Characterisation sets for the prenucleolus

Thesis Outline

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Abstract

We introduce a unified generalization of the prenucleolus, the percapita prenucleolus, and the weighted or q-prenucleolus, termed the $\mathbf{u}$-prenucleolus, by applying utility functions to the excesses. Using this framework, we also generalize the core, least-core, balanced games, essential coalitions, and dually essential coalitions into the $\mathbf{u}$-core, $\mathbf{u}$-least-core, $\mathbf{u}$-balanced games, $\mathbf{u}$-essential coalitions, and dually $\mathbf{u}$-essential coalitions, respectively. We demonstrate how various game-theoretic concepts and theorems operate within this new setup, including the definition of multiple characterization sets for the $\mathbf{u}$-prenucleolus – namely, $\mathbf{u}$-essential and dually $\mathbf{u}$-essential coalitions of $\mathbf{u}$-balanced games, and the intersection of $\mathbf{u}$-essential and dually $\mathbf{u}$-essential coalitions of games where the $\mathbf{u}$-least-core is a proper subset of the $\mathbf{u}$-core.

Summary

We introduce the **u**-prenucleolus (Dornai and Pintér, 2024), a common generalization of the prenucleolus (Schmeidler, 1969), the percapita prenucleolus (Grotte, 1970, 1972), the weighted or q-prenucleolus (Derks and Haller, 1999; Solymosi, 2019) and a special case of the general prenucleolus (Potters and Tijs, 1992; Maschler et al, 1992), by applying so-called utility functions to the excesses.

In addition, we study TU-games with restricted cooperation, where classical properties of the prenucleolus may no longer hold – such as single-valuedness. Building on Katsev and Yanovskaya (2013), who provided necessary and sufficient conditions for the non-emptiness and uniqueness of the prenucleolus, we extend these results to the **u**-prenucleolus.

Using the concept of utility functions, we introduce other generalizations like the **u**-core, **u**-least-core, **u**-balanced games, and **u**-essential coalitions. We generalize the Bondareva–Shapley theorem, showing that a game is **u**-balanced if and only if its **u**-core is nonempty. Similarly, we extend Huberman's theorem, proving that **u**-essential coalitions characterize the **u**-prenucleolus of **u**-balanced games. We identify sufficient conditions under which the **u**-prenucleolus and the **u**-core remain invariant, and present a class of games with a quadratic number of **u**-essential coalitions. In addition, we demonstrate that least essential coalitions – a subclass of **u**-essential coalitions – also characterize the prenucleolus of non-balanced games (Dornai and Pintér, 2022).

In the final chapter, we explore the dual of TU-games with utility functions, defining the $\mathbf{u}^*$-anti-nucleolus, $\mathbf{u}^*$-anti-prenucleolus, $\mathbf{u}^*$-anti-core and the $\mathbf{u}^*$-least-anti-core. We introduce dually-**u**-essential coalitions (Dornai and Pintér, 2025), extending the concept from Solymosi and Sziklai (2016) to TU-games with utility functions. We prove a variant of Huberman's theorem, showing that these coalitions also characterize the **u**-prenucleolus of **u**-balanced games.

Finally, we generalize a result by Granot et al (1998) regarding characterization sets of the prenucleolus. Leveraging this theorem, we generalize a result by Solymosi and Sziklai (2016), showing that the intersection of **u**-essential and dually-**u**-essential coalitions forms a characterization set for the **u**-prenucleolus under certain conditions.

1 Preliminaries

1.1 TU-games and solutions

Given a nonempty finite set of the players N and a function $v\colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$; then v is called a TU-*game* (henceforth game for short). Let $\mathcal{G}^N$ denote the class of games with player set N, moreover, let the set of coalitions be denoted by $\mathcal{P}(N) := \{S \subseteq N\}$ and the set of non-trivial coalitions be denoted by $\mathcal{P}^*(N) := \{S \subseteq N\colon S \neq \emptyset,\, S \neq N\}$. Let $\mathcal{D}_S$ denote the class of partitions of set $S \subseteq N$ except $\{S\}$.

Let $\mathcal{A} \subseteq \mathcal{P}(N)$ be such that $\emptyset, N \in \mathcal{A}$, then $\mathcal{A}$ is called a set of feasible coalitions. In this case, the function $v\colon \mathcal{A} \to \mathbb{R}$, $v(\emptyset) = 0$ is called a game with restricted cooperation. Let $\mathcal{G}^{N,\mathcal{A}}$ denote the class of games with restricted cooperation, where $\mathcal{A}$ is the set of feasible coalitions. If $\mathcal{A} = \mathcal{P}(N)$ then $\mathcal{G}^{N,\mathcal{A}} = \mathcal{G}^N$, therefore each introduced concept for games with restricted cooperation is a generalization of the related concept for classical games.

A set of coalitions $\mathcal{S} \subseteq \mathcal{A}$ is a balanced set system if there exist $\lambda_S \in \mathbb{R}_+$, $S \in \mathcal{S}$, called balancing weight system, such that

$$\sum_{S \in \mathcal{S}} \lambda_S \, \chi_S = \chi_N,$$

where $\chi_E \in \mathbb{R}^N$ is the characteristic vector of set E.

Let $\mathcal{A}^* := \mathcal{A} \setminus \{N, \emptyset\}$ denote the set of non-trivial feasible coalitions, and let $\mathcal{D}_S^{\mathcal{A}^*} := \{B \in \mathcal{D}_S\colon B \subseteq \mathcal{A}^*\}$ denote the non-trivial $\mathcal{A}^*$-partitions of set $S \in \mathcal{A}^*$.

A solution is a set-valued mapping from a set of games with player set N to $\mathbb{R}^N$. For example, the core (Shapley, 1955; Gillies, 1959), the kernel (Davis and Maschler, 1965), and the bargaining set (Aumann and Maschler, 1964). A value is a singleton valued solution, for example the Shapley-value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969).

Let $I(v) := \{x \in \mathbb{R}^N\colon \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\})\ \forall \{i\} \in \mathcal{A}\}$ and $I^*(v) := \{x \in \mathbb{R}^N\colon \sum_{i \in N} x_i = v(N)\}$ denote the set of imputations and preimputations of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ respectively.

Given a game with restricted cooperation $v \in \mathcal{G}^{N,\mathcal{A}}$, a coalition $S \in \mathcal{A}$ and a payoff a vector $x \in \mathbb{R}^N$, the *excess* of coalition S by the payoff vector x in the game v is $e(S,x) := v(S) - x(S)$, where $x(S) := \sum_{i \in S} x_i$.

The core of a game with restricted cooperation v is the set of preimputations for which the excess of every feasible coalition is non-positive:

$$\mathrm{core}(v) := \left\{x \in \mathbb{R}^N \colon x(N) = v(N) \text{ and } e_v(S,x) \leq 0,\ \forall S \in \mathcal{A}^*\right\}.$$

In case, the core is nonempty, we say the game is balanced.

