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On Generalized Turán Problems

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By

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Abstract

Given graphs H and F, the generalized Turán number, $\mathrm{ex}(n, H, F)$, is the maximum possible number of copies of H in an n-vertex graph without containing F as a subgraph.

We will study this function for various graphs and different settings. First, we determine the exact value of $\mathrm{ex}(n, H, F)$, where H is a triangle and F is the k-fan, F_k, which consists of k triangles sharing one common vertex, and we determine the extremal graphs achieving the value. Then we study the stability of this function for different graphs, which is the study of near extremal graphs. For instance, when F is a clique and H is a multipartite graph, and when F is a long odd cycle and H is a bipartite graph. Next, considering the class of regular graphs, we introduce the regular variant of this problem. Namely, the function $\mathrm{rex}(n, H, F)$ is the maximum possible number of copies of H in an n-vertex regular graph that does not contain F. We compare the behavior of this function with $\mathrm{ex}(n, H, F)$, providing examples to show similarities and differences, as well as obtaining some exact values. Analogously, a planar variant of this function is studied in the literature. This is denoted by $\mathrm{ex}_{\mathcal{P}}(n, H, F)$, and is the maximum number of copies of H in an n-vertex planar graph that does not contain F. We determine the exact values and extremal graphs for $\mathrm{ex}_{\mathcal{P}}(n, C_l, C_3)$ for $4 \leq l \leq 6$. Moreover, we determine sharp upper bounds for $\mathrm{ex}_{\mathcal{P}}(n, C_3, C_l)$, with $4 \leq l \leq 6$, and give constructions that achieve these bounds for infinitely many n.

List of Publications

This thesis is based on the following papers that we published during my PhD studies. Specifically, the contents of chapters two to five are based on these articles, respectively.

1. Xiutao Zhu, Yaojun Chen, Dániel Gerbner, Ervin Győri and Hilal Hama Karim, Maximum number of triangles in F_k-free graphs. *European Journal of Combinatorics*, **114** (103793), 2023.
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2. Dániel Gerbner and Hilal Hama Karim, Stability from graph symmetrization arguments in generalized Turán problems. *Journal of Graph Theory*, **104**(4), 681-692, 2024.
   https://doi.org/10.1002/jgt.23151
3. Dániel Gerbner and Hilal Hama Karim, 2024, Generalized regular Turán numbers. *Australasian Journal of Combinatorics*, **90**(3), 326-340.
   https://ajc.maths.uq.edu.au/pdf/90/ajc_v90_p326.pdf
4. Ervin Győri and Hilal Hama Karim, 2024, Generalized planar Turán numbers related to short cycles, *arXiv preprint*, arXiv:2405.08162.
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Contents

Chapter 1

Introduction

Turán-type problems are central and among the most widely studied topics in extremal combinatorics. Roughly speaking, extremal combinatorics deals with the quest for the optimal value (minimum or maximum) of some parameters of a combinatorial structure that satisfies certain properties. This is a well developed area of research in mathematics with many deep and subtle interactions with other areas of mathematics and computer science (see, for example, [2, 7, 69]). The typical Turán-type question is the following: What is the maximum number of edges in a graph on n vertices that does not contain a certain graph as a (not necessarily induced) subgraph? This is mostly known as, and this is how we refer to it in this work, ordinary (or classical) Turán problems. Unless otherwise stated, by a graph we mean a simple graph, that is, they contain no loops or parallel edges. According to particular properties imposed on the host graphs (for example, restricting to graphs that are regular, planar, edge ordered, etc.) respective variations of the Turán problem have also been introduced and studied. A natural generalization, which is known as generalized Turán problems, is to seek the maximum number of copies of another subgraph instead of the number of edges. This thesis mainly concerns the latter type of problems. In this Introduction, we aim at presenting a brief overview of this background, setting the notation, and describing the structure of the rest of the thesis.

1.1 Ordinary Turán problems

In 1907, Mantel [82] determined that the maximum number of edges in a graph on n vertices that does not contain a triangle is $\lfloor \frac{n^2}{4} \rfloor$, and this maximum is uniquely achieved by a balanced complete bipartite graph, i.e. with each partite set of order $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. Much later, in 1941, Paul Turán extended this result to complete graphs of any given order in his seminal paper [92], which triggered the start of this area of research. Namely, he proved that for a complete graph K_{r+1} of order $r+1$, the unique graph with the maximum number of edges among all n-vertex graphs that do not contain K_{r+1} is the complete r-partite graph with each partite set of order $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. This balanced complete r-partite n-vertex graph is denoted by $T(n,r)$ and is called the *Turán graph*.

More generally, given a family $\mathcal{F}$ of graphs, a graph G is said to be $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph, that is, no subgraph of G is isomorphic to any member of $\mathcal{F}$. The Turán number of $\mathcal{F}$ (also called the extremal number), $\mathrm{ex}(n, \mathcal{F})$ (simply, $\mathrm{ex}(n, F)$, if $\mathcal{F} = \{F\}$), is the maximum number of edges that an $\mathcal{F}$-free graph on n vertices can have. The n-vertex, $\mathcal{F}$-free graphs that contain $\mathrm{ex}(n, \mathcal{F})$ edges are called the *extremal graphs.*

The most general result in this area is the theorem of Erdős-Stone-Simonovits [30, 31](sometimes called the fundamental theorem of extremal graph theory [8]) that determines the limit $\pi(F) := \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{2}}$, for every graph F. This limit is called the *Turán density* of F and its existence is proven by Katona, Nemetz and Simonovits [70].

Theorem 1.1 (Erdős-Stone-Simonovits [30,31])**.** *For any graph F with chromatic number $\chi(F) = r$,*

$$\mathrm{ex}(n, F) = \left(\frac{r-2}{r-1}\right)\frac{n^2}{2} + o(n^2).$$

This determines the asymptotics of the Turán number of any graph F with chromatic number $\chi(F) \geq 3$. That is, it solves the Turán problem for such graphs asymptotically, and leaves the challenge only in figuring out the exact value. However, for bipartite graphs less is known. If F is bipartite, the theorem only gives $\mathrm{ex}(n, F) = o(n^2)$. Erdős and Gallai [28] determined the Turán number for paths on k vertices, P_k, for any k.

Theorem 1.2 (Erdős and Gallai [28])**.** *For every $k, n > 1$, $\mathrm{ex}(n, P_k) \leq \frac{k-2}{2}n$, where equality holds for $\frac{n}{k-1}$ disjoint copies of K_{k-1} when $(k-1)|n$.*

The famous Erdős-Sós conjecture [23] states that this bound is also true for any tree on k vertices. For even cycles C_{2k} on $2k$ vertices, Bondy and Simonovits [10] showed that $\mathrm{ex}(n, C_{2k}) \leq ckn^{1+\frac{1}{k}}$, where c is a constant and n is sufficiently large. For complete bipartite graphs $K_{s,t}$, the well-known theorem of Kővári-Sós-Turán [72] shows that $\mathrm{ex}(n, K_{s,t}) \leq c \cdot n^{2-1/s}$, for some constant c. For an excellent survey on this topic, we refer the reader to [33].

Theorem 1.1 shows that the extremal number of the forbidden graph F depends on $\chi(F)$. Clearly, the Turán graph $T(n, \chi(F) - 1)$ attains this asymptotic value. Moreover, Erdős and Simonovits [30, 89] showed that the structure of "almost" extremal graphs are also "close" to the Turán graph.

This is one of the central concepts in the area, known as *stability.* Informally, it is described as follows. if an n-vertex F-free graph G contains "almost" $\mathrm{ex}(n, F)$ edges, then its structure is "close" to an extremal graph. Erdős and Simonovits proved the following first stability result.

Theorem 1.3 (Erdős and Simonovits [30,89])**.** *Let F be a graph with $\chi(F) = r$. For every $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 = n(\varepsilon)$ such that for every $n > n_0$ the following holds. If G is an n-vertex F-free graph such that it contains at least $\mathrm{ex}(n, F) - \delta n^2$ edges, then G can be obtained from $T(n, r-1)$ by adding and/or deleting at most εn^2 edges.*

Furthermore, Simonovits [89,90] proved that for sufficiently large n, the Turán graph $T(n,r-1)$ is the unique extremal graph for graphs with chromatic number $r \geq 3$ that have a color critical edge (an edge is called color critical if deleting it reduces the chromatic number).

1.2 Variants of the Turán number

The ordinary Turán number was about the maximum number of edges among F-free graphs, where the class of all ordinary graphs is considered. One can seek the same maximum among different classes of graphs and, accordingly, different versions of the Turán problem have been introduced and studied.

Probably among the earliest considerations is the case of taking the class of bipartite graphs, especially due to the extra challenges in determining the Turán number of bipartite graphs and to its relation to the Zarankiewicz problem. We refer the reader to [33] on these problems.

Another variant is to consider the class of graphs on which some extra structure is defined. Pach and Tardos [84] studied the Turán problem for vertex ordered graphs. Gerbner, Methuku, Nagy, Pálvölgyi, Tardos and Vizer [45] introduced the Turán problem for edge ordered graphs. Interestingly, in each case a notion of chromatic number is defined to play the role of the usual chromatic number and obtain an analogue of Theorem 1.1. It is worth mentioning that Caragliano and Razbarov [17] took a general model theoretic approach to study the Turán problem for graphs with extra structures in a unified way. Gerbner, Hama Karim and Kucheriya [43] also studied them in a unified, and yet pure combinatorial, way.

Gerbner, Patkós, Tuza, and Vizer [51] introduced the *regular Turán numbers*, considering the class of regular graphs. Given a graph F, the regular Turán number, $\mathrm{rex}(n,F)$, is the maximum number of edges an n-vertex *regular* F-free graph can have. One immediately sees that $\mathrm{rex}(n,F) \leq \mathrm{ex}(n,F)$, since the class of regular graphs is part of all ordinary graphs. Of course, equality holds only when an extremal graph of $\mathrm{ex}(n,F)$ is regular. Obviously, when n is odd, there cannot be a regular bipartite graph on n vertices. Consequently, for odd n, $\mathrm{rex}(n,K_3)$ behaves very differently from $\mathrm{ex}(n,K_3)$. On the other hand, sometimes, through changing a small number of edges, an extremal graph of $\mathrm{ex}(n,F)$ can be made regular without containing F, giving $\mathrm{rex}(n,F) = (1+o(1))\mathrm{ex}(n,F)$. For more results on regular Turán numbers, we refer the reader to [11,12,50].

Perhaps the variant most widely studied is the planar Turán number, denoted by $\mathrm{ex}_{\mathcal{P}}(n,F)$, which is the maximum number of edges possible in an n-vertex F-free *planar* graph. This was initiated by Dowden [19] in 2015, who determined upper bounds for $\mathrm{ex}_{\mathcal{P}}(n,C_4)$ and $\mathrm{ex}_{\mathcal{P}}(n,C_5)$, together with constructions attaining these bounds for infinitely many n. Ghosh, Győri, Martin, Paulos and Xiao [53] determined $\mathrm{ex}_{\mathcal{P}}(n,C_6)$ introducing a contribution method. Applying this method, Győri, Wang and Zheng [65] obtained some other results and gave a new shorter proof for $\mathrm{ex}_{\mathcal{P}}(n,C_5)$ with a clearer extremal construction. Many other results for various other graphs have been obtained, see for example, [59,73–75].

There are yet other types of the Turán number. Examples include rainbow Turán numbers, introduced by Keevash, Mubayi, Sudakov and Verstraëte in 2007 [71], singular Turán numbers, introduced by Gerbner, Patkós, Tuza and Vizer in 2022 [51], etc.

1.3 Generalized Turán problems

Given graphs H and G, let $\mathcal{N}(H,G)$ denote the number of copies of H in G, that is, the number of subgraphs of G that are isomorphic to H. Given a graph H and a family of graphs $\mathcal{F}$, the *generalized Turán number* is denoted by $\mathrm{ex}(n,H,\mathcal{F})$, and defined as follows

$$\mathrm{ex}(n,H,\mathcal{F}) = \max\{\mathcal{N}(H,G) \ : G \text{ is } \mathcal{F}\text{-free and } |V(G)| = n\}.$$

If $\mathcal{F} = \{F\}$, we simply write $\mathrm{ex}(n,H,F)$. When H is K_2 (an edge), it is the classical Turán number, $\mathrm{ex}(n,F)$, of F.

The study of generalized Turán numbers appeared quite early, too. Not much after Turán's work, Zykov [95], in 1949, using a symmetrizarion technique (see Chapter 3 for a description), determined $\mathrm{ex}(n,K_k,K_{r+1})$, for $k \le r$. Independently, in 1962, Erdős [22] obtained the same result. Győri, Pach and Simonovits [61] studied $\mathrm{ex}(n,H,K_r)$, for various graphs H and $r \ge 3$. In 1984, Erdős [26] conjectured that the maximum number of C_5's in a triangle-free graph on n vertices is at most $(n/5)^5$. The first estimate was due to Győri [60] who gave the upper bound $\mathrm{ex}(n,C_5,K_3) \le 1.03(\frac{n}{5})^5$. The conjecture was proved in 2013 independently by Grzesik [56] and Hatami, Hladký, Král, Norine, and Razborov [67] using the newly developed technique of flag algebras by Razbarov in 2007. Bollobás and Győri [9] also studied $\mathrm{ex}(n,K_3,C_5)$. In fact, even before Turán's Theorem, in 1938, Erdős [21] proved that $\mathrm{ex}(n,P_3,C_4) \le \binom{n}{2}$ to solve a problem in number theory. In their recent survey on generalized Turán numbers [48], Gerbner and Palmer mention that this could have led Erdős to initiate the study of generalized Turán numbers even before the emergence of classical Turán numbers.

More recently, in 2014, Alon and Shikhleman took a general and systematic approach to study the generalized Turán number $\mathrm{ex}(n,H,F)$, improving some previously known estimates and proving some other new results. Among several results they obtained, is the characterization of pairs of graphs H and F, so that $\mathrm{ex}(n,H,F) = \Theta(n^{|V(H)|})$ (see Theorem 4.1).

Among the concepts and properties of the function $\mathrm{ex}(n,F)$ that have been extended to the generalized version, $\mathrm{ex}(n,H,F)$, is that of stability. The first such result is due to Ma and Qiu [81] who showed stability of the problem $\mathrm{ex}(n,K_k,K_{r+1})$, where $k \le r$. More on the related literature will be given in Chapter 3.

It is natural to study the generalized version of the other variants of the Turán number. Győri, Paulos, Salia, Tompkins and Zamora [62] introduced the generalized version of the planar Turán numbers, $\mathrm{ex}_{\mathcal{P}}(n,H,\mathcal{F})$, the maximum number of copies of a subgraph H in $\mathcal{F}$-free n-vertex planar graphs. In particular, they showed that for any $k \ge 5$, $\mathrm{ex}_{\mathcal{P}}(n,C_k,C_4) = \Theta(n^{\lfloor k/3 \rfloor})$, and in case $k = 5$, they proved $\mathrm{ex}_{\mathcal{P}}(n,C_5,C_4) = n - 4$, for all $n \ge 5$ (except $n = 6$). Also, Gerbner, Methuku, Mészáros and Palmer [44] initiated the study of generalized rainbow Turán numbers. In Chapter 4, we will introduce and study the generalized version of the regular Turán numbers.

A related concept to counting subgraphs (or edges) in a graph is that of homomorphisms. A homomorphism from a graph H to a graph G is an edge preserving map from $V(H)$ to $V(G)$. Let $\mathrm{Hom}(H, G)$ denote the set of all homomorphisms from H to G, and $\hom(H, G) := |\mathrm{Hom}(H, G)|$. A map $f : V(H) \to V(G)$ is an isomorphism if it is bijective and for every $u, v \in V(H)$, $uv \in E(H)$ if and only if $f(u)f(v) \in E(G)$. An automorphism of a graph H is an isomorphism from H to itself. We denote by $\mathrm{Aut}(H)$ the set of all automorphisms of H. Let $\mathrm{Inj}(H, G)$ denote the set of all injective homomorphisms from H in G, and $\mathrm{inj}(H, G) := |\mathrm{Inj}(H, G)|$. An injective homomorphism from H to G corresponds to a copy of H to G, and for each copy of H in G there are $|\mathrm{Aut}(H)|$ injective homomorphisms from H to G. Thus, $\mathrm{inj}(H, G) = |Aut(H)| \cdot \mathcal{N}(H, G)$. Thus, maximizing $\mathrm{inj}(H, G)$ is the same as maximizing $\mathcal{N}(H, G)$ in terms of the order of magnitude and they asymptotically differ by a factor of $|Aut(H)|$. Sometimes this approach is taken in dealing with extremal problems (see e.g., [6, 78, 83, 88]). Note that over F-free graphs G, maximizing $\hom(H, G)$ could behave very differently from $\mathcal{N}(H, G)$. For example, let $F := M_4$, a matching of size four. Then, $\mathrm{ex}(n, C_7, M_4) = O(1)$, since once we have a copy C of C_7, we have three independent edges, then any other edge with at least one end not in $V(C)$ results in a copy of M_4. However, let G be a graph consisting of a triangle and an independent set X of size $n - 3$, and all the edges between X and the triangle. Then, G is M_4-free and has $O(n)$ triangles. Since every triangle is a homomorphic image of C_7, the maximum of $\hom(C_7, G) = \Omega(n)$, over all M_4-free graphs G.

1.4 Notations and structure of this thesis

This thesis will focus on generalized Turán problems. In Chapter 2, we consider the graph F_k, known as the friendship graph (or a k-fan), which consists of k-triangles sharing a vertex, and determine the exact value of the generalized Turán number $\mathrm{ex}(n, K_3, F_k)$, for every $k \geq 2$ and sufficiently large n. That is, we determine the maximum number of triangles in n-vertex F_k-free graphs. The content of this chapter is published in [94], which is a joint work with Zhu, Chen, Gerbner and Győri.

Chapter 3 is devoted to the concept of stability in generalized Turán problems. Applying graph symmetrization techniques, we will obtain stability results for some generalized Turán problems. Namely, we consider complete l-partite graphs in K_{k+1}-free graphs, where $l \leq k$, and for sufficiently large k, any graph H instead of the complete multipartite graph. Finally, we obtain a stability result for bipartite graphs when a sufficiently long odd cycle is forbidden. This chapter is based on the joint work with Gerbner [42].

In Chapter 4, we consider the regular variant of the Turán problem, and we introduce the generalized regular Turán problem. Namely, we define the generalized regular Turán number, $\mathrm{rex}(n, H, F)$, the maximum number of copies of H possible in n-vertex F-free regular graphs. First, we will extend some general results about

$\mathrm{ex}(n, H, F)$ to this new version, and then provide examples to show different behaviors of $rex(n, H, F)$. Finally, we determine the exact value of $\mathrm{rex}(n, K_3, P_k)$, for sufficiently large n and any k. This chapter is based on our joint work with Gerbner [41].

Finally, in Chapter 5, we study some generalized planar Turán problems. As mentioned before, the pentagons vs triangles problem was among the very interesting ones and was very difficult to settle. Here, we investigate the same problems in the planar version. Specifically, we study $\mathrm{ex}_{\mathcal{P}}(n, K_3, C_l)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_l, K_3)$, for $4 \leq l \leq 6$, and determine their exact values together with extremal graphs that achieve them. The content of this chapter is based on our joint work with Ervin Győri [58].

Notation

Throughout, the notation we follow is fairly standard. $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G, respectively, and $e(G) := |E(G)|$ (sometimes just e if G is clear from the context). For a subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced on X, and $G \setminus X$ (or simply $G \setminus v$, if $X = \{v\}$) denotes the induced subgraph $G[V(G) \setminus X]$. We denote the complement of a graph G, by $\overline{G}$. For any vertex $v \in V(G)$ and subset $S \subseteq V(G)$, let $N_S(v)$ denote the neighbors of v in S and $d_S(v) = |N_S(v)|$. If $S = V(G)$, then $N(v) = N_S(v)$ and $d(v) = d_S(v)$. For vertices $x_1, \dots, x_k \in V(G)$, $N(x_1, \dots, x_k)$ denotes the common neighbors of all the vertices $x_1, \dots, x_k$. When xy is an edge, we may write $N(xy)$ for $N(x, y)$. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees in G, respectively. For two graphs G_1 and G_2, $G_1 \cup G_2$ is the vertex disjoint union of G_1 and G_2, kG_1 is the vertex disjoint union of k copies of G_1, and $G_1 + G_2$ is the graph obtained by taking $G_1 \cup G_2$ and joining all pairs v_1, v_2 with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ (Care needs to be taken here, $G + G \neq 2G$).

For subsets $A, B \subseteq V(G)$, the set of the edges of G between the vertices of A and B (i.e. those with one end in A and the other in B) is denoted by $E(A, B)$, and $e(A, B) := |E(A, B)|$. Given graphs H and G, we denote the number of copies of H in G (i.e. subgraphs of G isomorphic to H) by $\mathcal{N}(H, G)$, when the graph G is clear, we also use $\#H$ to denote the number of copies of H. For a positive integer t, we use $[t]$ to denote the set $\{1, 2, \dots, t\}$.

Given a graph H, a blow-up of H is a graph obtained by replacing each vertex of H by an independent set of vertices and each edge $uv \in E(H)$ by a complete bipartite graph between the independent sets replacing u and v. A blow-up is balanced, denoted by $H(m)$, if every independent set replacing each vertex is of size m, for some $m \in \mathbb{Z}^+$. In the respective chapters, we will introduce other notation that are used in them. For positive numbers x and y, the notation $x \gg y$ means "x is sufficiently larger than y", more precisely, $x > My$ for a sufficiently large constant M.

Chapter 2

Maximum number of triangles in F_k-free graphs

2.1 Introduction

The *friendship graph* (or *k-fan*) F_k consists of k triangles all intersecting in one common vertex v. Erdős, Füredi, Gould and Gunderson determined the Turán number of F_k.

Theorem 2.1 (Erdős, Füredi, Gould and Gunderson [27]). *For every $k \geq 1$ and $n \geq 50k^2$,*

$$\mathrm{ex}\left(n, F_k\right) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even}. \end{cases}$$

A particular line of research is to determine for a given graph H, what graphs F have the property that $\mathrm{ex}(n, H, F) = O(n)$. This was started by Alon and Shikhelman [3], who dealt with the case $H = K_3$, and was continued for other graphs in [37, 46].

An *extended friendship graph* consists of F_k for some $k \geq 0$ and any number of additional vertices or edges that do not create any additional cycles. Alon and Shikhelman [3] showed that $\mathrm{ex}(n, K_3, F) = O(n)$ if and only if F is an extended friendship graph. We remark that known results easily imply that if F is not an extended friendship graph, then $\mathrm{ex}(n, K_3, F) = \omega(n)$ and it is also easy to see that adding further edges to F without creating any cycle does not change the linearity of $\mathrm{ex}(n, K_3, F)$. Hence, the key part of their proof is the following theorem.

Theorem 2.2 (Alon and Shikhelman [3]). *For any k we have* $\mathrm{ex}(n, K_3, F_k) < (9k - 15)(k + 1)n$.

This upper bound for $\mathrm{ex}(n, K_3, F_k)$ is not tight. For instance, for $k = 2$, it was observed by Liu and Wang [76] that a hypergraph Turán theorem of Erdős and Sós [91] gives the exact result for $\mathrm{ex}(n, K_3, F_2)$. Let $\mathcal{F}_k$ denote the 3-uniform hypergraph (k-star) consisting of k hyperedges sharing exactly one vertex. Let $\mathrm{ex}_3(n, \mathcal{F}_k)$ denote the largest number of hyperedges that an $\mathcal{F}_k$-free n-vertex 3-uniform hypergraph can contain.

Theorem 2.3 (Erdős and Sós [91]). *For all* $n \geq 3$,

$$\mathrm{ex}_3(n, \mathcal{F}_2) = \begin{cases} n & \text{if } n = 4m, \\ n-1 & \text{if } n = 4m+1, \\ n-2 & \text{if } n = 4m+2 \text{ or } n = 4m+3. \end{cases}$$

Hence, it is interesting to determine the exact value of $\mathrm{ex}(n, K_3, F_k)$ for any F_k $(k \geq 3)$.

Throughout this chapter, let $\pi(G)$ denote the degree sequence of G. For $X, Y \subseteq V(G)$, $[X, Y]$ denotes the set of edges with one end in X and another in Y and $[x, Y] = [X, Y]$ if $X = \{x\}$. Recall that K_n and $\bar{K}_n$ denote the complete graph and the empty graph on n vertices, respectively.

We first define two graphs. Let $k \geq 4$ be even, $X = \{x_1, ..., x_{k-1}\}$ and $Y = \{y_1, ..., y_{k-1}\}$. The graph H'_k is a graph obtained from a complete bipartite graph with vertex classes X and Y. We subdivide the edge $x_i y_i$ once for $i \leq \frac{k}{2} - 1$, and then identify the $\frac{k}{2} - 1$ inserted vertices into one vertex z. The graph H_k is the complement of H'_k deleting the edge $zy_{k/2}$, which is shown in Figure 1.

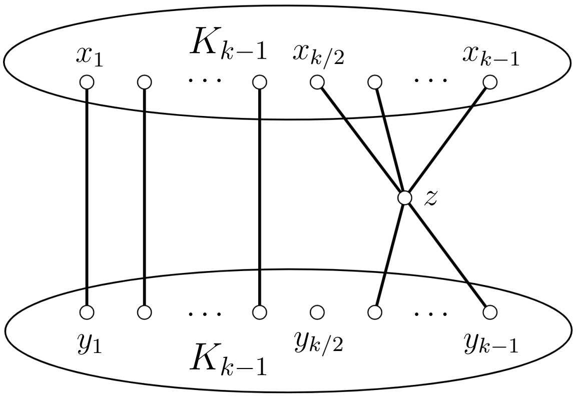


Figure 1. The graph H_k

It is clear that $|V(H_k)| = |V(H'_k)| = 2k - 1$ and $\pi(H_k) = \pi(H'_k) = (k-1, ..., k-1, k-2)$.

The main result of this chapter is the following.

Theorem 2.4. *Let* $k \geq 3$ *be an integer and* $n \geq 4k^3$*. If* k *is odd, then*

$$\mathrm{ex}\big(n, K_3, F_k\big) = (n - 2k)k(k-1) + 2\binom{k}{3},$$

and $\bar{K}_{n-2k} + 2K_k$ *is the unique extremal graph, and if* k *is even, then*

$$\mathrm{ex}\big(n, K_3, F_k\big) = (n - 2k + 1)k\left(k - \frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2,$$

and $\bar{K}_{n-2k+1} + H_k$ *is the unique extremal graph.*

Given a graph G, we let $\mathcal{T}(G)$ denote the 3-uniform hypergraph on the vertex set $V(G)$ where $\{u, v, w\}$ form a hyperedge if and only if uvw is a triangle in G. The key observation is that if G is F_k-free, then $\mathcal{T}(G)$ is $\mathcal{F}_k$-free. Therefore, $\mathrm{ex}(n, K_3, F_k) \leq \mathrm{ex}_3(n, \mathcal{F}_k)$. In the case $k = 2$, the upper bound obtained this way matches the lower bound provided by $\lfloor n/4 \rfloor$ vertex-disjoint copies of K_4, and in the case $n = 4m + 3$ we also have a triangle on the remaining vertices. This gives the exact value of $\mathrm{ex}(n, K_3, F_2)$.

The result of Erdős and Sós [91] was extended to arbitrary k by Chung and Frankl [16], after partial results [14, 15, 20].

Theorem 2.5 (Chung and Frankl [16]). *Let $k \geq 3$. If n is sufficiently large, then*

$$\mathrm{ex}_3(n, \mathcal{F}_k) = \begin{cases} (n-2k)k(k-1) + 2\binom{k}{3} & \text{if } k \text{ is odd,} \\[2ex] (n-2k+1)\frac{(2k-1)(k-1)-1}{2} + (2k-2)\binom{k-1}{2} & \\ \qquad + \binom{k-2}{2} - \frac{(k-2)(k-4)}{2} + \frac{k}{2} & \text{if } k \text{ is even.} \end{cases}$$

and $\mathscr{F}_k := \mathcal{T}\left(\bar{K}_{n-2k} + 2K_k\right)$ is the unique extremal 3-uniform hypergraph, when k is odd.

For odd k, this completes the proof of the upper bound. However, for even k, the construction giving the lower bound in the above theorem is obtained by taking all 3-sets of $\bar{K}_{n-2k+1} + H'_k$ that either intersect $V(H'_k)$ in one edge or contain two edges of H'_k together with the 3-sets x_1, z, y_i of $V(H'_k)$ with $1 \leq i \leq k/2$. This is not $\mathcal{T}(G)$ for some F_k-free graph G (observe that the 3-sets that contain two edges of H'_k and x_1, z, y_i's are not triangles). Despite this, the upper bound differs from the lower bound only by an additive constant $c(k)$. We will heavily use the tools provided by Chung and Frankl [14] to obtain the improvement needed in Theorem 2.4.

The rest of this chapter is organized as follows. In Section 2.2, we present some preliminary results. In Section 2.3, we study the local structure within the neighborhood of a vertex in an F_k-free graph, and some properties of a weight function, defined on the vertices of triangles, which is our main method for counting the number of triangles. These results can be used to prove Theorem 2.4 with the best coefficient of n but a weak constant $f(k)$. The very technical Section 2.4 is devoted to the precise proof of Theorem 2.4. In Section 2.5, we give some concluding remarks.

2.2 Preliminaries

As a preparation for proving our result, we first present some known theorems, and then we count the number of triangles in a graph with a given degree sequence, which are interesting in their own right.

Let $\nu(G)$ denote the number of edges of a maximum matching in a graph G. The following is a famous result, which is essentially due to Tutte, known as the Tutte-Berge formula.

Theorem 2.6 (Berge [5])**.** *Let $o(G \setminus X)$ denote the number of odd components of $G \setminus X$. Then*

$$\nu(G) = \frac{1}{2} \min \Big\{ |V(G)| - o(G \setminus X) + |X| : \ X \subseteq V(G) \Big\}.$$

Theorem 2.7 (Chung and Frankl [16])**.** *Let k be an even integer and H be a graph on $2k-1$ vertices and with $\pi(H) = (k-1, \dots, k-1, k-2)$. Then either*

$$\mathcal{N}(K_3, H) \geq \left(\frac{k}{2} - 1 \right)^2 - 1,$$

or

$$\mathcal{N}(K_3, H) = \left(\frac{k}{2} - 2 \right) \left(\frac{k}{2} - 1 \right) \text{ and } H = H_k'.$$

Theorem 2.8. *Let k be an even integer and H be a graph on $2k-1$ vertices with $\pi(H) = (k-1, \dots, k-1, k-2)$. Then*

$$\mathcal{N}(K_3, H) \leq 2\binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2,$$

equality holds if and only if $H = H_k$.

