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On Generalized Turán Problems

Thesis Booklet

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1 Introduction

Turán-type problems are central and among the most widely studied topics in extremal combinatorics. Roughly speaking, extremal combinatorics deals with the quest for the optimal value (minimum or maximum) of some parameters of a combinatorial structure that satisfies certain properties. This is a well developed area of research in mathematics with many deep and subtle interactions with other areas of mathematics and computer science (see, for example, [2, 4, 52]). The typical Turán-type question is the following: What is the maximum number of edges in a graph on n vertices that does not contain a certain graph as a (not necessarily induced) subgraph? This is mostly known as, and this is how we refer to it in this work, ordinary (or classical) Turán problems. Unless otherwise stated, by a graph we mean a simple graph, that is, they contain no loops or parallel edges. According to particular properties imposed on the host graphs (for example, restricting to graphs that are regular, planar, edge ordered, etc.) respective variations of the Turán problem have also been introduced and studied. A natural generalization, which is known as generalized Tuán problems, is to seek the maximum number of copies of another subgraph instead of the number of edges. This thesis mainly concerns the latter type of problems. Here, we aim to present a brief overview of this topic, set the notation, and describe the results of the thesis.

1.1 Ordinary Turán problems

Given a family $\mathcal{F}$ of graphs, a graph G is said to be $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph, that is, no subgraph of G is isomorphic to any member of $\mathcal{F}$. The Turán number of $\mathcal{F}$ (also called the extremal number), $\mathrm{ex}(n, \mathcal{F})$ (simply, $\mathrm{ex}(n, F)$, if $\mathcal{F} = \{F\}$), is the maximum number of edges that an $\mathcal{F}$-free graph on n vertices can have. The n-vertex, $\mathcal{F}$-free graphs that contain $\mathrm{ex}(n, \mathcal{F})$ edges are called the *extremal graphs.*

Paul Turán extending a result of Mantel [59] for triangles, proved that for a complete graph K_{r+1} of order $r + 1$, the unique graph with the maximum number of edges among all n-vertex graphs that do not contain K_{r+1} is the complete r-partite graph with each partite set of order $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. This balanced complete r-partite n-vertex graph is denoted by $T(n, r)$ and is called the *Turán graph.*

The most general result in this area is the theorem of Erdős-Stone-Simonovits [18, 19](sometimes called the fundamental theorem of extremal graph theory [5]) that determines the limit $\pi(F) := \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{2}}$, for every graph F. This limit is called the *Turán density* of F and its existence is proven by Katona, Nemetz and Simonovits [53].

Theorem 1.1 (Erdős-Stone-Simonovits [18, 19]). *For any graph F with chromatic number $\chi(F) = r$,*

$$\mathrm{ex}(n, F) = \left(\frac{r-2}{r-1}\right)\frac{n^2}{2} + o(n^2).$$

This determines the asymptotics of the Turán number of any graph F with chromatic number $\chi(F) \geq 3$. That is, it solves the Turán problem for such graphs asymptotically, and leaves the challenge only in figuring out the exact value. However, for bipartite graphs less is known. If F is bipartite, the theorem only gives $\mathrm{ex}(n, F) = o(n^2)$.

Theorem 1.1 shows that the extremal number of the forbidden graph F depends on $\chi(F)$. Clearly, the Turán graph $T(n, \chi(F) - 1)$ attains this asymptotic value. Moreover, Erdős and Simonovits [18, 64] showed that the structure of "almost" extremal graphs are also "close" to the Turán graph.

This is one of the central concepts in the area, known as *stability*. Informally, it is described as follows. if an n-vertex F-free graph G contains "almost" $\mathrm{ex}(n, F)$ edges, then its structure is "close" to an extremal graph. Erdős and Simonovits proved the following first stability result.

Theorem 1.2 (Erdős and Simonovits [18,64])**.** *Let F be a graph with $\chi(F) = r$. For every $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 = n(\varepsilon)$ such that for every $n > n_0$ the following holds. If G is an n-vertex F-free graph such that it contains at least $\mathrm{ex}(n, F) - \delta n^2$ edges, then G can be obtained from $T(n, r - 1)$ by adding and/or deleting at most εn^2 edges.*

1.2 Variants of the Turán number

The ordinary Turán number was about the maximum number of edges among F-free graphs, where the class of all ordinary graphs is considered. One can seek the same maximum among different classes of graphs and, accordingly, different versions of the Turán problem have been introduced and studied.

Probably among the earliest considerations is the case of taking the class of bipartite graphs, especially due to the extra challenges in determining the Turán number of bipartite graphs and to its relation to the Zarankiewicz problem. We refer the reader to [20] on these problems.

Another variant is to consider the class of graphs on which some extra structure is defined. Pach and Tardos [61] studied the Turán problem for vertex ordered graphs. Gerbner, Methuku, Nagy, Pálvölgyi, Tardos and Vizer [31] introduced the Turán problem for edge ordered graphs. Interestingly, in each case a notion of chromatic number is defined to play the role of the usual chromatic number and obtain an analogue of Theorem 1.1. It is worth mentioning that Caragliano and Razbaroz [10] took a general model theoretic approach to study the Turán problem for graphs with extra structures in a unified way. Gerbner, Hama Karim and Kucheriya [29] also studied them in a unified, and yet pure combinatorial, way.

Gerbner Patkós, Tuza, and Vizer [37] introduced the *regular Turán numbers*, considering the class of regular graphs. Given a graph F, the regular Turán number, $\mathrm{rex}(n, F)$, is the maximum number of edges an n-vertex *regular* F-free graph can have. One immediately sees that $\mathrm{rex}(n, F) \leq \mathrm{ex}(n, F)$, since the class of regular graphs is part of all ordinary graphs. Of course, equality holds only when an extremal graph of $\mathrm{ex}(n, F)$ is regular. Obviously, when n is odd, there cannot be a regular bipartite graph on n vertices. Consequently, for odd n, $\mathrm{rex}(n, K_3)$ behaves very differently from $\mathrm{ex}(n, K_3)$. On the other hand, some times, through changing a small number of edges, an extremal graph of $\mathrm{ex}(n, F)$ can be made regular without containing F, giving $\text{rex}(n, F) = (1 + o(1))\text{ex}(n, F)$. For more results on regular Turán numbers, we refer the reader to [8, 9, 36].

