Coupled Vibration of Thin Walled Beams

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Summary

Consistent and simple lumped mass matrices are formulated for the dynamic analysis of beams with arbitrary cross section. The development is based on a general beam theory which includes the effect of flexural-torsion coupling, the constrained torsion warping and the shear centre location. Numerical test are presented to demonstrate the importance of torsion warping constraints and the acceptable accuracy of the lumped mass matrix formulation.

Introduction

During the torsion of bars an out of section plane, axial warping displacement takes place which is assumed to depend on the change of the angle of twist. The torsional warping has no effect on stresses if the measure of warping is the same in each section including the ends. This implies that the torsional rotation is a linear function along the beam axis. If the torsional rotation is far from the linear distribution, as it is in torsional vibration modes, or the beam ends are constrained, the torsional warping may have an important effect on the static or dynamic response of the beam structure. In addition to the torsion warping effect, the coupling between the bending and the tosional free vibration modes occurs when the centroid (mass centre) and the shear centre (centre of twist) of the beam section are non-coincident.

In this paper an exactly integrated consistent and a lumped mass matrix are presented for the 7 DOF finite element beam model. The formulation includes the flexure-torsion coupling and the constrained warping effects.

The equation for free vibration of an elastic system undergoing small deformations and displacements can be expressed in the form

$$\underline{\underline{\mathbf{K}}}\;\underline{\mathbf{U}} + \underline{\underline{\mathbf{M}}}\;\underline{\ddot{\mathbf{U}}} = \underline{\mathbf{0}}\,,$$

where $\underline{\underline{\mathbf{K}}}$ and $\underline{\underline{\mathbf{M}}}$ are the assembled elastic stiffness and mass matrices, respectively, and $\underline{\mathbf{U}}(t)$ is the set of nodal displacements. The dot represents the time derivative.

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Kinematics of beam

Figure 1. shows the basic systems and notations. The local x axis of the right hand orthogonal system is parallel to the beam straight axis and passes trough the $\mathbf{N_1}$, $\mathbf{N_2}$ element nodes of the finite element mesh. The axes y and z are parallel to the principal axes, signed as r and s. The position of the centroid **C** and shear centre **T** relative to the node **N** in the plane of the section are given by the co-ordinates y_{NC}, $y_{CT,}$ and z_{NC}, z_{CT}.

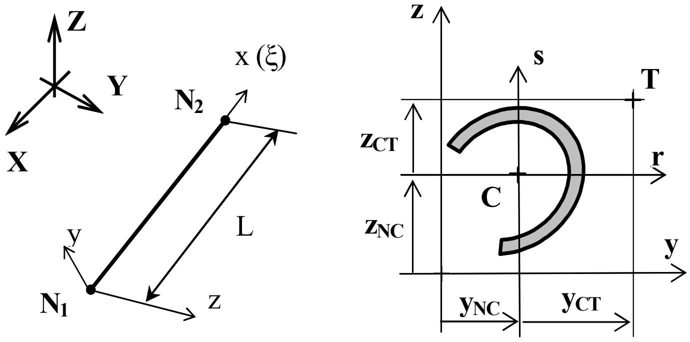


Fig 1. Beam section layout

The linear kinematics of an initially straight, prismatic beam element can be described on the assumption, that the in plane displacements of a point can be expressed by three parameters, the angle of twist $\theta^T{}_x$ about the longitudinal axis passing trough the **T** shear centre and the two $u^T{}_y$ and $u^T{}_z$ displacement components of point **T**. The axial displacement is the sum of the $u^C{}_x$ axial displacement of the **C** centroid, the $\theta^C{}_y$, $\theta^C{}_z$ rotations of planar section about the axes r and s, and the out of plane torsion warping displacement. Accordingly, the displacement vector is

$$\mathbf{u}(x,r,s,t) = [u_k] = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x^C + \Theta_y^C\, s - \Theta_z^C\, r - \vartheta\omega^T \\ u_y^T - \Theta_x^T\,(s - z_{CT}) \\ u_z^T + \Theta_x^T\,(r - y_{CT}) \end{bmatrix} . \tag{1}$$

where $\vartheta(x,t)$ is the warping parameter and $\omega^T(r,s)$ is the warping function, or – for thin walled sections – the sector area co-ordinate.