The ε-core of the game is the set of preimputations, for which the maximal excess is at most ε, that is

$$\mathrm{core}_\varepsilon(v) := \{x \in I^*(v) \colon \max_{S \in \mathcal{A}^*} e_v(S,x) \leq \varepsilon\}.$$

Moreover, the least-core of the game is its smallest nonempty ε-core, provided it exists. The least-core is always well-defined, when all coalitions are feasible. However, in games with restricted cooperation, it may happen that a smallest nonempty ε-core does not exist.

Theorem 1. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game. The least-core of the game is well-defined if and only if $\mathcal{A}^*$ contains a balanced set system.*

The vector $E_v(x) := [\ldots \geq e_v(S,x) \geq \ldots \colon S \in \mathcal{A}^*]$ is called excess vector. It consists of all the excesses in non-increasing order. The lexicographical ordering between $x, y \in \mathbb{R}^n$ is the following: $x \leq_L y$ if $x = y$ or if there exists k such that $x_k < y_k$ and for every $i < k$ it holds that $x_i = y_i$. The nucleolus is the set of imputations which lexicographically minimize the excess vectors over the set of imputations, that is,

$$N(v) := \{x \in I(v) \colon E_v(x) \leq_L E_v(y),\ \forall y \in I(v)\}.$$

Moreover, the prenucleolus is the set of preimputations which lexicographically minimize the excess vectors over the set of preimputations, that is,

$$N^*(v) := \{x \in I^*(v) \colon E_v(x) \leq_L E_v(y),\ \forall y \in I^*(v)\}.$$

We say that a set of coalitions $\mathcal{S} \subseteq \mathcal{A}^*$ forms a characterization set of the prenucleolus of the game $v \in \mathcal{G}^{N,\mathcal{A}}$, if for the game $v' = v|_{\mathcal{S} \cup \{N\}}$ it holds that $N^*(v) = N^*(v')$.

1.2 The dual of TU-games

Given a game $v \in \mathcal{G}^N$, then its dual is the game $v^* \colon 2^N \to \mathbb{R}$ such that $v^*(S) = v(N) - v(N \setminus S)$, for all $S \in \mathcal{P}(N)$.

Let $N \setminus \mathcal{A}$ denote the set of the complements of the sets from $\mathcal{A}$, that is, $N \setminus \mathcal{A} := \{N \setminus S \colon S \in \mathcal{A}\}$. Given a game with restricted cooperation $v \in \mathcal{G}^{N,\mathcal{A}}$, then its dual is the game with restricted cooperation $v^*\colon 2^{N\setminus\mathcal{A}} \to \mathbb{R}$, such that $v^*(S) = v(N) - v(N\setminus S)$, for all $S \in N\setminus\mathcal{A}$. Moreover, let anti- $I(v^*) := \{x \in I^*(v^*)\colon v^*(N\setminus\{i\}) \geq x(N\setminus\{i\}),\ \forall N\setminus\{i\} \in N\setminus\mathcal{A}\}$ denote the set of anti-imputations of the dual game v^*. Notice that $I(v) =$ anti-$I(v^*)$, for all $v \in \mathcal{G}^{N,\mathcal{A}}$.

Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$, a coalition $S \in N\setminus\mathcal{A}$ and a payoff vector $x \in \mathbb{R}^N$. Then the satisfaction of coalition S by the payoff vector x in the dual game v^* is $f_{v^*}(S,x) := x(S) - v^*(S)$. Let $F_{v^*}(x) := [\ldots \geq f_{v^*}(S,x) \geq \ldots\colon S \in N\setminus\mathcal{A}^*]$ denote the satisfaction vector of the dual game v^*. It consists of all the satisfactions in non-increasing order. The anti-nucleolus of the dual game is the set of anti-imputations which lexicographically minimize the satisfaction vectors over the set of anti-imputations, that is,

$$\text{anti-}N(v^*) := \left\{x \in \text{anti-}I(v^*)\colon F_{v^*}(x) \leq_L F_{v^*}(y),\ \forall y \in \text{anti-}I(v^*)\right\}.$$

Moreover, the anti-prenucleolus of the dual game is the set of preimputations which lexicographically minimize the satisfaction vectors over the set of preimputations, that is,

$$\text{anti-}N^*(v^*) := \left\{x \in I^*(v^*)\colon F_{v^*}(x) \leq_L F_{v^*}(y),\ \forall y \in I^*(v^*)\right\}.$$

The anti-core of the dual of the game v is the set of preimputations for which the satisfaction of any coalition from the complementer set of the set of feasible coalitions is non-positive, that is,

$$\text{anti-core}(v^*) := \left\{x \in I^*(v^*)\colon f_{v^*}(S,x) \leq 0,\ \forall S \in N\setminus\mathcal{A}^*\right\}.$$

Similarly, the anti-ε-core of the dual game is the set of preimputations for which the maximal satisfaction of the coalitions from the complementer set of the set of feasible coalitions is not greater then ε, that is,

$$\text{anti-core}_\varepsilon(v^*) = \{x \in I^*(v^*)\colon \max_{S\in N\setminus\mathcal{A}^*} f_{v^*}(S,x) \leq \varepsilon\}.$$

In addition, the least-anti-core of the dual game is its smallest non-empty anti-ε-core, if it exists. Theorem 1 also applies here, meaning, that the least-anti-core of the dual game is well-defined if and only if $\mathcal{A}^*$ contains a balanced set-system.

Solutions of primal and dual games are related to each other as follows:

$$\begin{aligned} N(v) &= \text{anti-}N(v^*)\,, \\ N^*(v) &= \text{anti-}N^*(v^*)\,, \\ \text{core}(v) &= \text{anti- core}(v^*)\,, \\ \text{least-core}(v) &= \text{least-anti-core}(v^*)\,. \end{aligned}$$

1.3 The lexicographic center algorithm

The lexicographic center algorithm Kopelowitz (1967); Maschler et al (1979) is one of the most well-known algorithms for computing the (pre)nucleolus. In this section, we will discuss the lexicographic center algorithm with the modifications by Huberman (1980).

Consider a game $v \in \mathcal{G}^N$ and the following problem:

$$\begin{aligned} & t \to \min \\ \text{s.t.}\quad & e(S,x) \le t, \quad S \in \mathcal{P}^*(N) \\ & x \in I^*(v) \\ & t \in \mathbb{R} \end{aligned} \tag{1}$$

It is easy to see, that (1) has an optimal solution. Let the optimal value of (1) be denoted by t_1 and

$$X_1 = \{x \in I^*(v) \colon e(S,x) \le t_1,\ \forall S \in \mathcal{P}^*(N)\}.$$

Let W_1 denote the fix-set of (1), that is

$$W_1 = \{S \in \mathcal{P}^*(N) \colon \exists c_S \in \mathbb{R},\ \text{such that } e(S,x) = c_S,\ \forall x \in X_1\}.$$

For all $k \ge 2$ consider the following LP:

$$\begin{aligned} & t \to \min \\ \text{s.t.}\quad & e(S,x) \le t, \quad S \in \mathcal{P}^*(N) \setminus (\cup_{r=1}^{k-1} W_r) \\ & x \in X_{k-1} \\ & t \in \mathbb{R} \end{aligned} \tag{2}$$

It is easy to see, that (2) has an optimal solution. Let the optimal value of (2) be denoted by t_k and

$$X_k = \{x \in X_{k-1} \colon e(S,x) \le t_k,\ \forall S \in \mathcal{P}^*(N) \setminus (\cup_{r=1}^{k-1} W_r)\}.$$

Let W_k denote the fix-set of (2), that is

$$W_k = \{S \in \mathcal{P}^*(N) \colon \exists c_S \in \mathbb{R}, \text{ such that } e(S,x) = c_S,\ \forall x \in X_k\}.$$

It is easy to see, that $t_k \geq t_{k+1}$, $X_k \supseteq X_{k+1}$ for all k and there exists a k^*, such that for all $l \geq k^*$ $X_l = X_{k^*}$.