Proof. One can easily check that the statement holds for $k = 2$, so in the rest of the proof we assume that $k \geq 4$. The proof will be similar to the proof of Goodman's theorem [55]. It is easy to see that

$$\mathcal{N}(K_3, H_k) = 2\binom{k-1}{3} + \binom{k/2}{2} + \binom{k/2-1}{2} = 2\binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2.$$

Let $\overline{H}$ be the complement of H. For any triple (x, y, z), if xyz is neither a triangle in H nor a triangle in $\overline{H}$, then it is easy to check exactly two of the three, say x and y, are such that $||[x, \{y, z\}]|| = 1$ and $||[y, \{x, z\}]|| = 1$ in H. Thus, we have

$$\begin{aligned}
\mathcal{N}(K_3, H) &= \binom{2k-1}{3} - \mathcal{N}\big(K_3, \overline{H}\big) - \frac{1}{2} \sum_v d(v)(2k-2-d(v)) \\
&= \binom{2k-1}{3} - (k-1)^3 - \frac{1}{2}k(k-2) - \mathcal{N}\big(K_3, \overline{H}\big) \\
&= 2\binom{k-1}{3} + \frac{(k-2)^2}{2} - \mathcal{N}\big(K_3, \overline{H}\big).
\end{aligned}$$

Obviously, it is sufficient to show $\mathcal{N}\big(K_3, \overline{H}\big) \geq (\frac{k}{2} - 1)^2$.

Note that $\overline{H}$ is a graph on $2k-1$ vertices with $\pi\big(\overline{H}\big) = (k-1, \dots, k-1, k)$. Let z be the vertex of degree k in $\overline{H}$.

If there is an edge $zz' \in E\big(\overline{H}\big)$ such that zz' is contained in at least two triangles, then $\mathcal{N}\big(K_3, \overline{H}\big) \geq \mathcal{N}\big(K_3, \overline{H} - zz'\big) + 2$. Because of $\pi\big(\overline{H} - zz'\big) = (k-1, \dots, k-$

$1, k-2)$, by Theorem 2.7, either $\mathcal{N}(K_3, \overline{H}-zz') \geq \left(\frac{k}{2}-1\right)^2 - 1$ or $\overline{H}-zz' = H_k'$. In the former case, we have

$$\mathcal{N}(K_3, H) \leq 2\binom{k-1}{3} + \left(\frac{k}{2}-1\right)^2 - 1.$$

In the latter case, it is easy to check that $\overline{H} = \overline{H_k}$, and hence $H = H_k$.

If each edge zz' in $E(\overline{H})$ is contained in at most one triangle, then $\Delta(\overline{H}[N(z)]) \leq 1$. Let $V_1 = N(z)$, $V_2 = V(H) - V_1$, $s = e(\overline{H}[V_1])$ and $t = e(\overline{H}[V_2])$. Count the edges between V_1 and V_2 in two ways, we have

$$k(k-1) - 2s = (k-1)^2 + 1 - 2t,$$

which implies $s - t = \frac{k}{2} - 1$. However, because $s \leq \frac{k}{2}$, we must have $s = \frac{k}{2}$ and $t = 1$, or $s = \frac{k}{2} - 1$ and $t = 0$. Each edge in $\overline{H}[V_1]$ can form a triangle with at least $k-3$ vertices in V_2, since each of its end points has at least $k-3$ neighbors in $V_2 \setminus \{z\}$, and hence, they have $k-4$ common neighbors. Together with z, this gives $k-3$ triangles. Similarly, if $t = 1$, the edge in V_2 is in at least $k-4$ triangles with vertices in V_1. Thus, we get

$$\mathcal{N}(K_3, \overline{H}) \geq \frac{k}{2}(k-3) + (k-4) > \left(\frac{k}{2}-1\right)^2$$

in the former case, and

$$\mathcal{N}(K_3, \overline{H}) \geq \left(\frac{k}{2}-1\right)(k-3) \geq \left(\frac{k}{2}-1\right)^2$$

in the latter case with equality only if $k = 4$. In which case, it is not difficult to check that $\overline{H} = \overline{H_k}$, and so $H = H_k$. □

Theorem 2.9. *Let k be an even integer and H be a graph on $2k-1-2s$ vertices with $\pi(H) = (k-1, \ldots, k-1, k-2)$. Then*

$$\mathcal{N}(K_3, H) \leq \frac{1}{6}(2k-1-2s)\Big((k-1)(k-2) - (k-1-2s)(2s+1)\Big) + \frac{1}{2} - s.$$

Proof. Let $\Lambda(H)$ denote the number of triples (x, y, z) having exactly two edges in H, say $xy, xz \in E(H)$ and $yz \notin E(H)$. Because $|N(y) \cap N(z)| \geq d(y) + d(z) - (|H| - 2)$ for every nonadjacent pair (y, z), and there are $|H| - (d(y)+1)$ nonadjacent pairs containing y for any $y \in V(H)$, we have

$$\Lambda(H) \geq \frac{1}{2}\Big((2k-1-2s)(k-1-2s) + 1\Big)(2s+1) - (k-2s).$$

On the other hand, since

$$(2k-2-2s)\binom{k-1}{2} + \binom{k-2}{2} = \Lambda(H) + 3\mathcal{N}(K_3, H),$$

we get

$$3\mathcal{N}(K_3, H) \leq \frac{1}{2}(2k-1-2s)\Big((k-1)(k-2) - (k-1-2s)(2s+1)\Big) + \frac{3}{2} - 3s.$$

This completes the proof. □

2.3 Some properties of F_k-free graphs and a weight function

Let G be an F_k-free graph, $uv \in E(G)$ and $N(uv) = N(u) \cap N(v)$. Clearly, $|N(uv)|$ is the number of triangles containing the edge uv in G. We classify the edges into the following three classes:

- Heavy edges: $\mathcal{H} = \big\{uv : |N(uv)| \geq 2k-1\big\}$,
- Medium edges: $\mathcal{M} = \big\{uv : k \leq |N(uv)| \leq 2k-2\big\}$, and
- Light edges: $\mathcal{L} = \big\{uv : 1 \leq |N(uv)| \leq k-1\big\}$.

For a fixed vertex $u \in V(G)$, let $G_u = G[N(u)]$ and

- $\mathcal{H}(u) = \big\{v : v \in N(u) \text{ and } uv \in \mathcal{H}\big\}$,
- $\mathcal{M}(u) = \big\{v : v \in N(u) \text{ and } uv \in \mathcal{M}\big\}$, and
- $\mathcal{L}(u) = \big\{v : v \in N(u) \text{ and } uv \in \mathcal{L}\big\}$.

This notation will be used throughout the rest of this chapter.

Since G is F_k-free, then $\nu(G_u) \leq k-1$ for any u. Thus, Theorem 2.6 implies

Observation 2.10. *There exists some $X \subseteq V(G_u)$ such that*

$$\sum_{i=1}^{\ell} \left\lfloor \frac{|C_i|}{2} \right\rfloor + |X| \leq k-1, \tag{2.1}$$

where $C_1, ..., C_\ell$ are all the components of $G_u - X$.

Lemma 2.11. *Let G be an F_k-free graph, $u \in V(G)$ and X a subset of $V(G_u)$ satisfying Equation (2.1). Then we have the following:*

(i) $\mathcal{H}(u) \subseteq X$. Moreover, $|\mathcal{H}(u)| \leq k-1$ and if equality holds, then $\mathcal{M}(u) = \emptyset$.
(ii) $|\mathcal{H}(u)| + \frac{1}{2}|\mathcal{M}(u)| \leq k - \frac{1}{2}$.

Proof. Let $C_1, ..., C_\ell$ be the components of $G_u - X$.

(i) Let $v \in \mathcal{H}(u)$, we know that $|N(uv)| \geq 2k-1$. If v lies in some component C_i, then $N(uv) \subseteq V(C_i) \cup X$ and so

$$\frac{1}{2}\big|(C_i \cap N(uv)) \cup \{v\}\big| + \big|X \cap N(uv)\big| \geq k,$$

which contradicts (2.1). Hence we have $\mathcal{H}(u) \subseteq X$.

By (2.1), we have $|\mathcal{H}(u)| \leq k-1$, and if $|\mathcal{H}(u)| = k-1$, then $X = \mathcal{H}(u)$ and $|C_i| = 1$ for $1 \leq i \leq \ell$. Let v be any vertex of $G_u - X$, then $N(uv) \subseteq X$ and hence $uv \in \mathcal{L}$, and so $\mathcal{M}(u) = \emptyset$.

(ii) Clearly, $N(uv) \subseteq X \cup V(C_i)$ if $v \in V(C_i)$. Thus, if there are two components, say C_1, C_2, such that $\mathcal{M}(u) \cap V(C_i) \neq \emptyset$, then $|X| + |C_i| \geq k+1$ for $i = 1, 2$. Hence we have $|X| + \lfloor |C_1|/2 \rfloor + \lfloor |C_2|/2 \rfloor \geq k$, which contradicts (2.1). Thus, we may assume $\mathcal{M}(u) \subseteq X \cup V(C_i)$. Note that $\mathcal{H}(u) \subseteq X$ as shown in (i), and

$$\begin{aligned}\big|\mathcal{H}(u)\big| + \frac{1}{2}\big|\mathcal{M}(u)\big| &= \big|\mathcal{H}(u)\big| + \frac{1}{2}\big|\mathcal{M}(u) \cap (X - \mathcal{H}(u))\big| + \frac{1}{2}\big|\mathcal{M}(u) \cap C_i\big| \\ &\leq |X| + \frac{1}{2}|C_i| \leq |X| + \left\lfloor \frac{|C_i|}{2} \right\rfloor + \frac{1}{2} \leq k - \frac{1}{2}.\end{aligned}$$

The proof of the lemma is complete. □

For each triangle $T = abc$ in G, assign T of weight 1 and define a distribution rule $w(T, \cdot)$ to distribute the weight 1 to its three vertices as below (suppose $|N(ab)| \geq |N(bc)| \geq |N(ac)|$):

$$\begin{aligned}
&w(T,a) = w(T,b) = w(T,c) = \tfrac{1}{3}, && \text{\textit{if } } E(T) \cap \mathcal{H} = \emptyset \text{\textit{ or }} E(T) \cap \mathcal{L} = \emptyset,\\
&w(T,a) = w(T,c) = \tfrac{1}{2},\ w(T,b) = 0, && \text{\textit{if }} ab \in \mathcal{H},\ bc \in \mathcal{H} \cup \mathcal{M} \text{ and } ac \in \mathcal{L},\\
&w(T,a) = w(T,b) = 0,\ w(T,c) = 1, && \text{\textit{if }} ab \in \mathcal{H} \text{ and } bc, ac \in \mathcal{L}.
\end{aligned}$$

Now, we define a weight function $f(u)$ for each vertex u of G as follows.

$$f(u) = \sum_{vx \in E(G_u)} w(uvx, u)$$

if u lies in at least one triangle, and $f(u) = 0$ otherwise. It is clear

$$\mathcal{N}(K_3, G) = \sum_{u \in V(G)} f(u).$$

Now, we first discuss some properties of the weight functions $w(T, \cdot)$ and $f(u)$.

Lemma 2.12. *Let uv be an edge of an F_k-free graph with $k \geq 3$. Then either*

$$\sum_{x \in N(uv)} w(uvx, u) = k - 1,$$

if $uv \in \mathcal{L}, |N(uv)| = k - 1, vx \in \mathcal{H}$ and $ux \in \mathcal{L}$ for any $x \in N(uv)$, or

$$\sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2}$$

otherwise.

Proof. Let $\mathcal{H}'(v)$, $\mathcal{M}'(v)$ and $\mathcal{L}'(v)$ be $\mathcal{H}(v)$, $\mathcal{M}(v)$ and $\mathcal{L}(v)$ intersecting with $N(uv)$, respectively. It is clear

$$\sum_{x \in N(uv)} w(uvx, u) = \sum_{x \in \mathcal{H}'(v)} w(uvx, u) + \sum_{x \in \mathcal{M}'(v)} w(uvx, u) + \sum_{x \in \mathcal{L}'(v)} w(uvx, u).$$

We distinguish three cases on the number of $|N(uv)|$.

Case 1. $uv \in \mathcal{H}$

By the definition of $w(T, \cdot)$, $w(uvx, u) \leq \frac{1}{2}$ if $x \in \mathcal{H}'(v) \cup \mathcal{M}'(v)$ and $w(uvx, u) = 0$ if $x \in \mathcal{L}'(v)$. Noting that $u \in \mathcal{H}(v) - \mathcal{H}'(v)$, we have $|\mathcal{H}'(v)| + \frac{1}{2}|\mathcal{M}'(v)| \leq k - \frac{3}{2}$ by Lemma 2.11(ii), and hence

$$\sum_{x \in N(uv)} w(uvx, u) \leq \frac{1}{2}|\mathcal{H}'(v)| + \frac{1}{2}|\mathcal{M}'(v)| \leq k - \frac{3}{2} - \frac{1}{2}|\mathcal{H}'(v)|, \tag{2.2}$$

which implies the result holds.

Case 2. $uv \in \mathcal{M}$

In this case, $|N(uv)| \leq 2k-2$. Since $uv \in \mathcal{M}$ implies $\mathcal{M}(v) \neq \emptyset$, by Lemma 2.11(i), we have $|\mathcal{H}'(v)| \leq |\mathcal{H}(v)| \leq k-2$. By the definition of $w(T,\cdot)$, $w(uvx,u) \leq \frac{1}{2}$ if $x \in \mathcal{H}'(v)$ and $w(uvx,u) \leq \frac{1}{3}$ if $x \in \mathcal{M}'(v) \cup \mathcal{L}'(v)$. Thus, we have

$$\sum_{x\in N(uv)} w(uvx,u) \leq \frac{1}{2}\big|\mathcal{H}'(v)\big| + \frac{1}{3}\big|\mathcal{M}'(v)\big| + \frac{1}{3}\big|\mathcal{L}'(v)\big| \leq k-1-\frac{k}{6}. \tag{2.3}$$

The upper bound $k-\frac{3}{2}$ follows from the assumption $k \geq 3$.

Case 3. $uv \in \mathcal{L}$

Because $uv \in \mathcal{L}$, we have $|N(uv)| \leq k-1$. By the definition of $w(T,\cdot)$, some triangles satisfy $w(uvx,u) = 1$ and other triangles satisfy $w(uvx,u) \leq \frac{1}{2}$. Thus we have

$$\sum_{x\in N(uv)} w(uvx,u) \leq (k-1) - \frac{1}{2}\left|\left\{uvx : w(uvx,u) \leq \frac{1}{2}\right\}\right|, \tag{2.4}$$

which implies $\sum_{x\in N(uv)} w(uvx,u) \leq k-\frac{3}{2}$ or $\sum_{x\in N(uv)} w(uvx,u) = k-1$, and the latter holds if and only if $|N(uv)| = k-1$, and all triangles uvx satisfy $w(uvx,u) = 1$, that is, $vx \in \mathcal{H}$ and $ux \in \mathcal{L}$ for any $x \in N(uv)$. □

Lemma 2.13. *Suppose G is an F_k-free graph and $k \geq 4$ is even. Let $u \in V(G)$, X be a subset of $V(G_u)$ satisfying (2.1) and $C_1, ..., C_\ell$ be the components of $G_u - X$ with $|C_1| \geq \cdots \geq |C_\ell|$. Then*

$$f(u) \leq k\left(k-\frac{3}{2}\right) - \frac{1}{2},$$

or $f(u) = k\left(k-\frac{3}{2}\right)$ and the following hold:

(i) If $X \neq \emptyset$, then X is an independent set and $d_{G_u}(v) = k-1$ for any $v \in X$;

(ii) $\pi(C_1) = (k-1,\dots,k-1,k-2)$, and either $G_u = C_1 \cup K_{k-1}$ with $|C_1| = k+1$, or $G_u - X = C_1 \cup (\ell-1)K_1$ with $|C_1| = 2k-1-2|X| \geq k+1$;

(iii) $E(G_u) \subseteq \mathcal{H}$, $[u, G_u] \subseteq \mathcal{L}$ and $\Delta(G_u) = k-1$.

Proof. By (2.1), $|C_i| \leq k$ for all $i \neq 1$. Let uvx be any triangle. Then

$$f(u) = \sum_{i=1}^{\ell} \sum_{vx\subseteq E(C_i)} w(uvx,u) + \sum_{\{v,x\}\cap X\neq\emptyset} w(uvx,u).$$

If the edge vx satisfies $\{v,x\} \cap X \neq \emptyset$, then by Lemma 2.12, we have

$$\sum_{\{v,x\}\cap X\neq\emptyset} w(uvx,u) \leq \sum_{v\in X}\sum_{x\in N(uv)} w(uvx,u) \leq \big|X\big|(k-1). \tag{2.5}$$

If $vx \in E(C_i)$ with $|C_i| \leq k$, then noting that k is even, uvx is a triangle for each $vx \in E(C_i)$ and $w(uvx,u) \leq 1$, we have

$$\sum_{vx \in E(C_i)} w(uvx,u) \leq \frac{1}{2} \sum_{v \in V(C_i)} \sum_{x \in N(uv) \cap C_i} w(uvx,u)$$

$$\leq \frac{1}{2}|C_i|\left(|C_i|-1\right) \leq \left\lfloor \frac{|C_i|}{2} \right\rfloor (k-1). \tag{2.6}$$

If $|C_1| \leq k$, that is, $|C_i| \leq k$ for $1 \leq i \leq \ell$, then by (2.1), (2.5) and (2.6), we have

$$f(u) \leq (k-1)\left(\sum_{i=1}^{\ell} \left\lfloor \frac{|C_i|}{2} \right\rfloor + |X|\right) \leq (k-1)^2 < k\left(k-\frac{3}{2}\right)-\frac{1}{2}.$$

The last inequality holds because of $k \geq 4$. So, we may assume that $|C_1| > k$.

If $\Delta(C_1) \geq k$, say $d_{C_1}(v) \geq k$ for some vertex v in C_1, then $|N(uv)| \geq k$ and so $uv \notin \mathcal{L}$. Since $\mathcal{H}(u) \subseteq X$ by Lemma 2.11(i), we have $v \notin \mathcal{H}(u)$ and hence $uv \in \mathcal{M}$. By (2.3), we have $\sum_{x \in N(uv) \cap C_1} w(uvx,u) \leq (k-1) - \frac{k}{6}$. Meanwhile, because $d_{C_1}(v) \geq k \geq 4$, there exists $v_i \in N(uv) \cap C_1$ for $1 \leq i \leq 4$. Note that $v \in \mathcal{M}(u)$ and $v \in N(uv_i)$, by Lemma 2.12, we have $\sum_{x \in N(uv_i) \cap C_1} w(uv_ix,u) \leq (k-1) - \frac{1}{2}$ for $1 \leq i \leq 4$, and

$$\sum_{vx \in E(C_1)} w(uvx,u) = \frac{1}{2} \sum_{v \in C_1} \sum_{x \in N(uv) \cap C_1} w(uvx,u) \leq \frac{1}{2}|C_1|(k-1) - \left(\frac{k}{12}+1\right). \tag{2.7}$$

If $\Delta(C_1) \leq k-1$, then

$$\sum_{vx \subseteq E(C_1)} w(uvx,u) = \frac{1}{2} \sum_{v \in C_1} \sum_{x \in N(uv) \cap C_1} w(uvx,u) \leq \left\lfloor \frac{1}{2}|C_1|(k-1) \right\rfloor. \tag{2.8}$$

Set $\mu(C_1) = \frac{k}{12}+1$ if $\Delta(C_1) \geq k$, $\mu(C_1) = \frac{1}{2}$ if $|C_1|$ is odd and $\Delta(C_1) \leq k-1$ and $\mu(C_1) = 0$ if $|C_1|$ is even and $\Delta(C_1) \leq k-1$, then (2.5)-(2.8) imply

$$f(u) \leq (k-1)\left(\frac{|C_1|}{2} + \sum_{i=2}^{\ell} \left\lfloor \frac{|C_i|}{2} \right\rfloor + |X|\right) - \mu(C_1). \tag{2.9}$$

Assume that $f(u) > k\left(k-\frac{3}{2}\right)-\frac{1}{2}$. By (2.1) and (2.9), we have $\mu(C_1) = \frac{1}{2}$. In this case, $|C_1| \geq k+1$ is odd and $\Delta(C_1) \leq k-1$. Note that if one of the equalities in (2.5), (2.6) and (2.8) does not hold, then the upper bound in (2.9) can be reduced by at least an extra $\frac{1}{2}$. This implies the equalities in (2.5), (2.6) and (2.8) hold.

It is clear that the equalities in (2.5) hold if and only if X is an independent set and $\sum_{x \in N(uv)} w(uvx,u) = k-1$ for any $v \in X$. By Lemma 2.12, we get that $d_{G_u}(v) = k-1$, $uv, ux \in \mathcal{L}$ and $vx \in \mathcal{H}$ for any $v \in X$.

Since $|C_1| \geq k+1$ is odd, $\Delta(C_1) \leq k-1$ and the equality in (2.8) holds, we can deduce that $\pi(C_1) = (k-1, k-1, \ldots, k-2)$, $E(C_1) \subseteq \mathcal{H}$ and $[u, C_1] \subseteq \mathcal{L}$.

Because equality (2.6) holds, recalling $|C_1| \geq k+1$ and k is even, by (2.1), we have $|C_i| \in \{1, k-1\}$ for $i \geq 2$ and each C_i is a clique with $E(C_i) \subseteq \mathcal{H}$ and $[u, C_i] \subseteq \mathcal{L}$. In addition, if $|C_i| = k-1$ for some $i \geq 2$, then by (2.1), $|C_1| = k+1$ and $X = \emptyset$, that is, $G_u - X = C_1 \cup K_{k-1} \cup (\ell-2)K_1$. If $|C_i| = 1$ for all $i \geq 2$, then $|C_1| = 2k-1-2|X|$. So, the statements (i), (ii) and (iii) hold. ☐

Remark. There is a similar lemma in Chung and Frankl's paper [16] when they deal with function $\mathrm{ex}_3\big(n, \mathcal{F}_k\big)$. However, in their lemma, they overlooked the case $G_u = C_1 \cup K_{k-1}$. Using our method in Section 4, it is not difficult to complete the proof of this missed case, too.

Definition 2.1. For any vertex $u \in V(G)$, the loss of u is the number

$$k\left(k - \frac{3}{2}\right) - f(u).$$

See the following simple observations about the losses.

Observation 2.14. *If some vertex $v \in X$ has $\sum_{x \in N(uv)} w(uvx, u) \leq (k-1) - c$, then the edge uv contributes c to the loss of u.*

Proof. It is a direct consequence of (2.5). ☐

Observation 2.15. *An edge $uv \in \mathcal{H}$ contributes $\frac{1}{2}$ to the loss of u. Moreover, a triangle uvx with $uv, vx \in \mathcal{H}$ contributes another $\frac{1}{2}$ to the loss of u.*

Proof. Since $uv \in \mathcal{H}$, by (2.2), $\sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2} - \frac{1}{2}|\mathcal{H}'(v)|$. Because $v \in X$ by Lemma 2.11, the edge uv contributes $\frac{1}{2}$ to the loss of u by Observation 2.14. Moreover, since a triangle uvx with $uv, vx \in \mathcal{H}$ satisfies $x \in \mathcal{H}'(v)$, so it contributes another $\frac{1}{2}$ to the loss of u by (2.2). ☐

Observation 2.16. *Let $uv \in \mathcal{M}$. If $\sum_{x \in N(uv)} w(uvx, u) \leq (k-1) - c$, then the edge uv contributes at least $\min\left\{\frac{c}{2}, \ \frac{k}{4} - \frac{1}{2}\right\}$ to the loss of u. Moreover, the edge uv contributes at least $\frac{k}{12}$ to the loss of u.*

Proof. If $v \in X$, then by (2.3) and Observation 2.14, uv contributes c to the loss of u. If $v \in V(C_1)$ and $|C_1| \geq k+1$, then by (2.7) and (2.8), uv contributes at least $\frac{c}{2}$ to the loss of u. If $v \in V(C_i)$ for some i with $|C_i| \leq k$, then by (2.6), there is a gap between $\lfloor |C_i|/2 \rfloor (k-1)$ and $\frac{1}{2}|C_i|\big(|C_i|-1\big)$, and for this gap, any edge uv' with $v' \in V(C_i)$ contributes

$$\frac{1}{|C_i|}\left(\left\lfloor \frac{|C_i|}{2} \right\rfloor (k-1) - \frac{1}{2}|C_i|\left(|C_i|-1\right)\right)$$

to the loss of u. On the other hand, because

$$\sum_{x \in N(uv) \cap C_i} w(uvx, u) \leq \frac{1}{2}\big(|C_i|-1\big) = \big(|C_i|-1\big) - \frac{1}{2}\big(|C_i|-1\big),$$

this reduces the right hand of (2.6) by an additional $\frac{1}{4}\big(|C_i|-1\big)$. Hence the total loss of u contributed by the edge uv is at least

$$\frac{1}{|C_i|}\left(\left\lfloor\frac{|C_i|}{2}\right\rfloor(k-1)-\frac{1}{2}|C_i|\big(|C_i|-1\big)\right)+\frac{1}{4}\big(|C_i|-1\big)\geq\frac{k}{4}-\frac{1}{2}.$$

Together with (2.3), $c\geq\frac{k}{12}$, it implies that the statements of the lemma are proved. ☐

2.4 Proof of the main result

We are now ready to prove the main result, which is Theorem 2.4.

Proof of Theorem 2.4. Let G be an extremal graph of $\mathrm{ex}(n, K_3, F_k)$.

If k is odd, then by Theorem 2.5, we have

$$\mathcal{N}(K_3,G)=e(\mathcal{T}(G))\leq\mathrm{ex}_3(n,\mathcal{F}_k)=(n-2k)k(k-1)+2\binom{k}{3},$$

and the unique extremal hypergraph is $\mathscr{F}_k$ for which equality holds. Because

$$\mathcal{N}(K_3,\bar{K}_{n-2k}+2K_k)=(n-2k)k(k-1)+2\binom{k}{3},$$

and $\mathcal{T}(\bar{K}_{n-2k}+2K_k)=\mathscr{F}_k$, we get

$$\mathcal{N}(K_3,G)=(n-2k)k(k-1)+2\binom{k}{3},$$

where equality holds if and only if $G=\bar{K}_{n-2k}+2K_k$.

The remaining part is devoted to the case when $k\geq 4$ is even. Because an edge not lying in a triangle makes no contribution to $\mathcal{N}(K_3,G)$, we may assume that each edge of G is covered by some triangles.

If $f(v)=k\left(k-\frac{3}{2}\right)$, then we call v a *good* vertex. Let $U_1=\{v: v\ is\ good\}$. Since $n\geq 4k^3$ and $f(v)\leq k\left(k-\frac{3}{2}\right)-\frac{1}{2}$ for any $v\notin U_1$ by Lemma 2.13, we have

$$\begin{aligned}nk\left(k-\frac{3}{2}\right)-\frac{1}{2}(n-|U_1|)\geq\sum_{v\in V(G)}f(v)&=\mathcal{N}(K_3,G)\\&\geq(n-2k+1)k\left(k-\frac{3}{2}\right)+2\binom{k-1}{3}+\left(\frac{k}{2}-1\right)^2,\end{aligned}$$

which implies

$$|U_1|\geq n-2(2k-1)k\left(k-\frac{3}{2}\right)>2k.$$

Moreover, if there exist $v,v'\in U_1$ such that $vv'\in E(G)$, then $vv'\in\mathcal{L}$ by Lemma 2.13. Let $vv'x$ be a triangle. Applying Lemma 2.13 to v, we have $v'x\in\mathcal{H}$ and using

Lemma 2.13 to v', we have $v'x \in \mathcal{L}$, a contradiction. Therefore, U_1 is an independent set.

Let $u \in U_1$ be given, $G_u = G[N(u)]$ as before and $U_2 = V(G) - V(G_u) - U_1$. We will prove Theorem 2.4 by showing G is an extremal graph only if $U_2 = \emptyset$. Since the proof is a little complicated and long, so we sketch it first in the following two paragraphs.

In the case when $N(u') = N(u)$ for any $u' \in U_1 - \{u\}$, our main idea for doing this is to partition the total weights of all vertices of G_u into two parts: One part comes from the triangles contained in G_u, which is exactly $\mathcal{N}(K_3, G_u)$, and another part is contributed by the triangles containing one or two vertices in U_2. And then we use discharge method to transfer the latter part to the vertices in U_2 such that $f(v) \leq k\left(k - \frac{3}{2}\right)$ is still valid for any $v \in U_2$ after transferring. Using this method, we show that if $U_2 \neq \emptyset$, then the total weight of G is less than the expected number.

In the case when there is some $u' \in U_1 - \{u\}$ such that $N(u) \neq N(u')$, we transform G into a graph G' such that G' and G have the same good vertices, and all good vertices of G' have the same neighborhood as $N(u)$, through an operation as follows: Delete all the edges between u' and $G_{u'}$ and add new edges joining u' to all vertices in G_u. Repeat this operation until all vertices in U_1 have the same neighborhood $N(u)$. Let G' be the resulting graph, $U_1' = \{v : v \textit{ is good in } G'\}$ and $U_2' = V(G') - V(G_u') - U_1'$. We will see that $\mathcal{N}(K_3, G') = \mathcal{N}(K_3, G)$, G' is also F_k-free and $U_1' = U_1$.

Firstly, since u' is good, by Lemma 2.13, we have $f(u') = e(G_{u'}) = k\big(k - \frac{3}{2}\big)$, which implies we destroy $k\big(k - \frac{3}{2}\big)$ triangles first and then add $k\big(k - \frac{3}{2}\big)$ new triangles, and so $\mathcal{N}(K_3, G') = \mathcal{N}(K_3, G)$. Moreover, $G_u' = G_u$. Secondly, since G_u has no kK_2 and $\Delta(G_u) = k - 1$ by Lemma 2.13, we can see that G' is also F_k-free after an easy check. Finally, because $|U_1| > 2k$, we have $E(G_u') \subseteq \mathcal{H}$, which implies $v \notin U_1'$ for any $v \in V(G_u')$ by Lemma 2.13. Furthermore, since $\Delta(G_u') = k - 1$ and U_1' is an independent set, $[U_1, G_u'] \subseteq \mathcal{L}$. Thus, we have $U_1 \subseteq U_1'$ by the definition of $w(T, \cdot)$. Suppose that there is some $v \in U_2$ in G such that $v \in U_1'$ in G'. Let $G_v' = G'[N(v)]$ and $X' \subseteq G_v'$ satisfy (2.1). Since $v \in U_1'$, $E(G_v') \subseteq \mathcal{H}$ and $[v, G_v'] \subseteq \mathcal{L}$ by Lemma 2.13. By the operation above, $E(G_v') \subseteq \mathcal{H}$ in G. Since $v \in U_2$, there is some $v' \in G_v'$ such that $vv' \notin \mathcal{L}$ in G, which means there is some $u' \in U_1$ such that $u'vv'$ is a triangle in G. Note that $V(G_v') \cup \{u'\} \subseteq N_G(v)$. If $v' \notin X'$, then by Lemma 2.13, it is easy to check that $G[\{u'\} \cup V(G_v')]$ contains kK_2, and so G has an F_k. Thus we have $v' \in X'$. In this case, by Lemma 2.13, $|\mathcal{H}(v')| = k - 1$ in G. Let $X'' \subseteq G_{v'}$ satisfy (2.1). By Lemma 2.11, $\mathcal{H}(v') \subseteq X''$ and hence $|X''| = k - 1$. Because $u'v$ is an edge in $G_{v'} - X''$, this contradicts (2.1). Thus, we have $U_1' = U_1$.