The geometric properties of the cross section are:

$$I_r = \int_A s^2\, dA\ , \quad I_s = \int_A r^2\, dA\ , \quad y_{CT} = -\frac{1}{I_r}\int_A s\omega^T\, dA\ , \quad z_{CT} = \frac{1}{I_s}\int_A r\omega^T\, dA\ ,$$

$$I_\omega = \int_A \omega^{T2}\, dA\ , \quad J = I_r + I_s + \int_A \left(s\frac{\partial\omega^T}{\partial r} - r\frac{\partial\omega^T}{\partial s} \right) dA\ , \tag{2}$$

$$I_P = \int_A \left[(r - y_{CT})^2 + (s - z_{CT})^2 \right] dA = I_s + I_r + A\left(y_{CT}{}^2 + z_{CT}{}^2 \right).$$

By using the displacement vector (1) and the *Vlaszov* and *Bernoulli* constraints as

$$\Theta_y^C(x,t) = -\frac{du_z^T}{dx} = -u_z'^T \ , \ \ \Theta_z^C(x,t) = \frac{du_y^T}{dx} = u_y'^T \ , \ \ \vartheta(x,t) = \frac{d\Theta_x^T}{dx} = \Theta_x'^T \ , \qquad (3)$$

the U strain and K kinetic energy stored in a linear elastic beam element of length L are:

$$U = \frac{1}{2}\int_0^L \left[EAu_x'^{C2} + EI_r u_z''^{T2} + EI_s u_y''^{T2} + GJ\Theta_x'^{T2} + EI_\omega \Theta_x''^{T2} \right] dx \ ,$$

$$K = \int_0^L \left[\dot{u}_x^{C2} + \dot{u}_y^{T2} + \dot{u}_z^{T2} + \frac{I_r}{A}\dot{u}_z'^{T2} + \frac{I_s}{A}\dot{u}_y'^{T2} + \frac{I_\omega}{A}\dot{\Theta}_x'^{T2} + \frac{I_p}{A}\dot{\Theta}_x^{T2} + \right. \qquad (4)$$
$$\left. +2\left(z_{CT}\dot{u}_y^T\dot{\Theta}_x^T - y_{CT}\dot{u}_z^T\dot{\Theta}_x^T\right)\right] \rho A \, dx \ ,$$

where E, G are the properties of isotropic elastic material and ρ is the mass density. The assumptions (3) imply that the shear deformations are neglected.

Element Matrices

The derivation of element matrices is based on the assumed displacement field. A linear interpolation is adopted for the axial displacement and a cubic for the lateral deflections and the twist:

$$\begin{aligned}
u_x^C &= u_{x1}^C(1-\xi) + u_{x2}^C\xi \ , \\
u_y^T(\xi) &= u_{y1}^T N_1(\xi) + \Theta_{z1}^C \, L\, N_2(\xi) + u_{y2}^T N_3(\xi) + \Theta_{z2}^C \, L\, N_4(\xi), \\
u_z^T(\xi) &= u_{z1}^T N_1(\xi) - \Theta_{y1}^C \, L\, N_2(\xi) + u_{z2}^T N_3(\xi) - \Theta_{y2}^C \, L\, N_4(\xi), \qquad (5)\\
\Theta_x^T(\xi) &= \Theta_{x1}^T N_1(\xi) + \vartheta_1 \, L\, N_2(\xi) + \Theta_{x2}^T N_3(\xi) + \vartheta_2 \, L\, N_4(\xi),
\end{aligned}$$

$$N_1 = 1 - 3\xi^2 + 2\xi^3 \ , \ \ N_2 = \xi - 2\xi^2 + \xi^3 \ , \ \ N_3 = 3\xi^2 - 2\xi^3 \ , \ \ N_4 = \xi^3 - \xi^2 \ , \ \ \xi = \frac{x}{L} \ .$$