Kopelowitz (1967); Maschler et al (1979) proved that the lexicographic center algorithm returns with the prenucleolus. The above described algorithm is a modification of the lexicographic center algorithm by Huberman (1980). These modifications do not change the result of the algorithm. Therefore, we can say, that Kopelowitz (1967) and Maschler et al (1979) proved the following theorem:

Theorem 2. *$N^*(v) = X_{k^*}$, for all $v \in \mathcal{G}^N$.*

1.4 Huberman's theorem

Huberman (1980) showed that the so-called essential coalitions give a characterization set for the nucleolus of balanced TU-games. Since in case of balanced games, the nucleolus and the prenucleolus coincide, the essential coalitions also give a characterization set for the prenucleolus. First, consider the definition of essential coalitions used by Huberman (1980).

Definition 3. *Let $v \in \mathcal{G}^N$ be a game. Then, a coalition $S \in \mathcal{P}^*(N)$ is essential, if either $|S| = 1$, or*

$$v(S) > \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v(T).$$

Let $\mathcal{E}_v$ denote the class of essential coalitions of the game v.

Here is Huberman (1980)'s theorem:

Theorem 4 (Huberman (1980))**.** *Let $v \in \mathcal{G}^N$ be a balanced game. Then $\mathcal{E}_v$ is a characterization set for the nucleolus, that is, the values $(v(S))_{S \in \mathcal{E}_v}$ determine the nucleolus of the game v.*

Huberman's theorem can be used to show that for certain classes of games the prenucleolus can be calculated in polynomial time (in the number of players), since there are only polynomial many essential coalitions.

For example, in case of matching games (see Example 5), it can be shown, that only the singletons and the two-element coalitions are essential; therefore, if the core is non-empty, the prenucleolus can be calculated in polynomial time.

Similarly, in case of assignment games, only the singletons and the pairs are essential. In addition, in case of assignment games, the core is always non-empty; therefore the prenucleolus can be calculated in polynomial time.

Example 5. In case of matching games, the value of singletons is 0, the values of pairs is given by $v(\{i,j\}) = a_{i,j}$ for all $i \neq j \in N$. For all other coalitions $S \in \mathcal{P}^*(N)$ $v(S) = \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{\{i,j\} \in \mathcal{B}} a_{i,j}$.

Therefore, for all $S \in \mathcal{P}^*$, $|S| > 2$ there exists a $\mathcal{B}^* \subseteq \mathcal{D}_S$ containing only singletons and pairs, such that:

$$v(S) = \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{\{i,j\} \in \mathcal{B}} a_{i,j} = \sum_{\{i,j\} \in \mathcal{B}^*} v(\{i,j\}) + \sum_{\{k\} \in \mathcal{B}^*} v(\{k\}),$$

so S is not essential.

Therefore, by applying Huberman's theorem, if v is balanced, the pairs and the singletons are enough to calculate the prenucleolus of v.

In this thesis, we provide multiple generalizations and variations of Huberman's theorem, that can be applied in different settings (Theorems 4, 5 and 6).

2 TU-games with utility functions

A well-known variant of the prenucleolus is the percapita prenucleolus (Grotte, 1970, 1972). The percapita prenucleolus differs from the preucleolus in a way that instead of using the excesses, it uses the so-called percapita excesses. The percapita excess of a coalition $S \in \mathcal{A}^*$ of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a payoff vector $x \in \mathbb{R}^N$ is $\frac{e_v(S,x)}{|S|}$. Similarly, the percapita excess vector is $E_v^{pc}(x) := (\frac{e_v(S,x)}{|S|})_{S \in \mathcal{A}^*} \in \mathbb{R}^{|\mathcal{A}^*|}$, where $E_v^{pc}(x)_i \geq E_v^{pc}(x)_j$ if $i \leq j$. Accordingly, the percapita prenucleolus is defined as follows: $N_{pc}^*(v) = \{x \in I^*(v) \colon E_v^{pc}(x) \leq_L E_v^{pc}(y),\ \forall y \in I^*(v)\}$.

Solymosi (2019) considers a further generalization of the percapita prenucleolus, where, instead of dividing the excess by the cardinality of S, it is divided by $q(S)$, where q is a positive real valued function over the feasible coalitions. Thereby, Solymosi (2019) introduced the notion of q-nucleolus N_q, which is defined as follows: $N_q^*(v) = \{x \in I^*(v) \colon E_v^q(x) \leq_L E_v^q(y),\ \forall y \in I^*(v)\}$, where the q-excess vector is defined as $E_v^q(x) := (\frac{e_v(S,x)}{q(S)})_{S \in \mathcal{A}^*}$, where $E_v^q(x)_i \geq E_v^q(x)_j$ if $i \leq j$.

Partially inspired by the above generalizations of the prenucleolus we generalize the prenucleolus further by introducing functions, called utility functions, applied to the excesses (Dornai and Pintér, 2024, 2005). Formally, see the following definition.

Definition 6. *A utility function* $\mathbf{u} \colon \mathcal{A}^* \times \mathbb{R} \to \mathbb{R}$ *is a family of functions* $(u_S)_{S \in \mathcal{A}^*}$ *such that* $u_S \colon \mathbb{R} \to \mathbb{R}$ *is strictly monotone increasing, continuous, and its domain is* $\mathbb{R}$*. Moreover, the ranges of* u_S *and* u_T *are the same for every* $S, T \in \mathcal{A}^*$*; let* $R_{\mathbf{u}}$ *denote this common range.*

Let the **u**-excess of a coalition $S \in \mathcal{A}^*$ by the payoff vector $x \in \mathbb{R}^N$ in the game v be as follows: $u_S \circ e_v(S,x) = u_S(v(S) - x(S))$. Moreover, let the **u**-excess vector be defined as $E_v(x) := (u_S(e_v(S,x)))_{S \in \mathcal{A}^*} \in \mathbb{R}^{|\mathcal{A}^*|}$, where $E_v(x)_i \geq E_v(x)_j$ if $i \leq j$.

We can now define the **u**-prenucleolus similarly to the percapita prenucleolus.