Since $U_1' = U_1$ implies $U_2' = U_2$, and $U_2' \neq \emptyset$ in this case, G' cannot be an extremal graph, and so does G since $\mathcal{N}(K_3, G') = \mathcal{N}(K_3, G)$. Therefore, it is sufficient to show G is an extremal graph only if $U_2 = \emptyset$ in the case when $N(u') = N(u)$ for any $u' \in U_1 - \{u\}$.

Let $X \subseteq G_u$ satisfy (2.1). By Lemma 2.13, $\Delta(G_u) = k - 1$. Moreover, G_u has the following structural properties.

Claim 2.1. *Let $v \in V(C_i)$, where C_i is some component of $G_u - X$.*

(1) If $d_{G_u}(v) = k-1$, then $N_{U_2}(v)$ is an independent set and $[v, U_2] \subseteq \mathcal{L}$.

(2) If $d_{C_i}(v) = k-2$, then $G[N_{U_2}(v)]$ is a star or a triangle, together with some isolated vertices. Moreover, if $v_1 \in N_{U_2}(v)$ and $vv_1 \in \mathcal{H} \cup \mathcal{M}$, then v_1 is the center of the star with at least 3 vertices, or lies on the triangle.

Proof. (1) Since $d_{G_u}(v) = k-1$ and $E(G_u) \subseteq \mathcal{H}$ by Lemma 2.13, we have $|\mathcal{H}(v)| \geq k-1$. Let $X' \subseteq V(G_v)$ satisfy (2.1). By Lemma 2.11, $X' = \mathcal{H}(v) \subseteq V(G_u)$ which in turn implies $G_v - X'$ has no edges by (2.1), and $\mathcal{M}(v) = \emptyset$, and so $[v, U_2] \subseteq \mathcal{L}$.

(2) Since $d_{C_i}(v) = k-2$ and $E(G_u) \subseteq \mathcal{H}$, we have $|\mathcal{H}(v)| \geq k-2$. Let $X' \subseteq V(G_v)$ satisfy (2.1). By Lemma 2.11, $\mathcal{H}(v) \subseteq X'$ and so $|X' \cap V(G_u)| \geq k-2$. Because $|X'| \leq k-1$, we have $|X' \cap U_2| \leq 1$, which implies $G_v - \mathcal{H}(v)$ has no $2K_2$ by (2.1), that is, $G[N_{U_2}(v)]$ is a star or a triangle, together with some isolated vertices. Moreover, if $vv_1 \in \mathcal{H} \cup \mathcal{M}$, then we have $d_{G_u}(v) = k-2$ by Lemma 2.11. Since $|N(vv_1)| \geq k$ and we have $|N(vv_1) \cap U_2| \geq 2$, that is, v_1 has at least 2 neighbors in $G[N_{U_2}(v)]$. Hence, v_1 is the center of the star or lies on a triangle in $G[N_{U_2}(v)]$. $\square$

Let C be the largest component of $G_u - X$. Then $\pi(C) = (k-1, ..., k-1, k-2)$ by Lemma 2.13. Let $z \in V(C)$ be the unique vertex with $d_C(z) = k-2$ and $N_C(z) = \{z_1, \ldots, z_{k-2}\}$. Since $\Delta(G_u) = k-1$, we have $|[V(C), X]| \leq 1$ and $[V(C), X] \subseteq [z, X]$. In addition, we have the following.

Claim 2.2. $|(\mathcal{H}(z) \cup \mathcal{M}(z)) \cap U_2| \leq 1$.

Proof. Assume $v_1, v_2 \in N_{U_2}(z)$ with $zv_1, zv_2 \notin \mathcal{L}$. By Claim 2.1(2), v_1v_2v is a triangle in $N_{U_2}(z)$ for some v. Since $|N(zv_i)| \geq k$ for $i = 1, 2$, $\{z_1, \ldots, z_{k-2}\} \subseteq N(v_1) \cap N(v_2)$ which contradicts Claim 2.1(1) since $v_1, v_2 \in N_{U_2}(z_1)$ and $v_1v_2 \in E(G)$. $\square$

If $|(\mathcal{H}(z) \cup \mathcal{M}(z)) \cap U_2| = 1$, we assume $zv_1 \in \mathcal{H} \cup \mathcal{M}$ and $N(zv_1) \cap U_2 = \{v_2, ..., v_t\}$. By Claim 2.1(2), $G[\{v_1, ..., v_t\}]$ is a K_3 or a star with center v_1. If $N(zv_1) \cap V(C) \neq \emptyset$, let $N(zv_1) \cap V(C) = \{z_1, \ldots, z_{t'}\}$, where $t' \leq k-2$.

Let us consider the total weight in $\sum_{v \in V(C)} f(v)$ coming from the triangles not contained in C. Since $[U_1, V(C)] \subseteq \mathcal{L}$, U_1 is an independent set, $|[V(C), X]| \leq 1$ and $[X, U_2] \subseteq \mathcal{L}$ by Lemma 2.13 and Claim 2.1, the weight is contributed by the triangles intersecting only with U_2. By Claims 2.1 and 2.2, only z is contained in some triangles with two vertices in U_2, and so the weight coming from such triangles is $\sum_{i \leq t} w(zv_1v_i, z) + \lambda(z)$, where $\lambda(z) = w(zv_2v_3, z)$ if $v_1v_2v_3$ is a triangle and $\lambda(z) = 0$ otherwise. Furthermore, by Claims 2.1 and 2.2, for any triangle containing two vertices in C and one vertex in U_2, only $w(zv_1z_i, z_i) \neq 0$ for $i \leq t'$ in the case $zv_1 \in \mathcal{H} \cup \mathcal{M}$. Therefore, the weight is

$$f_{U_2}(z) = \sum_{i \leq t} w(zv_1v_i, z) + \lambda(z) + \sum_{i=1}^{t'} w(zv_1z_i, z_i).$$

Claim 2.3. *If $zv_1 \notin \mathcal{H}$, then $f_{U_2}(z) \leq k-1$, and if $zv_1 \in \mathcal{H}$, then $f_{U_2}(z) \leq k - 1 + \frac{t'}{2} \leq \frac{3k}{2} - 2$ and zv_1 contributes at least $\frac{1}{2}(t'+1)$ to the loss of v_1.*

Proof. If $zv_1 \in \mathcal{L}$, then $\sum_{i\le t'} w(z_i zv_1, z_i) = 0$ and hence $f_{U_2}(z) = \sum_{i\le t} w(zv_1v_i, z) + \lambda(z) \le max\{3, k-1\}$ by Lemma 2.12.

If $zv_1 \in \mathcal{M}$, then $|N(zv_1)| = (t-1)+t' \le 2k-2$. By the definition of $w(T,\cdot)$ and Claim 2.1, we get $f_{U_2}(z) \le \frac{1}{2}(t-1)+\lambda+\frac{t'}{2}$. Note that $\lambda(z) = 0$ if $(t-1)+t' = 2k-2$, we have $f_{U_2}(z) \le k-1$.

If $zv_1 \in \mathcal{H}$, then $t > k$ and so $\lambda(z) = 0$. By Lemma 2.12, $\sum_{i\le t} w(zv_1v_i, z) \le k-1$ and so $f_{U_2}(z) \le k-1+\frac{t'}{2} \le \frac{3k}{2}-2$. Moreover, since the triangle $z_i zv_1$ satisfies $zz_i \in \mathcal{H}$ for $i \le t'$, by Observation 2.15, the edge zv_1 contributes at least $\frac{1}{2}(t'+1)$ to the loss of v_1. $\square$

By Lemma 2.13, either $|C| = 2k-1-2|X| \ge k+1$ or $|C| = k+1$. Moreover, since each edge of G is covered by triangles, if $X = \emptyset$, then G_u has no isolated vertices. That is, $G_u = C$ or $C \cup K_{k-1}$ if $X = \emptyset$.

We distinguish the following two cases separately according to $|C|$.

Case 1. $|C| = 2k-1-2|X|$

In this case, the structure of G are shown in Figure 2, where the thick edges are in $\mathcal{H}$ and the thin edges are in $\mathcal{L}$.

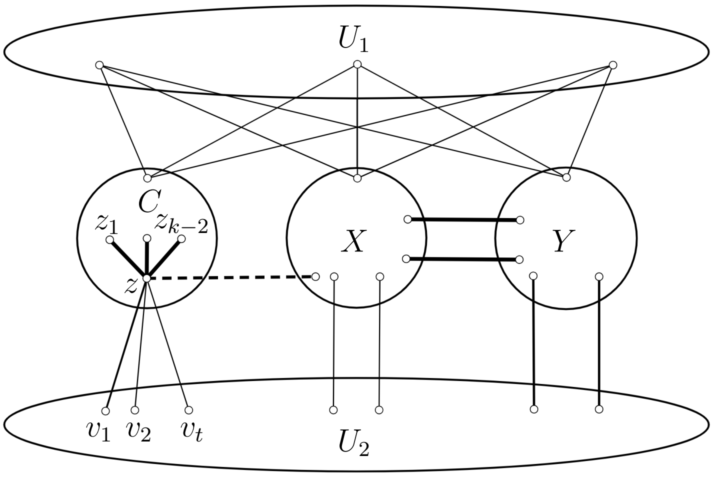


Figure 2. $N(u') = N(u)$ for any $u' \in U_1$

Case 1.1 $X = \emptyset$.

In this case, $G_u = C$ and $|C| = 2k-1$. If $U_2 = \emptyset$, then since $[U_1, G_u] \subseteq \mathcal{L}$ and $E(G_u) \subseteq \mathcal{H}$, by the definition of $w(T,\cdot)$, we have

$$\begin{aligned}\sum_{v\in V(G)} f(v) &= \sum_{v\in U_1} f(v) + \sum_{v\in V(C)} f(v)\\ &= \sum_{v\in U_1} f(v) + \mathcal{N}(K_3, C) = (n-2k+1)k\left(k-\frac{3}{2}\right) + \mathcal{N}(K_3, C).\end{aligned}$$

Because $|C| = 2k-1$ and $\pi(C) = (k-1,\dots,k-1,k-2)$, by Theorem 2.8,

$$\sum_{v\in V(G)} f(v) \le (n-2k+1)k\left(k-\frac{3}{2}\right) + 2\binom{k-1}{3} + \binom{k/2}{2} + \binom{k/2-1}{2}$$

and equality holds if and only if $G = \bar{K}_{n-2k+1} + H_k$, and so the result follows. Therefore, we may assume that $U_2 \neq \emptyset$.

In this case, we will try to transfer the weight $f_{U_2}(z)$ to the vertices in U_2 such that $f(v) \le k(k-\frac{3}{2})$ still valid after transferring.

(1) $zv_1 \in \mathcal{H}$.

By Claim 2.3, the edge zv_1 contributes at least $\frac{1}{2}(t'+1)$ to the loss of v_1. Transfer the weight $\sum_{i\le t'} w(z_izv_1, z_i)$ to v_1 to cover the loss of v_1 caused by the edge zv_1 and the weight $\sum_{i\le t} w(zv_1v_i, z)$ to $v_2,\dots,v_t(\lambda(z)=0$ in this case). After transferring, $f_{U_2}(z) = 0$, $f(v) \le k\left(k-\frac{3}{2}\right)$ is still valid for any $v \in U_2$ and $f(v_1) \le k\left(k-\frac{3}{2}\right) - \frac{1}{2}$. Therefore,

$$\sum_{v\in V(G)} f(v) < (n-2k+1)k\left(k-\frac{3}{2}\right) + \mathcal{N}(K_3, C). \tag{2.10}$$

(2) $zv_1 \in \mathcal{M}$.

By the definition of $w(T,\cdot)$, we have $w(z_izv_1, z_i) = \frac{1}{2}$ for $i \le t'$, and $w(zv_1v_i, z) \le \frac{1}{2}$ for $i \le t$. Let $|U_2| - t = t''$.

Suppose $t'' \ge t'$. Note that either $\lambda(z) = 1$, which implies $G[\{v_1,\dots,v_t\}]$ is a triangle and $v_2v_3 \in \mathcal{H}$, or $\lambda(z) \le \frac{1}{3}$. For the former case, $N(v_2v_3) - \{v_1, z\} \subset U_2 - \{v_1,v_2,v_3\}$ and hence $|U_2 - \{v_1,v_2,v_3\}| \ge 2k-3$ by Claim 2.1(1), which means $t'' > 2t'$. Thus we can transfer the weight $w(z_izv_1, z_i)$ of z_i $(i \le t')$ to the vertices in $U_2 - \{v_1,v_2,v_3\}$, the weight $\sum_{i\le t} w(zv_1v_i, z) + \lambda(z)$ to the vertices v_1, v_2, v_3 and some others in $U_2 - \{v_1, v_3, v_3\}$. For the latter case, we transfer the weights $w(z_izv_1, z_i)$ to the vertices in $U_2 - \{v_1,\dots,v_t\}$ and the weights $\sum_{i\le t} w(zv_1v_i, z) + \lambda(z)$ to the vertices of $\{v_1,\dots,v_t\}$. After the transferring, $f_{U_2}(z) = 0$, $f(v) \le k\left(k-\frac{3}{2}\right)$ is still valid for any $v \in U_2$, and (2.10) still holds.

Assume $t'' < t' \le k-2$. In this case, all edges of $G[\{v_1,\dots,v_t\}]$ are in $\mathcal{L}$ for otherwise we have $t'' \ge k-2$. Hence, $w(zv_1v_i, z) = \frac{1}{3}$ for $i \le t$ and $\lambda(z) \le \frac{1}{3}$. Recalling $\{z_1,\dots,z_{t'}\} \subseteq N(zv_1)$, by Claim 2.1(1), we get $N(v_i) \cap \{z_1,\dots,z_{t'}\} = \emptyset$ for $2 \le i \le t$, and if $v_iz'v$ is a triangle such that $z' \in V(C)$ and $v \in U_2$, then $v_iz'v = v_izv_1$, or $G[\{v_1,...,v_t\}] = K_3$ and $v_iz'v = v_2zv_3$. Thus, for $2 \le i \le t$, we have

$$\begin{aligned}
f(v_i) &= \sum_{z'z''\subseteq C} w\left(v_iz'z'', v_i\right) + \sum_{v',v''\in U_2} w\left(v_iv'v'', v_i\right) + w\left(v_iv_1z, v_i\right) + \eta \\
&\le \frac{1}{2}\Big(\left(2k-1-t'\right)(k-1) - t'(k-t')\Big) + \binom{t''+1}{2} + \frac{1}{3} + \frac{1}{3} \\
&= k\left(k-\frac{3}{2}\right) - \frac{1}{2}\Big(t'(k+t'-1) - (t''+1)t'' - 1\Big) + \frac{2}{3},
\end{aligned}$$

where $\eta = w(v_2zv_3, v_2)$ or $w(v_2zv_3, v_3)$ if v_2zv_3 is a triangle, and $\eta = 0$ otherwise.

Because the total loss of the vertices in U_2 is at least

$$\left(\frac{1}{2}\Big(t'(k+t'-1)-(t''+1)t''-1\Big)-\frac{2}{3}\right)(t-1)+\frac{1}{2}+\frac{t''}{2},$$

$$\geq \sum_{i\leq t} w(zv_1v_i,z)+\lambda(z)+\sum_{i\leq t'} w(z_izv_1,z_i)=\frac{t'}{2}+\frac{t}{3},$$

we can transfer these weights to vertices in U_2 and for $f_{U_2}(z)=0$, $f(v)\leq k\left(k-\frac{3}{2}\right)$ is still valid for any $v\in U_2$, and (2.10) still holds.

(3) $zv_1\in\mathcal{L}$.

In this case, we have $w(z_izv_1,z_i)=0$ for $i\leq t'$ by the definition of $w(T,\cdot)$. Since $zv_1\in\mathcal{L}$ and $\{v_2,...,v_t\}\subseteq N(zv_1)$, we have $t\leq k-1$. If $G[\{v_1,...,v_t\}]$ has an edge $v_iv_j\in\mathcal{H}$, then since $N(v_iv_j)-\{z,v_1\}\subseteq U_2-\{v_1,...,v_t\}$ by Claim 2.1(1), we have $|U_2-\{v_1,...,v_t\}|\geq 2k-3>k-1=t$, which implies $|U_2|>2t$. If $G[\{v_1,...,v_t\}]$ has no edge in $\mathcal{H}$, then $w(zv_1v_i,z)\leq\frac{1}{3}$ and $\lambda(z)\leq\frac{1}{3}$. Thus, by Lemma 2.13, we can we transfer the weight $\sum_{i\leq t} w(zv_1v_i,z)+\lambda(z)$ to the vertices of U_2 in the former case and to the vertices of $\{v_1,...,v_t\}$ in the latter case. So, $f_{U_2}(z)=0$, $f(v)\leq k\left(k-\frac{3}{2}\right)$ is still valid for any $v\in U_2$, and (2.10) still holds.

Thus, Theorem 2.8 and (2.10) hold.

Case 1.2 $X\neq\emptyset$.

Let $X=\{x_1,\ldots,x_s\}$ and $Y=V(G_u)-V(C)-X$. By Lemma 2.13, both X and Y are independent sets. Moreover, since $|C|=2k-1-2|X|=2k-1-2s\geq k+1$, we get that $s\leq\frac{k}{2}-1$.

By Claim 2.3, $f_{U_2}(z)\leq\frac{3k}{2}-2$. Moreover, if $\big|[V(C),X]\big|=1$, then $d_{G_u}(z)=k-1$. By Claim 2.1(1), $zv_1\in\mathcal{L}$ and $N_{U_2}(z)$ is an independent set. Therefore,

$$f_{U_2}(z)=0 \text{ if } \big|[V(C),X]\big|=1. \tag{2.11}$$

Consider the total loss of all the vertices in Y contributed by the edges in $[X,Y]$. Let xy be any edge with $x\in X$ and $y\in Y$, and $X_y\subseteq V(G_y)$ satisfy (2.1). Because $xy\in\mathcal{H}$, by Lemma 2.11, $x\in\mathcal{H}(y)\subseteq X_y$. Since X and Y are independent sets, $N(xy)\subseteq U_1\cup U_2$. Thus, if xyv is a triangle, then $xv\in\mathcal{L}$ by Claim 2.1(1), which implies $w(xyv,y)=0$, and so $\sum_{v\in N(xy)} w(xyv,y)=0$. By Observation 2.14, the edge xy contributes $k-1$ to the loss of y. Therefore, the total loss of all the vertices in Y, contributed by the edges in $[X,Y]$, is at least $\big|[X,Y]\big|\cdot(k-1)$.

Since X and Y are independent sets, $N_{U_2}(x)$ is an independent set by Claim 2.1(1), $\big|[V(C),X]\big|\leq 1$ and $[U_1,G_u]\subseteq\mathcal{L}$. So,

$$f(x)=\sum_{y\in N_Y(x)}\ \sum_{v\in N(xy)\cap U_2} w(xyv,x) \quad\text{for any } x\in X.$$

We try to transfer the weights $w(xyv,x)$ of $x\in X$ to the vertices y,v, such that the new weight of x is 0, and that of each other vertex remains no more than $k\big(k-\frac{3}{2}\big)$.

Fix an edge yv and let $N(yv)\cap X=\{x_1,\ldots,x_{s'}\}$. Then $x_iy\in\mathcal{H}$ and $x_iv\in\mathcal{L}$ for $1\leq i\leq s'$ by the arguments above. If $yv\in\mathcal{L}$, then $w(x_iyv,x_i)=0$, and so there is nothing to transfer. If $yv\notin\mathcal{L}$, then $w(x_iyv,x_i)=\frac{1}{2}$. Let $X'\subseteq V(G_v)$ satisfy (2.1).

If $yv \in \mathcal{H}$, then since $yv, x_iy \in \mathcal{H}$, by Observation 2.15, the edge yv contributes at least $\frac{1}{2}(s'+1)$ to the loss of v, and so we can transfer the weight $\sum_{i=1}^{s'} w(x_iyv, x_i) = \frac{s'}{2}$ to v to cover the loss caused by the edge yv.

Suppose $yv \in \mathcal{M}$. Note that $\sum_{v' \in N(x_iv)} w(x_ivv', v) \le (k-1) - \frac{1}{2}$ by (2.4) because $w(x_iyv, v) = \frac{1}{2}$. If $x_i \in X'$, then by Observation 2.14, the edge x_iv contributes $\frac{1}{2}$ to the loss of v. So we can transfer the weight $w(x_iyv, x_i)$ to v to cover the loss contributed by the edge vx_i. If $y \in X'$, then by (2.3) and Observation 2.14, the edge yv contributes $\frac{k}{6}$ to the loss of v, and $\frac{k}{12}$ to the loss of y by Observation 2.16, which means yv contributes at least $\frac{k}{4}$ to the total loss of y and v. Recalling $s' \le s \le \frac{k}{2} - 1$, we can transfer the weight $\sum_{i=1}^{s'} w(x_iyv, x_i) = \frac{s'}{2}$ to y, v to cover the loss contributed by the edge yv. If neither $x_i \in X'$ nor $y \in X'$, then the edge x_iy lies in some component C' of $G_v - X'$. Remember $\sum_{v' \in N(x_iv)} w(x_ivv', v) \le (k-1) - \frac{1}{2}$, the edge x_iv contributes at least $\frac{1}{4}$ to the loss of v. Since $yv \in \mathcal{M}$, by Observation 2.16, it contributes at least $\frac{k}{6}$ to the total loss of y, v. Thus, the total loss of y and v is at least $\frac{s'}{4} + \frac{k}{6} > \frac{s'}{2}$, and so we can transfer the weight $\sum_{i=1}^{s'} w(x_iyv, x_i) = \frac{s'}{2}$ to y, v such that the weights of y, v are still no more than $k(k - \frac{3}{2})$.

If $zv_1 \in \mathcal{H}$, then by Claim 2.3, we can transfer $\frac{t'}{2}$ from $f_{U_2}(z)$ to v_1 to cover the loss of v_1 caused by zv_1 such that $f_{U_2}(z) \le k-1$. After this possible transferring, we always have $f_{U_2}(z) \le k-1$. Thus, recalling the total loss of all the vertices in Y contributed by the edges in $[X, Y]$ is at least $\big|[X,Y]\big| \cdot (k-1)$, (2.11) and $f(x) = 0$ for each $x \in X$ after transferring and Theorem 2.9, the total weight of G is

$$\begin{aligned}\sum_{v \in V(G)} f(v) &\le (n-2k+1+s)k\left(k - \frac{3}{2}\right) + \mathcal{N}(K_3, C) + (k-1) - s(k-1)^2 \\ &\le (n-2k+1)k\left(k-\frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2}-1\right)^2 \\ &\quad + \frac{k^2}{4} + \left(-k^2 + \frac{7k}{2} - \frac{11}{3}\right)s + (2k-2)s^2 - \frac{4}{3}s^3 \\ &= (n-2k+1)k\left(k-\frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2}-1\right)^2 + \varphi(s,k).\end{aligned}$$

Because $1 \le s \le \frac{k}{2} - 1$, after an easy calculation, we get $\varphi(k,s) < 0$ except $\varphi(2,6) = 1$, $\varphi(1,4) = 3$ so the result holds if $(s,k) \ne (2,6), (1,4)$.

Now, consider the two exceptions. Let yv be any edge with $y \in Y$ and $v \in U_2$.

Suppose that $(s,k) = (2,6)$. Then $X = \{x_1, x_2\}$. Let $U = N_{U_2}(x_1) \cup N_{U_2}(x_2)$. Assume that $yv \in \mathcal{H} \cup \mathcal{M}$. If both x_1yv and x_2yv are triangles, then since $N_{U_2}(x_1)$ and $N_{U_2}(x_2)$ are independent sets by Claim 2.1, $N(yv) - \{x_1, x_2\} \subseteq U_2 - U$. Clearly, $|U_2 - U| \ge |N(yv) - \{x_1, x_2\}| \ge k-2$. If $|N(yv) \cap \{x_1, x_2\}| \le 1$ for any $yv \in \mathcal{H} \cup \mathcal{M}$, then since $k = 6$, yv contributes at least $\frac{1}{2}$ to the loss of y by Observations 2.15 and 2.16, and so we can transfer $w(x_iyv, x_i)$ only to y to cover the loss of y contributed by yv if x_iyv is a triangle, such that $f(x_i) = 0$ for $i = 1, 2$ after the possible transferring. Moreover, since $|N(yv) \cap U_2| \ge k-1$, $|U_2| \ge k$. Thus, note that no weights are transferred to the vertices in $U_2 - U$ in the former case and in U_2 in the latter case,

U_2 has at least $k-2$ vertices whose weights are at most $k(k-\frac{3}{2})-\frac{1}{2}$ after transferring, which implies

$$\sum_{v\in V(G)} f(v) \leq (n-2k+1)k\left(k-\frac{3}{2}\right)+2\binom{k-1}{3}+\left(\frac{k}{2}-1\right)^2+\varphi(s,k)-\frac{1}{2}(k-2)$$
$$< (n-2k+1)k\left(k-\frac{3}{2}\right)+2\binom{k-1}{3}+\left(\frac{k}{2}-1\right)^2.$$

If $[Y,U_2]\subseteq \mathcal{L}$, then $f(x_i)=0$ for $i=1,2$ by Claim 2.1 and the definition of $w(T,\cdot)$. Since $\varphi(2,6)=1$, if $|U_2|\geq 3$, then replace $\frac{1}{2}(k-2)$ with $\frac{1}{2}\cdot 3$ in the above inequality, we get the desired result. If $|U_2|\leq 2$, then $f(y)\leq 1$ for any $y\in Y$. It is easy to see the total weight of G is less than the expected number.

Suppose that $(s,k)=(1,4)$. Then $X=\{x_1\}$. We will transfer the weight $f(x_1)$ and $f_{U_2}(z)$ to other vertices in a bit different way. Note that $f(x_1)$ comes from the triangles x_1yv with $yv\in\mathcal{H}\cup\mathcal{M}$. Since $x_1y\in\mathcal{H}$ for any $y\in Y$, by Lemma 2.11, $x_1\in X_y$. If $yv\in\mathcal{H}$, then by Observation 2.15, the edge yv contributes $\frac{1}{2}$ to the loss of y. Assume that $yv\in\mathcal{M}$. If $v\in X_y$, then by (2.3) and Observation 2.14, yv contributes at least $\frac{k}{6}=\frac{2}{3}>\frac{1}{2}$ to the loss of y. If v lies in a component C' of G_y-X_y, then $x_1\in X_y$ and (2.1) imply $|C'|\leq 2k-3$, and so

$$\sum_{v'\in N(yv)\cap C'} w(yvv',y)\leq \frac{1}{2}(k-2)+\frac{1}{3}\Big(2k-4-(k-2)\Big)=(k-1)-\left(\frac{k}{6}+\frac{2}{3}\right),$$

and so yv contributes $\frac{1}{2}\left(\frac{k}{6}+\frac{2}{3}\right)>\frac{1}{2}$ to the loss of y by Observation 2.16. Therefore, the edge yv contributes at least $\frac{1}{2}$ to the loss of y. Transfer the weight $w(x_1yv,x_1)=\frac{1}{2}$ to y such that the new weight of y is no more than $k\left(k-\frac{3}{2}\right)-(k-1)$, where the loss $k-1$ is contributed by the edge x_1y. Transfer the weight $f_{U_2}(z)$ to the vertices in U_2 in the same way used in Case 1.1. After the transferring, the weight of G satisfies

$$\sum_{v\in V(G)} f(v) < (n-2k+1)k\left(k-\frac{3}{2}\right)+2\binom{k-1}{3}+\left(\frac{k}{2}-1\right)^2,$$

and so the proof of Case 1 is complete.

Case 2. $|C|=k+1$.

In this case, $X=\emptyset$ and $G_u=C\cup K_{k-1}$. Since $\pi(C)=(k-1,...,k-1,k-2)$ by Lemma 2.13 and $|C|=k+1$, C is the complement of $\frac{1}{2}(k-2)K_2\cup P_3$, and so we have

$$\mathcal{N}(K_3,G_u)=\binom{k-1}{3}+\binom{k+1}{3}-\frac{1}{2}(k-2)(k-1)-2(k-2)-1.$$

Set $V(K_{k-1})=\{p_1,\ldots,p_{k-1}\}$. We first discuss some properties of these vertices.

Claim 2.4. *If $q_1,q_2\in U_2$ such that $p_iq_1,p_iq_2\in\mathcal{H}\cup\mathcal{M}$, then*
(1) $G[N_{U_2}(p_i)]$ consists of a triangle $q_1q_2q_3$ and some isolated vertices;
(2) $|N(p_iq_1)|=|N(p_iq_2)|=k$ and $\{p_1,...,p_{k-1}\}\subseteq N(q_1)\cap N(q_2)$;
(3) If $v\in U_2$ such that $p_jv\in\mathcal{H}\cup\mathcal{M}$, then $v\in\{q_1,q_2,q_3\}$. Moreover, if $v=q_3$, then $|N(p_jq_s)|=k$ for $1\leq j\leq k-1$ and $1\leq s\leq 3$.

Proof. (1) Since $d_{G_u}(p_i) = k-2$ and $p_iq_1, p_iq_2 \in \mathcal{H}\cup\mathcal{M}$, by Claim 2.1(2), $G[N_{U_2}(p_i)]$ is a triangle containing q_1, q_2, say $q_1q_2q_3$, together with some isolated vertices.

(2) Since $|N(p_iq_1)| \geq k$ and $N(p_iq_1)\cap U_2 = \{q_2, q_3\}$ by (1), $\{p_1, ..., p_{k-1}\} \subseteq N(q_1)$. By the symmetry of q_1 and q_2, $\{p_1, ..., p_{k-1}\} \subseteq N(q_2)$, and so the result follows.