Perhaps the variant most widely studied is the planar Turán number, denoted by $\text{ex}_{\mathcal{P}}(n, F)$, which is the maximum number of edges possible in an n-vertex F-free *planar* graph. This was initiated by Dowden [12] in 2015, who determined upper bounds for $\text{ex}_{\mathcal{P}}(n, C_4)$ and $\text{ex}_{\mathcal{P}}(n, C_5)$, together with constructions attaining these bounds for infinitely many n.

There are yet other types of the Turán number. Examples include rainbow Turán numbers, introduced by Keevash, Mubayi, Sudakov and Verstraëte in 2007 [54], singular Turán numbers, introduced by Gerbner, Patkós, Tuza and Vizer in 2022 [37], etc.

1.3 Generalized Turán problems

Given graphs H and G, let $\mathcal{N}(H, G)$ denote the number of copies of H in G, that is, the number of subgraphs of G that are isomorphic to H. Given a graph H and a family of graphs $\mathcal{F}$, the *generalized Turán number* is denoted by $\text{ex}(n, H, \mathcal{F})$, and defined as follows

$$\text{ex}(n, H, \mathcal{F}) = \max\{\mathcal{N}(H, G) \ : G \text{ is } \mathcal{F}\text{-free and } |V(G)| = n\}.$$

If $\mathcal{F} = \{F\}$, we simply write $\text{ex}(n, H, F)$. When H is K_2 (an edge), it is the classical Turán number, $\text{ex}(n, F)$, of F.

The study of generalized Turán numbers appeared quite early, too. Not much after Turán's work, Zykov [68], in 1949, using a symmetrizarion technique, determined $\text{ex}(n, K_k, K_{r+1})$, for $k \leq r$. Independently, in 1962, Erdős [14] obtained the same result. Győri, Pach and Simonovits [45] studied $\text{ex}(n, H, K_r)$, for various graphs H and $r \geq 3$. In 1984, Erdős [17] conjectured that the maximum number of C_5's in a triangle-free graph on n vertices is at most $(n/5)^5$. The first estimate was due to Győri [44] who gave the upper bound $\text{ex}(n, C_5, K_3) \leq 1.03(\frac{n}{5})^5$. The conjecture was proved in 2013 independently by Grzesik [41] and Hatami, Hladký, Král, Norine, and Razborov [50] using the newly developed technique of flag algebras by Razbarov in 2007. Bollobás and Győri [6] also studied $\text{ex}(n, K_3, C_5)$. In fact, even before Turán's Theorem, in 1938, Erdős [13] proved that $\text{ex}(n, P_3, C_4) \leq \binom{n}{2}$ to solve a problem in number theory. In their recent survey on generalized Turán numbers [34], Gerbner and Palmer mention that this could have led Erdős to initiate the study of generalized Turán numbers even before the emergence of classical Turán numbers.

More recently, in 2014, Alon and Shikhleman took a general and systematic approach to study the generalized Turán number $\text{ex}(n, H, F)$, improving some previously known estimates and proving some other new results. Among several results they obtained, is the characterization of pairs of graphs H and F, so that $\text{ex}(n, H, F) = \Theta(n^{|V(H)|})$

Among the concepts and properties of the function $\text{ex}(n, F)$ that have been extended to the generalized version, $\text{ex}(n, H, F)$, is that of stability. The first such result is due to Ma and Qiu [58] who showed stability of the problem $\text{ex}(n, K_k, K_{r+1})$, where $k \leq r$.

It is natural to study the generalized version of the other variants of the Turán number. Győri, Paulos, Salia, Tompkins and Zamora [46] introduced the generalized version of the planar Turán numbers, $\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F})$, the maximum number of copies of a subgraph H in $\mathcal{F}$-free n-vertex planar graphs. In particular, they showed that for any $k \geq 5$, $\mathrm{ex}_{\mathcal{P}}(n, C_k, C_4) = \Theta(n^{\lfloor k/3 \rfloor})$, and in case $k = 5$, they proved $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_4) = n - 4$, for all $n \geq 5$ (except $n = 6$). Also, Gerbner, Methuku, Mészáros and Palmer [30] initiated the study of generalized rainbow Turán numbers.

1.4 Description of the results and notations

The focus of the thesis is on generalized Turán problems. First, we consider a generalized Turán problem. The graph F_k, known as the friendship graph (or a k-fan) consists of k-triangles sharing a vertex. We determine the exact value of the generalized Turán number $\mathrm{ex}(n, K_3, F_k)$, for every $k \geq 2$ and sufficiently large n. That is, we determine the maximum number of triangles in n-vertex F_k-free graphs. This result is published in [67], which is a joint work with Zhu, Chen, Gerbner and Győri.

Then, we study the concept of stability in generalized Turán problems. Applying graph symmetrization techniques, we will obtain stability results for some generalized Turán problems. Namely, we consider complete l-partite graphs in K_{k+1}-free graphs, where $l \leq k$, and for sufficiently large k, any graph H instead of the complete multipartite graph. Finally, we obtain a stability result for bipartite graphs when a sufficiently long odd cycle is forbidden. These results are published in [28].

Next, we consider the regular variant of the Turán problem, and we introduce the generalized regular Turán numbers. The generalized regular Turán number, $\mathrm{rex}(n, H, F)$, is the maximum number of copies of H possible in n-vertex F-free regular graphs. First, we will extend some general results about $\mathrm{ex}(n, H, F)$ to this new version, and then provide examples to show different behaviors of $rex(n, H, F)$. Finally, we determine the exact value of $\mathrm{rex}(n, K_3, P_k)$, for sufficiently large n and any k. These results are published in [27].