Definition 7. *The* **u***-prenucleolus is the set of preimputations, which lexicographically minimizes the* **u***-excess vectors over the set of preimputations. Formally,*

$$N^*_{\mathbf{u}}(v) := \{x \in I^*(v) \colon E^{\mathbf{u}}_v(x) \leq_L E^{\mathbf{u}}_v(y)\ \forall y \in I^*(v)\}.$$

Example 8. Some examples of utility functions:

- If **u** is the identity function, then the **u**-penucleolius is the prenucleolus.
- If **u** is defined for all $S \in \mathcal{A}^*$ as $u_S(t) = \frac{t}{|S|}$, then the **u**-prenucleolus is the percapita prenucleolus.
- We can also define **u** as a shift by a constant c. In this case $u_S(t) = t + c$, and for any game $v \in \mathcal{G}^{N,\mathcal{A}}$ the **u**-prenucleolus is the prenucleolus of the game v', where $v'(S) = v(S) + c$ for all $S \in \mathcal{A}^*$, and $v'(N) = v(N)$. Since the prenucleolus is invariant for shifting, in this case the prenucleolus and the **u**-prenucleolus of the game are the same.
- Note that **u** is not necessarily a family of linear functions. For example $u_S(t) = \arctan(t)$ for all $S \in \mathcal{A}^*$ can also be a utility function.

Next, we introduce a generalization of the core (Shapley, 1955; Gillies, 1959):

Definition 9. *Given a utility function* **u***, the* **u**-core *of a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *is defined as follows:*

$$\mathbf{u}\text{-core}(v) := \left\{x \in \mathbb{R}^N \colon x(N) = v(N) \text{ and } u_S \circ e_v(S,x) \leq 0, \forall S \in \mathcal{A}^*\right\}.$$

Notice that, if $\mathcal{A} = \mathcal{P}(N)$ and **u** is the identity function, then the **u**-core is the core.

The **u**-ε-core of the game is the set of preimputations, for which the maximal **u**-excess is at most ε, that is

$$\text{core}^{\mathbf{u}}_{\varepsilon}(v) := \{x \in I^*(v) \colon \max_{S \in \mathcal{A}^*} u_S \circ e_v(S,x) \leq \varepsilon\}.$$

Moreover, the **u**-least-core of the game is its smallest nonempty **u**-ε-core, provided it exists.

Theorem 10. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* **u**. *The* **u***-least-core of the game is well-defined if and only if* $\mathcal{A}^*$ *contains a balanced set system.*

3 Thesis Points

The principal results of the dissertation are highlighted as Thesis 1 to 9.

3.1 A generalization of the Bondareva–Shapley Theorem holds for TU-games with utility functions

Let $\mathfrak{B}$ denote the class of balanced set systems of $\mathcal{A}$.

Definition 11. *Given a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$*, the game* v *is* $\mathbf{u}$*-balanced, if either* $R_{\mathbf{u}} \subseteq \mathbb{R}_- \setminus \{0\}$ *or if* $0 \in R_{\mathbf{u}}$ *and*

$$\max_{\mathcal{B} \in \mathfrak{B}} \left(\lambda_N v(N) + \sum_{S \in \mathcal{B} \setminus \{N\}} \lambda_S \left(v(S) - \mathbf{u}_S^{-1}(0) \right) \right) \leq v(N), \tag{3}$$

where $(\lambda_S)_{S \in \mathcal{B}}$ *is the balancing weight system of the balanced coalition system* $\mathcal{B}$*.*

Notice that, if $\mathcal{A} = \mathcal{P}(N)$ and $\mathbf{u}$ is the identity function, then the $\mathbf{u}$-balancedness pins down to balancedness.

The following theorem is a generalization of the Bondareva–Shapley theorem (Bondareva, 1963; Shapley, 1967; Faigle, 1989) to games with utilities.

Thesis 1. *Given a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$*, the* $\mathbf{u}\text{-core}(v) \neq \emptyset$ *if and only if* v *is* $\mathbf{u}$*-balanced.*

The proof of Thesis 1 relies on the strong duality theorem. For the proof see Dornai and Pintér (2024).

3.2 A generalization of theorems on the non-emptiness and cardinality of the prenucleolus by Katsev and Yanovskaya holds for the u-prenucleolus

When dealing with TU-games with restricted cooperation, the prenucleolus in not necessarily single-valued. This also stands for the $\mathbf{u}$-prenucleolus. We showed, that generalizations of two theorems by Katsev and Yanovskaya (2013) about the non-emptiness and cardinality of the prenucleolus also hold for the $\mathbf{u}$-prenucleolus.

The following lemma offers a key observation for proving Theorem 2.

Lemma 12. *Let* $v \in \mathcal{G}^{N,\mathcal{A}}$ *be a game, and* $\mathbf{u^1}$, $\mathbf{u^2}$ *be utility functions. Let* $X \subseteq I^*(v)$*, then*

$$\min_{x \in X} \max_{S \in \mathcal{A}^*} u_S^1 \circ (v(S) - x(S))$$

exists if and only if

$$\min_{x \in X} \max_{S \in \mathcal{A}^*} u_S^2 \circ (v(S) - x(S))$$

exists.

Thesis 2. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and $\mathbf{u}$ be a utility function. Then $\mathcal{A}^*$ is a balanced set of coalitions, if and only if the $\mathbf{u}$-prenucleolus of the game is nonempty.*

For the proof see Dornai and Pintér (2024).

For a family of coalitions $\mathcal{A} \subseteq \mathcal{P}(N)$ let $X(\mathcal{A})$ denote the $|\mathcal{A}| \times |N|$ dimensional matrix, where its row vectors are the characteristic vectors of the sets from $\mathcal{A}$.

Thesis 3. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, where $\mathcal{A}^*$ is a balanced set of coalitions, and a utility function $\mathbf{u}$, the $\mathbf{u}$-prenucleolus of the game v is a singleton if and only if rank$(X(\mathcal{A})) = |N|$.*

For the proof see Dornai and Pintér (2024).

3.3 The u-essential coalitions characterize the u-prenucleolus of u-balanced games

When generalizing Huberman's theorem, we need to "redefine" the essential coalitions (Definition 3).

Definition 13. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system and utility function $\mathbf{u}$, a coalition $S \in \mathcal{A}^*$ is $\mathbf{u}$-essential, if either $\mathcal{D}_S^{\mathcal{A}^*} = \emptyset$ or if $\exists x \in \mathbf{u}$-least-core(v) such that*

$$u_S \circ e_v(S,x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}} \sum_{T \in \mathcal{B}} u_T \circ e_v(T,x).$$

Let $\mathcal{E}_v^{\mathbf{u}}$ denote the class of $\mathbf{u}$-essential coalitions of the game v.

Let $X_0 = I^*(v)$ and $W_0 = \emptyset$. For $k \geq 1$, let us consider the following problem:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_S \circ e_v(S,x) \leq t, \quad S \in \mathcal{A}^* \setminus (\cup_{r=1}^{k-1} W_r) \\ & x \in X_{k-1} \\ & t \in \mathbb{R} \end{aligned} \tag{4}$$

If (4) has an optimal solution, let t_k denote the optimum of (4).