(3) Suppose $v \notin \{q_1, q_2, q_3\}$. By Claim 2.1(2) and (2), $G[N_{U_2}(p_j)]$ is a triangle vq_1q_2 together with some isolated vertices. Because $|N(p_jv)| \geq k$ and $N(p_jv)\cap U_2 = \{q_1, q_2\}$, we have $\{p_1, ..., p_{k-1}\} \subseteq N(v)$. Thus, $v \in N(p_iq_1)$ and hence $|N(p_iq_1)| \geq k+1$ which contradicts (2). Therefore, $v \in \{q_1, q_2, q_3\}$. Moreover, if $v = q_3$, then since $|N(p_jq_3)| \geq k$ and $N(p_jq_3)\cap U_2 = \{q_1, q_2\}$, we can deduce $\{p_1, ..., p_{k-1}\} \subseteq N(q_3)$. After an easy check, we get that $|N(p_jq_s)| = k$ for $1 \leq j \leq k-1$ and $1 \leq s \leq 3$. ☐

By Lemma 2.11, we have $|(\mathcal{H}(p_i)\cup\mathcal{M}(p_i))\cap U_2| \leq 3$ for all $1 \leq i \leq k-1$. Suppose $|(\mathcal{H}(p_i)\cup\mathcal{M}(p_i))\cap U_2| = 3$ for some i. By Claim 2.4, $G[N_{U_2}(p_i)]$ is a triangle $q_1q_2q_3$ together with some isolated vertices and $p_iq_1, p_iq_2, p_iq_3 \in \mathcal{M}$. Furthermore, we have $p_jq_s \in E(G)$ and $|N(p_jq_s)| = k$ for $1 \leq j \leq k-1$ and $1 \leq s \leq 3$, and $q_1q_2, q_1q_3, q_2q_3 \in \mathcal{H}\cup\mathcal{M}$. Because

$$\sum_{v\in N(p_iq_s)} w(p_iq_sv, q_s) = \frac{k}{3} \leq \frac{k}{2} = k-1-\left(\frac{k}{2}-1\right),$$

the edge p_iq_s contributes at least $\frac{k}{4}-\frac{1}{2}$ to the loss of q_s by Observation 2.16. Hence, all the edges p_jq_s, $1 \leq j \leq k-1$ and $1 \leq s \leq 3$, contribute at least $3(k-1)\left(\frac{k}{4}-\frac{1}{2}\right)$ to the total loss of q_1, q_2 and q_3. On the other hand, note that $G[N_{U_2}(p_j)]$ is the triangle $q_1q_2q_3$ together with some isolated vertices by Claim 2.4(1) and $[p_j, U_2-\{q_1,q_2,q_3\}] \subseteq \mathcal{L}$ by Claim 2.4(3). So, the weight of p_j contributed by the triangles not in K_{k-1} is

$$\sum_{1\leq r<s\leq 3} w(p_iq_rq_s, p_i) + \sum_{p_j\neq p_i}\sum_{1\leq s\leq 3} w(p_ip_jq_s, p_i) = 3\cdot\frac{1}{3} + 3(k-2)\cdot\frac{1}{3} = k-1.$$

By Claim 2.3, $f_{U_2}(z) \leq \frac{3k}{2}-2$. Therefore, the total weight of G is at most

$$\begin{aligned}&(n-2k)k\left(k-\frac{3}{2}\right) + \mathcal{N}(K_3, G_u) + \left(\frac{3k}{2}-2\right) + (k-1)^2 - 3(k-1)\left(\frac{k}{4}-\frac{1}{2}\right)\\ <&(n-2k+1)k\left(k-\frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2}-1\right)^2,\end{aligned}$$

a contradiction. So we assume that $|(\mathcal{H}(p_i)\cup\mathcal{M}(p_i))\cap U_2| \leq 2$ for $1 \leq i \leq k-1$.

Fix p_i and let $N_{U_2}(p_i) = \{q_1, \ldots, q_r\}$. By Claim 2.1, we may assume that $G[N_{U_2}(p_i)]$ is a star with the center q_1 or a triangle $q_1q_2q_3$, and some isolated vertices. Let

$$f_{U_2}(p_i) = \sum_{v,v'\in U_2} w(p_ivv', p_i) + \sum_{p_j\neq p_i}\sum_{v\in N(p_ip_j)\cap U_2} w(p_ip_jv, p_j).$$

It is clear that $\sum_{i=1}^{k-1} f_{U_2}(p_i)$ is the total weight of $V(K_{k-1})$ contributed by the triangles not contained in K_{k-1}. We will complete the proof by showing that $f_{U_2}(p_i) \leq \frac{3k}{4}-\frac{1}{2}$ and $f_{U_2}(z) < \frac{3k}{4}-\frac{1}{2}$, after some appropriate weight transferring.

If $|(\mathcal{H}(p_i) \cup \mathcal{M}(p_i)) \cap U_2| = 2$, say $p_iq_1, p_iq_2 \notin \mathcal{L}$, then by Claim 2.4, $N_{U_2}(p_i)$ is a triangle $q_1q_2q_3$ together with some isolated vertices, $|N(p_iq_1)| = |N(p_iq_2)| = k$ and so

$$f_{U_2}(p_i) = \sum_{v,v' \in \{q_1,q_2,q_3\}} w(p_ivv', p_i) + \sum_{p_j \neq p_i} w(p_ip_jq_1, p_j) + \sum_{p_j \neq p_i} w(p_ip_jq_2, p_j).$$

Since

$$\sum_{v \in N(p_iq_s)} w(p_iq_sv, q_s) \leq \frac{k}{2} = (k-1) - \left(\frac{k}{2} - 1\right) \text{ for } s = 1, 2,$$

the edge p_iq_s contributes at least $\frac{k}{4} - \frac{1}{2}$ to the loss of q_s for $s = 1, 2$ by Observation 2.16. Because $\sum_{p_j \neq p_i} w(p_ip_jq_s, p_j) \leq \frac{1}{2}(k-2)$, transfer $\frac{k}{4} - \frac{1}{2}$ from $\sum_{p_j \neq p_i} w(p_ip_jq_s, p_j)$ to q_s for $s = 1, 2$. After transferring, we have

$$f_{U_2}(p_i) \leq \frac{3}{2} + 2 \cdot \frac{1}{2}(k-2) - 2\left(\frac{k}{4} - \frac{1}{2}\right) = \frac{k+1}{2} \leq \frac{3k}{4} - \frac{1}{2}. \tag{2.12}$$

Now, let $|(\mathcal{H}(p_i) \cup \mathcal{M}(p_i)) \cap U_2| \leq 1$. Assume $\mathcal{H}(p_i) \cup \mathcal{M}(p_i) \subseteq \{q_1\}$ by Claim 2.1(2). Because $p_ip_j \in \mathcal{H}$ and $p_iq_s \in \mathcal{L}$, $w(p_ip_jq_s, p_j) = 0$ for $2 \leq s \leq r$ and hence

$$f_{U_2}(p_i) = \sum_{j=2}^{r} w(p_iq_1q_j, p_i) + \lambda(p_i) + \sum_{p_j \neq p_i} w(p_ip_jq_1, p_j),$$

where $\lambda(p_i) = w(p_iq_2q_3, p_i)$ if $q_1q_2q_3$ is a triangle and $\lambda(p_i) = 0$ otherwise. Using the same proof as that of Claim 2.3, we have

Claim 2.5. *$f_{U_2}(p_i) \leq k-1$ if $p_iq_1 \notin \mathcal{H}$, and $f_{U_2}(p_i) \leq k - 1 + \frac{\ell}{2} \leq \frac{3k}{2} - 2$ and p_iq_1 contributes at least $\frac{1}{2}(\ell+1)$ to the loss of q_1 if $p_iq_1 \in \mathcal{H}$, where $|N(p_iq_1) \cap V(K_{k-1})| = \ell$.*

In order to show $f_{U_2}(p_i) \leq \frac{3k}{4} - \frac{1}{2}$ in this case and $f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}$, we need to consider the structure of $G[U_2]$.

If $vv' \in \mathcal{M}$ is an edge in $G[U_2]$, then by Observation 2.16, vv' contributes $\frac{k}{12}$ to the loss of v and v', respectively, that is, vv' contributes $\frac{k}{6}$ to the total loss of vertices in U_2. On the other hand, by Claims 2.3 and 2.5, we can transfer some weight from $f_{U_2}(z)$ and $f_{U_2}(p_i)$ to v_1 and q_1, respectively, such that $f_{U_2}(z) \leq k-1$ and $f_{U_2}(p_i) \leq k-1$, and $f(v_1) \leq k\left(k - \frac{3}{2}\right)$ and $f(q_1) \leq k\left(k - \frac{3}{2}\right)$ still hold. This together with (2.12) implies that after transferring some weights to the vertices in U_2, the total weight in $\sum_{v \in V(G_u)} f(v)$ coming from the triangles not in G_u is at most $k(k-1)$. Therefore, if $G[U_2]$ has $\frac{3k}{2} - 2$ edges in $\mathcal{M}$, then we have

$$\begin{aligned}\sum_{v \in V(G)} f(v) &\leq (n-2k)k\left(k - \frac{3}{2}\right) + \mathcal{N}(K_3, G_u) + k(k-1) - \left(\frac{3k}{2} - 2\right) \cdot \frac{k}{6} \\ &< (n-2k+1)k\left(k - \frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2,\end{aligned}$$

a contradiction. Hence, $G[U_2]$ contains at most $\frac{3k}{2} - 3$ edges in $\mathcal{M}$. Moreover, we have

Claim 2.6. *Let $qq' \in \mathcal{H}$ be an edge in $G[U_2]$. If $\mathcal{M}(q) \cap \mathcal{M}(q') \cap \{p_1, ..., p_{k-1}\} = \emptyset$, then qq' contributes $\frac{3k}{8}$ to the total loss of q and q'. Furthermore, $G[U_2]$ contains at most $\frac{3k}{2} - 2$ edges in $\mathcal{H} \cup \mathcal{M}$.*

Proof. By Claim 2.2 and the assumption, $\mathcal{M}(q) \cap \mathcal{M}(q') \cap \{z, p_1, ..., p_{k-1}\} = \emptyset$. Noting that $G[U_2]$ has at most $\frac{3k}{2} - 3$ edges in $\mathcal{M}$, we have $|\mathcal{M}(q)| + |\mathcal{M}(q')| \leq k + \frac{3k}{2} - 3$. Thus, by (2.2) and Lemma 2.11(ii), we get

$$\begin{aligned}&\sum_{v \in N(qq')} w(qq'v, q) + \sum_{v \in N(qq')} w(q'qv, q') \\ &\leq \frac{1}{2} \left(|\mathcal{H}(q) - \{q'\}| + |\mathcal{M}(q)| + |\mathcal{H}(q') - \{q\}| + |\mathcal{M}(q')| \right) \leq 2(k-1) - \frac{3k}{8},\end{aligned}$$

and so the conclusion follows by Observation 2.14.

In addition, recall $|(\mathcal{H}(p_j) \cup \mathcal{M}(p_j)) \cap U_2| \leq 2$ for all $1 \leq j \leq k-1$, by Claim 2.4, $G[U_2]$ has at most one edge $q_1'q_2'$ such that $\mathcal{M}(q_1') \cap \mathcal{M}(q_2') \cap \{p_1, ..., p_{k-1}\} \neq \emptyset$. For any other $\mathcal{H}$-edge qq' in $G[U_2]$, qq' contributes at least $\frac{3k}{8} > \frac{k}{6}$ to the total loss of q and q', which, together with the possible edge $q_1'q_2'$, implies $G[U_2]$ contains at most $\frac{3k}{2} - 2$ edges in $\mathcal{H} \cup \mathcal{M}$. □

Now, let us re-consider $f_{U_2}(z)$ and $f_i = f_{U_2}(p_i)$ based on Claim 2.6. For convenience, let $a \in \{z, p_1, \ldots, p_{k-1}\}$, $N_{U_2}(a) = \{b_1, \ldots, b_m\}$ is a star with center b_1 or a triangle $b_1b_2b_3$ in $G[N_{U_2}(a)]$, $(\mathcal{H}(a) \cup \mathcal{M}(a)) \cap U_2 \subseteq \{b_1\}$ and $N(ab_1) \cap V(G_u) = \{a_1, \ldots, a_\ell\}$. Recall the expressions of $f_{U_2}(z)$ and $f_i = f_{U_2}(p_i)$, we have

$$f_{U_2}(a) = \sum_{j=2}^{m} w(ab_1b_j, a) + \lambda(a) + \sum_{j=1}^{\ell} w(ab_1a_j, a_j).$$

If $ab_1 \in \mathcal{H}$, then since $aa_i \in \mathcal{H}$, we can transfer the weight $w(ab_1a_i, a_i)$ to b_1 to cover the loss caused by the edge ab_1 by Observation 2.15. For the weight $w(ab_1b_j, a)$, we have $w(ab_1b_j, a) \leq \frac{1}{2}$ with equality only if $b_1b_j \in \mathcal{H} \cup \mathcal{M}$. Thus, by Claim 2.6, after transferring, we have

$$f_{U_2}(a) = \sum_{j=2}^{m} w(ab_1b_j, a) + \lambda(a) \leq max\left\{2, \frac{3k}{4} - 1\right\} < \frac{3k}{4} - \frac{1}{2}.$$

Assume $ab_1 \in \mathcal{M}$. Consider $w(ab_1b_j, a)$. If $b_1b_j \in \mathcal{H}$, then $w(ab_1b_j, a) = \frac{1}{2}$, and b_1b_j contributes $\frac{3k}{8}$ to the total loss of b_1 and b_j by Claim 2.6. Since $a \in \{z, p_1, \ldots, p_{k-1}\}$, there are at most k such triangles, and so we can transfer $\frac{3}{8}$ of each $w(ab_1b_j, a)$ to b_1 and b_j to cover the total loss of b_1 and b_j caused by the edge b_1b_j. If $b_1b_j \in \mathcal{M} \cup \mathcal{L}$, then $w(ab_1b_j, a) = \frac{1}{3}$ and ab_1 contributes $\frac{k}{12}$ to the loss of b_1 by Observation 2.16. Thus, we can transfer $\frac{1}{12}$ of each $w(ab_1b_j, a)$ to cover the loss of b_1 caused by the edge ab_1. After transferring, we have $w(ab_1b_j, a) \leq \frac{1}{4}$ and so

$$f_{U_2}(a) \leq \frac{1}{4}|N(ab_1) \cap U_2)| + \frac{1}{2}|N(ab_1) \cap V(G_u)| \leq \frac{3k}{4} - 1 < \frac{3k}{4} - \frac{1}{2}.$$

If $ab_1 \in \mathcal{L}$, then $w(ab_1a_j, a_j) = 0$ for $1 \leq i \leq \ell$. If $b_1b_j \in \mathcal{H}$, then $w(ab_1b_j, a) = 1$. By Claim 2.6, we can transfer $\frac{3}{8}$ to cover the total loss of b_1 and b_j caused by the edge b_1b_j, with at most one exceptional edge in $G[U_2]$. If $b_1b_j \in \mathcal{M} \cup \mathcal{L}$, then $w(ab_1b_j, a) = \frac{1}{3}$. After transferring, we have $w(ab_1b_j, a) \leq \frac{5}{8}$ with at most one exception and so

$$f_{U_2}(a) = \sum_{j=2}^{m} w(ab_1b_j, a) + \lambda(a) + \leq \frac{5}{8}(k-2) + 1 < \frac{3k}{4} - \frac{1}{2}.$$

By the three inequalities above, we have $f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}$. Moreover, combining the three inequalities with (2.12), we have $f_{U_2}(p_i) \leq \frac{3k}{4} - \frac{1}{2}$. Thus, after appropriate weight transferring, we have $f_{U_2}(z) + \sum_{i=1}^{k-1} f_i < k(\frac{3k}{4} - \frac{1}{2})$. Hence, the total weight of G is

$$\begin{aligned}\sum_{v \in V(G)} f(v) &< (n-2k)k\left(k - \frac{3}{2}\right) + \mathcal{N}(K_3, G_u) + k\left(\frac{3k}{4} - \frac{1}{2}\right) \\ &\leq (n-2k+1)k\left(k - \frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2.\end{aligned}$$

The proof of Theorem 2.4 is complete. □

2.5 Conclusion

Theorem 2.4 is proved for $n \geq 4k^3$, but notice that the statement does not hold for small n. For example, take at most five disjoint copies of K_{2k} then the number of copies K_3 is more than the extremal number in the theorem. It would be nice to determine the sharp bound for n when this generalized Turán number is correct.

It is natural to ask what happens if we count larger cliques. Gerbner [40] showed that $\mathrm{ex}(n, K_r, F_k) = O(n)$ for every k and r, but the constants in the upper bound are large. We conjecture that the extremal graph for $\mathrm{ex}(n, K_r, F_k)$ is still $\bar{K}_{n-v(H)} + H$, where H is a graph with $V(H) = k-1, \Delta(H) = k-1$.

Let H^T be the graph obtained by replacing each edge of H with a triangle, e.g., the friendship graph can be considered as a S_k^T. So it is also interesting to ask what if we replace each edge of any other graph H with a triangle? For example, consider the extremal function $\mathrm{ex}(n, K_3, P_k^T)$, $\mathrm{ex}(n, K_3, C_k^T)$.

We have determined the largest number of triangles in F-free graphs when F is a friendship graph, but not when F is an extended friendship graph. Alon and Shikhelman [3] showed that in that case $c_1|V(F)|^2 n \leq \mathrm{ex}(n, K_3, F) \leq c_2|V(F)|^2 n$ for absolute constants c_1 and c_2. Better bounds were obtained for some forests, including exact results for stars [13], paths [79] and forests consisting only of path components of order different from 3 [93].

Chapter 3

Stability from symmetrization in generalized Turán problems

3.1 Introduction

As mentioned in the first chapter, the first generalized Turán result is due to Zykov [95], who showed that $\mathrm{ex}(n, K_k, K_{r+1}) = \mathcal{N}(K_k, T(n,r))$. Let us briefly describe his proof. Given two vertices u and v of a graph G, we say that we *symmetrize* u to v if we delete all the edges incident to u and add all the edges of form uw where w is a neighbor of v. In other words, we change the neighborhood of u to the neighborhood of v.

If u and v are non-adjacent and G is K_{r+1}-free, then the resulting graph G' is also K_{r+1}-free. Indeed, a copy of K_{r+1} would contain u as all the new edges are incident to u. Then this copy cannot contain v, as v is not adjacent to u. But then deleting u and adding v to this copy, we find another copy of K_{r+1} that is also present in G, contradicting our assumption.

The other key property is that if u is contained in x copies of K_k and v is contained in y copies of K_k, then this symmetrization removes x and adds y copies of K_k. Therefore, by applying the symmetrization if $x \leq y$, the total number of copies of K_k does not decrease.

The proof goes on by applying several such symmetrization steps. One can show that this process terminates, i.e., at one point we arrive to a graph where symmetrization cannot change the graph. This means that non-adjacent vertices have the same neighborhood, thus being non-adjacent is an equivalence relation, i.e., the resulting graph is complete multipartite (obviously with at most r parts). We omit the proof that the process terminates, as it is not relevant to our work. We also omit for the same reason showing that among complete multipartite graphs with at most r parts, $T(n,r)$ contains the most copies of K_k.

Győri, Pach and Simonovits [61] generalized this argument, showing that for any complete multipartite graph H, $\mathrm{ex}(n, H, K_{r+1}) = \mathcal{N}(H, T)$ for some complete r-partite graph T (which is not necessarily the Turán graph). One could think that this is the limit of Zykov's symmetrization argument in this topic, since only cliques have the property that symmetrization cannot create them, and only complete multipartite graphs H have the property that symmetrizing either u to v or v to u does not decrease the number of copies of H. However, some more advanced applications have appeared in the literature. One can symmetrize only to vertices v satisfying some property that ensures that no F will be created [49, 76]. Another example is [39], where the neighborhood of u is changed to the common neighborhood of a set of vertices.

Here, we will present three stability results on generalized Turán problems that are obtained by using symmetrization. Stability refers to the phenomenon that an F-free n-vertex graph with almost $\mathrm{ex}(n, H, F)$ copies of H is close to the extremal graph. There are different kinds of stability, depending on what "almost" and "close" mean in the previous sentence. Here, we deal with one specific kind of stability.

The *edit distance* of two n-vertex graphs G and G' is the number of edges needed to be deleted and added to G in order to obtain a graph isomorphic to G'. Let $h = |V(H)|$. Given a graph F with chromatic number $r+1$ and another graph H with chromatic number at most r, we say that H is *F-Turán-stable* if the following holds. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if an n-vertex F-free graph G contains at least $\mathrm{ex}(n, H, F) - \delta n^h$ copies of H, then the edit distance of G and $T(n, r)$ is at most εn^2. We say that H is *weakly F-Turán-stable* if the same holds with $T(n, r)$ replaced by some appropriate complete r-partite graph T.

Using these notions, the classical Erdős-Simonovits stability theorem [24, 25, 90] shows that K_2 is K_{r+1}-Turán-stable for every $r \geq 2$. The first stability result concerning generalized Turán problems is due to Ma and Qiu [81], who showed that K_k is K_{r+1}-Turán-stable for every $r \geq k$. Turán-stable graphs were introduced by Gerbner in [39]. It was shown in [39] that if H is (weakly) F-Turán-stable, it often implies that H is (weakly) F'-Turán-stable for some other graphs F'.

Another important feature of these notions is that they often imply exact results on $\mathrm{ex}(n, H, F)$. We say that H is *F-Turán-good* if for each sufficiently large n we have $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T(n, r))$. We say that H is *weakly F-Turán-good* if for each sufficiently large n we have that $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T)$ for some complete r-partite graph T. It was shown in [39] that if H is weakly F-Turán-stable and F has a color-critical edge (an edge whose deletion decreases the chromatic number), then $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T)$ for some complete r-partite graph T. Some other exact results were shown in [38].

Gerbner in [39] claimed that a result of Liu, Pikhurko, Sharifzadeh and Staden [77] implies that complete multipartite graphs are weakly K_{r+1}-Turán-stable. It turned out that this does not follow from their result. Even though our problem is in many sense simpler than theirs, it still does not fit into their general framework. Here, we present a proof of this statement. Note that the case $H = C_4$ was proved in [68].

Theorem 3.1. *Let H be a complete k-partite graph and $r \geq k$. Then H is weakly K_{r+1}-Turán-stable.*

Morrison, Nir, Norin, Rzażewski and Wesolek [83] showed (proving a conjecture of Gerbner and Palmer [47]) that for every graph H, if r is large enough, then H is K_{r+1}-Turán-good. They ask whether this can imply K_{r+1}-Turán-stability. We partly answer this question in the affirmative.

Theorem 3.2. *For every graph H, if $r \geq 300h^9$, then H is K_{r+1}-Turán-stable.*

Finally, we obtain a theorem of a similar flavor.

Theorem 3.3. *For every bipartite graph H, if $k > (2h)^{h+1}$, then H is weakly C_{2k+1}-Turán-stable.*

We note that the above theorem implies that H is weakly C_{2k+1}-Turán-good, since C_{2k+1} has a color-critical edge. Gerbner [36] showed that for every k, there exists a graph that is C_{2k+1}-Turán-good and not weakly $C_{2\ell+1}$-Turán-good for any $\ell < k$.

Let us briefly summarize what is implied by our theorems. First, the F-Turán-stability implies F'-Turán-stability for other graphs, as observed in [39] in the case $F = K_{r+1}$. We present the following more general version. We will be using the graph removal lemma [85] in the proof of the next proposition as well as that of Theorem 3.3. Recall that it states that if a graph G contains $o(n^{|V(H)|})$ copies of a graph H, then by removing $o(n^2)$ edges we can remove all the copies of H from G.

Proposition 3.4. *Let $\chi(F) = \chi(F')$ and assume that F' is contained in a blow-up of F. If H is weakly F-Turán-stable, then H is weakly F'-Turán-stable.*

Proof. Let G' be an F'-free n-vertex graph with at least $\mathrm{ex}(n, H, F') - o(n^h)$ copies of H. Let $\chi(F) = \chi(F') = r + 1$. Since H is weakly F-Turán-stable, for any F-free graph G_0 with $\mathrm{ex}(n, H, F)$ copies of H, there is a complete r-partite graph T with edit distance $o(n^2)$ to G_0. This implies that $\mathcal{N}(H, T) = \mathrm{ex}(n, H, F) - o(n^h)$. Note that $\mathrm{ex}(n, H, F') \geq \mathcal{N}(H, T)$ because T is F'-free.

By a result of Alon and Sikhelman [3] we have $\mathrm{ex}(n, F, F') = o(n^{|V(F)|})$. By the removal lemma, there are $o(n^2)$ edges in G' contained in some copy of F. We delete those edges to obtain an F-free graph G. Those edges are in $o(n^h)$ copies of H, thus $\mathcal{N}(H, G) = \mathcal{N}(H, G') - o(n^h) \geq \mathrm{ex}(n, H, F') - o(n^h) - o(n^h) \geq \mathcal{N}(H, T) - o(n^h) \geq \mathrm{ex}(n, H, F) - o(n^h)$ using the already established bounds. Since H is weakly F-Turán-stable, there is a complete r-partite graph T' with edit distance $o(n^2)$ to G. Then the edit distance of T' and G' is also $o(n^2)$, completing the proof. ☐

Note that the above proposition remains true if we replace weakly Turán-stability by Turán stability, since if we replace both T and T' by the Turán graph in the above proof, all the arguments are still valid. There are multiple ways in [36] and [38] to obtain new (weakly) F-Turán-stable graphs from other F-Turán-stable graphs. Our results give new building blocks to those methods. Finally, as we have mentioned, our new stability results give new exact results using theorems from [36] and [38].

The rest of this chapter is organized as follows. Each theorem is proved in its own section, and we finish the chapter with a conclusion providing some remarks and potential further work.

3.2 Complete multipartite graphs

In this section we prove Theorem 3.1. Let us denote the edit distance between two graphs G_1 and G_2 on the same vertex set by $\Delta_1(G_1, G_2) := |E(G_1) \triangle E(G_2)|$. Let H

be a complete k-partite graph on h vertices and $k \le r$. We first prove a lemma that will be used in the proof.

Lemma 3.5. *For every $r \ge k$ there is $\gamma > 0$ and $\zeta > 0$ such that the following holds. For sufficiently large n, if P is a complete s-partite K_{r+1}-free graph on n vertices such that $\mathcal{N}(H,P) \ge \mathrm{ex}(n,H,K_{r+1}) - \zeta \cdot n^h$, then $s = r$ and the size of every part of P is at least γn.*

Proof. Let $1/s \gg \gamma \gg \zeta > 0$. Assume that there is some part V_j with $|V_j| < \gamma n$. Let V_1 be the largest part, $|V_1| = pn$, thus $p \ge 1/s$. Let $H[A \cup B]$ be an induced bipartite subgraph of H with $|A| = a$ and $|B| = b$ such that $A \subseteq V_1$ and $B \subseteq V_j$. Let us move $\lfloor pn/2 \rfloor$ vertices of V_1 to V_j. Then the number of copies of such bipartite subgraphs between V_1 and V_j increases by at least

$$\binom{\lfloor pn/2 \rfloor}{a}\binom{\lceil pn/2 \rceil}{b} - \left[\binom{\gamma n}{a}\binom{pn}{b} + \binom{pn}{a}\binom{\gamma n}{b}\right] \ge c_1 n^{a+b} - \gamma c_2 n^{a+b} \ge \frac{c_1}{2} n^{a+b},$$

where c_1 and c_2 are constants. The last inequality holds because c_1 depends on p, a and b but not γ, and $\gamma \ll p$. Note that if a copy of H does not intersect both V_1 and V_j, then their number will not be affected. Consequently, the number of copies of H increases by at least cn^h (where c is a constant), a contradiction.

This argument proves $s = r$, too. Since we know $s \le r$ as P is K_{r+1}-free, and if $s < r$ we may think of $r - s$ parts being empty, so contain $< \gamma n$ vertices. ☐

Proof of Theorem 3.1. Let $\gamma \gg \alpha \gg \beta \gg \mu \gg \varepsilon \gg \delta$ be a sequence of positive numbers such that each depends on H, r and the previous numbers in the sequence, and is sufficiently small. We will also assume that n is sufficiently large.

Assume the theorem does not hold, and hence there is a graph G with $|V(G)| = n$, for a sufficiently large n, such that $\mathcal{N}(H,G) \ge \mathrm{ex}(n,H,K_{r+1}) - \delta n^h$ while $\Delta_1(G,F) > \varepsilon n^2$, for every complete r-partite graph F. Now we apply Zykov symmetrization repeatedly. It was shown in [61] that we can pick symmetrization steps in such a way that the number of copies of H does not decrease, no K_{r+1} is created and the process eventually terminates, which results in a complete multipartite graph. This gives a sequence of graphs $G = G_0, G_1, \dots, G_l$, where $\mathcal{N}(H,G_{i+1}) \ge \mathcal{N}(H,G_i)$ for all $i = 0,1,\dots,l-1$, and G_l is a complete s-partite graph. In particular, $\mathcal{N}(H,G_l) \ge \mathcal{N}(H,G) \ge \mathrm{ex}(n,H,K_{r+1}) - \delta n^h$, and hence, by Lemma 3.5, $s = r$, and every part of G_l has size at least γn.

Let t be the largest i such that $\Delta_1(G_i,F) > \varepsilon n^2$ for every complete r-partite graph F. Let T be the closest graph to G_t and T' be the closest graph to G_{t+1} among all complete r-partite graphs on the vertex set $V(G)$. Observe that $\varepsilon n^2 < \Delta_1(G_t,T) \le \Delta_1(G_t,T') \le \Delta_1(G_t,G_{t+1}) + \Delta_1(G_{t+1},T') \le 2n + \varepsilon n^2 \le 2\varepsilon n^2$. Let $G' := G_t$.

Note that this implies $e(G') - e(T) \le 2\varepsilon n^2$. Since each edge is in at most n^{h-2} copies of H, we have T has at least $\mathcal{N}(H,G') - 2\varepsilon n^h$ copies of H. Thus, $\mathcal{N}(H,T) \ge \mathrm{ex}(n,H,K_{r+1}) - (\delta + 2\varepsilon)n^h$. By Lemma 3.5 again, we have that each part of T has size at least γn.

Let $V_1, V_2, \dots, V_r$ be the parts of T.

Claim 3.1. *In the graph G', every vertex v has at least βn non-neighbors in its part.*

Proof of the claim. Assume indirectly that $v \in V_j$ has fewer than βn non-neighbors in V_j (thus v has at least βn neighbors in V_j). Assume first that v has at least βn neighbors in each other part and for every $i = 1, 2, \ldots, r$, choose βn neighbors of v from V_i, to form an induced subgraph G'' of G'. Clearly, G'' does not contain a K_r, as together with v, this would create a K_{r+1} in G'. Now, we have that G'' is a K_r-free graph and $|V(G'')| := n' = \beta r n$. Then, by Turán's theorem we have

$$e(G'') \leq \left(\frac{r-2}{r-1}\right)\frac{(n')^2}{2} = \left(\frac{r-1}{r} - \frac{1}{r(r-1)}\right)\frac{(n')^2}{2} = \frac{r-1}{r}\frac{(n')^2}{2} - cn^2,$$

where $c = \frac{r}{2(r-1)}\beta^2$. On the other hand, as G'' is an induced subgraph of G', the vertex set of G'' consists of r sets, each of them is a subset of size βn of a different part of T, and $\Delta_1(G', T) \leq 2\varepsilon n^2$, we have

$$e(G'') \geq e(T[V(G'')]) - 2\varepsilon n^2 = \frac{r-1}{r}\frac{(n')^2}{2} - 2\varepsilon n^2.$$

Thus, we have $cn^2 \leq 2\varepsilon n^2$, which contradicts the assumption that ε is sufficiently small compared to β.

Assume now that v has fewer than βn neighbors in V_i. Then we move v to V_i. We claim that the complete r-partite graph T^* we obtain this way is closer to G' than T, a contradiction. Indeed, v was incident to more than $|V_j| - \beta n$ edges inside its part V_j and more than $|V_i| - \beta n$ missing edges with other endpoint inside V_i in T. In T^*, v is incident to fewer than βn edges inside its part, now V_i, and fewer than βn missing edges with other endpoints inside V_j. ☐

Now there are two cases to consider, and we shall see both of them lead to contradictions.

Case 1. Every vertex has fewer than αn non-neighbors in each of the parts other than its own part.

First, this implies there are no edges inside any of the parts, for if there is an edge inside some part, then we will have a K_{r+1} in G'. To see this, let v_0v_1 be an edge inside some part, we may say V_1. Then, v_0 and v_1 have at least $\gamma n - 2\alpha n$ common neighbors in V_2, choose v_2 among them, and then continuing this way, we can choose the vertex v_{r-1} from V_{r-1} forming a K_r in the first $r-1$ parts. The so far chosen r vertices have at least $\gamma n - r\alpha n$ common neighbors in V_r, from which we can choose a vertex v_r to form a K_{r+1}.