Finally, we study some generalized planar Turán problems. As mentioned before, the pentagons vs triangles problem was among the very interesting ones and was very difficult to settle. Here, we investigate the same problems in the planar version. Specifically, we study $\mathrm{ex}_{\mathcal{P}}(n, K_3, C_l)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_l, K_3)$, for $4 \leq l \leq 6$, and determine their exact values together with extremal graphs that achieve them. The content of this part is based on our joint work with Ervin Győri [43].

Notation

Throughout, the notation we follow is fairly standard. $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G, respectively, and $e(G) := |E(G)|$ (sometimes just e if G is clear from the context). For a subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced on X, and $G \setminus X$ (or simply $G - v$, if $X = \{v\}$) denotes the induced subgraph $G[V(G) \setminus X]$. We denote the complement of a graph G, by $\overline{G}$. For any vertex $v \in V(G)$ and subset $S \subseteq V(G)$, let $N_S(v)$ denote the neighbors of v in

S and $d_S(v) = |N_S(v)|$. If $S = V(G)$, then $N(v) = N_S(v)$ and $d(v) = d_S(v)$. For vertices $x_1, \dots, x_k \in V(G)$, $N(x_1, \dots, x_k)$ denotes the common neighbors of all the vertices $x_1, \dots, x_k$. For two graphs G_1 and G_2, $G_1 \cup G_2$ is the vertex disjoint union of G_1 and G_2, kG_1 is the vertex disjoint union of k copies of G_1, and $G_1 + G_2$ is the graph obtained by taking $G_1 \cup G_2$ and joining all pairs v_1, v_2 with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

For subsets $A, B \subseteq V(G)$, the set of the edges of G between the vertices of A and B is denoted by $E(A, B)$, and $e(A, B) := |E(A, B)|$. Given graphs H and G, we denote the number of copies of H in G (i.e. subgraphs of G isomorphic to H) by $\mathcal{N}(H, G)$, when the graph G is clear, we also use $\#H$ to denote the number of copies of H. For a positive integer t, we use $[t]$ to denote the set $\{1, 2, \dots, t\}$.

Given a graph H, a blow-up of H is a graph obtained by replacing each vertex of H by an independent set of vertices and each edge $uv \in E(H)$ by a complete bipartite graph between the independent sets replacing u and v. A blow-up is balanced, denoted by $H(m)$, if every independent set replacing each vertex is of size m, for some $m \in \mathbb{Z}^+$. In the respective chapters, we will introduce other notation that are used in them.

2 Maximum number of triangles in F_k-free graphs

The *friendship graph* (or *k-fan*) F_k consists of k triangles all intersecting in one common vertex v.

A particular line of research is to determine, for a given graph H, which graphs F have the property that $\text{ex}(n, H, F) = O(n)$. This was started by Alon and Shikhelman [3], who dealt with the case $H = K_3$, and was continued for other graphs in [24, 32].

An *extended friendship graph* consists of F_k for some $k \geq 0$ and any number of additional vertices or edges that do not create any additional cycles. Alon and Shikhelman [3] showed that $\text{ex}(n, K_3, F) = O(n)$ if and only if F is an extended friendship graph. We remark that known results easily imply that if F is not an extended friendship graph, then $\text{ex}(n, K_3, F) = \omega(n)$ and it is also easy to see that adding further edges to F without creating any cycle does not change the linearity of $\text{ex}(n, K_3, F)$. Hence, the key part of their proof is the following theorem.

Theorem 2.1. *(Alon and Shikhelman [3]) For any k we have* $\text{ex}(n, K_3, F_k) <$ $(9k - 15)(k + 1)n$.

This upper bound for $ex(n, K_3, F_k)$ is not tight. For instance, for $k = 2$, it was observed by Liu and Wang [55] that a hypergraph Turán theorem of Erdős and Sós [66] gives the exact result for $\text{ex}(n, K_3, F_2)$. Let $\mathcal{F}_k$ denote the 3-uniform hypergraph (k-star) consisting of k hyperedges sharing exactly one vertex. Let $\text{ex}_3(n, \mathcal{F}_k)$ denote the largest number of hyperedges that an $\mathcal{F}_k$-free n-vertex 3-uniform hypergraph can contain.

Theorem 2.2. *(Erdős and Sós [66]) For all $n \geq 3$,*

$$\mathrm{ex}_3(n, \mathcal{F}_2) = \begin{cases} n & \text{if } n = 4m, \\ n-1 & \text{if } n = 4m+1, \\ n-2 & \text{if } n = 4m+2 \text{ or } n = 4m+3. \end{cases}$$

Hence, it is interesting to determine the exact value of $\mathrm{ex}(n, K_3, F_k)$ for any F_k ($k \geq 3$).

Throughout this section, let $\pi(G)$ denote the degree sequence of G. For $X, Y \subseteq V(G)$, $[X, Y]$ denotes the set of edges with one end in X and another in Y and $[x, Y] = [X, Y]$ if $X = \{x\}$. Recall that K_n and $\bar{K}_n$ denote the complete graph and the empty graph on n vertices, respectively.

We first define two graphs. Let $k \geq 4$ be even, $X = \{x_1, ..., x_{k-1}\}$ and $Y = \{y_1, ..., y_{k-1}\}$. The graph H'_k is a graph obtained from a complete bipartite graph with vertex classes X and Y. We subdivide the edge $x_i y_i$ once for $i \leq \frac{k}{2} - 1$, and then identify the $\frac{k}{2} - 1$ inserted vertices into one vertex z. The graph H_k is the complement of H'_k deleting the edge $zy_{k/2}$, which is shown in Figure 1.