Let X_k be defined as follows

$$X_k = \{x \in X_{k-1} \colon u_S \circ e_v(S,x) \leq t_k, \ \forall S \in \mathcal{A}^* \setminus (\cup_{r=1}^{k-1} W_r)\}.$$

Furthermore, let

$$W_k = \{S \in \mathcal{A}^* \colon \exists c_S \in \mathbb{R}, \text{ such that } u_S \circ e_v(S,x) = c_S,\ \forall x \in X_k\}.$$

Consider the following optimization problem as well:

$$\begin{aligned} & t \to \min \\ \text{s.t.}\quad & u_S \circ e_v(S,x) \leq t, \quad S \in \mathcal{E}_v^{\mathbf{u}} \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{5}$$

Let t_1' be the optimum of problem (5) and X_1' be the set of optimal solutions of problem (5) except t, that is, $X_1' = \{x \in I^*(v) \colon u_S \circ e_v(S,x) \leq t_1' \ \forall S \in \mathcal{E}_v^{\mathbf{u}}\}$.

Lemma 14 is a core observation for many other lemmata and theorems. It shows that we can use the **u**-excesses of only the **u**-essential coalitions to give an upper estimate of the **u**-excess of a not **u**-essential coalition.

Lemma 14. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system. Let $S \in \mathcal{A}^* \setminus \mathcal{E}_v^{\mathbf{u}}$. Then for every $x \in \mathbf{u}\text{-least-core}(v)$ there exists $\mathcal{B}^* \in \mathcal{D}_S^{\mathcal{A}^*}$ such that $u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e_v(T,x)$ and $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}}$.*

Next, we introduce the following notion: for a class of coalitions $\mathcal{S} \subseteq \mathcal{A}^*$ and $t \in \mathbb{R}$ let $X(\mathcal{S},t) := \{x \in I^*(v) \colon u_S \circ e_v(S,x) \leq t, \forall S \in \mathcal{S}\}$.

Lemmata 15 and 16 show that some very important geometrical properties of the sets $X(\mathcal{A}^*,t)$ and $X(\mathcal{E}_v^{\mathbf{u}},t)$ remain unchanged by the introduction of utility functions (compared to using the identity utility function).

Lemma 15. *Given a utility function $\mathbf{u}$ and a $\mathbf{u}$-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system, for every $t_1 \leq t$ it holds that both $X(\mathcal{A}^*,t)$ and $X(\mathcal{E}_v^{\mathbf{u}},t)$ are nonempty, convex and closed.*

Lemma 16. *Given a utility function $\mathbf{u}$ and a game $v \in \mathcal{G}^{N,\mathcal{A}}$, take $x_1, x_2 \in \mathbb{R}^N$ and $S \in \mathcal{A}$. If $u_S \circ e_v(S,x_1) < u_S \circ e_v(S,x_2)$, then for every $\lambda \in (0,1)$ it holds that*

$$u_S \circ e_v(S,x_1) < u_S \circ e_v(S, \lambda x_1 + (1-\lambda)x_2) < u_S \circ e_v(S,x_2).$$

The following proposition states, that the **u**-essential coalitions characterize the **u**-least-core of **u**-balanced games.

Proposition 17. *Given a utility function $\mathbf{u}$, a $\mathbf{u}$-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system, the following holds: $t_1 = t_1'$ and $X_1 = X_1'$.*

The following theorem generalizes Huberman (1980)'s theorem (Theorem 7 on page 420 of Huberman (1980)):

Thesis 4. *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, such that* $\mathcal{A}^*$ *is a balanced set system, and let*

$$Y_1 = \{x \in I^*(v)\colon u_S \circ e_v(S,x) \leq t_1,\ \forall S \in \mathcal{E}_v^{\mathbf{u}}\},$$

and for all $k \geq 2$ *let* Y_k *be defined as follows:*

$$Y_k = \{x \in X_{k-1}\colon u_S \circ e_v(S,x) \leq t_k,\ \forall S \in \mathcal{E}_v^{\mathbf{u}} \setminus (\cup_{r=1}^{k-1} W_r)\},$$

where t_k *is the optimum of* (4)*, if it exists and* $-\infty$ *otherwise. Then,* $X_k = Y_k$ *for all* $k \geq 1$*.*

In other words, Theorem 4 claims that the **u**-essential coalitions give a characterization set for the **u**-prenucleolus of **u**-balanced games.

For the proof see Dornai and Pintér (2024) and Dornai and Pintér (2005).

A subclass of **u**-essential coalitions is called least-essential coalitions. In this case, the utility function applies a uniform shift of $(-t_1)$ to each non-trivial coalition. In Dornai and Pintér (2022), we show that least-essential coalitions characterize the prenucleolus in both balanced and non-balanced games. This result extends the applicability of a variation of essential coalitions to settings beyond balanced games.

The above results were published in Dornai and Pintér (2024), Dornai and Pintér (2022) and Dornai and Pintér (2005).

3.4 The dually-u-essential coalitions characterize the u-prenucleolus of u-balanced games

The idea of applying utility functions to games can be extended to dual games. Since the feasible coalitions of the dual game are the complementer set of the feasible coalitions of the primal game, we define the **u**-satisfaction of the dual game using the u_S functions corresponding to their complementers.

Definition 18. *The dual of a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with utility function* $\mathbf{u}$ *is the game* $v^* \in \mathcal{G}^{N,N\setminus\mathcal{A}}$ *with utility function* $\mathbf{u}^* := [u_{N\setminus S}]_{S\in N\setminus\mathcal{A}^*}$*.*

Notice that the dual of the dual game with a utility function is the primal game with the same utility function. This follows from the identities $v^{**} = v$ and $\mathbf{u}^{**} = ([u_{N\setminus S}]_{S\in N\setminus\mathcal{A}^*})^* = [u_{N\setminus(N\setminus S)}]_{S\in N\setminus(N\setminus\mathcal{A}^*)} = [u_S]_{S\in\mathcal{A}^*}$. Thus, applying the dual operation twice restores both the original game and the original utility function.

Definition 19. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$*. Then the* $\mathbf{u}^*$*-satisfaction of a coalition* $S \in N \setminus \mathcal{A}$ *by the payoff vector* $x \in \mathbb{R}^N$ *in the dual* *game* v^* *is* $u_{N\setminus S} \circ f_{v^*}(S,x)$*, and the* $\mathbf{u}^*$*-satisfaction vector is* $F^{\mathbf{u}^*}_{v^*}(x) := [\ldots \geq u_{N\setminus S} \circ f_{v^*}(S,x) \geq \ldots : S \in N \setminus \mathcal{A}^*]$.

The $\mathbf{u}^*$-anti-nucleolus and $\mathbf{u}^*$-anti-prenucleolus is defined with the $\mathbf{u}^*$-satisfaction vectors:

Definition 20. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$*. Then the* $\mathbf{u}^*$*-anti-nucleolus of* v^* *is the set of* $\mathbf{u}^*$*-anti-imputations, which lexicographically minimize the* $\mathbf{u}^*$*-satisfaction vectors over the set of* $\mathbf{u}^*$*-anti-imputations, that is,*

$$\text{anti-}N_{\mathbf{u}^*}(v^*) = \left\{x \in \mathbf{u}^*\text{-anti-}I(v^*)\colon F^{\mathbf{u}^*}_{v^*}(x) \leq_L F^{\mathbf{u}^*}_{v^*}(y),\ \forall y \in \mathbf{u}^*\text{-anti-}I(v^*)\right\},$$

where $\mathbf{u}^*\text{-anti-}I(v^*) := \{x \in I^*(v^*)\colon u_{\{i\}} \circ f_{v^*}(N \setminus \{i\}, x) \leq 0,\ \forall N \setminus \{i\} \in N \setminus \mathcal{A}^*\}$.