Therefore, G' differs from T only by missing some edges between the different parts. First we show that each edge uv with u and v from different parts is contained in at least $(\gamma n/2)^{h-2}$ copies of H in T.

Indeed, for a copy of H that contains u and v, we need to choose the other $h-2$ vertices of H. As we choose each one from some part of T and each part of T contains at least γn vertices, there are at least $\gamma n - h \geq \gamma n/2$ choices for each of the other $h-2$ vertices. Here, $-h$ generously excludes the previously chosen vertices.

Let us now bound $\mathcal{N}(H,T)-\mathcal{N}(H,G')$. The more than εn^2 missing edges give at least $\varepsilon(\gamma/2)^{h-2}n^h$ missing copies of H, but some H may be counted multiple times. Those copies contain at least 2 missing edges. There are two cases. For two missing independent edges, we can present the upper bound $4\varepsilon^2 n^h$ by picking two edges and four vertices in at most $4\varepsilon^2 n^4$ ways, and then each other vertex in at most n ways. For two missing edges sharing a vertex, we can present the upper bound $4\varepsilon\alpha n^h$ by picking a missing edge, an endpoint of that edge, another missing edge from the same vertex, and then each other vertex in at most n ways. We obtained that $\mathcal{N}(H,G') < \mathcal{N}(H,T) - \varepsilon((\gamma/2)^{h-2} - 4\varepsilon - 4\alpha)n^h < \mathcal{N}(H,T) - \delta n^h \leq \mathrm{ex}(n,H,K_{r+1}) - \delta n^h \leq \mathcal{N}(H,G)$. This is a contradiction, since we obtained G' from G through the symmetrization steps.

Case 2. There are some vertices that have at least αn non-neighbors in some of the other parts.

Case 2.1. Each vertex has at most βn neighbors in its own part. Let B be the set of such vertices with at least αn non-neighbors in some other part. As $\Delta_1(G',T) \leq 2\varepsilon n^2$, we have $|B|\alpha n \leq 2\varepsilon n^2$, which means $|B| \leq \frac{2\varepsilon}{\alpha}n$. Hence, $|V_i \setminus B| = \gamma n - |B_i| \geq \gamma n - \frac{2\varepsilon}{\alpha}n = \gamma' n$, where $\gamma' = \gamma - \frac{2\varepsilon}{\alpha}$. Note that, for every i, each vertex $v \in V_i \setminus B$ has fewer than αn non-neighbors in any of the other parts. As in the previous case, this implies that there are no edges inside $V_i \setminus B$, for every i.

For each $v \in B := \bigcup_{i=1}^r B_i$, delete the at most βn edges incident to it inside its part and add the at least αn edges between v and the other parts that are missing in G'. Note that after all these changes for every vertex in B are done, we obtain an r-partite graph G^*. In G^*, a vertex from B has no non-neighbors in the other parts and a vertex outside B has fewer than αn non-neighbors in any of the other parts. Let us now observe the change in the number of copies of H. For each $v \in B$, we can choose one of its new neighbors in at least αn ways, and then we choose $h-2$ vertices from other parts. In each part V_i, we have at least γn vertices, and the $h-1$ vertices picked from the other parts each have at most αn non-neighbors in V_i, thus we can pick each vertex at least $\gamma n - (h-1)\alpha n \geq \gamma n/2$ ways. Therefore, the number of new copies of H is at least $\gamma^{h-2}\alpha n^{h-1}/2^{h-1}$, and the number of copies deleted is at most βn^{h-1}, since we have to pick v, one of the (at most) βn neighbors of it in its part, and other vertices.

We repeat this for every vertex u in B. Note that again we add at least αn new edges, but it is possible that some of the at most βn edges incident to u in its part were already deleted. This does not affect our calculations; the total number of deleted copies of H is at most $\beta|B|n^{h-1}$. Note that a new copy of H can be counted multiple times, but at most $|E(H)| \leq h^2$ times. Therefore, the total number of new copies of H is at least $\gamma^{h-2}\alpha|B|n^{h-1}/2^{h-1}h^2$. So the increase is at least $c_0\alpha n^{h-1}$ for some constant c_0, which means $\mathcal{N}(H,G^*) \geq \mathcal{N}(H,G') + |B|c_0\alpha n^{h-1}$.

Note that if $\Delta_1(G^*,T) \geq \varepsilon n^2/2$, then we can use the same argument as in Case 1. The only difference is that G' was missing at least εn^2 edges, while G^* is missing at least $\varepsilon n^2/2$ edges. The same calculation gives $\mathcal{N}(H,G^*) < \mathcal{N}(H,T) - \varepsilon((\gamma/2)^{h-2} - \varepsilon - 2\alpha)n^h < \mathcal{N}(H,T) - \delta n^h$, a contradiction. Thus, $\Delta_1(G^*,T) \leq \varepsilon n^2/2$. Hence

$$\varepsilon n^2 < \Delta_1(G',T) \leq \Delta_1(G',G^*) + \Delta_1(G^*,T) \leq \Delta_1(G',G^*) + \varepsilon n^2/2,$$

which implies that the number of edges we added and removed from G' is $\Delta_1(G', G^*) > \varepsilon n^2/2$. Observe that the number of edges added and removed is at most $2\alpha|B|n$. Therefore, the increase of the number of copies of H is at least $|B|c_0\alpha n^{h-1} \geq c_0\varepsilon n^h/4$ if $|B| > 0$, which is a contradiction. This completes the proof.

Case 2.2. There is a vertex with more than βn neighbors in its part, let B' denote the set of such vertices. As $\Delta_1(G', T) \leq 2\varepsilon n^2$, we have $|B'|\beta n \leq 2\varepsilon n^2$, which means $|B'| \leq \frac{2\varepsilon}{\beta}n$. Similar to the proof of Claim 3.1, we can show that v has fewer than μn neighbors in V_j for some j.

Recall that B denotes the set of vertices not in B' with at least αn non-neighbors in some of the other parts. For them, we do the same change as in Case 2.1. For each vertex $v \in B'$, we pick an $i = i_v$ such that v has at most μn neighbors in V_i. Then we move v to V_i, delete the at most μn edges from v to V_i, and add all the edges from v to the part that used to contain v. Then we repeat this for the other vertices in B'.

The resulting graph G^{**} is r-partite. Let us now observe the change in the number of copies of H. First, as in Case 2.1, this increase is at least $c_0\varepsilon n^h/2$ if $|B| > 0$. For a vertex $u \in B'$, we added new edges connecting it to its non-neighbors in its part. By Claim 3.1, there are at least βn such vertices. However, it is possible that some of these βn non-neighbors of u were moved out during this procedure. This happens at most $|B| + |B'| \leq \frac{2\varepsilon}{\alpha}n + \frac{2\varepsilon}{\beta}n \leq \beta n/2$ times, thus we added at least $\beta n/2$ new edges. Therefore, we created at least $\gamma^{h-2}\beta n^{h-1}/2^h$ new copies of H. When considering all the vertices in B', we may count a copy of H multiple times, but clearly at most h^2 times, thus the number of new copies of h is at least $\gamma^{h-2}\beta|B'|n^{h-1}/2^h h^2$. We deleted at most μn edges, thus at most μn^{h-1} copies of H. So the increase is at least $c_1\beta n^{h-1}$ for some constant c_0, which means $\mathcal{N}(H, G^{**}) \geq \mathcal{N}(H, G') + |B|c_1\beta n^{h-1}$. Then we can finish as in Case 2.1. □

3.3 Every graph is eventually K_{r+1}-Turán-stable

In this section, we give a proof of Theorem 3.2. Our proof follows the argument in [83], with a more careful analysis at some places. We will talk about injective homomorphisms from H to G instead of copies of H in G. A homomorphism from a graph H to a graph G is an edge preserving map from $V(H)$ to $V(G)$. Let $\mathrm{inj}(H, G)$ denote the number of injective homomorphisms from H to G, then $\mathrm{inj}(H, G) = a(H)\mathcal{N}(H, G)$, where $a(H)$ is the number of automorphisms of H. For $\beta > 0$, a graph G is said to be β-dense if $\deg(v) \geq (1-\beta)|V(G)|$, for every $v \in V(G)$. Recall that we denote by h the number of vertices in H. Theorem 3.2 is a direct consequence of the following theorem.

Theorem 3.6. *For every graph H, if $r \geq 300h^9$, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that every K_{r+1}-free n-vertex graph G with $\mathrm{inj}(H, G) \geq \mathrm{inj}(H, T(n, r)) - \delta n^h$ has edit distance at most εn^2 to $T(n, r)$.*

The authors of [83] present an informal outline of the proof. The first idea is that most of the maps from H to G that are not injective homomorphisms fail due to a single edge being mapped to a non-edge. Therefore, an approximate bound on $\mathrm{inj}(H, G)$ can be given that depends on $e(G)$. As the Turán graph contains the most edges among K_{r+1}-free graphs, one can obtain the result if the error terms in "approximately" can be controlled. The same holds in our setting.

A key part where they use that G maximizes $\mathrm{inj}(H, G)$ among K_{r+1}-free graphs is the symmetrization. They prove that the minimum degree in G is large by showing that one can symmetrize a low-degree vertex to a vertex in the most copies of H and increasing the number of copies of H. This is not a contradiction in our setting. However, we can repeatedly apply such symmetrization steps to remove all the low degree vertices, and the edit distance of the resulting graph is close to the edit distance of the original graph to $T(n, r)$.

We will use the following stability theorem of Füredi [32].

Theorem 3.7 (Füredi [32]). *Every n-vertex K_{r+1}-free graph G contains an r-partite subgraph G_0 such that $e(G) - e(G_0) \le e(T(n, r)) - e(G)$.*

We will also use the following lemma from [83].

Lemma 3.8 (Morrison, Nir, Norin, Rzążewski, Wesolek [83]). *For every graph H with at least one edge, every $0 < \beta \le 1/4$, and every r-partite β-dense n-vertex graph G we have*

$$\mathrm{inj}(H, T(n, r)) - \mathrm{inj}(H, G) \ge 2e(H)(1 - 3\beta h^3)(e(T(n, r)) - e(G))n^{h-2}.$$

For two graphs H and G, and a vertex $v \in V(G)$, we denote by $\mathrm{inj}(v)$ the number of injective homomorphisms from H to G that contain v in their images. To avoid redundant repetition of arguments that have been presented in [83], we state them as facts here and refer to them in the proof. As always, in the following statements $h = |V(H)|$ and $n = |V(G)|$.

Fact 1. If G is δ-dense, then $\mathrm{inj}(H, G) \ge (1 - \delta h^2)n^h$ (see Corollary 2.2 in [83]). In particular, as the Turán graph $T(n, r)$ is $2/r$-dense, we have $\mathrm{inj}(H, T(n, r)) \ge (1 - 2h^2/r)n^h$.

Fact 2. For every vertex $v \in V(G)$, $\mathrm{inj}(v) \le hn^{h-1} - (n - deg(v))n^{h-2}$.

Fact 3. If G' is obtained from G by symmetrizing a vertex v to another vertex u in G, then $\mathrm{inj}(H, G') \ge \mathrm{inj}(H, G) - inj(v) + inj(u) - h^2 n^{h-2}$. Also, if G is K_{r+1}-free, then so is G'.

Fact 4. Let G' be a spanning r-partite subgraph of G. Then,

$$\mathrm{inj}(H, G') \ge \mathrm{inj}(H, G) - 2e(H)(e(G) - e(G'))n^{h-2}.$$

Proofs of facts 2, 3, and 4, can be found in Section 3, Proof of Theorem 1.3 in [83]

Proof of Theorem 3.6. Let $\beta = 1/100h^6$, we are given $\varepsilon > 0$, and let $\delta = \delta(\varepsilon, r, H)$ be small enough. Assume that the statement does not hold, i.e., we have an n-vertex graph G with $\mathrm{inj}(H,G) \geq \mathrm{inj}(H,T(n,r)) - \delta n^h$ and edit distance more than εn^2 to $T(n,r)$. We can also assume that n is sufficiently large.

First we show that we can replace G by a β-dense graph G'. By Fact 2 above, for any vertex v of G, we have

$$\mathrm{inj}(v) \leq hn^{h-1} - (n - \deg(v))n^{h-2}.$$

By Fact 1 and our assumption on $\mathrm{inj}(H,G)$, we have that $\mathrm{inj}(H,G) \geq (1 - 2h^2/r - \delta)n^h$. Then, by averaging, there is a vertex v_0 in G with

$$\mathrm{inj}(v_0) \geq h(1 - 2h^2/r - \delta)n^{h-1}.$$

If $\deg(v) \leq n - \beta n$, then $\mathrm{inj}(v) \leq hn^{h-1} - \beta n^{h-1}$. Now we symmetrize v to v_0 to obtain a graph G_1. By Fact 3, G_1 is K_{r+1}-free and $\mathrm{inj}(H,G_1) \geq \mathrm{inj}(H,G) - \mathrm{inj}(v) + \mathrm{inj}(v_0) - h^2n^{h-2}$. Since $r \geq 300h^9$, if n is large enough and δ is small enough, then this implies $\mathrm{inj}(H,G_1) \geq \mathrm{inj}(H,G) + \beta n^{h-1}/4$.

We repeat this, as long as there is a vertex of degree at most $n - \beta n$. This can happen at most $4\delta n/\beta$ times by our assumption on δ. Therefore, the resulting graph G' has edit distance more than $(\varepsilon - 4\delta/\beta)n^2 \geq \varepsilon n^2/2$ from $T(n,r)$. In the last bound we use that δ is sufficiently small.

Now we proceed by taking an r-partite subgraph G'' of G' with the most edges. Theorem 3.7 gives that $e(G'') \geq 2e(G') - e(T(n,r))$. Clearly $e(T(n,r)) - e(G'')$ is at least half the edit distance of G' and $T(n,r)$. Indeed, to obtain G' from $T(n,r)$, first we delete at least $e(T(n,r)) - e(G'')$ edges, then we add at most $e(T(n,r)) - e(G'')$ (otherwise we obtained a K_{r+1}-free graph with more edges than $T(n,r)$, contradicting Turán's theorem).

By Fact 4, we have

$$\mathrm{inj}(H,G'') \geq \mathrm{inj}(H,G') - 2e(H)(e(G') - e(G''))n^{h-2}. \tag{3.1}$$

We will show that G'' is $2/13h^3$-dense. The proof of this statement is exactly the same as in [83], we include it for sake of completeness. We have that $e(G'') \geq e(G') - (n^2/2 - e(G')) \geq (1 - 1/50h^6)n^2/2$. Let $V_1, \ldots, V_r$ be the parts of G''. As $e(G'') \leq n^2 - \sum_{i=1}^r |V_i|^2$, we have that $\max_{1\leq i\leq r} |V_i| \leq n/7h^3$. By maximality of G'', we have that $\deg_{G''}(v) \geq \deg_{G'}(v) - \max|V_i| \geq n - \beta n - n/7h^3 \geq n - 2n/13h^3$.

Now we can apply Lemma 3.8 to G'', which gives

$$\mathrm{inj}(H,T(n,r)) \geq \mathrm{inj}(H,G'') + 2e(H)(1 - 6h^3/13h^3)(e(T(n,r)) - e(G''))n^{h-2}.$$

Applying the bounds $e(T(n,r)) - e(G'') \geq \varepsilon n^2/4$ and (3.1), we obtain

$$\begin{aligned}\mathrm{inj}(H,T(n,r)) &\geq \mathrm{inj}(H,G') - 2e(H)(e(G') - e(G''))n^{h-2}\\ &\quad + 2e(H)(e(T(n,r)) - e(G''))n^{h-2} + e(H)\varepsilon n^h/52\\ &\geq \mathrm{inj}(H,G') + e(H)(e(G'') + e(T(n,r)) - 2e(G'))n^{h-2} + \varepsilon n^h/52\\ &> \mathrm{inj}(H,G) + \delta n^h,\end{aligned}$$

where the last inequality uses $e(G'') \geq 2e(G') - e(T(n,r))$ and that δ is sufficiently small. Clearly this contradicts our assumption on $\mathrm{inj}(H,G)$, completing the proof. $\square$

3.4 Cycles and bipartite graphs

Given a graph F, denote by $\mathcal{F}^\star$ the family of graphs obtained from F by taking two non-adjacent vertices u and v of F and connecting both of them to $N(u) \cup N(v)$. We say that a (possibly infinite) sequence of graphs $F_1, \dots, F_i, \dots$ is *nice* if for any i, any $F \in \mathcal{F}_i^\star$ contains some F_j with $j < i$.

Observe that by definition, F_1 is a clique, say K_t. Indeed, if F_1 contains a non-adjacent pair of vertices, then there is $F \in \mathcal{F}_1^\star$ (that is, $\mathcal{F}_1^*$ would not be empty), and then the definition requires the existence of some F_j with $j < 1$ that is contained in F, while there are none. Moreover, if the sequence contains graphs with chromatic number $m < t$, then the first graph with chromatic number at most m must also be a clique. For if that graph F_j is not a clique, then it has two vertices u, v in the same color class in a proper m-coloring of F_j. When we connect these two vertices to $N(u) \cup N(v)$, then the resulting graph F has chromatic number m and is contained in $\mathcal{F}_j^\star$. Recall that all the graphs before F_j have chromatic number larger than m, and hence cannot be contained in F.

The prime example is the sequence of odd cycles C_{2k+1} (with the natural ordering). Indeed, adding any chord to an odd cycle creates a shorter odd and a shorter even cycle. Let us show some other examples. If (F_i) and (F'_j) form nice sequences, consider the graphs $F_i + F'_j$, which are obtained by taking a copy of F_i and F_j and adding all the edges between them. There is a natural partial ordering: $F_i + F'_j$ is before $F_k + F'_\ell$ if they are not equal and $i \le k$, $j \le \ell$. Extending this to any total ordering, we obtain a nice sequence.

A family $\mathcal{F}$ of graphs is *nice* if its elements can be ordered to form a nice sequence. Let $\mathcal{C}_{2k+1}^{\text{odd}} = \{C_3, C_5, \dots, C_{2k+1}\}$.

Proposition 3.9. *Let $\mathcal{F}$ be nice and G be an $\mathcal{F}$-free graph. If G' is obtained from G by symmetrizing a vertex u to a non-adjacent vertex v, then G' is $\mathcal{F}$-free.*

Proof. Consider the smallest i such that F_i is in G'. A copy of F_i must contain both u (since only u is incident to new edges) and v (since otherwise u could be replaced by v in the copy of F_i). Since u and v have the same neighborhood in G', the graph F we obtain from F_i by connecting both u and v to $N(u) \cup N(v)$ is a subgraph of G'. Since $F \in \mathcal{F}_i^\star$, we have that F_j is a subgraph of G' for some $j < i$, contradicting our choice of i. $\square$

So far, this does not extend the known results from [61] much. Recall that symmetrizing one of u to v or v to u does not decrease the number of copies of H if H is complete multipartite. This shows that if H is a complete r-partite graph, then $\mathrm{ex}(n, H, \mathcal{F}) = \mathcal{N}(H, T)$ for some complete multipartite n-vertex graph T. However, if the smallest chromatic number of graphs in $\mathcal{F}$ is m, then $\mathcal{F}$ contains K_m, thus any $\mathcal{F}$-free graph is K_m-free, which already implies that a complete $(m-1)$-partite n-vertex graph is the extremal, as discussed in the introduction. Yet, surprisingly we can use Proposition 3.9 to prove Theorem 3.3.

The other ingredient we use is a well-known result of Andrásfai, Erdős, and Sós [4]: if a $\mathcal{C}_{2k+1}^{\text{odd}}$-free n-vertex graph is not bipartite, then it contains a vertex of degree at most $2n/(2k+1)$. This assumption is missing for other nice sequences.

Recall that Theorem 3.3 states that for every bipartite graph H, if $k > (2h)^{h+1}$, then H is weakly C_{2k+1}-Turán-stable. In other words, for any $\varepsilon > 0$ there is $\delta > 0$ such that if a C_{2k+1}-free n-vertex graph G contains at least $\mathrm{ex}(n, H, C_{2k+1}) - \delta n^h$ copies of H, then the edit distance of G and some bipartite graph B is at most εn^2.

***Proof of Theorem** 3.3.* If H has isolated vertices, they do not affect anything, thus we can assume H does not have them.

Given ε, we let $\alpha > 0$ be sufficiently small, then δ sufficiently small compared to α and ε, i.e., $\delta \ll \alpha \ll \varepsilon$. Let n be sufficiently large and G be an n-vertex C_{2k+1}-free graph with at least $\mathrm{ex}(n, H, C_{2k+1}) - \delta n^h$ copies of H. A result of Alon and Shikhelman [3] determines when $\mathrm{ex}(n, H_1, F_1) = o(n^{|V(H_1)|})$, for any two graphs H_1 and F_1. It implies that there are $o(n^{2m+1})$ copies of C_{2m+1} for each $m < k$ in G. Therefore, by the removal lemma, there are at most $\alpha n^2/k$ edges in G such that each copy of C_{2m+1} contains at least one of those edges. Let G' be the graph obtained by deleting those edges for each $m < k$, then clearly G' is $\mathcal{C}_{2k+1}^{\text{odd}}$-free and $\mathcal{N}(H, G') \geq \mathcal{N}(H, G) - \alpha n^{|V(H)|}$.

Let βn^h be the largest number of copies of H in bipartite graphs on n vertices. To obtain a lower bound on β, we consider $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. We pick the h vertices one by one, each at least $\lfloor n/2 \rfloor - h$ ways. Then we may count a copy multiple times, but at most $h!$ times. Therefore, $\beta \geq (\lfloor n/2 \rfloor - h)^h / h! n^h > 1/(2h)^h$, using that n is sufficiently large.

By the theorem of Andrásfai, Erdős, and Sós we mentioned before the proof, G' contains a vertex of degree at most $2n/(2k+1)$. Let $u, v \in G'$ be non-adjacent vertices, then by Proposition 3.9, symmetrizing u to v results in a $\mathcal{C}_{2k+1}^{\text{odd}}$-free graph.

Clearly, $\mathcal{N}(H, G') \geq (\beta - \alpha - \delta) n^h$, thus there is a vertex v of G' in at least $(\beta - \alpha - \delta) h n^{h-1}$ copies of H. Let u be a vertex of degree at most $2n/(2k+1)$, then u is in at most $2hn^{h-1}/(2k+1)$ copies of H. Indeed, we pick which vertex is mapped to u, there are at most $2n/(2k+1)$ choices for at least one neighbor and at most n choices for the other vertices. Now we symmetrize u to v to obtain G'' (in the case u and v are adjacent, we delete the edge between them first). We have $\mathcal{N}(H, G'') \geq \mathcal{N}(H, G') + (\beta - \alpha - \delta) h n^{h-1} - 2hn^{h-1}/(2k+1) - n^{h-2}$, where the last term is for those copies of H that contain both u and v. By choosing α and δ small enough and n large enough, we have $\mathcal{N}(H, G'') \geq \mathcal{N}(H, G') + \beta n^{h-1}/2$.

We apply this procedure repeatedly, until we obtain a bipartite graph B. If there are at most $(\varepsilon - \alpha)n/2$ symmetrization steps, then the edit distance of G' and B is at most $(\varepsilon - \alpha) n^2$, since we change at most $2n$ edges at each step. Therefore, the edit distance of G and B is at most εn^2.

If there are more than $(\varepsilon - \alpha)n/2 > \varepsilon n/3$ symmetrization steps, then we gained more than $\varepsilon \beta n^h/6$ copies of H, thus $\mathcal{N}(H, B) \geq \mathcal{N}(H, G') + \varepsilon \beta n n^{h-1}/6 \geq \mathcal{N}(H, G) + \varepsilon \beta n^h/6 - \alpha n^h$. By picking $\delta < \varepsilon \beta/6 - \alpha$, we obtain a contradiction with the stability assumption. $\square$

3.5 Conclusion

In our statements and proofs we aimed at simplicity rather than strongest possible results. In particular, we did not try to optimize the thresholds in Theorems 3.2 or 3.3. In the case of Theorem 3.2, we used the threshold $300h^9$ from [83], where the Turán-goodness was proved under the same assumptions. Note that there the threshold was not optimized either. These two thresholds may be the same, and it is possible that the threshold could be as low as $h+1$. In the case of Theorem 3.3, our threshold is exponential in h. We expect that the sharp threshold is actually smaller than h.

Another possible direction could be to study some other notions of stability. In all our bounds, δ depends linearly on ε. Therefore, we actually proved *perfect stability*, as described in [77]. Another strengthening in [77] is that instead of Zykov symmetrization, they study a more general version. They only assume that at most εn^2 edges change at each step, the value of the graph parameter does not decrease, and after some steps we arrive to a complete multipartite graph. It is possible that for some graph H, a symmetrization can be defined such that $\mathcal{N}(H,G)$ does not decrease and the graph remains K_{r+1}-free, although we could not find any other example. Our Theorem 3.1 also holds for such graphs H, since we use symmetrization only to obtain G'.

Chapter 4

Generalized regular Turán numbers

4.1 Introduction

In this chapter we define and study the generalized version of the regular Turán numbers. Recall that

$$\mathrm{rex}(n, H, F) := \max\{\mathcal{N}(H, G) : \ G \text{ is an } F\text{-free regular } n\text{-vertex graph}\}.$$

Our goal is to show some examples where $\mathrm{rex}(n, H, F)$ behaves similarly to $\mathrm{ex}(n, H, F)$ and also show some examples where they differ significantly. Similarly to the regular turán numbers, in the generalized version, we have the obvious observation that $\mathrm{rex}(n, H, F) \leq \mathrm{ex}(n, H, F)$.

Alon and Shikhelman [3] proved the following result.

Theorem 4.1 (Alon and Shikhelman [3])**.** *For any graphs F and H, we have that* $\mathrm{ex}(n, H, F) = \Theta(n^{|V(H)|})$ *if and only if F is not a subgraph of any blow-up of H.*

We extend this theorem to the regular setting. Note that the above simple observation gives the "if" part of the above Theorem, We will give regular constructions for the lower bound of the "only if" part, proving the following result.

Theorem 4.2. *For any graphs F and H, we have that* $\mathrm{rex}(n, H, F) = \Theta(n^{|V(H)|})$ *if and only if F is not a subgraph of any blow-up of H.*

Another result of Alon and Shikhelman [3] is that $\mathrm{ex}(n, K_3, F) = O(n)$ if and only if F is an extended friendship graph. Recall that in Chapter 2, we determined the exact value of this problem for the friendship graph. Recall that in an extended friendship graph, every cycle is a triangle and there is a vertex v such that every pair of triangles intersect in v (or equivalently, its 2-core is empty or a friendship graph). We extend this result as well to the regular setting.

Theorem 4.3. $\mathrm{rex}(n, K_3, F) = O(n)$ *if and only if F is an extended friendship graph.*

Let us turn to problems where adding the regularity changes the situation. It is well-known and easy to see that for any forest F, any graph with minimum degree at least $|V(F)|$ contains F. This implies that $\mathrm{rex}(n,F) \le (|V(F)|-1)n/2$. Let H be a connected graph, then the vertices of H have an ordering such that each but the first vertex has a neighbor that is earlier in the ordering. The copies of H in an F-free r-regular graph can be counted by picking the vertices in the above order. The first vertex can be picked n ways, and then each other vertex can be picked at most r ways among the neighbors of at least one of the vertices picked earlier. This shows that $\mathrm{rex}(n,H,F)=O(n)$, since $r<|V(F)|$ (more specifically, $\mathrm{rex}(n,H,F)\le nr^{|V(H)|-1}$). On the other hand, for $l<k$, $\mathrm{ex}(n,P_\ell,P_k)=\Theta(n^{\lceil \ell/2\rceil})$ by a theorem of Győri, Salia, Tompkins and Zamora [64].

Another example where the order of magnitude of $\mathrm{rex}(n,H,F)$ is much smaller than that of $\mathrm{ex}(n,H,F)$ is given by even cycles. When C_{2k} is forbidden, the regularity does not have to be constant, but it is $O(n^{1/k})$ by a theorem of Bondy and Simonovits [10]. Therefore, $\mathrm{rex}(n,C_\ell,C_{2k})=O(n^{1+\frac{\ell-1}{k}})$, while we have $\mathrm{ex}(n,C_\ell,C_{2k})=\Theta(n^{\lfloor \ell/2\rfloor})$ if $3\le \ell\ne 2k$ [54].

Note that we have $\mathrm{ex}(n,C_\ell,C_{2k+1})=\Theta(n^\ell)$ if ℓ is even or $\ell>2k+1$, as shown by the blow-up of C_ℓ, which is regular (or can be made regular by deleting a small number of edges). Interestingly, in the remaining case $3<\ell<2k+1$ is odd, we have $\mathrm{ex}(n,C_\ell,C_{2k+1})=\Theta(n^{\lfloor \ell/2\rfloor})$ [54], while the above argument does not give any non-trivial bound. In this case, the lower bound construction is obtained by blowing up every other vertex of C_l, which is far from regular. It is a natural question to ask whether $\mathrm{rex}(n,C_\ell,C_{2k+1})$ is significantly smaller in this case. We can answer this question in the negative.

Proposition 4.4. *If* $3<\ell<2k+1$ *is odd, then* $\mathrm{rex}(n,C_\ell,C_{2k+1})=\Theta(n^{\lfloor \ell/2\rfloor})$.

So far, we considered only the order of magnitude of $\mathrm{rex}(n,H,F)$. Let us turn to exact and asymptotic results. As shown in [11, 12], for $k\ge 3$ we have $\mathrm{rex}(n,K_{k+1})=(1+o(1))|E(T(n,k))|$ (we have $\mathrm{rex}(n,K_{k+1})=|E(T(n,k))|$ if k divides n). The exact value of $\mathrm{rex}(n,K_{k+1})$ was determined for all sufficiently large n in [50]. Forbidding K_3 is very different from forbidding larger cliques in the regular Turán problem. If n is even, then $T(n,2)$ is the regular n-vertex triangle-free graph with the most edges. If n is odd, then a regular n-vertex triangle-free graph with the most edges is obtained by deleting some edges of an n-vertex blow-up of C_5, as shown in [11, 12].

Given H with $\chi(H)\le k$, there has been a lot of research studying whether $\mathrm{ex}(n,H,K_{k+1})=\mathcal{N}(H,T(n,k))$ for sufficiently large n. For example, it is the case when H is a complete "balanced" l-partite graph with $3\le l\le k$, see e.g. [34, 35, 61]. There have been two types of counterexamples found (where even $\mathrm{ex}(n,H,K_{k+1})=(1+o(1))\mathcal{N}(H,T(n,k))$ does not hold). If H is a very unbalanced bipartite graph, then an unbalanced complete k-partite graph may contain more copies of H than the Turán graph. For some graphs H, there are n-vertex K_{k+1}-free graphs that contain more copies of H than any n-vertex complete k-partite graph. For example, let H be obtained from a path on vertices v_1,v_2,v_3,v_4,v_5,v_6 by adding s additional leaves connected to v_2 and s additional leaves connected to v_5. Then some unbalanced blowup of C_5 contains more copies of H than any bipartite graph, see [61]. Examples for $k>2$ can be found in [57]. In each of the known constructions, most of the vertices of H would belong to two different classes of k-partite graphs, but they can belong to the same class of the blow-up of another graph. Then that class has many vertices.