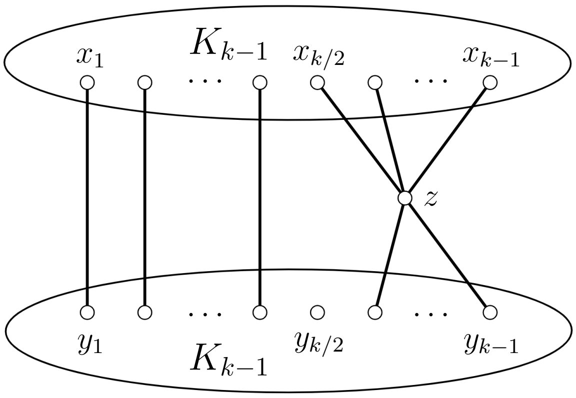


Figure 1. The graph H_k

It is clear that $|H_k| = |H'_k| = 2k-1$ and $\pi(H_k) = \pi(H'_k) = (k-1, ..., k-1, k-2)$.

Our main result here is the following.

Theorem 2.3. *Let $k \geq 3$ be an integer and $n \geq 4k^3$. If k is odd, then*

$$\mathrm{ex}\big(n, K_3, F_k\big) = (n-2k)k(k-1) + 2\binom{k}{3},$$

and $\bar{K}_{n-2k} + 2K_k$ is the unique extremal graph, and if k is even, then

$$\mathrm{ex}\big(n, K_3, F_k\big) = (n-2k+1)k\left(k - \frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2,$$

and $\bar{K}_{n-2k+1} + H_k$ is the unique extremal graph.

Our proof requires some preliminary results, among them we prove the following theorem, which can be interesting on its own.

Theorem 2.4. *Let k be an even integer and H be a graph on $2k-1$ vertices with $\pi(H) = (k-1, \ldots, k-1, k-2)$, then*

$$\mathcal{N}(K_3, H) \leq 2\binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2,$$

equality holds if and only if $H = H_k$.

3 Stability from symmitrization in generalized Turán problems

As mentioned in the first chapter, the first generalized Turán result is due to Zykov [68], who showed that $\mathrm{ex}(n, K_k, K_{r+1}) = \mathcal{N}(K_k, T(n, r))$ using the so called *symmetrization* tecknique. Given two vertices u and v of a graph G, we say that we *symmetrize* u to v if we delete all the edges incident to u and add all the edges of form uw where w is a neighbor of v. In other words, we change the neighborhood of u to the neighborhood of v.

If u and v are non-adjacent and G is K_{r+1}-free, then the resulting graph G' is also K_{r+1}-free. Indeed, a copy of K_{r+1} would contain u, since all the new edges are incident to u. Then this copy cannot contain v, as v is not adjacent to u. But then deleting u and adding v to this copy, we find another copy of K_{r+1} that is also present in G, contradicting our assumption.

The other key property is that if u is contained in x copies of K_k and v is contained in y copies of K_k, then this symmetrization removes x and adds y copies of K_k. Therefore, by applying the symmetrization if $x \leq y$, the total number of copies of K_k does not decrease. One can show that this process terminates, i.e., at one point we arrive to a graph where symmetrization cannot change the graph. This means that non-adjacent vertices have the same neighborhood, thus being non-adjacent is an equivalence relation, i.e., the resulting graph is complete multipartite (obviously with at most r parts).

Győri, Pach and Simonovits [45] generalized this argument, showing that for any complete multipartite graph H, $\mathrm{ex}(n, H, K_{r+1}) = \mathcal{N}(H, T)$ for some complete r-partite graph T (which is not necessarily the Turán graph). One could think that this is the limit of Zykov's symmetrization argument in this topic, since only cliques have the property that symmetrization cannot create them, and only complete multipartite graphs H have the property that symmetrizing either u to v or v to u does not decrease the number of copies of H. However, some more advanced applications have appeared in the literature. One can symmetrize only to vertices v satisfying some property that ensures that no F will be created [35, 55]. Another example is [26], where the neighborhood of u is changed to the common neighborhood of a set of vertices.

Here, we will present three stability results on generalized Turán problems that are obtained by using symmetrization. Stability refers to the phenomenon that an F-free n-vertex graph with almost $\mathrm{ex}(n, H, F)$ copies of H is close to the extremal graph. There are different kinds of stability, depending on what "almost" and "close" mean in the previous sentence. Here, we deal with one specific kind of stability.

The *edit distance* of two n-vertex graphs G and G' is the number of edges needed to be deleted and added to G in order to obtain a graph isomorphic to G'. Let $h = |V(H)|$. Given a graph F with chromatic number $r+1$ and another graph H with chromatic number at most r, we say that H is *F-Turán-stable* if the following holds. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if an n-vertex F-free graph G contains at least $\mathrm{ex}(n, H, F) - \delta n^h$ copies of H, then the edit distance of G and $T(n, r)$ is at most εn^2. We say that H is *weakly F-Turán-stable* if the same holds with $T(n, r)$ replaced by any complete r-partite graph T.

Using these notions, the classical Erdős-Simonovits stability theorem [15, 16, 65] shows that K_2 is K_{r+1}-Turán-stable for every $r \geq 2$. The first stability result concerning generalized Turán problems is due to Ma and Qiu [58], who showed that K_k is K_{r+1}-Turán-stable for every $r \geq k$. Turán-stable graphs were introduced by Gerbner in [26]. It was shown in [26] that if H is (weakly) F-Turán-stable, it often implies that H is (weakly) F'-Turán-stable for some other graphs F'.

Another important feature of these notions is that they often imply exact results on $\mathrm{ex}(n, H, F)$. We say that H is *F-Turán-good* if for each sufficiently large n we have $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T(n, r))$. We say that H is *weakly F-Turán-good* if for each sufficiently large n we have that $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T)$ for some complete r-partite graph T. It was shown in [26] that if H is weakly F-Turán-stable and F has a color-critical edge (an edge whose deletion decreases the chromatic number), then $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T)$ for some complete r-partite graph T. Some other exact results were shown in [25].