Moreover, the $\mathbf{u}^*$*-anti-prenucleolus of* v^* *is the set of preimputations which lexicographically minimize the* $\mathbf{u}^*$*-satisfaction vectors over the set of preimputations, that is,*

$$\text{anti-}N^*_{\mathbf{u}^*}(v^*) = \left\{x \in I^*(v^*)\colon F^{\mathbf{u}^*}_{v^*}(x) \leq_L F^{\mathbf{u}^*}_{v^*}(y),\ \forall y \in I^*(v^*)\right\}.$$

Similarly, the $\mathbf{u}^*$-anti-core of the dual game is also defined with the $\mathbf{u}^*$-satisfactions:

Definition 21. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$*. Then the* $\mathbf{u}^*$*-anti-core of the dual game* v^* *is the set of preimputations for which the* $\mathbf{u}^*$*-satisfactions are non-positive:*

$$\mathbf{u}^*\text{-anti-core}(v^*) = \left\{x \in I^*(v^*)\colon u_{N\setminus S} \circ f_{v^*}(S,x) \leq 0,\ \forall S \in N \setminus \mathcal{A}^*\right\}.$$

If the $\mathbf{u}^*$-anti-core of a game is not empty then we say that the game is $\mathbf{u}^*$-anti-balanced.

Similar relations hold for solutions of TU-games with utility functions as for TU-games:

Theorem 22. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$*. Then the following hold:*

1. $N_{\mathbf{u}}(v) = \text{anti-}N_{\mathbf{u}^*}(v^*)$,
2. $N^*_{\mathbf{u}}(v) = \text{anti-}N^*_{\mathbf{u}^*}(v^*)$,
3. $\mathbf{u}\text{-core}(v) = \mathbf{u}^*\text{-anti-core}(v^*)$,
4. $\mathbf{u}\text{-least-core}(v) = \mathbf{u}^*\text{-least-anti-core}(v^*)$.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game with a utility function $\mathbf{u}$. The modified lexicographic center algorithm for calculating the $\mathbf{u}^*$-anti-prenucleolus of the dual game v^* solves the following optimization problems iteratively:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_{N\setminus S} \circ f_{v^*}(S,x) \leq t, \quad S \in (N \setminus \mathcal{A}^*) \setminus (\cup_{r=0}^{k-1} W_r^d) \\ & x \in X_{k-1}^d \\ & t \in R_{\mathbf{u}}, \end{aligned} \tag{6}$$

where $X_0^d := I^*(v^*)$ and $W_0^d := \emptyset$ and for $k \geq 1$ t_k^d denotes the optimum of (6) if it exists, and

$$\begin{aligned} X_k^d &:= \{x \in X_{k-1}^d \colon u_{N\setminus S} \circ f_{v^*}(S,x) \leq t_k^d,\ \forall S \in (N \setminus \mathcal{A}^*) \setminus (\cup_{r=0}^{k-1} W_r^d)\}\,, \\ W_k^d &:= \{S \in N \setminus \mathcal{A}^* \colon \exists c_S \in \mathbb{R},\ \text{such that } u_{N\setminus S} \circ f_{v^*}(S,x) = c_S,\ \forall x \in X_k^d\}\,. \end{aligned}$$

It is easy to see that $t_k^d \geq t_{k+1}^d$ and $X_k^d \supseteq X_{k+1}^d$ for all $k \in \mathbb{N}_+$, and there exists k^* such that for all $l \geq k^*$ it holds that $X_{k^*}^d = X_l^d$. Since for every $x \in I^*(v), S \in \mathcal{A}^*$ $u_S \circ e_v(S,x) = u_S \circ f_{v^*}(N \setminus S, x)$, we have that $t_i^d = t_i$ and $X_i^d = X_i$ for all $i \in \mathbb{N}_+$. In addition, we learned in Theorem 22, that the $\mathbf{u}$-prenucleolus of the primal game equals the $\mathbf{u}^*$-anti-prenucleolus of the dual game. Therefore, we can apply the result of Maschler et al (1992), saying, that $X_{k^*}^d = \text{anti-}N_{\mathbf{u}^*}^*(v^*)$.

Definition 23. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system with a utility function $\mathbf{u}$. Then a coalition $S \in N \setminus \mathcal{A}^*$ is $\mathbf{u}^*$-anti-essential if $\mathcal{D}_S^{N\setminus\mathcal{A}^*} = \emptyset$ or if there exists $x \in \mathbf{u}^*\text{-least-anti-core}(v^*)$ such that*

$$u_{N\setminus S} \circ f_{v^*}(S,x) > \max_{\mathcal{B} \in \mathcal{D}_S^{N\setminus\mathcal{A}^*}} \sum_{T \in \mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)\,.$$

The complementer set of the class of $\mathbf{u}^*$-anti-essential coalitions (of game v^*) is called the class of dually-$\mathbf{u}$-essential coalitions (of game v). Let $\mathcal{E}_{v^*}^{a-\mathbf{u}^*}$ denote the class of $\mathbf{u}^*$-anti-essential coalitions (of v^*) and $\mathcal{E}_v^{d-\mathbf{u}}$ denote the class of dually-$\mathbf{u}$-essential coalitions (of v).

In words, the dually-$\mathbf{u}$-essential coalitions of a game are the complements of the $\mathbf{u}^*$-anti-essential coalitions of the dual of the game.

The following lemma is a key observation to prove our main result about $\mathbf{u}$-anti-essential and dually-$\mathbf{u}$-essential coalitions.

Lemma 24. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system with a utility function $\mathbf{u}$. Let $S \in N \setminus \mathcal{A}^*$ be a non $\mathbf{u}^*$-anti-essential coalition (of*

v^*). Then for every $x \in \mathbf{u}^*$-least-anti-core(v^*) there exists $\mathcal{B}^* \in \mathcal{D}_S^{N \setminus \mathcal{A}^*}$ such that $u_{N \setminus S} \circ f_{v^*}(S,x) \leq \sum_{T \in \mathcal{B}^*} u_{N \setminus T} \circ f_{v^*}(T,x)$ and $\mathcal{B}^* \subseteq \mathcal{E}_{v^*}^{a-\mathbf{u}^*}$.

The following proposition claims, that the $\mathbf{u}$-anti-essential coalitions characterize the $\mathbf{u}$-anti-least-core of the dual game.