Both counterexamples are very far from being regular. This suggests that maybe there are no regular counterexamples at all.

Conjecture 4.5. *Let $k \geq 3$ and $\chi(H) \leq k$. Then, $\mathrm{rex}(n, H, K_{k+1}) = (1 + o(1))\mathcal{N}(H, T(n, k))$. Moreover, if n is sufficiently large and is divisible by k, then $\mathrm{rex}(n, H, K_{k+1}) = \mathcal{N}(H, T(n, k))$.*

We prove Conjecture 4.5 for complete k-partite graphs H.

Proposition 4.6. *Let $k \geq 3$ and H be a complete k-partite graph. Then, $\mathrm{rex}(n, H, K_{k+1}) = (1 + o(1))\mathcal{N}(H, T(n, k))$. Moreover, if n is sufficiently large and is divisible by k, then $\mathrm{rex}(n, H, K_{k+1}) = \mathcal{N}(H, T(n, k))$.*

We also prove that the moreover part of Conjecture 4.5 holds in the case $k = 2$. In the case n is odd, the situation is very different, but we can describe the structure of the extremal graph.

Proposition 4.7. *Let H be a bipartite graph. If n is even and sufficiently large, then $\mathrm{rex}(n, H, K_3) = \mathcal{N}(H, T(n, 2))$. If H is a tree and n is odd and sufficiently large, then $\mathrm{rex}(n, H, K_3) = \mathcal{N}(H, G^*)$, where G^* is a regular graph obtained by deleting some edges of an n-vertex blow-up of C_5.*

Finally, we determine the exact value for $\mathrm{rex}(n, K_3, P_k)$, when n is large enough and P_k is a path on k vertices. To ease the notation and describe the extremal graphs, we first define some graphs for each fixed value of k. In case k is even, let $\mathcal{G}_{k-1}$ denote the graphs obtained from K_{k-1} by removing the edges of a triangle-free 2-regular subgraph, i.e., the union of vertex-disjoint cycles of length more than 3 such that the total length of the cycles is $k - 1$. Also, let $G_{k-2} := K_{k-2} - M$, a clique on $k-2$ vertices in which a perfect matching is removed. Observe that each of the above graphs is $(k - 4)$-regular and P_k-free. If k is odd, let $G'_{k-1} := K_{k-1} - M$, a clique on $k - 1$ vertices in which a perfect matching is removed. Note that $\mathcal{N}(K_3, G_{k-1}) = 8\binom{k/2-1}{3}+3-k/2$ for any graph $G_{k-1} \in \mathcal{G}_{k-1}$, $\mathcal{N}(K_3, G_{k-2}) = 8\binom{k/2-1}{3}$ and $\mathcal{N}(K_3, G'_{k-1}) = (k - 1)(k - 3)(k - 5)/6 = 8\binom{(k-1)/2}{3}$. One can obtain these computations from the fact that the complements of these graphs contain no triangles, the degrees of their vertices and the following formula that is basically proven by Goodman [55].

$$\mathcal{N}(K_3, G) + \mathcal{N}(K_3, \overline{G}) = \binom{n}{3} - \frac{1}{2} \sum_{v \in V(G)} \deg(v)(n - 1 - \deg(v)),$$

where $\overline{G}$ denotes the complement of G. Recall that $H \cup F$ denotes the disjoint union of two graphs H and F, and by mF we mean m disjoint copies of the graph F.

Theorem 4.8. *Let P_k be a path on k vertices and n be large enough. Then:*

1. *If* $(k-1)|n$, *then* $\mathrm{rex}(n,K_3,P_k) = \frac{n}{k-1}\binom{k-1}{3}$, *and the unique extremal graph is* $\frac{n}{k-1}K_{k-1}$.
2. *Assume that* $(k-1)\nmid n$, $k\geq 6$ *and either* $k-2$ *divides* n *or* k *is odd. Let* $n = a(k-2)+b$ *with* $b<k-2$. *Then we have* $\mathrm{rex}(n,K_3,P_k) = (a-b)\binom{k-2}{3} + 8b\binom{\frac{k-1}{2}}{3}$, *and the unique extremal graph is* $(a-b)(K_{k-2})\cup bG'_{k-1}$.
3. *If* $k\geq 6$ *is even, and* n *is neither divisible by* $k-1$ *nor by* $k-2$. *Let* $n=a(k-3)+b$, *with* $b<k-3$. *Then*

   $$\mathrm{rex}(n,K_3,P_k) = (a-\ell-\lfloor b/2\rfloor)\binom{k-3}{3} + \ell\mathcal{N}(K_3,G_{k-2}) + \lfloor b/2\rfloor\,\mathcal{N}(K_3,G_{k-1}),$$

   and the extremal graphs are formed by adding $\lfloor b/2\rfloor$ *graphs from* $\mathcal{G}_{k-1}$ *to* $(a-\ell-\lfloor b/2\rfloor)K_{k-3}\cup\ell G_{k-2}$, *where* $\ell=0$ *if* b *is even and* $\ell=1$ *otherwise.*
4. *If neither* 3, *nor* 4 *divides* n, *then* $\mathrm{rex}(n,K_3,P_5) = \lfloor n/3\rfloor - 1$, *and the unique extremal graph is formed by adding a* C_4 *or a* C_5 *to* $(\lfloor n/3\rfloor-1)K_3$.

   In all the cases not listed above, $\mathrm{rex}(n,K_3,P_k)=0$.

In Section 4.2 we present some preliminary results that will serve as tools in the proofs. Section 4.3 is devoted to the proofs of the general results that rather concern the behavior of the function $\mathrm{rex}(n,H,F)$ compared to $\mathrm{ex}(n,H,F)$. Finally, Section 4.4 contains the proof of Theorem 4.8.

4.2 Tools

We will use the following well-known theorem of Erdős and Sachs [29].

Theorem 4.9 (Erdős and Sachs [29]). *For every* r *and* g, *there exists an* r*-regular graph of girth at least* g.

In fact we rely on the following simple corollaries of the above theorem.

Lemma 4.10. ***(i)*** *For any* r *and* k, *if* n *is sufficiently large and* nr *is even, then there is an* n*-vertex* r*-regular graph with girth at least* k.

(ii) *For any* r, k *and* i, *if* n *is sufficiently large and* $nr-i$ *is even, then there is an* n*-vertex graph with girth at least* k *that contains* i *vertices of degree* $r-1$ *and each other vertex has degree* r. *Moreover, we can have that the vertices of degree* $r-1$ *are at distance at least* $k-1$.

Proof. Let us start by proving **(i)**. We know such a graph G_1 exists on m vertices for some m. If r is even, we take $r/2$ vertex-disjoint copies of G_1 and remove an edge from each. We add a new vertex and connect it to the endpoints of the removed edges. The resulting graph G_2 satisfies the desired properties on $\frac{rm}{2}+1$ vertices. For each $n \geq rm^2$, we can write n as $a(\frac{rm}{2}+1)+bm$, thus we can create an n-vertex graph by taking vertex-disjoint copies of G_1 and G_2.

If r is odd, since nr is even, we must have n is even. In this case, we take r vertex-disjoint copies of G_1 and remove an edge from each. We add two new vertices u, v and connect u to one of the endpoints of each removed edge and v to the other endpoint. The resulting graph G_2' satisfies the desired properties on $rm+2$ vertices. For each even $n \geq r^2m$, we can write n as $a(rm+2)+bm$, thus we can create an n-vertex graph by taking vertex-disjoint copies of G_1 and G_2'.

We continue with the proof of **(ii)**. If i is even, we take a graph guaranteed by **(i)** and remove $i/2$ independent edges such that the endpoints of these edges are at distance at least $k-1$. If n is sufficiently large, we can greedily find such edges. Indeed, we take an edge u_1v_1, then at most $2r-2$ other vertices are adjacent to u or v, and at most $2(r-1)^j$ vertices are at distance j from u or v. Altogether there are at most $2(r-1)^k$ vertices at distance at most $k-1$ from u or v. We take a vertex u_2 different from those at most $2(r-1)^k$ vertices and an arbitrary neighbor v_2 of u_2. Repeating this, we can find $i/2$ edges if we can pick a vertex $u_{i/2}$ that is not among the $i-2$ vertices picked earlier and the at most $(i-2)(r-1)^k$ vertices at distance at most $k-1$ from the vertices picked earlier. In other words, we can pick the desired edges if $n > i-2+(i-2)(r-1)^k$. Note that the distance of u_i and v_i is at least $k-1$ after removing the edge u_iv_i because of the girth condition.

If i is odd, observe that both n and r are odd. Let G_3 be an $(r-1)$-regular m-vertex graph for some odd m, with girth at least k. Let G_4 be an r-regular graph on m' vertices for some m' sufficiently large with girth at least k. Note that G_3 and G_4 exist by **(i)**. We take $(m-i)/2$ copies of G_4 and remove an edge from each. This way we obtain $m-i$ vertices of degree $r-1$, we connect each of them to a different vertex of G_3. The resulting graph has exactly i vertices of degree $r-1$ and each other vertex has degree r.

In each of the above constructions, we removed an edge uv from some copy of a graph of girth at least k, then we added some edges incident to u and v and outside vertices. After removing uv, the distance of u and v becomes at least $k-1$, thus this way we do not create cycles of length less than k.

The i vertices of degree $r-1$ (in the copy of G_3) can be chosen to be at distance at least $k-1$, by a similar reasoning as in the case i is even. □

Corollary 4.11. *For any sequence (a_n) of positive integers with $a_n = \omega(1)$, we can take for every sufficiently large n an n-vertex graph G_n that satisfies the assumptions of Lemma 4.10 with $r \leq a_n$ and $r = \omega(1)$.*

The following observation is a simple corollary of Hall's theorem, which will be used in our proofs.

Observation 4.12. *For every $k \leq n$, there exists a k-regular bipartite graph with both parts of order n.*

A theorem of Andrásfai, Erdős, and Sós [4], states that a non-bipartite triangle-free graph on n vertices contains a vertex of degree at most $2n/5$. Using this, we prove a stability result on $\mathrm{rex}(2n+1, K_3)$, which may be interesting on its own. When we talk about V_{i+j} in the statement or the proof, then $+$ is meant modulo 5.

Lemma 4.13. *Let G be a d-regular n-vertex triangle-free graph with n odd. Let $d \geq 2n/5 - o(n)$. Then $V(G)$ contains disjoint sets $V_1, \dots V_5$ such that $|V_i| = n/5 - o(n)$ and from V_i there is no edge to V_i, V_{i+2} and V_{i+3}, and $n/5 - o(n)$ edges go to V_{i+1} and V_{i+4}. In particular G is obtained by deleting some edges of an n-vertex blow-up of C_5.*

Proof. Observe that G cannot be bipartite, thus $d \leq 2n/5$ by the result of Andrásfai, Erdős, and Sós [4]. Let C_{2k+1} be a shortest odd cycle in G and C be a copy of C_{2k+1}. Then every vertex outside C is adjacent to at most two vertices of C. This implies that there are at most $2(n-2k-1)$ edges between C and the other vertices. On the other hand, there are at least $(2k+1)d - (2k+1) \geq 2(2k+1)n/5 - o(n)$ edges between C and the other vertices by our assumption on the degrees of the vertices of C (which is d). Here we use that there are $2k+1$ edges inside C, since it is the shortest odd cycle and so does not have any chords.

This shows that $k \leq 2$. Since G is triangle-free, we have $k = 2$. Furthermore, $n - o(n)$ vertices outside C have two neighbors in C, otherwise there are at most $2(n-2k-1) - \Omega(n) < (2k+1)d - (2k+1)$ edges between C and the other vertices. Let $v_1, \dots, v_5$ be the vertices of C in the cyclic order. Observe that no vertex can be adjacent to both v_i and v_{i+1}, thus $n - o(n)$ vertices are each, for some i, adjacent to v_i and v_{i+2}. We place those vertices to V_{i+1}. Let $U = V(G) \setminus (V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5)$, then $|U| = o(n)$.

Let $u \in V_i$. As u has a common neighbor with every vertex of V_i, V_{i+2} and V_{i+3}, there are no neighbors of u in $V_i \cup V_{i+2} \cup V_{i+3}$, thus all the neighbors of u are in V_{i+1} and V_{i+4} and U. In particular, $|V_{i+1}| + |V_{i+4}| \geq 2n/5 - o(n)$. This holds for every non-adjacent pair of classes. If $|V_i| \leq n/5 - \alpha n$, then $|V_{i+2}|, |V_{i+3}| \geq n/5 + \alpha n - o(n)$. Then $|V_{i+1}| + |V_{i+4}| \leq 2n/5 - \alpha n - o(n)$, thus $\alpha = o(1)$, completing the proof. ☐

4.3 General results

In this section, we provide proofs of Theorems 4.2 and 4.3, and Propositions 4.4, 4.6 and 4.7. Let us start with the proof of Theorem 4.2. Recall that it states that $\mathrm{rex}(n, H, F) = \Theta(n^{|V(H)|})$ if and only if F is not a subgraph of a blow-up of H.

Proof of Theorem 4.2. If F is a subgraph of a blow-up of H, then $\mathrm{rex}(n, H, F) \leq \mathrm{ex}(n, H, F) = o(n^{|V(H)|})$, where we use Theorem 4.1. Assume now that F is not a subgraph of any blow-up of H. If H is the empty graph, the statement follows. Observe that otherwise, since each bipartite graph is a subgraph of the (sufficiently large) blow-up of a single edge, we also have that F has chromatic number at least 3. We can also assume that there are no isolated vertices in F.

In the analogous statement for $\mathrm{ex}(n, H, F)$, this is the trivial direction, as the blow-up $H(m)$ with $m = \lfloor n/|V(H)| \rfloor$ is F-free and contains $\Omega(n^{|V(H)|})$ copies of H. However, we have two problems here: the first is that $H(m)$ is not regular if H is not regular, and the second is that we may need to add some vertices of degree 0 to obtain an n-vertex graph.

Let Δ be the largest degree in H. Let H_i be a graph with girth more than $3|V(F)|$ that has a set S of i vertices of degree 2Δ and all the other vertices of degree

$2\Delta+1$, such that the vertices of S are of distance at least $|V(F)|$. Such a graph exists by Lemma 4.10 where the number of vertices is large enough compared to Δ and $|V(F)|$, but constant compared to n.

For each $1 \le i \le \Delta$ and each vertex v of H of degree i, we take a copy of $H_{2\Delta+1-i}$ and join v to the $2\Delta+1-i$ vertices of this copy of degree 2Δ. This way we obtain a $(2\Delta+1)$-regular graph H' on constant many vertices.

Claim 4.1. *The blow-up $H'(m)$ is F-free for any m.*

Proof of Claim. We can assume that m is large enough compared to $|V(F)|$. Let us assume that there is a copy of F in $H'(m)$, that we will denote with F^*. Let H^* denote an arbitrary copy of an H_i in H', for some i. Let F_0 denote a connected component of the intersection of F^* with the blow-up of H^*. Observe that F_0 is bipartite, since any cycle inside F_0 has length at most $|V(F)|$ and any cycle inside H^* has length at least $3|V(F)|$. Also observe that F_0 contains at most one vertex of S. Indeed, otherwise F_0 would contain a path between two vertices of S inside H^*, but such a path contains more than $|V(F)|$ vertices, a contradiction. Let u be the vertex of H that is joined to v in H' and u' be an arbitrary neighbor of u in H.

Now we can delete F_0 and embed it to the complete bipartite graph between the blow-ups of u and u', using only vertices that were not in F^*. This can be done since F_0 is bipartite and m is large enough. We repeat this for every subgraph of F outside $H(m)$. At the end, we obtain a copy of F in $H(m)$, a contradiction. □

Let us return to the proof of the theorem. We are done if $|V(H')|$ divides n, as we can pick m to be $n/|V(H')|$. To prove the theorem for every n, we do the following. Let H'' denote the vertex-disjoint union of $H'(2m)$ and $C_{2|V(F)|+1}((2\Delta+1)m)$. Note that H'' is F-free, since every subgraph of $C_{2|V(F)|+1}((2\Delta+1)m)$ on at most $|V(F)|$ vertices is bipartite. If there is a copy of F in H'', then the components that are in $C_{2|V(F)|+1}((2\Delta+1)m)$ are bipartite, and hence could be easily replaced by copies in $H'(2m)$ (we can find such copies in the blow-up of any edge). This way we find a copy of F in $H'(2m)$, a contradiction.

Clearly, H'' is $2(2\Delta+1)m$-regular for any m, and the number of vertices have the same parity as m. Let us pick the largest m such that $n-|V(H'')|$ is even. Observe that $n-|V(H'')|$ is a constant. Now we modify the $C_{2|V(F)|+1}((2\Delta+1)m)$ subgraph. Note that this is similar to the way the odd cycles were modified in [12].

Let $A_1,\dots,A_{2|V(F)|+1}$ be the blown-up parts of the cycle in this order. We take a pair of neighboring parts, say A_i and A_{i+1}, and add $b=(n-|V(H'')|)/2$ vertices to each of A_i and A_{i+1}. We add them in such a way that we still have a complete bipartite graph between any pair of consecutive blown-up parts A_j, A_{j+1}, i.e., we connect the new vertices of A_i to each vertex of A_{i-1} and A_{i+1}, and connect the new vertices of A_{i+1} to each vertex of A_i and A_{i+2}. Then we remove the edges of a spanning bipartite graph B between A_{i-1} and A_i such that each vertex of A_{i-1} has degree b and each vertex of A_i has degree $\left\lfloor \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rfloor$ or $\left\lceil \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rceil$ in B. This can be done the following way. We cover A_i by $|A_{i-1}|$ sets of size b (each of them are the b neighbors of a vertex in A_{i-1}), such that each vertex of A_i is covered by as equal as possible number of such sets. Then we take a bijection between the vertices of A_{i-1} and these sets, and delete the edges between the vertices of A_{i-1} and the vertices of the sets they are mapped to.

We remove the edges of a copy of B between A_{i+1} and A_{i+2} as well such that the vertices of degree b are in A_{i+2}.

At this point the vertices outside A_i and A_{i+1} have degree $2(2\Delta+1)m$. The part A_i consists of a set A_i' of vertices of degree $2(2\Delta+1)m+b-\left\lfloor \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rfloor$ and a set A_i'' of vertices of degree $2(2\Delta+1)m+b-\left\lceil \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rceil$. Similarly, A_{i+1} consists of a set A_{i+1}' of vertices of degree $2(2\Delta+1)m+b-\left\lfloor \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rfloor$ and a set A_{i+1}'' of vertices of degree $2(2\Delta+1)m+b-\left\lceil \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rceil$. Observe that by the analogous construction, we have that $|A_i'| = |A_{i+1}'|$. We pick a perfect matching M' between A_i' and A_{i+1}', and extend it to a perfect matching M between A_i and A_{i+1}. We delete the edges of M.

Then the resulting graph between A_i and A_{i+1} is $2(2\Delta+1)m+b-1$-regular, thus we can delete matchings between A_i and A_{i+1} till we obtain a $\left((2\Delta+1)m+\left\lfloor \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rfloor\right)$-regular graph between A_i and A_{i+1}. After that, we add the edges of M that are not in M'. Observe that vertices of A_i' have $\left((2\Delta+1)m+\left\lfloor \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rfloor\right)$ neighbors in A_{i+1} and $(2\Delta+1)m-\left\lfloor \frac{(2\Delta+1)mb}{(2\Delta+1)m+b} \right\rfloor$ neighbors in A_{i-1}. Vertices of A_i'' have one more neighbor in A_{i+1} and one less neighbor in A_{i-1}. The same holds for vertices in A_{i+1}. Let G denote the resulting n-vertex graph. Then G is $2(2\Delta+1)m$-regular and contains at least $m^{|V(H)|} = \Theta(n^{|V(H)|})$ copies of H, completing the proof. □

Let us continue with the proof of Theorem 4.3. Recall that it states that $\mathrm{rex}(n, K_3, F) = O(n)$ if and only if F is an extended friendship graph.

Proof of Theorem 4.3. If F is an extended friendship graph, then $\mathrm{rex}(n, K_3, F) \le \mathrm{ex}(n, K_3, F) = O(n)$.

Assume that F is not an extended friendship graph. Then it either contains two vertex-disjoint triangles or a longer cycle C_k with $k \ge 4$. In the first case, we take the K_3-free graph on $n-1$ vertices with regularity $r = \Omega(n)$ due to Caro and Tuza [12]. In particular, it contains an induced copy K of $K_{r/2,r/2}$. We remove a perfect matching from K, and add a new vertex v, connected to the vertices of K. The resulting graph is r-regular, and contains $r^2/4$ triangles that all contain v, completing our proof.

Let us assume now that F contains C_k. Let n be sufficiently large. We pick $r = \omega(1)$ such that r is small enough to have a $2r$-regular graph of girth more than $3k^2$ on m vertices whenever $m \ge 11n/36$ and such that $n > r^{3|V(F)|}$. We take an r-regular m-vertex graph G_0 of girth at least $3k^2$ where $m = \lfloor n/4 \rfloor$, using Corollary 4.11. We consider G_0^k as an auxiliary graph. Recall that the kth power G_0^k of a graph G_0 is obtained by joining vertices of distance at most k.

It is easy to see that G_0^k is r'-reg, where $r' = r + r(r-1) + \cdots + r(r-1)^{k-1}$. We take a proper $r'+1$-edge-coloring of G_0^k. Since G_0 is a subgraph of G_0^k, we obtain a proper edge-coloring of G_0. For each color i, we partition the edges of color i to some number of r-sets and a set of order at most r. For each such set, we add a new vertex and connect it to the at most $2r$ vertices that are incident to those at most r edges. This way we obtain G_1.

The vertices of G_0 have degree r in G_0, thus they are incident to edges of r colors, hence their degree is $2r$ in G_1. The newly added vertices have degree $2r$, except $r'+1$ vertices, that are connected to the endpoints of less than r edges. Let us assume that the sum of degrees in G_1 is $2r|V(G_1)|-\ell$. Note that ℓ is even since each vertex has an even degree and at most $2r(r'+1)$. There are rm edges from $V(G_0)$ to $V(G_1)\setminus V(G_0)$ and at least $2r(|V(G_1)\setminus V(G_0)|-r'-1)$ edges from $V(G_1)\setminus V(G_0)$ to $V(G_0)$, thus $|V(G_1)\setminus V(G_0)|\le \frac{m}{2}+r'+1$, hence $V(G_1)\le 3n/8+O(1)\le 4n/9$.

Now we make G_1 regular. We take a copy of a $2r$-regular graph G_0' and remove $\ell/2$ independent edges the following way. First we delete an arbitrary edge u_1v_1. The number of vertices at distance at most $|V(F)|+1$ from u_1 is at most $r''=2r+2r(2r-1)+\dots+2r(2r-1)^{|V(F)|+1}$. Then we pick a vertex u_2 that is at distance at least $|V(F)|+1$ from u_1 and a neighbor v_2 of u_2. Then we delete the edge u_2v_2 and these four vertices are at distance at least $|V(F)|$ from each other. We repeat this, always picking vertices u_i that are at distance at least $|V(F)|+2$ from each of $u_1,\dots,u_{i-1}$. This is doable if $\ell r''<m$, which holds by our assumption on r.

The resulting graph Q_1 is of girth more than k with a set S of ℓ vertices of degree $2r-1$ and all the other vertices of degree $2r$. The vertices of S are at distance at least $|V(F)|$. We join each vertex v of G_1 to $2r-d(v)$ vertices of degree $2r-1$ in this new graph. The resulting graph G_2 is $2r$-regular on at most $25n/36$ vertices.

Finally, we add a $2r$-regular graph of girth more than k on $n-|V(G_2)|$ vertices. This exists by the choice of r. □

Now, we turn to the proof of Proposition 4.4. Recall that it states that if $3<\ell<2k+1$ is odd, then $\mathrm{rex}(n,C_\ell,C_{2k+1})=\Theta(n^{\lfloor \ell/2\rfloor})$.

Proof of Proposition 4.4. The upper bound is shown by $\mathrm{ex}(n,C_\ell,C_{2k+1})=\Theta(n^{\lfloor \ell/2\rfloor})$ in [54].

Let us turn to the lower bound. We start with an unbalanced blow-up of C_ℓ, where we blow up $(\ell-1)/2$ independent vertices to m-sets, and keep the other vertices (note that this construction shows the analogous bound for $\mathrm{ex}(n,C_\ell,C_{2k+1})$, but it is far from regular). Let H denote this graph, then the largest degree is $2m$ in H. We take two vertex-disjoint copies of H, denoted by $2H$. We add sets $A_1,A_2,\dots,A_{2k}$ of new vertices of order $2m-2$. We take two blown-up parts A and A' of order m of $2H$ arbitrarily. We add sets $A_1,A_2,\dots,A_{2k}$ of new vertices of order $2m-2$. We take all the possible edges between A and A_1, then an m-regular graph between A_i and A_{i+1} for each $i\le 2k-1$ (this exists because of Observation 4.12), and then take all the possible edges between A_{2k} and A'. It is easy to see that each vertex of A, A' and each A_i has degree $2m$ and no C_{2k+1} is created this way. We repeat this by taking $4k(m-1)$ new vertices as long as there are at least two blown up classes of order m in $2H$.

We are left with several vertices of degree $2m$ and exactly two adjacent vertices u,v and u',v' of degree $m+1$ in both copies of H (they are the adjacent vertices of the original ℓ-cycles that were not blown up). We take sets B, B' of order $m-1$ and $B_1,\dots,B_{2k}$ of order $2m-1$. We take all the edges between u and B and between

B and B_1. Then we take an $(m+1)$-regular graph between B_{2i+1} and B_{2i+2}, for each $0 \leq i \leq k-1$, and an $(m-1)$-regular graph between B_{2i} and B_{2i+1}, for each $1 \leq i \leq k-1$. Finally, we take all the edges between B_{2k} and B' and between B' and v. We do the same to deal with u' and v' in the other copy of H.

It is left to add $n - |V(H')|$ vertices without ruining these properties. Observe that we added at most $16\ell k m$ vertices to H. We pick m to be the largest odd number below $\lfloor n/40k\ell \rfloor$, thus H' has at most $n/2$ vertices. If $n - |V(H')|$ is even, we can pick a bipartite $2m$-regular graph on those vertices, completing the proof. If $n - |V(H')|$ is odd, we additionally pick a copy of $C_{2k+3}(m)$, and then pick a bipartite $2m$-regular graph on the remaining vertices, completing the proof. □

Let us continue with the proof of Proposition 4.6. Recall that it states that if H is a complete k-partite graph and $k \geq 3$, then $\mathrm{rex}(n, H, K_{k+1}) = (1+o(1))\mathcal{N}(H, T(n,k))$, and without the error term if k divides n. Let $T^*(n,k)$ denote an arbitrary n-vertex K_{k+1}-free regular graph with $\mathrm{rex}(n, K_{k+1})$ edges. Recall that if $k|n$, then $T^*(n,k)$ is $T(n,k)$, otherwise it is obtained from $T(n,k)$ by changing a small number of edges, as shown in [11, 12].

Proof of Proposition 4.6. The lower bound follows from $T^*(n,k)$, since as mentioned above it is very close to the Turán graph. We will prove the upper bound for $k \geq 2$. Note that if $k = 2$ the case n is even is equivalent to the even case of Proposition 4.7.

Let $H = K_{s_1,\dots,s_k}$ with $s_1 \leq s_2 \leq \dots \leq s_k$. For simplicity, we will deal with labeled copies of H, clearly the same n-vertex K_{k+1}-free regular graph maximizes (asymptotically) the number of labeled copies of H as the number of copies. We will show that for $k \geq 2$, any n-vertex K_{k+1}-free graph contains at most $k!\prod_{i=1}^k \binom{(n/k)}{s_i}$ copies of H. Clearly $T(n,k)$ satisfies this with equality if k divides n, and $T^*(n,k)$ gives the correct asymptotics, thus this upper bound completes the proof for other values of n as well.

We apply induction on k and on $\sum_{i=1}^k s_i$. The base case $k = 2$ follows from Proposition 4.7 if n is even. Moreover, the exact same proof works for the case n is odd, without changing a single word (but gives a bound that is not sharp). The other base case $\sum_{i=1}^k s_i = k$ follows from Zykov's theorem [95], which states that $\mathrm{ex}(n, K_r, K_{k+1}) = \mathcal{N}(H, T(n,k))$.

Let G be an n-vertex K_{k+1}-free r-regular graph, then $r \leq (k-1)n/k$ by Turán's theorem. We consider two cases. Assume first that $s_1 = 1$ and let H' be the graph we obtain by deleting the first class from H. Then we first pick a vertex v of G corresponding to the single vertex in the first class, at most n ways. Then we pick a labeled copy of H' in the neighborhood of v, at most $(k-1)!\prod_{i=2}^k \binom{(r/(k-1))}{s_i}$ by the induction on k. This way we picked the labeled copies of H at most $n(k-1)!\prod_{i=2}^k \binom{(r/(k-1))}{s_i} \leq nk!/k\prod_{i=2}^k \binom{(k-1)n/k)/(k-1)}{s_i} = k!\frac{n}{k}\prod_{i=2}^k \binom{n/k}{s_i} = k!\binom{n/k}{s_1}\prod_{i=2}^k \binom{n/k}{s_i}$.

Assume now that $s_1 > 1$ and let H'' be the graph we obtain by deleting a K_k from H. We first pick an unlabeled copy K of K_k, then a labeled copy of H'' from the remaining vertices, and then add the labels to the vertices of K. By Zykov's theorem, the number of unlabeled copies of K_k is maximized by the Turán graph, thus it is at most n^k/k^k. The number of labeled copies of H'' is at most $k!\prod_{i=1}^k \binom{(n-k)/k}{s_i-1}$ by induction on $\sum_{i=1}^k s_i$. Afterwards, we add the vertices of K to the vertices of H''. Observe that each vertex of K has a copy of K_{k-1} in their neighborhood in H''. As G is K_{k+1}-free, the vertices of a K_{k-1} cannot be adjacent to two adjacent vertices. This implies that each copy of K_{k-1} has at most one common neighbor in K. Therefore, each vertex of K can belong to at most one of the classes of H. This means that the number of labels the vertices of K can receive is at most $\prod_{i=1}^k s_i$, hence the number of labeled copies of H is at most $\frac{1}{\prod_{i=1}^k s_i}\frac{n^k}{k^k}k!\prod_{i=1}^k\binom{(n-k)/k}{s_i-1} = k!\prod_{i=1}^k\binom{n/k}{s_i}$. This completes the proof. □

Let us continue with Proposition 4.7. Recall that it extends the above proposition to the case $k = 2$ if n is even. If n is odd, then it deals with the case H is a tree and claims that the extremal graph G^* is a regular graph obtained by deleting some edges of a blow-up of C_5.

Proof of Proposition 4.7. Let n be even, consider a component H_0 of H and an ordering of the vertices of H_0 such that each but the first vertex has an earlier neighbor. Such an ordering exists by first picking an arbitrary vertex and then each time picking a neighbor of a vertex already picked.