Define the order of the element 2x7=14 local displacements at the two ends as

$$\underset{(14,1)}{\underline{\mathbf{U}}^C}(t) = \left[u_{x1}^C, \ u_{y1}^T, u_{z1}^T, \Theta_{x1}^T, \ \Theta_{y1}^C, \ \Theta_{z1}^C, \ \vartheta_1, \ u_{x2}^C, \ u_{y2}^T, u_{z2}^T, \ \Theta_{x2}^T, \Theta_{y2}^C, \ \Theta_{z2}^C, \ \vartheta_2\right]^T . (6)$$

Substituting interpolation (5) into (4) the expression for the potential and kinetic energy may be defined in terms of (6) local variables as

$$U = \frac{1}{2}\underline{\mathbf{U}}^{CT}\underline{\mathbf{k}}^C\underline{\mathbf{U}}^C \ , \qquad K = \frac{1}{2}\dot{\underline{\mathbf{U}}}^{CT}\underline{\mathbf{m}}^C\dot{\underline{\mathbf{U}}}^C \ .$$

The exactly integrated consistent mass matrix $\underline{\mathbf{m}}^C$ is given in *Appendix* **A**. The $\underline{\mathbf{k}}^C$ stiffness matrix – apart from sign conventions – is identical to the matrix published in [3], page 89.

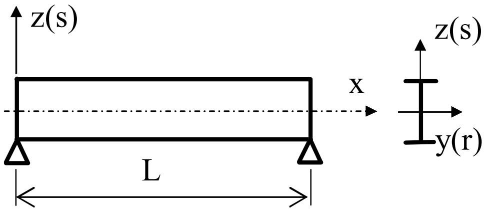
The lumped mass matrix can be derived from the kinetic energy expression for an element which undergoes a rigid body like motion and rotation. The element lumped mass matrix is given in *Appendix* **B**. Here the lumped mass, due to the shear centre location, is not a diagonal matrix. Nevertheless, it is computationally much more economical then the corresponding consistent mass detailed in *Appendix* **A**.

Numerical examples

To illustrate the importance of internal and external warping constraints and the performance of the lumped mass matrix, the results of two test problems are detailed herein: (1) simply supported beam and (2) a cantilever.

(1) Simply supported beam: Properties used in this example are listed on Fig. 2. Closed form solution for the torsion vibration frequency in Hz with free end warping is known as:

$$\alpha_n = \frac{n}{2L}\sqrt{\frac{GJ}{\rho\left(I_r + I_s\right)}}\sqrt{\left(1 + n^2\frac{\pi^2 EI_\omega}{L^2 GJ}\right)\Big/\left(1 + n^2\frac{\pi^2 I_\omega}{L^2\left(I_r + I_s\right)}\right)}\ , \quad n = 1, 2, .. \quad (7)$$



$L = 2000$ mm
$A = 2848\ mm^2$
$I_r = 1{,}943\ e^7\ mm^4$
$I_s = 1{,}423\ e^6\ mm^4$
$J = 6{,}847\ e^4\ mm^4$
$I_\omega = 1{,}274\ e^{10}\ mm^4$
$E = 2{,}0\ e^5\ N/mm^2$
$G = 0{,}8\ e^5\ N/mm^2$
$\nu = 0{,}25$
$\rho = 8{,}0\ e^{-9}\ Nsec^2/mm^4$

Fig 2. Simply supported beam with doubly symmetric I section.

The beam was analysed with different *m* number of elements. At the two end nodes in addition to the hinged support conditions the 7-th warping parameter was left free. The results in Tables 1a, b show that the torsional frequencies – even for a coarse mesh and lumped mass – are in good agreement with the (7) analytical solution.

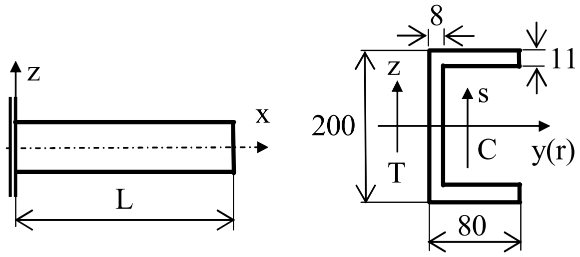
n	m = 2	m = 4	m = 8	m = 16	m = 20	analytical
1	66,485	66,349	66,340	66,339	66,339	66,339
2		214,29	213,63	213,59	213,59	213,59
3		462,00	454,91	454,41	454,39	454,37