Gerbner in [23] claimed that a result of Liu, Pikhurko, Sharifzadeh and Staden [56] implies that complete multipartite graphs are weakly K_{r+1}-Turán-stable. This is not true. Even though our problem is in many sense simpler than theirs, it still does not fit into their general framework. Here we present a proof of this statement. Note that the case $H = C_4$ was proved in [51].

Theorem 3.1. *Let H be a complete k-partite graph and $r \geq k$. Then H is weakly K_{r+1}-Turán-stable.*

Morrison, Nir, Norin, Rzażewski and Wesolek [60] showed (proving a conjecture of Gerbner and Palmer [33]) that for every graph H, if r is large enough, then H is K_{r+1}-Turán-good. They ask whether this can be improved to K_{r+1}-Turán-stability. We partly answer this question in the affirmative.

Theorem 3.2. *For every graph H, if $r \geq 300h^9$, then H is K_{r+1}-Turán-stable.*

Finally, we obtain a theorem of a similar flavor.

Theorem 3.3. *For every bipartite graph H, if $k > (2h)^{h+1}$, then H is weakly C_{2k+1}-Turán-stable.*

We note that the above theorem implies that H is weakly C_{2k+1}-Turán-good, since C_{2k+1} has a color-critical edge. Gerbner [23] showed for every k a graph that is C_{2k+1}-Turán-good and not weakly $C_{2\ell+1}$-Turán-good for any $\ell < k$.

Furthermore, F-Turán-stability implies F'-Turán-stability for other graphs, as observed in [26] in the case $F = K_{r+1}$. We present the following more general version.

Proposition 3.4. *Let $\chi(F) = \chi(F') = r + 1$ and assume that F' is contained in a blowup of F. If H is weakly F-Turán-stable, then H is weakly F'-Turán-stable.*

There are multiple ways in [23] and [25] to obtain new (weakly) F-Turán-stable graphs from other F-Turán-stable graphs. Our results give new building blocks to those methods. Finally, as we have mentioned, our new stability results give new exact results using theorems from [23] and [25].

4 Generalized regular Turán numbers

Here, we define and study the generalized version of the regular Turán numbers. Let $\mathrm{rex}(n, H, F) := \max\{\mathcal{N}(H, G) :\ G \text{ is an } F\text{-free regular } n\text{-vertex graph}\}$. Our goal is to show some examples where $\mathrm{rex}(n, H, F)$ behaves similarly to $\mathrm{ex}(n, H, F)$ and also show some examples where they differ significantly.

Extending a result of Alon and Shikhelman [3] to the regular version, we prove the following.

Theorem 4.1. *For any graph F and H, we have that $\mathrm{rex}(n, H, F) = \Theta(n^{|V(H)|})$ if and only if F is not a subgraph of a blow-up of H.*

Another result of Alon and Shikhelman [3] is that $\mathrm{ex}(n, K_3, F) = O(n)$ if and only if F is an extended friendship graph. In an extended friendship graph, every cycle is a triangle and there is a vertex v such that every pair of triangles intersect in v (or equivalently, its 2-core is empty or a Friendship graph). We extend this theorem as well to our setting.

Theorem 4.2. $\mathrm{rex}(n, K_3, F) = O(n)$ *if and only if F is an extended friendship graph.*

Let us turn to problems where adding the regularity changes the situation. It is well-known and easy to see that for any forest F, any graph with minimum degree at least $|V(F)|$ contains F. This implies that $\mathrm{rex}(n, F) \le (|V(F)| - 1)n$. Let H be a connected graph, then the vertices of H have an ordering such that each but the first vertex has a neighbor that is earlier in the ordering. The copies of H in an F-free r-regular graph can be counted by picking the vertices in the above order. The first vertex can be picked n ways, and then each other vertex can be picked at most r ways among the neighbors of at least one of the vertices picked earlier. This shows that $\mathrm{rex}(n, H, F) = O(n)$, since $r < |V(F)|$. On the other hand, $\mathrm{ex}(n, P_\ell, P_k) = \Theta(n^{\lceil \ell/2 \rceil})$ by a theorem of Győri, Salia, Tompkins and Zamora [48].

Another example where the order of magnitude of $\mathrm{rex}(n, H, F)$ is much smaller than that of $\mathrm{ex}(n, H, F)$ is given by even cycles. When C_{2k} is forbidden, the regularity does not have to be constant, but it is $O(n^{1/k})$ by a theorem of Bondy and Simonovits [7]. Therefore, $\mathrm{rex}(n, C_\ell, C_{2k}) = O(n^{1+\frac{\ell-1}{k}})$, while we have $\mathrm{ex}(n, C_\ell, C_{2k}) = \Theta(n^{\lfloor \ell/2 \rfloor})$ if $3 \le \ell \ne 2k$ [39].

Note that we have $\mathrm{ex}(n, C_\ell, C_{2k+1}) = \Theta(n^\ell)$ if ℓ is even or $\ell > 2k+1$, as shown by the blow-up of C_ℓ. Interestingly, in the remaining case $3 < \ell < 2k+1$ is odd, we have $\mathrm{ex}(n, C_\ell, C_{2k+1}) = \Theta(n^{\lfloor \ell/2 \rfloor})$ [39], while the above argument does not give any non-trivial bound. It is a natural question to ask whether $\mathrm{rex}(n, C_\ell, C_{2k+1})$ is significantly smaller in this case. We can answer this question in the negative.