Let G be an n-vertex r-regular triangle-free graph, then $r \leq n/2$. Moreover, either G is bipartite (with both parts of order $n/2$, thus G is contained in $T(n,2)$, completing the proof), or $r \leq 2n/5$. Assume that $r \leq 2n/5$. There are at most n ways to pick the first vertex and at most $2n/5$ ways to pick each subsequent vertex. In $T(n,2)$ there are n and $(1+o(1))n/2$ ways to do this. The copies of H_0 may be counted multiple times, but the number of times is a fixed constant c depending only on the automorphisms of H and not the host graph. Therefore, G contains at most $2\left(\frac{2n}{5}\right)^{|V(H_0)|}/c$, while $T(n,2)$ contains $(2+o(1))\left(\frac{n}{2}\right)^{|V(H_0)|}/c$ copies of H_0. Then we pick the other components of H. Similarly, there are more ways to pick each component in $T(n,2)$ than in G if n is large enough. Therefore, $T(n,2)$ contains more copies of H for sufficiently large n, completing the proof.

Assume now that n is odd and H is a forest. Let G be an n-vertex r-regular triangle-free graph, then $r \leq 2n/5$. Moreover, by Lemma 4.13, either G is obtained by deleting some edges of an n-vertex blow-up of C_5, or $r \leq 2n/5 - \varepsilon n$ for some $\varepsilon > 0$. In the first case, we are done. In the second case, we can proceed similarly to the argument in the case where n is even. G contains at most $2\left(\frac{2n}{5} - \varepsilon n)\right)^{|V(H_0)|}/c$ copies of H_0, while G^* contains $2\left(\frac{2n}{5}\right)^{|V(H_0)|}/c$ copies of H_0. The same holds for other components, thus G^* contains more copies of H for sufficiently large n, completing the proof in this case. □

4.4 Triangles and Paths

Here, we prove Theorem 4.8. First, let us mention some results that will be used. By a theorem of Erdős and Gallai [28], if a connected graph has at least k vertices and minimum degree $\lfloor k/2 \rfloor$, then it contains a P_k. Gerbner, Patkós, Tuza and Vizer [50] gave the exact value of $\mathrm{regex}(n,T)$ for any tree T and large n. Note that

$\mathrm{regex}(n,F) = \max\{d \ : \ \text{there is an } n\text{-vertex, } d\text{-regular, } F\text{-free graph } G\}$. Clearly we have that $2 \cdot \mathrm{rex}(n,F) = n \cdot \mathrm{regex}(n,F)$.

Theorem 4.14 (Gerbner, Patkós, Tuza, Vizer [50])**.** *Let T be a tree on t vertices and $n > n_0(T)$. Then*

$$\mathrm{regex}(n,T) = \begin{cases} t-2 & \text{if } (t-1)|n \text{ or } T \text{ is a star and } t \text{ or } n \text{ is even,} \\ t-3 & \text{if the above fails, and either } t \text{ is odd, or } (t-2)|n, \\ & \text{or } T \text{ is a star or } T \text{ is an almost-star and } n \text{ is even.} \\ t-4 & \text{otherwise.} \end{cases}$$

A tree is an *almost-star* if in its proper 2-coloring, one of the classes consists of at most two vertices (thus a path on at least 6 vertices is not an almost-star).

Now we are ready to present the proof of Theorem 4.8, which determines $\mathrm{rex}(n,K_3,P_k)$ if $k \geq 7$ and n is sufficiently large.

Proof of Theorem 4.8. Let G be an n-vertex r-regular P_k-free graph containing the maximum number of triangles. Observe that each vertex of G is in at most $\binom{r}{2}$ triangles, and hence $\mathcal{N}(K_3,G) \leq \frac{n}{3}\binom{r}{2}$, with equality only when each vertex is in a clique K_{r+1}. Note that in each of the constructions described in the introduction, the r-regular n-vertex graph contains $n/(r+1) - O(1)$ copies of K_{r+1}, hence contains $\frac{n}{3}\binom{r}{2} - O(1)$ triangles. This implies that, for large n, a graph with smaller regularity cannot contain more triangles than our construction. Therefore, r is at least the regularity of the claimed unique construction. Using Theorem 4.14, in each of the cases we know that r is at most the regularity of the claimed unique construction, and it is left to show that no other r-regular graph can contain at least as many triangles as our construction, in each case.

As G is P_k-free, we have $r \leq k-2$ by Theorem 4.14. Therefore, $\mathcal{N}(K_3,G) \leq \frac{n}{3}\binom{k-2}{2} = \frac{n}{k-1}\binom{k-1}{3}$, with equality only when $(k-1)|n$ and G is $n/k-1$ disjoint copies of K_{k-1}, proving the first case. If n is not divisible by $(k-1)$ and $r = k-2$, then by Theorem 4.14 we have that P_k is a star, i.e., $k \leq 3$. Consequently, $r < k-2$.

In the second case, we have k is odd or $k-2$ divides n and we can assume $r = k-3$. We can write n as $a(k-2)+b$, where $0 \leq b \leq k-3$. Since $k \geq 6$, we have $k-3 \geq k/2$, and hence, by the result of Erdős and Gallai, each component has at most $k-1$ vertices. Thus, each component is either a K_{k-2} or G'_{k-1}, for these are the only $(k-3)$-regular graphs on at most $k-1$ vertices. This means $G = xK_{k-2} + yG'_{k-1}$, which gives $x(k-2) + y(k-1) = n = a(k-2)+b$, implying

$$y = a'(k-2)+b, \ \text{where } a' = a-x-y.$$

If $y > b$, then $a' \geq 1$, and hence, $y \geq (k-2)+b$, then we can replace $k-2$ copies of G'_{k-1} by $k-1$ copies of K_{k-2}, increasing the number of triangles as $\mathcal{N}(K_3,K_{k-2}) > \mathcal{N}(K_3,G'_{k-1})$, contradicting the choice of G. Also, if $y < b$, then as $k-2 > 0$, we must have $a' < 0$, which implies $y < 0$, a contradiction. Therefore, $y = b$ and $x = a-b$, which proves the second case.

Note that if k is even, then K_{k-1} does not contain a perfect matching, and hence, if $(k-2) \nmid n$, then $r < k-3$, leading to the third case.

Assume now $r = k - 4$ and $k \geq 6$ is even. We can write n as $a(k-3) + b$, where $0 \leq b \leq k - 4$. First recall that each graph in $\mathcal{G}_{k-1}$ contains the same number of triangles. By the same reasoning of the previous case, we may assume that G consists of $xK_{k-3} + yG_{k-2}$ and z copies of graphs from $\mathcal{G}_{k-1}$. This gives $x(k-3) + y(k-2) + z(k-1) = n = a(k-3) + b$, implying

$$2z \geq a'(k-3) - y + b, \text{ where } a' = a - x - y - z.$$

Note that $\mathcal{N}(K_3, K_{k-3}) + \mathcal{N}(K_3, G_{k-1}) > 2\mathcal{N}(K_3, G_{k-2})$, for any graph $G_{k-1} \in \mathcal{G}_{k-1}$, and hence, whenever there are two copies of G_{k-2} in G, we can replace them by a copy of K_{k-3} and a copy of G_{k-1}, increasing the number of triangles. Therefore, we have that y is either 0 or 1. If $z > \lfloor b/2 \rfloor$, then we have $a' \geq 1$, which means $z \geq (k-3)/2 - y/2 + b/2$. If b is even we may assume $y = 0$, and hence, in both cases of b being odd or even, we still have $z \geq \lceil (k-3)/2 \rceil + \lfloor b/2 \rfloor$. We can then replace $\lceil (k-3)/2 \rceil$ copies of G_{k-1} by $\lfloor (k-1)/2 \rfloor$ copies of K_{k-3} and a copy of G_{k-2}, increasing the number of triangles, which contradicts the extremality of G. Again, due to compatibility of the number of vertices, z cannot be less than $\lfloor b/2 \rfloor$, proving the third case.

Finally, if $k = 5$, the only connected regular P_k-free graph with regularity at least 3 is K_4. In the case of regularity 2, the only connected 2-regular graph that contains a triangle is K_3. If 3 does not divide n, we need to add at least one longer cycle to a graph consisting of vertex-disjoint triangles. If $k = 4$, the only connected regular P_k-free graph that contains a triangle is K_3, thus if n not divisible by 3, then $\mathrm{rex}(n, P_4) = 0$. In the remaining cases $k \leq 3$, the triangle contains P_k, thus $\mathrm{rex}(n, P_k) = 0$, completing the proof. $\square$

Chapter 5

Generalized planar Turán numbers related to short cycles

5.1 Introduction

In this chapter, we study some generalized planar problems. We consider triangle-free planar graphs and count cycles of lengths $4, 5$ or 6. Similarly, we determine the maximum possible number of triangles in planar graphs when such a cycle is forbidden. As mentioned in Chapter 1, the generalized version of the planar Turán numbers, $\mathrm{ex}_{\mathcal{P}}(n, H, \mathcal{F})$, were introduced by Győri, Paulos, Salia, Tompkins and Zamora [62] in 2021. Interestingly, maximizing the number of copies of a given subgraph H in planar graphs can be viewed as a special case of generalized planar Turán problems by taking $\mathcal{F} = \emptyset$. Such a problem has been studied much earlier. In 1979, Hakimi and Schmeichel [66] determined the maximum number of triangles and 4-cycles in planar graphs. Alon and Caro [1] determined the maximum number of copies of $K_{2,t}$ in planar graphs. Győri, Paulos, Salia, Tompkins and Zamora [63] determined the maximum number of 5-cycles in planar graphs, proving that $\mathrm{ex}_{\mathcal{P}}(n, C_5, \emptyset) = 2n^2 - 10n + 12$, for every $n = 6$ or $n \geq 8$. Although exact results for longer cycles are not known, Cox and Martin [18] developed a general technique to count subgraphs in planar graphs, and conjectured that the maximum number of an even cycle C_{2k} is asymptotically $(n/k)^k$. This conjecture was proved by Lv, Győri, He, Salia, Tompkins and Zhu [80].

Theorem 5.1 (Lv *et al.* [80]). *For every $k \geq 3$, $\mathrm{ex}_{\mathcal{P}}(n, C_{2k}, \emptyset) = \left(\frac{n}{k}\right)^k + o(n^k)$.*

Another direction of research is maximizing the number of induced subgraphs in planar graphs. Ghosh, Győri, Janzer, Paulos, Salia, Zamora [52], and independently Savery [87], determined the maximum number of induced 5-cycles in planar graphs. Savery [86] extended this to induced 6-cycles.

Now, let us state our results that we prove in this chapter.

Theorem 5.2. *For every $n \geq 4$, $\mathrm{ex}_{\mathcal{P}}(n, C_4, C_3) = \binom{n-2}{2}$, and the unique extremal graph is $K_{2,n-2}$.*

Theorem 5.3. *For every* $n \geq 5$, $\mathrm{ex}_{\mathcal{P}}(n, C_4, C_5) = \binom{n-2}{2}$*. Furthermore, for* $n = 5$*, the extremal graphs are* $K_{2,n-2}$*,* $K_2 + \overline{K_{n-2}}$*,* $K_4 \cup K_1$ *or* K_4 *with a pendant edge, and for* $n \geq 6$*, the extremal graphs are* $K_{2,n-2}$ *or* $K_2 + \overline{K_{n-2}}$*.*

Note that for $n = 4$, we trivially have $\mathrm{ex}_{\mathcal{P}}(n, C_4, C_5) = 3$, and K_4 is the extremal graph.

Theorem 5.4. *For every* $n \geq 4$*, we have* $\mathrm{ex}_{\mathcal{P}}(n, C_3, C_4) \leq \frac{5}{7}(n-2)$*, and this bound is sharp for infinitely many values of* n*.*

The following theorem is a bit different, as we determine the maximum number of triangles in K_4-free planar graphs.

Theorem 5.5. *For every* $n \geq 3$*, we have* $\mathrm{ex}_{\mathcal{P}}(n, C_3, K_4) \leq \frac{7}{3}n - 6$*, and this bound is sharp for all* n *divisible by* 3*.*

Theorem 5.6. *For every* $n \geq 5$*,* $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_3) = \lfloor (n-3)/2 \rfloor \cdot \lceil (n-3)/2 \rceil$*.*

An extremal graph for $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_3)$ can easily be seen to be a C_5 on which two non-adjacent vertices are blown-up in a balanced way. Somewhat surprisingly, there are many other extremal graphs.

For each $n \geq 5$, define a class $\mathcal{J}_n$ of n-vertex planar graphs as follows. Take a regular pentagon, replace one of its vertices, x, by an independent set of vertices C, and replace the edge yz opposite to x by a tree with color classes A and B, such that $|C|$ and $|A \cup B| - 1$ are as equal as possible (see Figure 5.1(a)). We prove the following stronger theorem.

Theorem 5.7. *For every* $n \geq 5$*,* $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_3) = \lfloor (n-3)/2 \rfloor \cdot \lceil (n-3)/2 \rceil$*, and* $\mathcal{J}_n$ *is the set of all extremal graphs.*

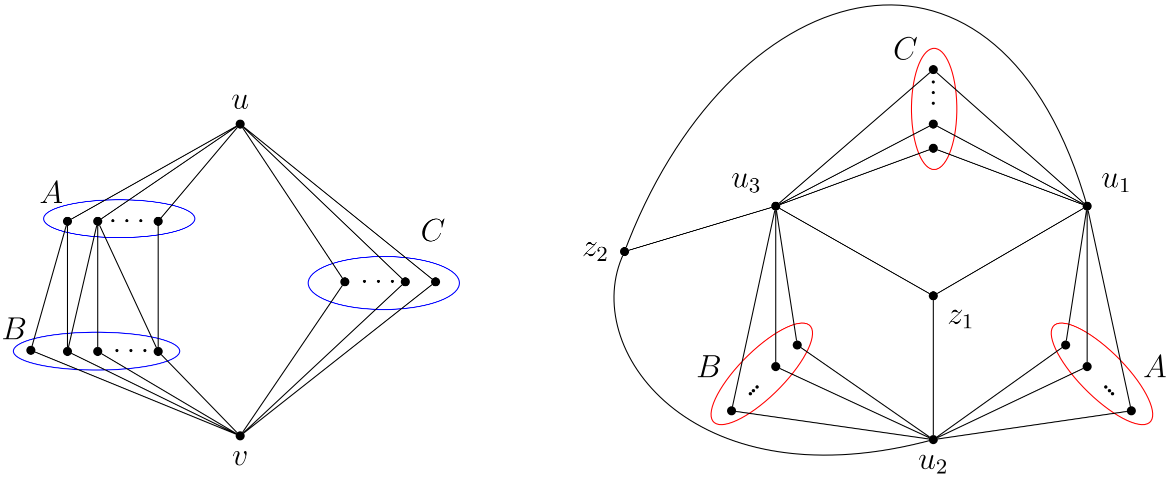
In the other direction, maximizing the number of triangles pentagon-free graphs, we prove the following theorem.

Theorem 5.8. *For every* $n \geq 11$*,* $\mathrm{ex}_{\mathcal{P}}(n, C_3, C_5) \leq \left\lfloor \frac{8n-22}{5} \right\rfloor$*, and this bound is sharp for infinitely many values of* n*.*

Then, we turn to triangles and 6-cycles, and first maximize the number of triangles in C_6-free graphs.

Theorem 5.9. *For every* $n \geq 18$*,* $\mathrm{ex}_{\mathcal{P}}(n, C_3, C_6) \leq \frac{35n-98}{18}$*, and this bound is sharp for infinitely many values of* n*.*

Finally, we determine the maximum number of 6-cycles while forbidding triangles, together with the unique extremal graph achieving the maximum value. Along the proof, we will be considering the number of paths of length four (i.e. of five vertices) between two vertices. This is an interesting problem on its own, and we prove a theorem as follows.



(a) $\mathcal{J}_n$ (b) H_n

Figure 5.1: The extremal graphs for $\mathrm{ex}_{\mathcal{P}}(n, C_5, C_3)$ and $\mathrm{ex}_{\mathcal{P}}(n, C_6, C_3)$

Theorem 5.10. *Let G be a triangle-free planar graph on $n \geq 5$ vertices. For any two vertices $u, v \in V(G)$, there are at most $(\frac{n-1}{2})^2 - 2$ paths of length four connecting them.*

Before stating the next theorem, we present the following construction. For every $n \geq 6$, define a graph H_n as follows. The vertex set of H_n consists of $\{u_1, u_2, u_3\} \cup \{z_1, z_2\} \cup A \cup B \cup C$, such that each of these sets is an independent set of vertices, they are pairwise disjoint, each of the u_i's is adjacent to each of the z_i's, every vertex in A is adjacent to both of u_1 and u_2, every vertex in B is adjacent to both of u_2 and u_3, and every vertex in C is adjacent to both u_1 and u_3. Moreover, the sizes of A, B and C are as equal as possible (see Figure 5.1(b)). It is easy to see that H_n contains $h(n)$ 6-cycles (nevertheless, we will give an explanation for this in Section 5.4), where $h(n)$ is defined as follows.

$$h(n) = \begin{cases} \frac{n^3}{27} + \frac{n^2}{9} - 2n + 2, & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{n^3}{27} + \frac{n^2}{9} - 2n + \frac{50}{27}, & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{n^3}{27} + \frac{n^2}{9} - \frac{17n}{9} + \frac{55}{27}, & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

Theorem 5.11. *Let n be sufficiently large. Then, $\mathrm{ex}_{\mathcal{P}}(n, C_6, C_3) = h(n)$, and the unique extremal graph is H_n.*

In fact, Theorems 5.7 and 5.11 are the highlights of this chapter. The proof of Theorem 5.11 makes it possible to prove the following interesting variant of Theorem 5.10.

Theorem 5.12. *Let G be a triangle-free planar graph on n vertices. For any three distinct vertices $u_1, u_2, u_3 \in V(G)$, the total number of paths of length four joining* *any two of these three vertices is at most* $3\left(\frac{n+1}{3}\right)^2 - 6$. *Moreover, for each* $n \equiv 2$ *(mod 3), the graph* H_n *attains this bound.*

In this chapter, some further notation is used. We say that a cycle C in a plane graph G is not empty if there are vertices of G in the region bounded by C, and C is said to be a separating cycle if there are vertices in both the interior and the exterior regions of C. For a separating cycle C of G, a subgraph H is said to be a *crossing* H, if it uses vertices in both the interior and exterior regions of C. We let $P_l(x, y)$ denote a path P_l connecting the two vertices x and y (that is, x and y are the end points of the path). Given a vertex v, we denote by $C_l(v)$ a cycle of length l that contains v. Besides $\mathcal{N}(H, G)$, when the graph G is clear, we also use $\#H$ to denote the number of copies of H.

We will prove the results concerning 4-cycles in Section 5.2. Section 5.3 contains the proof of the results about 5-cycles and triangles, and in Section 5.4 we give the proofs of the results related to 6-cycles.

5.2 Triangles and 4-cycles

In this Section, we prove Theorems 5.2 - 5.5. Let us start with Theorem 5.2

Proof of Theorem 5.2. Obviously, the complete bipartite graph $K_{2,n-2}$ is a planar triangle-free graph that contains $\binom{n-2}{2}$ copies of C_4. Let G be a triangle-free planar graph on $n \geq 4$ vertices. We prove that $\mathcal{N}(C_4, G) \leq \binom{n-2}{2}$, and equality holds if and only if G is $K_{2,n-2}$.

The proof is by induction on n. For $n \leq 5$ the statement holds obviously. Assume $n > 5$. Let G_0 be a maximum $K_{2,k}$ subgraph of G. Let $\{u, v\}$ be the class of G_0 of size two, and $\{x_1, x_2, \ldots, x_k\}$ be its class of size k. Since G is triangle-free, these sets are independent in G, too. Note that the subgraph G_0 has k faces each of which is a C_4. If a face of G_0 is not a face in G, i.e. it contains some vertices of G in its interior, then there cannot be any crossing C_4,i.e. a 4-cycle that uses vertices from both the interior and the exterior of the face. Indeed such a crossing C_4 must use exactly one vertex within the face, and since G is planar and triangle-free, it must be of the form $yuzv$, for some vertex y within the face and z not in the face. Then, y is a common neighbor of u and v different from x_i's, contradicting the maximality of G_0. Note that if all the 4-cycles of G are faces of G, then due to Euler's formula, $\mathcal{N}(C_4, G) \leq n - 2 < \binom{n-2}{2}$ (since $n > 5$), so we may assume that G contains 4-cycles that are not faces.

Claim 5.1. *If* $V(G) \setminus V(G_0) \neq \emptyset$*, then* G *has a separating* C_4 *for which there are no crossing* 4*-cycles.*

Proof of the claim. If $k = 2$, then G_0 is a C_4 and has two faces. We may choose it to be a separating C_4, i.e. both faces are non-empty, since otherwise all the 4-cycles of G are faces, a contradiction.

If $k > 2$, then G_0 has a non-empty face, since $V(G) \setminus V(G_0) \neq \emptyset$. Hence, obviously that face is a separating C_4. In both cases the separating C_4 is a face of G_0, and hence as explained above, for this separating C_4 there are no crossing 4-cycles in G. $\square$

Now, let us complete the proof of the theorem. If $V(G) \setminus V(G_0) = \emptyset$, then $k = n-2$, G is the graph $K_{2,n-2}$ and $\mathcal{N}(C_4, G) = \binom{n-2}{2}$. Otherwise, G has a separating C_4 by Claim 5.1, call it C, for which there are no crossing 4-cycles in G. Let L be the set of vertices of G in the interior of C, with $|L| = l$. Let $G_1 := G[L \cup V(C)]$ and $G_2 := G[V(G) \setminus L]$. Thus,

$$\begin{aligned}\mathcal{N}(C_4, G) &= \mathcal{N}(C_4, G_1) + \mathcal{N}(C_4, G_2) - 1\\ &\leq \binom{(l+4)-2}{2} + \binom{(n-l)-2}{2} - 1\\ &= \binom{l+2}{2} + \binom{(n-2)-l}{2} - 1\\ &= \binom{n-2}{2} + l^2 + 4l - nl\\ &< \binom{n-2}{2}.\end{aligned}$$

where the term -1 is to avoid double counting C, the first inequality is due to the induction hypothesis, and the last one is justified by the fact that $n \geq l+5$ and $l \geq 1$ (since C is a separating cycle), which gives $l^2+4l-nl \leq l^2+4l-(l+5)l = -l < 0$. $\square$

Proof of Theorem 5.3. The mentioned extremal graphs establish the lower bounds. Let G be a C_5-free planar graph on $n \geq 4$ vertices. Observe that a triangle and a C_4 cannot share exactly one edge, since otherwise they would form a C_5. Obviously, a triangle does not share all of its three edges with a C_4.

Claim 5.2. *For every triangle T in G, either T has at least one edge not contained in any C_4, or T is in a K_4.*

Proof. Let $T := uvw$ be a triangle in G that is not in any K_4 of G. Suppose T shares edges with a 4-cycle C. Then, T shares two edges with C, without loss of generality, say uv and vw. Thus, C is $uvwx$, for some vertex x in G. Now, assume that the edge uw of T is also contained in some 4-cycle C'. If $V(C') = V(C)$, then C' must be $uwvx$, which means $V(C)$ induces a K_4 in G and T is in a K_4, a contradiction. Otherwise, since C' contains two edges of T, it contains uw and another edge, say vw. Hence, C' is $uwvy$ for some y different from x. But then $uyvwx$ is a C_5 in G, a contradiction. $\square$

We now consider two cases.

Case 1. G contains a K_4.

Let K be a copy of a K_4 in G. Then, no 4-cycles outside k can use more than a vertex of K, as otherwise they form a C_5. Also, for the same reason, no edge of K is in a triangle with a vertex not in K. We proceed by induction on n. If $n = 4$, then G is a K_4 and $\mathcal{N}(C_4, G) = 3$, so the statement is true. Also, for the above reasons, if

$n = 5$ or 6, each vertex in $V(G) \setminus V(K)$ can be adjacent to at most one vertex of K. Thus, the vertices not in K cannot be in any 4-cycles. Hence, $\mathcal{N}(C_4, G) = 3 \leq \binom{n-2}{2}$.

Note that if $n = 5$ equality holds above, and hence G can be a $K_4 \cup K_1$ or a K_4 with a vertex joined to one of its vertices. However, for $n = 6$, the inequality is strict, which implies that the extremal graph cannot contain a K_4. Now, let $n \geq 7$. Then, one of the triangles of K is a separating cycle, denote it by T. Let L be the set of vertices in the region bounded by $T := uvw$. Let $G_1 := G[L \cup V(T)]$ and $G_2 := G \setminus L$. Let $n_i = |V(G_i)|$, for $i = 1, 2$. Note that $n_1 + n_2 = n + 3$. We claim that there is no crossing 4-cycle that uses vertices both of L and of $V(G) \setminus (L \cup \{u, v, w\})$. Indeed, such a C_4 must use one vertex in each side and two vertices of T, which means an edge of T, and in turn of K, is in a triangle with a vertex not in K, which is a contradiction. Thus,

$$\mathcal{N}(C_4, G) = \mathcal{N}(C_4, G_1) + \mathcal{N}(C_4, G_2)$$

If $n_1 = 4$, then $|L| = 1$, say $L = \{x\}$. Then, G_1 contains at most three C_4's, and hence x is in at most three 4-cycles. Since $n \geq 7$, $n_2 = n - 1 \geq 6$, by induction hypothesis, $\mathcal{N}(C_4, G_2) \leq \binom{n_2-2}{2} = \binom{n-3}{2}$. Then,

$$\mathcal{N}(C_4, G) = 3 + \mathcal{N}(C_4, G_2) \leq 3 + \binom{n-3}{2} < \binom{n-2}{2}.$$

If $n_2 = 4$, we similarly get the same result. So assume that each of n_1 and n_2 is at least 5. Then, by induction hypothesis, for each $i = 1, 2$, $\mathcal{N}(C_4, G_i) \leq \binom{n_i-2}{2}$. Thus,

$$\begin{aligned}
\mathcal{N}(C_4, G) &= \mathcal{N}(C_4, G_1) + \mathcal{N}(C_4, G_2) \\
&\leq \binom{n_1-2}{2} + \binom{n_2-2}{2} \\
&< \binom{(n_1-2) + (n_2-2) - 1}{2} = \binom{n-2}{2}.
\end{aligned}$$

Case 2. G is K_4-free.

By claim 5.2, every triangle of G has at least one edge that is not contained in any C_4. By deleting such edges from each triangle, we obtain a triangle-free subgraph $G' \subseteq G$ without reducing the number of 4-cycles. Thus, applying Theorem 5.2, we have $\mathcal{N}(C_4, G) = \mathcal{N}(C_4, G') \leq \mathrm{ex}_{\mathcal{P}}(n, C_4, C_3) = \binom{n-2}{2}$, which shows that $\mathrm{ex}_{\mathcal{P}}(n, C_4, C_5) \leq \mathrm{ex}_{\mathcal{P}}(n, C_4, C_3) = \binom{n-2}{2}$.

It remains to show that $K_{2,n-2}$ and $K_2 + \overline{K_{n-2}}$ are the only extremal graphs. Assume G in the above argument has $\binom{n-2}{2}$ 4-cycles. Then, the triangle-free subgraph $G' \subseteq G$ has $\binom{n-2}{2}$ copies of C_4, too. Then, again by Theorem 5.2, G' is isomorphic to $K_{2,n-2}$. If $|E(G) \setminus E(G')| \geq 2$, then at least one of the edges of G has its both endpoints in the partite set of size $n - 2$, forming a C_5, which is a contradiction. Thus, $|E(G) \setminus E(G')| \leq 1$, and hence, either $G = G' = K_{2,n-2}$ or G has one edge in the partite set of size 2, which means $G = K_2 + \overline{K_{n-2}}$. ☐

Proof of Theorem 5.4. If G is a planar C_4-free graph, then no two triangles in G can share an edge, as otherwise a C_4 would be formed. Then, every edge of G is in at most one triangle, which gives

$$3\mathcal{N}(C_3, G) \leq e(G) \leq \mathrm{ex}_{\mathcal{P}}(n, C_4).$$

Dowden [19] showed that $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15}{7}(n-2)$, which implies $\mathcal{N}(C_3, G) \leq \frac{5}{7}(n-2)$, and hence,

$$\mathrm{ex}_{\mathcal{P}}(n, C_3, C_4) \leq \frac{5}{7}(n-2).$$

They also provided constructions for every $n \equiv 30$ $(mod\ 70)$ that attain $\mathrm{ex}_{\mathcal{P}}(n, C_4)$, implying the sharpness of their result. In fact, every edge in their extremal graphs is in a C_3 (see Theorem 2 in [19]), and hence they provide graphs that achieve the stated upper bound in our case as well, implying the sharpness of the bound. □

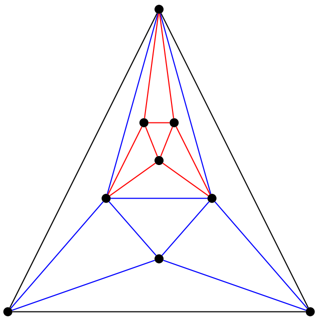


Figure 5.2: A K_4-free graph attaining maximum number of triangles.

Proof of Theorem 5.5. The proof is by induction on n. Let G be a K_4-free planar graph on n vertices. We can easily check that the result holds for $n = 3, 4$ and 5, so assume $n \geq 6$. Further, we may assume that G is connected as otherwise we may apply induction on the components. If every C_3 in G is a face, then using Euler's formula, we have $\mathcal{N}(C_3, G) = f = e + 2 - n \leq (3n-6) + 2 - n \leq 7n/3 - 6$. Let C be a triangle that is not a face, then it contains vertices both in its interior and its exterior. Let C contain m vertices inside and $n - m$ vertices outside. Let G_1 be the induced subgraph of G on the vertices of C and those in its interior, and G_2 be the induced subgraph on the vertices on C and those in its exterior. Note that $|V(G_1)| = m + 3, |V(G_2)| = n - m$ and C is the only triangle that is in both of G_1 and G_2. Applying the induction hypothesis, we obtain

$$\begin{aligned} \mathcal{N}(C_3, G) &= \mathcal{N}(C_3, G_1) + \mathcal{N}(C_3, G_2) - 1 \\ &\leq \left(\frac{7}{3}(m+3) - 6\right) + \left(\frac{7}{3}(n-m) - 6\right) - 1 \\ &= \frac{7}{3}n - 6. \end{aligned}$$

The sharpness of the bound is shown by the following construction. Starting from a triangle, insert a triangle inside and in the region between the two triangles join all the vertices via a 6-cycle, keeping the graph planar. Now, we obtain G_{k+1} from G_k by inserting a triangle in a face of G_k (all of which are triangles) in the same way, as shown in Figure 5.2. It is not difficult to show that for all k, G_k is K_4-free. Indeed, G_1 is just a triangle. Assume G_k is K_4-free. Suppose G_{k+1} contains a K_4. This K_4 cannot contain vertices of the triangle added at the last step to build G_{k+1} from G_k. For it cannot be only one vertex, since none of the new vertices has degree three in G_k, it cannot be an edge of the newly added triangle, as we do not join it completely to two vertices of the boundary of the face we put it in, and finally it cannot be the hole triangle, as we do not join a vertex of G_k to all vertices of the new triangle. Thus, the K_4 must be entirely contained in G_k, a contradiction. ☐

5.3 Triangles and Pentagons

In this section we prove Theorem 5.7 and Theorem 5.8. For the first one we follow a similar idea as in the proof of Theorem 5.2. Basically, given an n-vertex graph G we take a subgraph G_0 that is a slight modification of a member of $\mathcal{J}_n$, and then prove that G has the maximum number of C_5's if G_0 contains all vertices of G. First we prove a lemma that will be useful and used repeatedly throughout.

