István and Bercsey Tibor. Csavarfelületek geometriai hibáinak modellezése. *GÉP*, LVII:57–60, 2006.

[2] Groma István and Bercsey Tibor. Csavarfelületek geometriai hibáinak modellezése. In *OGÉT 2007: XV. Nemzetközi Gépész Találkozó*, pages 57–60, Cluj-Napoca, 2007.

[3] Groma István and Bercsey Tibor. Csavarfelületek gyártási hibáinak modellezése valószínűségi változók használatával. *GÉP*, LVIII:51–54, 2007.

[4] Groma István and Bercsey Tibor. Modeling shape errors of worm gears. In *2007 ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, Las Vegas, 2007.

[5] Groma István and Bercsey Tibor. Modelling shape inaccuracies of worm milling cutter. In *Proceeding of the 12th International Conference on Tools*, pages 163–168, Miskolc, 2007.

[6] Groma István and Bercsey Tibor. Csavarfelületek gyártási hibáinak modellezése valószínűségi változók bevezetésével. *GÉP*, LIX:31–33, 2008.

[7] Groma István and Bercsey Tibor. Evolvens fogazat megmunkálási hibáinak geometriai modellezése. *GÉP*, LVIII:40–43, 2008.

[8] Groma István and Bercsey Tibor. Modeling errors in worm gear manufacture with random variables. In *Proceedings of 6th Conference on Mechanical Engineering*, Budapest, 2008.

[9] Groma István and Bercsey Tibor. Hengeres és kúpos csigák gyártási alakhibáinak modellezése. *Gépgyártás*, XLIX(4–5):17–22, 2009.

[10] Groma István and Bercsey Tibor. Modeling errors in worm gear manufacture with random variables. In *Proceedings of The JSME International Conference on Motion and Power Transmissions*, pages 143–147, Sendai, 2009.

[11] Groma István and Bercsey Tibor. Modelling Shape Inaccuracies of Worm Gears. *Journal of Machine Manufacturing*, XLIX(6):9–13, 2009.

[12] Groma István and Bercsey Tibor. Modelling Tooth-Shape Errors Using Random Variables. *Periodica Polytechnica – Mechanical Engineering*, 53(2), 2009.

[13] Groma István, Bercsey Tibor, and Horák Péter. Modeling errors in worm gear manufacturing with random variables. In *Dresdner Maschinenelemente Kolloquium*, pages 171–182, Dresden, 2007.

# 7 Other publications

[14] Bercsey Tibor, Rick Tamás, and Groma István. Erőforráshelyes konstrukciós folyamattervezés és optimálás. *GÉP*, 11–12(2):141–144, 2005.

[15] Bercsey Tibor, Rick Tamás, and Groma István. Produktstrukturbasierte Produktentwicklungs- Prozessmodellierung und Optimierung mit genetischen Algorithmen. In *7. Magdeburger Maschinenbau-Tage*, pages 44–49, Magdeburg, 2005.

[16] Bercsey Tibor, Rick Tamás, and Groma István. Ressourcengerechte Produktentwicklungs-Prozessmodellierung und Optimierung mit Genetischen Algorithmen. In *Design for X: Beiträge zum 16. Symposium*, pages 59–66, Nuremberg, 2005.

[17] Bercsey Tibor, Rick Tamás, and Groma István. Tervezési folyamatok hozzárendelése heurisztikus módszerrel. *GÉP*, 8–9(2):153–155, 2006.

[18] Bercsey Tibor, Rick Tamás, and Groma István. LP Modell und heuristische Methoden für Ressourcen gerechte Produktentwicklungs-Prozessplanung. In *8. Magdeburger Maschinenbau-Tage & 7. MAHREG Innovationsforum*, pages 181–186, Magdeburg, 2007.

[19] Bercsey Tibor, Rick Tamás, Groma István, and Gránicz Ádám. Ga-based flexible and effective task scheduling and resource allocation. In *10th World Multiconference on Systemics, Cybernetics and Informatics*, pages 119–124, Orlando, 2006.

[20] Bercsey Tibor, Rick Tamás, Groma István, and Gránicz Ádám. Ga-based flexible and effective task scheduling and resource allocation. In *Design for X: Beiträge zum 17. Symposium*, pages 95–104, Nürnberg, 2006.

[21] Bercsey Tibor, Rick Tamás, Groma István, and Gránicz Ádám. Ga-based flexible and effective task scheduling and resource allocation. In *8th International Dependency Structure Matrix (DSM) Conference*, Seattle, October 2006.

[22] Bercsey Tibor, Rick Tamás, Groma István, and Gránicz Ádám. Task scheduling and resource allocation with multi-varibale heuristics. In *16th International Conference on Engineering Desing: Design Society*, pages 615–616, Paris, 2007.

[23] Groma István. The advantage of sparse symmetric matrix product calculation on distributed systems. In *Magdeburger Maschinenbau-Tage*, pages 290–296, Magdeburg, 2005.

[24] Groma István and Bercsey Tibor. Product development process scheduling with multivariable heuristic method. In *Proceedings of the 7th IPD Workshop*, Magdeburg, 2008.

[25] Groma István, Bercsey Tibor, and Rick Tamás. Product development process optimisation with heuristics methods, international design conference. In *Proceedings of the DESIGN 2008: 10th International Design Conference*, pages 229–236, Dubrovnik, 2008.

[26] Rick Tamás and Groma István. Design structure based tasks scheduling using genetic algorithms. In *Scientific Bulletin Serie C, Volume XIX: International Multidisciplinary Conference 2005*, pages 231–238, Baia Mare, 2005.