Proposition 25. *Consider a utility function $\mathbf{u}$, and a $\mathbf{u}$-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system. Let $t_1^{d'} := \min\{t \colon X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t) \neq \emptyset\}$, and $X_1^{d'} := X^d(N \setminus \mathcal{A}^*, t_1^{d'})$. Then the following hold: $t_1^d = t_1^{d'}$ and $X_1^d = X_1^{d'}$.*

The following theorem states that the $\mathbf{u}^*$-anti-essential coalitions form a characterization set for the $\mathbf{u}^*$-anti-prenucleolus of v^*, if v is $\mathbf{u}$-balanced.

Thesis 5. *Consider a utility function $\mathbf{u}$, and a $\mathbf{u}$-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system; moreover, for $k \geq 1$ let*

$$Y_k^d := \{x \in X_{k-1}^d \colon u_{N \setminus S} \circ f_{v^*}(S,x) \leq t_k^d \ \forall S \in \mathcal{E}_{v^*}^{a-\mathbf{u}^*}\} .$$

Then $X_k^d = Y_k^d$ for all $k \geq 1$.

In other words, the $\mathbf{u}^*$-anti-essential coalitions (of v^*) form a characterization set for the $\mathbf{u}^*$-anti-prenucleolus of v^*.

For the proof see Dornai and Pintér (2025).

The following theorem establishes the connection between characterization sets for the $\mathbf{u}$-prenucleolus of the primal game and characterization sets for the $\mathbf{u}^*$-anti-prenucleolus of the dual game.

Theorem 26. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$. Then a set system is a characterization set for the $\mathbf{u}$-prenucleolus of v if and only if its complementer set system is a characterisation set for the $\mathbf{u}^*$-anti-prenucleolus of v^*.*

The following theorem is a corollary of Theorems 5 and 26.

Thesis 6. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$, such that $\mathcal{A}^*$ contains a balanced set system with a utility function $\mathbf{u}$, such that v is $\mathbf{u}$-balanced. Then the dually-$\mathbf{u}$-essential coalitions form a characterization set for the $\mathbf{u}$-prenucleolus of v.*

For the proof see Dornai and Pintér (2025).

The above results were published in Dornai and Pintér (2025).

3.5 Generalizations of a theorem by Granot et al (1998) about characterization sets of the prenucleolus hold for the u-prenucleolus in case of TU-games with restricted cooperation

First, we provide a generalization of Theorem 2.3 on p. 362 in Granot et al (1998). In case of games with restricted cooperation, the original conditions do not imply the equality of the considered **u**-prenucleoli, but only less:

Thesis 7. *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with the utility function* $\mathbf{u}$*, and a set of coalitions* $\mathcal{F} \subseteq \mathcal{A}^*$*,* $\mathcal{F} \neq \emptyset$*. Let* $v' \in \mathcal{G}^{N,\mathcal{F}\cup\{N,\emptyset\}}$ *be* $v' = v|_{\mathcal{F}\cup\{N,\emptyset\}}$*. If* $x \in N^*_{\mathbf{u}}(v')$*, and for every* $S \in \mathcal{A}^* \setminus \mathcal{F}$ *there exists* $\mathcal{F}_{S,x} \subseteq \mathcal{F}$*,* $\mathcal{F}_{S,x} \neq \emptyset$ *such that*

1. $u_S \circ e_v(S,x) \leq u_T \circ e_v(T,x)$ *for all* $T \in \mathcal{F}_{S,x}$*,*
2. $\chi_S \in \mathrm{Lin}\{\chi_T \colon T \in \mathcal{F}_{S,x} \cup \{N\}\}$*,*

then $x \in N^*_{\mathbf{u}}(v)$*.*

For the proof see Dornai and Pintér (2025).

To prove this theorem we need a generalization of Kohlberg's theorem (Kohlberg, 1971).

Maschler et al (1992) proved a generalisation of Kohlberg's theorem for the general prencucleolus in Theorem 7.2 on pages 102-104 of Maschler et al (1992). The **u**-prenucleolus is a special case of the general prenucleolus, therefore the following generalization of Kohlberg's theorem is a corollary of the result by Maschler et al (1992):

Theorem 27 (Generalization of Kohlberg's theorem)**.** *Given a game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, a utility function* $\mathbf{u}$*, and* $x \in I^*(v)$*:* x *is an element of the* $\mathbf{u}$*-prenucleolus (*$x \in N^*_{\mathbf{u}}(v)$*) if and only if* $\mathcal{D}_{\mathbf{u}}(\alpha, x)$ *is a balanced set of coalitions for every* α *such that* $\mathcal{D}_{\mathbf{u}}(\alpha, x) \neq \emptyset$*, where* $\mathcal{D}_{\mathbf{u}}(\alpha, x^*) := \{S \in \mathcal{A}^* \colon u_S \circ e_v(S, x^*) \geq \alpha\}$*.*

Notice that by Theorem 7 we have that $N^*_{\mathbf{u}}(v') \subseteq N^*_{\mathbf{u}}(v)$. Therefore, if $N^*_{\mathbf{u}}(v)$ is a singleton and $N^*_{\mathbf{u}}(v')$ has at least one element, then we know, that the two are equal. However, in case of games with restricted cooperation, the **u**-prenucleolus is not necessarily a singleton. Therefore, in case of games with restricted cooperation, we need an extra condition to get $N^*_{\mathbf{u}}(v') = N^*_{\mathbf{u}}(v)$.

Thesis 8. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with a utility function* $\mathbf{u}$ *such that* v *is* $\mathbf{u}$*-balanced, and a set sytem* $\mathcal{F} \subseteq \mathcal{A}^*$*,* $\mathcal{F} \neq \emptyset$*. Let* $v' \in \mathcal{G}^{N,\mathcal{F}\cup\{N,\emptyset\}}$ *be such that* $v' =$

$v|_{\mathcal{F}\cup\{N,\emptyset\}}$ and $X \subseteq I^*(v)$ be such that $N^*_{\mathbf{u}}(v), N^*_{\mathbf{u}}(v') \subseteq X$. If for every $x \in X$, $S \in \mathcal{A}^* \setminus \mathcal{F}$ there exists $\mathcal{F}_{S,x} \subseteq \mathcal{F}$, $\mathcal{F}_{S,x} \neq \emptyset$ such that

1. *$u_S \circ e_v(S,x) \leq u_T \circ e_v(T,x)$ for all $T \in \mathcal{F}_{S,x}$,*
2. *$\chi_S \in \mathrm{Lin}\{\chi_T \colon T \in \mathcal{F}_{S,x} \cup \{N\}\}$,*
3. *for all $\alpha \in \mathbb{R}$ if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha)$ is balanced then $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(x,\alpha)$ is balanced,*

then $\mathcal{F}$ is a characterization set for $N^*_{\mathbf{u}}(v)$.

For the proof see Dornai and Pintér (2025).

The above results were published in Dornai and Pintér (2025).

3.6 The intersection of u-essential and dually-u-essential coalitions characterize the u-prenucleolus if the u-least-core is a proper subset of the u-core

Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$ such that v is $\mathbf{u}$-balanced. Then for every $S \in \mathcal{A}^* \setminus (\mathcal{E}^{d-\mathbf{u}}_v \cap \mathcal{E}^{\mathbf{u}}_v)$ it holds that either S is not $\mathbf{u}$-essential or it is not dually-$\mathbf{u}$-essential.