Table 1a. Convergence of torsional frequencies (Hz), with *consistent* mass matrix.

	m = 2	m = 4	m = 8	m = 16	m = 20	analytical
1	65,601	66,309	66,338	66,339	66,339	66,339
2		211,83	213,51	213,58	213,59	213,59
3		425,59	453,54	454,33	454,36	454,37

Table 1b. Convergence of torsional frequencies (Hz), with *lumped* mass matrix.
7 DOF results with free end warping

(2) Cantilever: Sectional properties of the U shape section are given on Fig. 3. For comparison the frequencies of the beam like modes obtained by COSMOS/M thick shell finite element model are listed in "SHELL" column in Table 2. The cantilevers were modelled by using 1280 (U section) and 1600 (I section) four-noded COSMOS/M thick shell elements.



L = 2000 mm
$A = 3157\ mm^2$
$I_r = 1,972\ e^7\ mm^4$
$I_s = 1,955\ e^6\ mm^4$
$J = 1,025\ e^5\ mm^4$
$I_\omega = 1,227\ e^{10}\ mm^4$
$y_{CT} = -48,7$ mm

Fig 3. Cantilever with a channel U section.

The comparisons of results in Table 2 show the significant effect of warping inertia and internal (col. A, $\vartheta(0)$ free) and external (col. B, $\vartheta(0)$ fixed) warping constraints on t*i* torsion vibrations. The modes – except the by*i* bending modes – in consequence of the eccentric position of the shear centre exhibit strong flexural bending coupling. All the numerical results prove the good accuracy of the simple lumped mass matrix.

	mode	A(cons)	*A(lump)*	mode	B(cons)	*B(lump)*	SHELL
1	by1	17,375	*17,355*	by1	17,375	*17,355*	17,31
2	bz+t1	23,314	*23,306*	bz+t1	30,198	*30,182*	29,57
3	bz+t2	63,876	*63,791*	bz+t2	66,630	*66,535*	65,34
4	bz+t3	98,621	*98,324*	by2	108,66	*108,23*	105,83
5	by2	108,66	*108,23*	bz+t3	119,90	*119,45*	116,14
6	bz+t4	234,78	*233,44*	bz+t4	279,27	*277,40*	269,83
7	by3	303,20	*301,25*	by3	303,20	*301,25*	

Table 2. Test problem 2, frequencies (Hz), 7 DOF results with free end warping (A) and constrained end warping (B). by*i*, bz*i* bending, t*i* torsion, a*i* longitudinal modes.

Reference

1 Banerjee, J.R., Guo, S. and Howson, W.P. (1996): "Exact dynamic stiffness matrix of a bending-torsion coupled beam including warping" *Computers and Structures*. Vol.59. pp.613-621.

2 Kim, S.B. and Kim,M.Y. (2000): "Improved formulation for spatial stability and free vibration of thin-walled tapered beams and space frames" *Engineering Structures.*Vol.22. pp.446-458.

3 Kitipornchai, S.and Chan, S.L. (1989): "Stability and non-linear finite element analysis of thin walled structures" in *Finite element applications to thin-walled structures*, ed. Bull, J.W. Elsevier.

Appendix A*:* The 14x14 *consistent* mass matrix. $\underset{(14,14)}{\underline{\mathbf{m}}^{C}} = \rho A L \begin{bmatrix} \underline{\mathbf{m}}_1 & \underline{\mathbf{m}}_{12} \\ \underline{\mathbf{m}}_{12}^{T} & \underline{\mathbf{m}}_2 \end{bmatrix}$,

$$\underset{(7,7)}{\underline{\mathbf{m}}_1} = \left[\begin{array}{ccc|ccc|c}
2a & 0 & 0 & 0 & 0 & 0 & 0 \\
 & b + k\, i_s^2 & 0 & b\, z_{CT} & 0 & f + m\, i_s^2 & f\, z_{CT} \\
 & & b + k\, i_r^2 & -b\, y_{CT} & -f - m\, i_r^2 & 0 & f\, y_{CT} \\
\hline
 & & & b\, i_p^2 + k\, i_\omega^4 & -f\, y_{CT} & f\, z_{CT} & f\, i_p^2 + m\, i_\omega^4 \\
 & & & & h + 4e\, i_r^2 & 0 & h\, y_{CT} \\
 & & & & & h + 4e\, i_s^2 & h\, z_{CT} \\
\hline
 & & & & & & h\, i_p^2 + 4e\, i_\omega^4
\end{array}\right]$$