Proposition 4.3. *If $3 < \ell < 2k+1$ is odd, then $\mathrm{rex}(n, C_\ell, C_{2k+1}) = \Theta(n^{\lfloor \ell/2 \rfloor})$.*

So far, we considered only the order of magnitude of $\mathrm{rex}(n, H, F)$. Let us turn to exact and asymptotic results. As shown in [8, 9], for $k \geq 3$ we have $\mathrm{rex}(n, K_{k+1}) = (1 + o(1))|E(T(n,k))|$ (we have $\mathrm{rex}(n, K_{k+1}) = |E(T(n,k))|$ if k divides n). The exact value of $\mathrm{rex}(n, K_{k+1})$ was determined for all sufficiently large n in [36]. Let $T^*(n,k)$ denote an arbitrary n-vertex K_{k+1}-free regular graph with $\mathrm{rex}(n, K_{k+1})$ edges. Forbidding K_3 is very different from forbidding larger cliques in the regular Turán problem. If n is even, then $T(n, 2)$ is the regular n-vertex triangle-free graph with the most edges. If n is odd, then a regular n-vertex triangle-free graph with the most edges is obtained by deleting some edges of an n-vertex blow-up of C_5, as shown in [8, 9].

Given H with $\chi(H) \leq k$, there have been a lot of research studying whether $\mathrm{ex}(n, H, K_{k+1}) = \mathcal{N}(H, T(n,k))$ for sufficiently large n. For example, it is the case when H is a complete l-partite graph with $3 \leq l \leq k$, see e.g. [21, 22, 45]. There have been two types of counterexamples found (where even $\mathrm{ex}(n, H, K_{k+1}) = (1 + o(1))\mathcal{N}(H, T(n,k))$ does not hold). If H is a very unbalanced bipartite graph, then an unbalanced complete k-partite graph may contain more copies of H than the Turán graph. For some graphs H, there are n-vertex K_{k+1}-free graphs that contain more copies of H than any n-vertex complete k-partite graph. For example, let H be obtained from a path on vertices $v_1, v_2, v_3, v_4, v_5, v_6$ by adding s additional leaves connected to v_2 and s additional leaves connected to v_5. Then some unbalanced blowup of C_5 contains more copies of H than any bipartite graph, see [45]. Examples for $k > 2$ can be found in [42]. In each of the known constructions, most of the vertices of H would belong to two different classes of k-partite graphs, but they can belong to the same class of the blow-up of another graph. Then that class has many vertices.

Both counterexamples are very far from being regular. This suggests that maybe there are no regular counterexamples at all.

Conjecture 4.4. *Let $k \geq 3$ and $\chi(H) \leq k$. Then, $\mathrm{rex}(n, H, K_{k+1}) = (1 + o(1))\mathcal{N}(H, T(n,k))$. Moreover, if n is sufficiently large and is divisible by k, then $\mathrm{rex}(n, H, K_{k+1}) = \mathcal{N}(H, T(n,k))$.*

We prove Conjecture 4.4 for complete k-partite graphs H.

Proposition 4.5. *Let $k \geq 3$ and H be a complete k-partite graph. Then, $\mathrm{rex}(n, H, K_{k+1}) = (1 + o(1))\mathcal{N}(H, T(n,k))$. Moreover, if n is sufficiently large and is divisible by k, then $\mathrm{rex}(n, H, K_{k+1}) = \mathcal{N}(H, T(n,k))$.*

We also prove that the moreover part of Conjecture 4.4 holds in the case $k = 2$. In the case n is odd, the situation is very different, but we can describe the structure of the extremal graph.

Proposition 4.6. *Let H be a bipartite graph. If n is even and sufficiently large, then* $\mathrm{rex}(n, H, K_3) = \mathcal{N}(H, T(n, 2))$. *If H is a tree and n is odd and sufficiently large, then* $\mathrm{rex}(n, H, K_3) = \mathcal{N}(H, G^*)$, *where G^* is a regular graph obtained by deleting some edges of an n-vertex blow-up of C_5.*

Finally, we determine the exact value for $\mathrm{rex}(n, K_3, P_k)$, when n is large enough and P_k is a path on k vertices. To ease the notation and describe the extremal graphs, we define some graphs first. Let $\mathcal{G}_{k-1}$ denote the graphs obtained from K_{k-1} by removing the edges of a triangle-free 2-regular subgraph, i.e., the union of vertex-disjoint cycles of length more than 3 such that the total length of the cycles is $k-1$. In the case k is even, let $G_{k-2} := K_{k-2} - M$, a clique on $k-2$ vertices in which a perfect matching is removed. Note that each of the above graphs is $(k-4)$-regular and P_k-free. If k is odd, let $G'_{k-1} := K_{k-1} - M$, a clique on $k-1$ vertices in which a perfect matching is removed. Note that $\mathcal{N}(K_3, G_{k-1}) = 8\binom{k/2-1}{3} + 3 - k/2$ for any graph $G_{k-1} \in \mathcal{G}_{k-1}$, $\mathcal{N}(K_3, G_{k-2}) = 8\binom{k/2-1}{3}$ and $\mathcal{N}(K_3, G'_{k-1}) = (k-1)(k-3)(k-5)/6 = 8\binom{(k-1)/2}{3}$. One can obtain these computations from the fact that the complements of these graphs contain no triangles, the degrees of their vertices and the following formula that is basically proven by Goodman [40].

$$\mathcal{N}(K_3, G) + \mathcal{N}(K_3, \overline{G}) = \binom{n}{3} - \frac{1}{2} \sum_{v \in V(G)} \deg(v)(n - 1 - \deg(v)),$$

where $\overline{G}$ denotes the complement of G. Recall that $H \cup F$ denotes the disjoint union of two graphs H and F, and by mF we mean m disjoint copies of the graph F.