Case 1: If S is not $\mathbf{u}$-essential then by Lemma 14 for every $x \in \mathbf{u}$-least-core(v) there exists $\mathcal{B}_S \in \mathcal{D}^{\mathcal{A}^*}_S$, $\mathcal{B}_S \subseteq \mathcal{E}^{\mathbf{u}}_v$ such that

$$u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{B}_S} u_T \circ e_v(T,x)\,.$$

Case 2: If S is $\mathbf{u}$-essential, that is, it is not dually-$\mathbf{u}$-essential then for every $x \in \mathbf{u}$-least-core(v)

$$\begin{aligned} u_S \circ e_v(S,x) &= u_S \circ (v(S) - x(S)) \\ &= u_S \circ (v(N) - v^*(N \setminus S) - (x(N) - x(N \setminus S))) \\ &= u_S \circ (x(N \setminus S) - v^*(N \setminus S)) = u_S \circ f_{v^*}(N \setminus S, x)\,, \end{aligned} \tag{7}$$

and by Lemma 24 there exists $\mathcal{B} \in \mathcal{D}^{N\setminus\mathcal{A}^*}_{N\setminus S}$, $\mathcal{B} \subseteq \mathcal{E}^{a-\mathbf{u}^*}_v$ such that $u_S \circ f_{v^*}(N \setminus S, x) \leq \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)$. Then

$$\begin{aligned} u_S \circ f_{v^*}(N \setminus S, x) &\leq \sum_{T \in \mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x) \\ = \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ e_v(N \setminus T, x) &= \sum_{T \in N\setminus\mathcal{B}} u_T \circ e_v(T,x). \end{aligned} \tag{8}$$

Let $\mathcal{B}_S := N \setminus \mathcal{B} \subseteq \mathcal{E}_v^{d-\mathbf{u}}$. Then by Eq. (7) and (8) we have that

$$u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}_S} u_T \circ e_v(T, x)\,.$$

For an $x \in \mathbf{u}\text{-least-core}(v)$, for all $S \in \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ fix one such $\mathcal{B}_S$ partition (Case 1) or anti-partition (Case 2). Let a directed graph $\Gamma_{v,\mathbf{u}}(x)$ be defined as follows: the nodes are the coalitions of $\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, and there is a directed edge from S to S', $S, S' \in \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, if S' appears in the partition or in the anti-partition of S, that is, if $S' \in \mathcal{B}_S \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$.

Remark 28. Note that since the considered partitions and anti-partitions can be arbitrarily chosen, the graph $\Gamma_{v,\mathbf{u}}(x)$ can be defined multiple ways. However, this does not make any problem in our analysis, since we need only an instance of such graphs, we do not need to consider all of them.

The following lemma shows that the above defined graph is acyclic, if $\mathbf{u}\text{-core}(v) \neq \mathbf{u}\text{-least-core}(v)$. We use the acyclic nature of this graph to prove the following lemmata, proposition and theorem.

Lemma 29. *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, such that* $\mathcal{A}^*$ *contains a balanced set system with the utility function* $\mathbf{u}$*, and* $x \in \mathbf{u}\text{-least-core}(v)$*. If* $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ *then* $\Gamma_{v,\mathbf{u}}(x)$ *is acyclic.*

We need the following lemma to show that Point *3.* of Theorem 8 holds for the intersection of the set of $\mathbf{u}$-essential and the set of dually-$\mathbf{u}$-essential coalitions, if $\mathbf{u}\text{-least-core}(v)$ is a proper subset of $\mathbf{u}\text{-core}(v)$.

Lemma 30. *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, such that* $\mathcal{A}^*$ *contains a balanced set system with a utility function* $\mathbf{u}$*, and* $x \in \mathbf{u}\text{-least-core}(v)$*. If* $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ *then the following holds: if* $\mathcal{A}^*$ *is balanced then* $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ *is balanced. In addition, for all* $\alpha \in \mathbb{R}$ *it holds that if* $\mathcal{D}_{\mathbf{u}}^{\mathcal{A}^*}(\alpha, x)$ *is balanced then* $\mathcal{D}_{\mathbf{u}}^{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}(\alpha, x)$ *is balanced.*

The following lemma shows that Points *1.* and *2.* of Theorem 8 hold for the intersection of the set of $\mathbf{u}$-essential and the set of dually-$\mathbf{u}$-essential coalitions, if $\mathbf{u}\text{-least-core}(v)$ is a proper subset of $\mathbf{u}\text{-core}(v)$.

Lemma 31. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, such that* $\mathcal{A}^*$ *contains a balanced set system with a utility function* $\mathbf{u}$ *such that the game is* $\mathbf{u}$*-balanced, and a coalition* $S \in \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$*. If* $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ *then for every* $x \in \mathbf{u}\text{-least-core}(v)$ *there exists* $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ *such that* $u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e_v(T, x)$ *and* $\chi_S \in \mathrm{Lin}\{\chi_T \colon \mathcal{B}^* \cup \{N\}\}$*.*

In order to apply Theorem 8 in the proof of that $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ is a characterization set for the **u**-prenucleolus we must find a set $X \subseteq I^*(v)$ such that $N_{\mathbf{u}}^*(v), N_{\mathbf{u}}^*(v') \subseteq X$, where $v' \in \mathcal{G}^{N,\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$ is such that $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$.

We know that $N_{\mathbf{u}}^*(v) \subseteq \mathbf{u}\text{-least-core}(v)$ and $N_{\mathbf{u}}^*(v') \subseteq \mathbf{u}\text{-least-core}(v')$. The following proposition state that $\mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-least-core}(v')$, hence we can apply Theorem 8 with $X = \mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-least-core}(v')$.

Proposition 32. *Consider a* **u**-*balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$, *such that* $\mathcal{A}^*$ *contains a balanced set system with the utility function* **u**, *and* $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$. *If* $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ *then* $\mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-least-core}(v')$.

Thesis 9. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$, *such that* $\mathcal{A}^*$ *contains a balanced set system with a utility function* **u** *such that* v *is* **u**-*balanced. If* $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ *then* $\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ *is a characterization set for the* **u**-*prenucleolus.*

For the proof see Dornai and Pintér (2025).

The above results were published in Dornai and Pintér (2025).

4 List of publications

1. Dornai Zs, Pintér M (2005) Corrigendum to “TU-games with utilities: the prenucleolus and its characterization set”. International Journal of Game Theory 54(29), 10.1007/s00182-025-00943-5

2. Dornai Zs, Pintér M (2025) Characterizations of the u-prenucleolus by dually-u-essential coalitions. Annals of Operations Research 349:1575-1607

3. Dornai Zs., Pintér M (2024) TU-games with utilities: the prenucleolus and its characterization set. International Journal of Game Theory 53:1005–1032

4. Dornai Zs, Pintér M (2022) Lényeges koalíciók nem kiegyensúlyozott játékok esetén. Alkalmazott Matematikai Lapok 39:59–75

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