$$\underset{(7,7)}{\underline{\mathbf{m}}_2} = \left[\begin{array}{ccc|ccc|c}
2a & 0 & 0 & 0 & 0 & 0 & 0 \\
 & b + k\, i_s^2 & 0 & b\, z_{CT} & 0 & -f - m\, i_s^2 & -f\, z_{CT} \\
 & & b + k\, i_r^2 & -b\, y_{CT} & f + m\, i_r^2 & 0 & -f\, y_{CT} \\
\hline
 & & & b\, i_p^2 + k\, i_\omega^4 & f\, y_{CT} & -f\, z_{CT} & -f\, i_p^2 - m\, i_\omega^4 \\
 & & & & h + 4e\, i_r^2 & 0 & h\, y_{CT} \\
 & & & & & h + 4e\, i_s^2 & h\, z_{CT} \\
\hline
 & & & & & & h\, i_p^2 + 4e\, i_\omega^4
\end{array}\right]$$

$$\underset{(7,7)}{\underline{\mathbf{m}}_{12}} = \left[\begin{array}{ccc|ccc|c}
a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c - k\, i_s^2 & 0 & c\, z_{CT} & 0 & -g + m\, i_s^2 & -g\, z_{CT} \\
0 & 0 & c\text{-}k\, i_r^2 & -c\, y_{CT} & g - j\, i_r^2 & 0 & -g\, y_{CT} \\
\hline
0 & c\, z_{CT} & -c\, y_{CT} & c\, i_p^2 - k\, i_\omega^4 & g\, y_{CT} & -g\, z_{CT} & -g\, i_p^2 + m\, i_\omega^4 \\
0 & 0 & -g + m\, i_r^2 & -g\, y_{CT} & -j - e\, i_r^2 & 0 & -j\, y_{CT} \\
0 & g - m\, i_s^2 & 0 & g\, z_{CT} & 0 & -j - e\, i_s^2 & -j\, z_{CT} \\
\hline
0 & g\, z_{CT} & g\, y_{CT} & g\, i_p^2 - m\, i_\omega^4 & -j\, y_{CT} & -j\, z_{CT} & -j\, i_p^2 - e\, i_\omega^4
\end{array}\right]$$

$$a = \frac{1}{6}\ ,\quad b = \frac{13}{35}\ ,\quad c = \frac{9}{70}\ ,\quad e = \frac{1}{30}\ ,\quad f = \frac{11L}{210}\ ,\quad g = \frac{13L}{420}\ ,\quad h = \frac{L^2}{105}\ ,\quad k = \frac{6}{5L^2}\ ,$$

$$j = \frac{L^2}{140}\ ,\quad m = \frac{1}{10L}\ ,\quad i_r^2 = \frac{I_r}{A}\ ,\quad i_s^2 = \frac{I_s}{A}\ ,\quad i_p^2 = i_r^2 + i_s^2 + {y_{CT}}^2 + {z_{CT}}^2\ ,\quad i_\omega^4 = \frac{I_\omega}{A}\ .$$

Appendix B: The 14x14 *lumped* mass matrix.

$$\underset{(14,14)}{\underline{\mathbf{m}}^{C}} = \begin{bmatrix} \underline{\mathbf{m}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{m}} \end{bmatrix}\ ,\qquad \underset{(7,7)}{\underline{\mathbf{m}}} = \frac{\rho A L}{2} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & z_{CT} & 0 & 0 & 0 \\
 & & 1 & -y_{CT} & 0 & 0 & 0 \\
 & & & i_p^2 & 0 & 0 & 0 \\
 & & & & i_r^2 & 0 & 0 \\
 & & & & & i_s^2 & 0 \\
 & & & & & & i_\omega^4
\end{bmatrix}$$