Theorem 4.7. *Let P_k be a path on k vertices and n be large enough. Then:*

1. *If $(k-1)|n$, then* $\mathrm{rex}(n, K_3, P_k) = \frac{n}{k-1}\binom{k-1}{3}$, *and the unique extremal graph is $\frac{n}{k-1}K_{k-1}$.*
2. *Assume that $(k-1) \nmid n$, $k \geq 6$ and either $k-2$ divides n or k is odd. Let $n = a(k-2)+b$ with $b < k-2$. Then we have* $\mathrm{rex}(n, K_3, P_k) = (a-b)\binom{k-2}{3} + 8b\binom{\frac{k-1}{2}}{3}$, *and the unique extremal graph is $(a-b)(K_{k-2}) \cup bG'_{k-1}$.*
3. *If $k \geq 6$ is even, and n is neither divisible by $k-1$ nor by $k-2$. Let $n = a(k-3) + b$, with $b < k-3$. Then*
$$\mathrm{rex}(n, K_3, P_k) = (a - \ell - \lfloor b/2 \rfloor)\binom{k-3}{3} + \ell\mathcal{N}(K_3, G_{k-2}) + \lfloor b/2 \rfloor \mathcal{N}(K_3, G_{k-1}),$$
and the extremal graphs are formed by adding $\lfloor b/2 \rfloor$ graphs from $\mathcal{G}_{k-1}$ to $(a - \ell - \lfloor b/2 \rfloor)K_{k-3} \cup \ell G_{k-2}$, where $\ell = 0$ if b is even and $\ell = 1$ otherwise.
4. *If neither 3, nor 4 divides n, then* $\mathrm{rex}(n, K_3, P_5) = \lfloor n/3 \rfloor - 1$, *and the unique extremal graph is formed by adding a C_4 or a C_5 to $(\lfloor n/3 \rfloor - 1)K_3$.*

In all the cases not listed above, $\mathrm{rex}(n, K_3, P_k) = 0$.

5 Generalized planar Turán numbers related to short cycles

The generalized version of the planar Turán numbers, $\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F})$, were introduced by Győri, Paulos, Salia, Tompkins and Zamora [46] in 2021. Interestingly, maximizing the number of copies of a given subgraph H in planar graphs can be viewed as a special case of generalized planar Turán problems by taking $\mathcal{F} = \emptyset$. Such a problem has been studied much earlier. In 1979, Hakimi and Schmeichel [49] determined the maximum number of triangles and 4-cycles in planar graphs. Alon and Caro [1] determined the maximum number of copies of $K_{2,t}$ in planar graphs. Győri, Paulos, Salia, Tompkins and Zamora [47] determined the maximum number of 5-cycles in planar graphs, proving that $\mathrm{ex}_{\mathcal{P}}(n, C_5, \emptyset) = 2n^2 - 10n + 12$, for every $n = 6$ or $n \geq 8$. Although exact results for longer cycles are not known, Cox and Martin [11] developed a general technique to count subgraphs in planar graphs, and conjectured that the maximum number of an even cycle C_{2k} is asymptotically $(n/k)^k$. This conjecture was proved by Lv, Győri, He, Salia, Tompkins and Zhu [57].

Theorem 5.1 (Lv *et al.* [57])**.** *For every $k \geq 3$, $\mathrm{ex}_{\mathcal{P}}(n, C_{2k}, \emptyset) = \left(\frac{n}{k}\right)^k + o(n^k)$.*

Another direction of research is maximizing the number of induced subgraphs in planar graphs. Ghosh, Győri, Janzer, Paulos, Salia, Zamora [38], and independently Savery [63], determined the maximum number of induced 5-cycles in planar graphs. Savery [62] extended this to induced 6-cycles.

Now, let us state our results that we prove in this chapter.

Theorem 5.2. *For every $n \geq 4$, $\mathrm{ex}_{\mathcal{P}}(n, C_4, C_3) = \binom{n-2}{2}$, and the unique extremal graph is $K_{2,n-2}$.*

Theorem 5.3. *For every $n \geq 5$, $\mathrm{ex}_{\mathcal{P}}(n, C_4, C_5) = \binom{n-2}{2}$. Furthermore, for $n = 5$, the extremal graphs are $K_{2,n-2}$, $K_2 + \overline{K_{n-2}}$, $K_4 \cup K_1$ or K_4 with a pending edge, and for $n \geq 6$, the extremal graphs are $K_{2,n-2}$ or $K_2 + \overline{K_{n-2}}$.*

Note that for $n = 4$, we trivially have $\mathrm{ex}_{\mathcal{P}}(n, C_4, C_5) = 3$, and K_4 is the extremal graph.

Theorem 5.4. *For every $n \geq 4$, we have $\mathrm{ex}_{\mathcal{P}}(n, C_3, C_4) \leq \frac{5}{7}(n-2)$, and this bound is sharp for infinitely many values of n.*

The following theorem is a bit different, as we determine the maximum number of triangles in K_4-free planar graphs.

Theorem 5.5. *For every $n \geq 3$, we have $\mathrm{ex}_{\mathcal{P}}(n, C_3, K_4) \leq \frac{7}{3}n - 6$, and this bound is sharp for all n divisible by 3.*

Theorem 5.6. *For every $n \geq 5$, $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_3) = \lfloor (n-3)/2 \rfloor \cdot \lceil (n-3)/2 \rceil$.*

An extremal graph for $\mathrm{ex}_p(n, C_5, C_3)$ can easily be seen to be a C_5 on which two non-adjacent vertices are blown-up in a balanced way. Somewhat surprisingly, there are many other extremal graphs.

For each $n \geq 5$, define a class $\mathcal{J}_n$ of n-vertex planar graphs J_n as follows. Take a regular pentagon, replace one of its vertices, x, by an independent set of vertices C, and replace the edge yz opposite to x by a tree with color classes A and B, such that $|C|$ and $|A \cup B| - 1$ are as equal as possible (see Figure 1(a)). We prove the following stronger theorem.

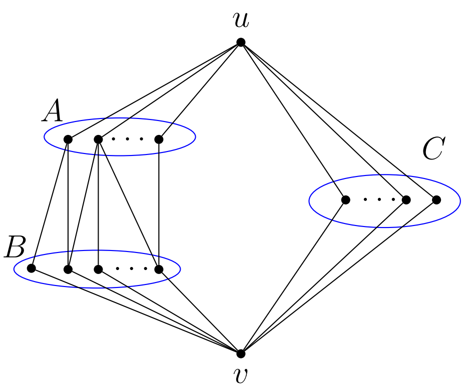
Theorem 5.7. *For every $n \geq 5$, $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_3) = \lfloor (n-3)/2 \rfloor \cdot \lceil (n-3)/2 \rceil$, and $\mathcal{J}_n$ is the set of all extremal graphs.*

In the other direction, maximizing the number of triangles pentagon-free graphs, we prove the following theorem.

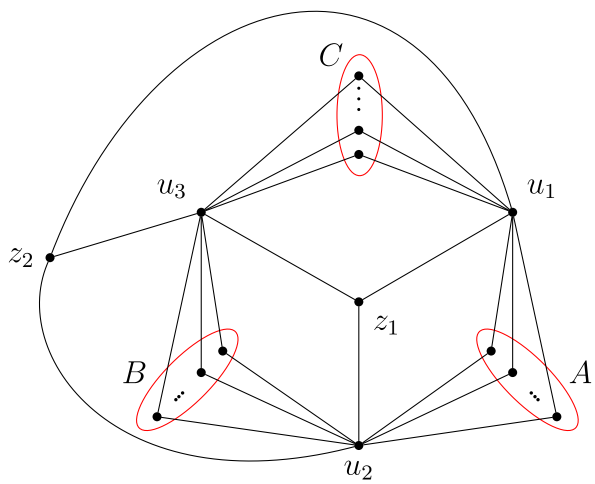
Theorem 5.8. *For every $n \geq 11$, $\mathrm{ex}_{\mathcal{P}}(n, C_3, C_5) \leq \left\lfloor \frac{8n-22}{5} \right\rfloor$, and this bound is sharp for infinitely many values of n.*

Then, we turn to triangles and 6-cycles, and first maximize the number of triangles in C_6-free graphs.

Theorem 5.9. *For every $n \geq 18$, $\mathrm{ex}_{\mathcal{P}}(n, C_3, C_6) \leq \frac{35n-98}{18}$, and this bound is sharp for infinitely many values of n.*



(a) A graph J_n



(b) The graph H_n

Figure 1: The extremal graphs for $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_3)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_6, C_3)$

Finally, we determine the maximum number of 6-cycles while forbidding triangles, together with the unique extremal graph achieving the maximum value. Along the proof, we will be considering the number of paths of length four (i.e. of five vertices) between two vertices. This is an interesting problem on its own, and we prove a theorem as follows.

Theorem 5.10. *Let G be a triangle-free planar graph on $n \geq 5$ vertices. For any two vertices $u, v \in V(G)$, there are at most $(\frac{n-1}{2})^2 - 2$ paths of length four connecting them.*

Before stating the next theorem, we present the following construction. For every $n \geq 6$, define a graph H_n as follows. The vertex set of H_n consists of $\{u_1, u_2, u_3\} \cup \{z_1, z_2\} \cup A \cup B \cup C$, such that each of these sets is an independent set of vertices, they are pairwise disjoint, each of the u_i's is adjacent to each of the z_i's, every vertex in A is adjacent to both of u_1 and u_2, every vertex in B is adjacent to both of u_2 and u_3, and every vertex in C is adjacent to both u_1 and u_3. Moreover, the sizes of A, B and C are as equal as possible (see Figure 1(b)). It is easy to see that H_n contains $h(n)$ 6-cycles (nevertheless, we will give an explanation for this in Section **??**), where $h(n)$ is defined as follows.

$$h(n) = \begin{cases} \frac{n^3}{27} + \frac{n^2}{9} - 2n + 2, & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{n^3}{27} + \frac{n^2}{9} - 2n + \frac{50}{27}, & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{n^3}{27} + \frac{n^2}{9} - \frac{17n}{9} + \frac{55}{27}, & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

Theorem 5.11. *Let n be sufficiently large. Then, $\mathrm{ex}_{\mathcal{P}}(n, C_6, C_3) = h(n)$, and the unique extremal graph is H_n.*

In fact, Theorems 5.7 and 5.11 are the highlights of this chapter. The proof of Theorem 5.11 makes it possible to prove the following interesting variant of Theorem 5.10.

Theorem 5.12. *Let G be a triangle-free planar graph on n vertices. For any three distinct vertices $u_1, u_2, u_3 \in V(G)$, the number of paths of length four joining all the three pairs of them is at most $3\left(\frac{n+1}{3}\right)^2 - 6$. Moreover, for each $n \equiv 2$ (mod 3), the graph H_n attains this bound.*

List of Publications

This thesis is based on the following articles.

1. Xiutao Zhu, Yaojun Chen, Dániel Gerbner, Ervin Győri and Hilal Hama Karim, Maximum number of triangles in F_k-free graphs. *European Journal of Combinatorics*, **114** (103793), 2023.
https://doi.org/10.1016/j.ejc.2023.103793

2. Dániel Gerbner and Hilal Hama Karim, Stability from graph symmetrization arguments in generalized Turán problems. *Journal of Graph Theory*, **104**(4), 681-692, 2024.
https://doi.org/10.1002/jgt.23151

3. Dániel Gerbner and Hilal Hama Karim, 2024, Generalized regular Turán numbers. *Australasian Journal of Combinatorics*, **90**(3), 326-340.
https://ajc.maths.uq.edu.au/pdf/90/ajc_v90_p326.pdf

4. Ervin Győri and Hilal Hama Karim, 2024, Generalized planar Turán numbers related to short cycles, *arXiv preprint*, arXiv:2405.08162. https://doi.org/10.48550/arXiv.2405.08162

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