Lemma 5.13. *Let G be a triangle-free planar graph. Let $u, v \in V(G)$, $A \subseteq N(u)$ and $B \subseteq N(v)$. Then $|E(A,B)| \leq |A| + |B|$. Moreover, if there is a path joining u and v without using a vertex from $A \cup B$, then $G[A \cup B]$ is acyclic, and $|E(A,B)| \leq |A| + |B| - 1$.*

Proof. Consider the bipartite subgraph H with partite sets $A \cup \{v\}$ and $B \cup \{u\}$. Note that a common neighbor of u and v cannot be adjacent to any vertex of $A \cup B$, since it would form a triangle, that is, it has degree zero in H, so we may assume that A and B are disjoint. Since H is planar, $|E(H)| \leq 2|V(H)| - 4 = 2(|A| + |B|)$. Since $|A|$ edges are incident to u and $|B|$ edges are incident to v, then $|E(A,B)| \leq 2(|A| + |B|) - |A| - |B| = |A| + |B|$.

For the second part, since G is triangle-free, both A and B are independent sets, hence $G[A \cup B]$ is bipartite. Thus, it cannot contain an odd cycle. Assume that it contains an even cycle $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ of length $2k$, for $k \geq 2$. Recall that there is a path P connecting u and v that does not use the vertices in $A \cup B$. Then, the sets $\{u, b_1, b_2\}$ and $\{v, a_1, a_2\}$ with the edges $ua_1, ua_2, b_1v, b_1a_1, b_1a_2, b_2v, b_2a_2$, the path $b_2a_3b_3 \dots a_kb_ka_1$ on the cycle and the path P joining u and v form a subdivision of $K_{3,3}$ in G, contradicting the planarity of G. This completes the proof. ☐

Corollary 5.14. *Let G be a triangle-free planar graph. For every two vertices $u, v \in V(G)$, the number of paths of length three $P_4(u,v)$ connecting them is at most $|N(u) \setminus N(v)| + |N(v) \setminus N(u)|$. Furthermore, if u and v are adjacent or have a common neighbor, then there are at most $|N(u) \setminus N(v)| + |N(v) \setminus N(u)| - 1$ such paths.*

Proof. Since G is triangle-free, no path $P_4(u,v)$ can contain a common neighbor of u and v, as otherwise a triangle is created. Every path $P_4(u,v)$ is determined by an edge in $E(N(u) \setminus N(v), N(v) \setminus N(u))$, so the result follows from Lemma 5.13. ☐

Proof of Theorem 5.7. The proof is by induction on n. Let G be a triangle free plane graph on n vertices. The result is obviously true for $n = 5$, and we can easily check it is also true for $n = 6, 7$. So, assume $n \geq 8$. Let u and v be two non-consecutive vertices on a C_5 in G, and then take G_0 to be the subgraph of G that consists of all the 5-cycles of G that contain u and v. Note that since G is triangle-free, then u and v are not adjacent, and G_0 consists of all the paths of length 2 and 3 connecting u and v in G. Observe also that a common neighbor of u and v cannot be on a path of length three joining them, as otherwise a triangle would be created. Therefore, G_0 contains u, v, their common neighbors, denote it by Z, a set of vertices $X \subseteq N(u) \setminus N(v)$ and a set of vertices $Y \subseteq N(v) \setminus N(u)$ that are on paths of length three joining u and v. By Lemma 5.13, $G[X \cup Y]$ is a forest with no isolated points, since each path of length three from u to v is determined by an edge of $G[X \cup Y]$ Furthermore, since G is triangle free, all the sets X, Y and Z are independent. Hence, the faces of G_0 are cycles of length 4, 5 or 6, since any face is either bounded by two paths connecting u and v, which have lengths two or three, forming cycles of length $4, 5$ or 6, or by u, two vertices from X and a common neighbor of them in X, forming a 4-cycle, or by v, two vertices from Y and a common neighbor of them in X, again forming a 4-cycle.

We choose G_0 so that $|Z|$ is the largest. In particular, for every $y \in Y$, $d_X(y) := |N(y, u)| \leq |Z|$ and for every $x \in X$, $d_Y(x) := |N(x, v)| \leq |Z|$.

Observe also that G_0 is an induced subgraph of G. Since adding any edge to G_0, either creates a triangle or will be in a C_5 containing u and v, which means that it must already be in G_0.

We then consider the following cases.

Case 1. $V(G) \setminus V(G_0) = \emptyset$.

Then, $G_0 = G$, since G_0 is an induced subgraph of G. Thus, $\mathcal{N}(C_5, G) = \mathcal{N}(C_5, G_0)$. We show that $\mathcal{N}(C_5, G_0) \leq \lfloor (n-3)/2 \rfloor \cdot \lceil (n-3)/2 \rceil$. Paths of length two connecting u and v are corresponding to their common neighbors, that is, there are $|Z|$ such paths. Paths of length three connecting u and v must use an edge between X and Y, and by Corollary 5.14, there are at most $|X| + |Y| - 1$ such paths. Therefore,

$$\mathcal{N}(C_5, G) = |Z|(|X| + |Y| - 1) \leq |Z|(n - 3 - |Z|) \leq \lfloor (n-3)/2 \rfloor \cdot \lceil (n-3)/2 \rceil,$$

and equality holds only when $|C|$ and $|A| + |B| - 1$ are as equal as possible, with $|E(X, Y)| = |X| + |Y| - 1$, which means G is in $\mathcal{J}_n$.

Case 2. $V(G) \setminus V(G_0) \neq \emptyset$.

Then, some faces of G_0 are not empty in G. We may assume that such a face forms a separating cycle of G, as otherwise G_0 is a 5-cycle and every 5-cycle of G is a face of G. Then, $\mathcal{N}(C_5, G)$ is at most the number of faces, which means $\mathcal{N}(C_5, G) \leq n - 2$, since G is a triangle-free. This is fewer than $\lfloor \frac{n-3}{2} \rfloor \lceil \frac{n-3}{2} \rceil$, if $n \geq 9$, and for smaller values we can easily check that G has much fewer 5-faces under these conditions.

Assume that a cycle C_t, which is the boundary of a face of G_0, is a separating cycle in G. Let G_1 be the subgraph of G induced by the vertices in the interior region of C_t together with $V(C_t)$, and G_2 be the subgraph induced by the exterior vertices of C_t together with $V(C_t)$. Let $|n_1 = V(G_1)|$ and $n_2 = |V(G_2)|$ (so $n_1 + n_2 = n + t$). Denote by $Cross(C_5)$ the number of crossing 5-cycles in G for C_t. Then,

$$\mathcal{N}(C_5, G) = \mathcal{N}(C_5, G_1) + \mathcal{N}(C_5, G_2) + Cross(C_5) - c_t, \tag{5.1}$$

where $c_t = 1$, if $t = 5$ and $c_t = 0$, otherwise.

Then we have the following possibilities for the separating cycle C_t.

Subcase 2.1. $t = 4$, and C_t is a 4-cycle uz_1vz_2 for some $z_1, z_2 \in Z$.

Then, there is no crossing C_5 in G for C_t. Since if a crossing C_5 uses one interior vertex, say w, then it is adjacent to two vertices of C_t. It cannot be adjacent to two adjacent vertices, since G is triangle-free and it cannot be adjacent to u and v, since otherwise $w \in Z \subseteq V(G_0)$, which means C_t is not the boundary of a face of G_0. Hence, x must be adjacent to z_1 and z_2 in which case, because of the other faces of G_0, we cannot use two exterior vertices and get a crossing C_5, and if a third vertex (u or v) on C_t with an exterior vertex are used, this would create a triangle, a contradiction. Similarly, if the crossing C_5 uses two interior vertices, say w_1 and w_2, then it uses the edge w_1w_2, and two vertices on C_t, and an exterior vertex. It cannot use w_1w_2 in a path joining u and v, since this would mean $w_1, w_2 \in V(G_0)$, which means C_t is not the boundary of a face of G_0. Also, if w_1w_2 is in a path joining two adjacent vertices of C_t, then the exterior vertex forms a triangle. Finally, if w_1w_2 is in a path joining z_1 and z_2, then because of the other faces of G_0, we cannot obtain a crossing C_5. Thus, no crossing C_5 is possible, that is $Cross(C_5) = 0$.

Now, let us compute the number of 5-cycles again, using Equation 5.1, and the induction hypothesis. Note that $n_1 + n_2 = n + 4$, $c_t = 0$, and each of n_1 and n_2 is at least 5.

$$\begin{aligned}
\mathcal{N}(C_5, G) &\leq \left(\frac{n_1 - 3}{2}\right)^2 + \left(\frac{n_2 - 3}{2}\right)^2 \\
&= \left(\frac{n_1 + n_2 - 7}{2}\right)^2 - \left(\frac{n_1 n_2}{2} - 2n_1 - 2n_2 + 31/4\right) \\
&= \left(\frac{n - 3}{2}\right)^2 - \frac{n_1 n_2}{2} + 2(n + 4) + 31/4 < \left(\frac{n - 3}{2}\right)^2 - 1,
\end{aligned}$$

where the last inequality is because $n_1n_2/2$ attains the minimum value when n_2 (or n_1) is the smallest possible, and $n \geq 8$.

Subcase 2.2. $t = 4$ and C_t is a 4-cycle ux_1yx_2, for some $x_1, x_2 \in X$, and $y \in Y$.

Then, there are $(n_1 - 5)(n_2 - 7)/2$ crossing 5-cycles. To see this, a crossing C_5 cannot contain only one vertex in the interior of C_t, for similar reasons as before. Assume that it contains two interior vertices, say a and b. Then, it uses the edge ab and a path of length three joining two of the vertices of C_t, and it uses an external vertex that is a common neighbor of the two vertices on C_t. Thus, the two vertices of C_t cannot be adjacent since it gives a triangle, and cannot be x_1 and x_2, since there cannot be a crossing C_5 because of the other faces of G_0. Hence, such a crossing C_5 consists of a path $P_4(u, y)$ joining u and y using the interior vertices, for which by Corollary

5.14, there are $n_1-4-1=n_1-5$ choices, and an exterior vertex that is a common neighbor of u and y, for which there are $d_X(y)-2$ choices (the -2 is for excluding x_1 and x_2). Note that $d_X(y)+|Y|+|Z|\leq |X|+|Y|+|Z|=|V(G_0)|-2\leq n_2-2$, and given that $d_X(y)\leq |Z|$, and $|Y|\geq 1$, we obtain $d_X(y)\leq (n_2-3)/2$. So, $Cross(C_5)\leq (n_1-5)(d_X(y)-2)\leq (n_1-5)(\frac{n_2-3}{2}-2)=\frac{(n_1-5)(n_2-7)}{2}$.

Note that $n_1+n_2=n+4$, and each of them is at least 5, and $n\geq 8$. Applying Equation 5.1 and induction hypothesis, we obtain

$$\begin{aligned}\mathcal{N}(C_5,G)&\leq \left(\frac{n_1-3}{2}\right)^2+\left(\frac{n_2-3}{2}\right)^2+\frac{(n_1-5)(n_2-7)}{2}\\&=\left(\frac{n_1+n_2-7}{2}\right)^2-\frac{1}{4}(6n_1+2n_2-39)\\&=\left(\frac{n-3}{2}\right)^2-\frac{1}{4}(4n_1+2(n+4)-39)<\left(\frac{n-3}{2}\right)^2-1.\end{aligned}$$

Subcase 2.3. $t=4$ and C_t is a 4-cycle vy_1xy_2, for some $x\in X$ and $y_1,y_2\in Y$. This is similar to the previous case.

Subcase 2.4. $t=5$ and C_t is a 5-cycle $uzvyx$, for some $x\in X$, $y\in Y$ and $z\in Z$.

We claim that the number of crossing C_5's for C_t is at most $\frac{(n_1-5)(n_2-5)}{2}$. Let C_5 be a crossing one. Again, it uses either one or two vertices in the interior region of C_t. Assume that it uses one vertex, say w. Then w is adjacent to two vertices on C_t, which are not adjacent, not u and v, not u and y, and not v and x, as in these cases either we get a triangle in G or C_t would not be the boundary of a face of G_0. Thus, w is adjacent to z and one of x and y (cannot be both as wxy would be a triangle), without loss, say y. Then, the other faces of G_0 do not allow using two exterior vertices to obtain a crossing C_5, and hence u or v on C_t must be used with an exterior vertex, w'. If v is used, then vyw' would be a triangle. So, it uses u, and w' is a common neighbor of u and y. In this case, a path of length three joining u and y in G_1 together with a path of length two joining them in G_2 are used. note that a $P_4(u,y)$ in G_1 does not use the vertices x, and the only one that uses v is $uzvy$ which is on C_t and so do not produce a crossing C_5. Hence, there are n_1-4 remaining vertices in G_1 apart from u and y themselves, that are on such paths. By Corollary 5.14, the number of those paths is n_1-5. Observe that if a path $P_4(u,y):=uaby$ exists, and forms a crossing C_5 with an exterior vertex w', then w' must be a common neighbor of y and u. In this case $d_X(y)>1$, and hence the cycle $uw'yba$ separates x and v, and hence they cannot have any common neighbor apart from y. That is, $d_Y(x)=1$, and there is no crossing C_5 that uses an $P_4(v,x)$ and an exterior vertex. Note also that, if a crossing C_5 uses two interior vertices, then it has to use a path of length three joining two vertices on C_t and an exterior vertex. But again the two vertices on C_t can only be either u and y or v and x, and such paths are already counted. Similarly as before $d_X(y)\leq (n_2-3)/2$, so the number of crossing 5-cycles is $(n_1-5)(d_X(y)-1)\leq (n_1-5)((n_2-3)/2-1)=(n_1-5)(n_2-5)/2$, proving our claim.

Now, we apply induction hypothesis, and Equation 5.1.

$$
\begin{aligned}
\mathcal{N}(C_5, G) &\leq \left(\frac{n_1-3}{2}\right)^2 + \left(\frac{n_2-3}{2}\right)^2 + \frac{(n_1-5)(n_2-5)}{2} - 1 \\
&= \left(\frac{n_1+n_2-8}{2}\right)^2 = \left(\frac{n-3}{2}\right)^2.
\end{aligned}
$$

Note that in counting the crossing 5-cycles, we assumed that all interior vertices (in G_1) are on paths of length three joining u and y (or v and x), but in this case they are divided into a set A of neighbors of u and another (disjoint from A) set B of neighbors of y, and in view of Lemma 5.13, $G[A \cup B]$ is acyclic, and the only C_5's in G_1 are formed by these paths and uxy, and hence $\mathcal{N}(C_5, G_1) = (n_1 - 5) + 1$ (+1 for C_t itself), which gives a much smaller bound than above. Therefore, we may assume that there are fewer crossing 5-cycles and hence $N(C_5, G) \leq (n-3)^2/4 - 1$.

Thus, we may assume that all faces of G_0 that are of the types mentioned so far are empty in G. We are left with dealing with a possible 6-face of G_0 that is not empty in G.

Subcase 2.5. $t = 6$ and C_t is a 6-cycle $ux_1y_1vy_2x_2$, for some $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Assume that a crossing C_5 contains two exterior vertices, so it uses only one interior vertex, say w. Then, w is adjacent to two vertices on C_t and the crossing C_5 uses only these two vertices on C_t. We cannot have uwv, uwy_i, vwx_i, for $i = 1, 2$, since then $w \in V(G_0)$, and the two boundary vertices cannot be two adjacent vertices on C_t, since they form a triangle with w. The remaining possibilities are $x_1wx_2, x_1wy_2, x_2wy_1$ or y_1wy_2. None of these can be used for a crossing C_5 with two exterior vertices because of the other faces of G_0.

Thus, the crossing C_5 uses only one exterior vertex. That is, it uses a path of length three in G_1 joining two vertices of C_t and an exterior vertex that is a common neighbor of them. The two vertices on C_t cannot be adjacent, because they would form a triangle with the exterior vertex, and they cannot be u and v as the path of length three joining them in G_1 would be part of G_0. Also, the pairs x_1, x_2 and y_1, y_2, and for $i, j = 1, 2$, $i \neq j$, the vertices x_i and y_j have no exterior vertex as a common neighbor because of the other faces of G_0. Thus, the crossing C_5 must use a $P_4(u, y_i)$ or a $P_4(v, x_i)$, for $i = 1, 2$ in G_1.

Recall that C_t is the boundary of a face of G_0, which means no interior vertex is a common neighbor of u and v, and C_t has no chords, since any chord either creates a triangle or would be an edge of G_0, contradicting that C_t is a face of G_0. In particular, there is no common neighbors of u and v in G_1.

For each $i \in \{1, 2\}$, if there is a path $P_4(u, y_i)$ in G_1, then there is no path $P_4(v, x_i)$ in G_1. Indeed, without loss of generality, assume that there is a $P_4(u, y_1)$, say $uaby_1$. Then, neither a nor b can be on a $P_4(v, x_1)$. Since if a is on such a path, then it is either adjacent to v or to x_1, in the first case it would be a common neighbor of u and v in G_1 and in the second case ux_1a forms a triangle. Similarly, if b is on such a path, then it creates a triangle with x_1y_1 or with y_1v. Moreover, the path $uaby_1$ divides the region inside C_t into two regions R_1 and R_2 bounded by cycles $uaby_1x_1$ and $uaby_1vy_2x_2$, respectively. Each interior vertex of G_1 is in one of these two regions. If a vertex x lies in R_1, then it cannot be adjacent to v, so if it is on a $P_4(v, x_1)$, then it must be with one of a and b, however, none of them can be on such a path, so neither w. Likewise, a vertex that lies in R_2 cannot be on a $P_4(v, x_1)$. Thus, if there are paths $P_4(u, y_1)$, then there are either paths $P_4(u, y_2)$ or paths $P_4(v, x_2)$.

Therefore, one of the following four possibilities holds:

- There are paths $P_4(u, y_1)$ and $P_4(u, y_2)$
- There are paths $P_4(u, y_1)$ and $P_4(v, x_2)$
- There are paths $P_4(v, x_1)$ and $P_4(v, x_2)$
- There are paths $P_4(v, x_1)$ and $P_4(u, y_2)$.

We deal with the first two cases, since the last two are exactly symmetric to the first two.

- **Suppose that there are paths $P_4(u, y_1)$ and paths $P_4(u, y_2)$.**

That is, there are no paths $P_4(v, x_2)$ in G_1. Observe that if x_2 is in a path $P_4(u, y_1)$, then the path must be u, x_2wy_1 for some vertex w in the interior of C_t. But then vy_1wx_2 is a path $P_4(v, x_2)$ in G_1, a contradiction. In particular, x_2 and y_1 have no common neighbors in this case. It is then easy to see that no other vertices on C_t can be on a $P_4(u, y_1)$ apart from u and y_1 themselves. Likewise, apart from u and y_2, no vertex of C_t can be on a $P_4(u, y_2)$.

First, assume that y_1 and y_2 have no common neighbors in the interior of C_t. Let L_1 and L_2 be the sets of interior vertices of C_t that are on paths $P_4(u, y_1)$ and paths $P_4(u, y_2)$, respectively. We claim that $|L_1 \cap L_2| \leq 1$. If a neighbor of y_1 is in $L_1 \cap L_2$, then as it is on a $P_4(u, y_1)$, it must have a common neighbor with u, so it cannot be adjacent to u, since it would form a triangle. However, since it is also on a $P_4(u, y_2)$, it must be adjacent to y_2, which means it is a common neighbor of y_1 and y_2, a contradiction. Similarly, a vertex in $L_1 \cap L_2$ cannot be a neighbor of y_2. Thus, $L_1 \cap L_2 \subseteq N(u)$. Let $w \in L_1 \cap L_2$, then w has a common neighbor b with y_1 and a common neighbor c with y_2. This divides the region bounded by C_t into three regions R_1, R_2 and R_3 that are bounded by $uwby_2x_1$, $uwcy_2x_2$ and wby_1vy_2c, respectively. Any other neighbor of u in the interior of C_2 lies in R_1 or R_2, and in either case it is easy to see that they cannot be in $L_1 \cap L_2$. Thus, $|L_1| + |L_2| \leq n_1 - 6 + 1 = n_1 - 5$. Also, by Corollary 5.14, we have $\#P_4(u, y_1) \leq |L_1| - 1$ and $\#P_4(u, y_2) \leq |L_2| - 1$.

Thus, we have the following upper bound for the number of crossing C_5's for C_t. Note that $x_2 \notin N_X(y_1)$, which gives $d_X(y_1) \leq |X| - 1$. Since $d_X(y_1) + |Y| + |Z| \leq (|X| - 1) + |Y| + |Z| = |V(G_0)| - 3 \leq n_2 - 3$, we obtain that $d_X(y_1) + |Z| \leq n_2 - 3 - |Y| \leq n_2 - 5$, since $|Y| \geq 2$. As $d_X(y_1) \leq |Z|$, we must have $d_X(y_1) \leq \frac{n_2-5}{2}$.

Similarly, $d_X(y_2) \leq \frac{n_2-5}{2}$.

$$\begin{aligned} Cross(C_5) &= \#P_4(u,y_1)\cdot(d_X(y_1)-1) + \#P_4(u,y_2)\cdot(d_X(y_2)-1) \\ &\leq (|L_1|-1)(\frac{n_2-5}{2}-1) + (|L_2|-1)(\frac{n_2-5}{2}-1) \\ &= (|L_1|+|L_2|-2)(\frac{n_2-7}{2}) \\ &\leq \frac{(n_1-7)(n_2-7)}{2} \end{aligned}$$

We can now apply Equation 5.1, and induction hypothesis. Recall that $n_1+n_2=n+6$ and $c_t=0$. Furthermore, since the 6-cycle C_t is a face for both of G_1 and G_2, then none of them is isomorphic to any member of $\mathcal{J}_{n_i}$. Therefore, the induction hypothesis in this case implies $\mathcal{N}(C_5,G_i) \leq \frac{(n_i-3)^2}{4}-1$. Hence,

$$\begin{aligned} \mathcal{N}(C_5,G) &\leq \left(\frac{n_1-3}{2}\right)^2 - 1 + \left(\frac{n_2-3}{2}\right)^2 - 1 + \frac{(n_1-7)(n_2-5)}{2} \\ &= \left(\frac{n_1+n_2-9}{2}\right)^2 - \left(\frac{n_1}{2}+\frac{n_2}{2}-\frac{27}{4}\right) \qquad (5.2) \\ &< \left(\frac{n-3}{2}\right)^2 - 1. \end{aligned}$$

Note that the last inequality is because we may assume that each of n_1 and n_2 is at least 8, since otherwise the interior or exterior of C_t would contain only one vertex, and hence there would be no crossing C_5.

Now suppose that y_1 and y_2 have common neighbors in the interior of C_t. If no common neighbor of y_1 and y_2 is contained in a path $P_4(u,y_1)$ and a path $P_4(u,y_2)$, then the previous argument applies again, and we are done. Let w be a common neighbor of y_1 and y_2 that is contained in a $P_4(u,y_1)$ and a $P_4(u,y_2)$. Then, it has a common neighbor with u. Let $A := \{a_1,\ldots,a_t\}$ be the common neighbors of w and u. We may order them so that the region bounded by ua_1wa_t contains all the vertices of A, and the regions R_1 and R_2 bounded the 5-cycles $ua_1wy_1x_1$ and $ua_twy_2x_2$, respectively, do not contain a common neighbor of u and w in their interior. Observe that any other common neighbor of y_1 and y_2 lies in the region R_3 bounded by the 4-cycle y_1wy_2v. And no vertex in the interior of R_3 can be on a $P_4(u,y_1)$ or a $P_4(u,y_2)$, since they are neither adjacent to u, nor have a common neighbor with u. Observe also that by the choice of G_0, $|A| \leq |Z|$.

We consider each of the regions R_1, R_2, R_3 and the regions bounded by two vertices of A with u and w. If any of them are not empty, that is, contains vertices in its interior, then we apply induction by taking its boundary cycle as a separating cycle, to obtain the desired upper bound. If they are all empty, then we know the structure of G_1, from which we can compute the number of 5-cycles.

Assume R_1 is not empty. Then the boundary cycle of R_1, $C := ua_1wy_1x_1$ is a separating cycle in G. Let G' be the subgraph of G induced by the vertices that are on C or in the interior of R_1. Let G'' be the subgraph of G that is induced by the vertices that are on C and those in the interior of R_1. Let $n' = |V(G')|$ and $n'' = |V(G'')|$, and then $n' + n'' = n + 5$. Let $Cross(C_5)$ be the number of crossing C_5's for C in G. Then,

$$\mathcal{N}(C_5, G) = \mathcal{N}(C_5, G') + \mathcal{N}(C_5, G'') - 1 + Cross(C_5) \quad \star$$

The situation is now similar to Subcase 2.4, any crossing C_5 for C is formed by a path of length three $P_4(u, w)$ connecting u and w in G' and a vertex in $A \setminus \{a_1\}$, or a path $P_4(u, y_1)$ connecting u and y_1 in G and a vertex of $N_X(y_1) \setminus \{x_1\}$. Moreover, either paths $P_4(u, w)$ exist in G' or paths $P_4(u, y_1)$ exist, but not both. And the number of paths $P_4(u, w)$ (same for $P_4(u, y_1)$) in G' is at most $n' - 5$. So we have $Cross(C_5) \leq (n' - 5)(|A| - 1)$ or $Cross(C_5) \leq (n' - 5)(d_X(y_1) - 1)$.

Note that $|A|+|X|+|Y|+|Z| \leq n''-2$, which implies $|A|+|Z| \leq n''-2-|X|-|Y| = n'' - 6$, since each of X and Y contains at least two vertices. As $|A| \leq |Z|$, we obtain that $|A| \leq (n''-6)/2$. Similarly, $d_X(y_1)+|A|+|Y|+|Z| \leq (|X|-1)+|A|+|Y|+|Z| \leq n'' - 3$, that is, $d_X(y_1) + |Z| \leq n - 3 - |A| - |Y| \leq n'' - 6$, since $|A| \geq 1$ and $|Y| \geq 2$. Again, since $d_X(y_1) \leq |Z|$, this gives $d_X(y_1) \leq (n'' - 6)/2$. Thus, we have obtained that $Cross(C_5) \leq (n' - 5)(\frac{n''-6}{2} - 1) = \frac{(n'-5)(n''-8)}{2}$.

Now, applying induction hypothesis on G' and G'', from Equation $\star$, we obtain

$$\begin{aligned}
\mathcal{N}(C_5, G) &\leq \left(\frac{n'-3}{2}\right)^2 + \left(\frac{n''-3}{2}\right)^2 - 1 + \frac{(n'-5)(n''-8)}{2} \\
&= \left(\frac{n'+n''-8}{2}\right)^2 - \frac{3(n'-5)}{2} \\
&< \left(\frac{n-3}{2}\right)^2 - 1,
\end{aligned}$$

since $n' + n'' = n + 5$ and $n' \geq 6$.

If R_2 is not empty, we similarly get the same conclusion.

Suppose that R_3 is not empty. Thus, the 4-cycle $C := y_1wy_2v$ is a separating cycle in G. Again, let G', G'', n' and n'' be defined similarly to above except this time for R_3 and its boundary 4-cycle C. Then, we have $n' + n'' = n + 4$ and the term (-1) does not appear in Equation $(\star)$. It is easy to see that for this separating cycle C, there are no crossing C_5 in G. Thus, applying induction hypothesis, with exactly the same computations as in Subcase 2.1 we obtain the desired result.

Hence, we may assume that all the regions R_1, R_2 and R_3 are empty. If all vertices of G that are in the interior of the 6-face C_t of G_0 are not in $A \cup \{w\}$, then some region bounded by a 4-cycle ua_iwa_j contains them. That is, the 4-cycle $C := ua_iwa_j$ is a separating cycle in G. We may assume that no vertex of A is in the interior region bounded by C. Then, define G', G'', n' and n'' as before. It is easy to see that any crossing C_5 for C consists of a path of length three $P_4(u, w)$ joining u and w in G' and a vertex of $A \setminus \{a_i, a_j\}$, and hence in this case $Cross(C_5) \leq (n' - 5)(|A| - 2)$, and with similar arguments as before we get that $|A| \leq (n''-6)/2$. Thus,

$$\begin{aligned}\mathcal{N}(C_5, G) &= \mathcal{N}(C_5, G') + \mathcal{N}(C_5, G'') + Cross(C_5)\\ &\leq \left(\frac{n'-3}{2}\right)^2 + \left(\frac{n''-3}{2}\right)^2 + \frac{(n'-5)(n''-10)}{2}\\ &< \left(\frac{n'+n''-7}{2}\right)^2 - \frac{12n'+2n''-69}{4} < \left(\frac{n-3}{2}\right)^2 - 1,\end{aligned}$$

since $n' \geq 10$, as it contains $X \cup Y \cup Z$ besides $V(C)$.

Therefore, we may assume that all vertices in the interior of C_t are in $A \cup \{w\}$. Any 5-cycle of G that contains a vertex in A must also contain w and either a vertex in $N_X(y_1)$ or a vertex in $N_X(y_2)$. Also, no 5-cycle of G can contain two vertices from A. So, each vertex of A is in $d_X(y_1) + d_X(y_2)$ 5-cycles in G. Thus, the number of 5-cycles in G that contain $A \cup \{w\}$ is at most $|A| \cdot (d_X(y_1) + d_X(y_2))$, since w can only be in a C_5 with one of the vertices of A. However, if we delete all the vertices in $A \cup \{w\}$ and add them to Z, that is, join each of them to both u and v, and add the edge y_1x_2, mimicking the structure of graphs in $\mathcal{J}_n$, each of them will be in at least $d_X(y_1) + d_X(y_2) + 1$ 5-cycles. That is, we would have at least $(|A|+1)(d_X(y_1) + d_X(y_2) + 1)$ 5-cycles that contain them. This argument shows that G contains fewer 5-cycles than any member of $\mathcal{J}_n$, completing the proof in this case.

- **Suppose that there are paths $P_4(u, y_1)$ and paths $P_4(v, x_2)$.**

That is, in this case, there are no paths $P_4(u, y_2)$. First, assume that x_2 and y_1 have no common neighbors in G_1. Then x_2 is not on any path $P_4(u, y_1)$ and y_2 is not on any path $P_4(v, x_2)$. Let L_1 and L_2 be the set of vertices in the interior region bounded by C_t that are on paths $P_4(u, y_1)$ and paths $P_4(v, x_2)$, respectively. Then, $|L_1 \cap L_2| = 0$. Since, if $uaby_1$ is a path $P_4(u, y_1)$, then none of a and b can be on any path $P_4(v, x_2)$. Since a cannot be adjacent to v, as otherwise it would be a common neighbor of u and v, which contradicts the fact C_t is a boundary of a face of G_0, and it cannot be adjacent to x_2, since uax_2 would be a triangle. Also if b is on a $P_4(v, x_2)$, then it is either adjacent to v, forming a triangle, or adjacent to x_2, and then it is a common neighbor of y_1 and x_2, contradicting our assumption. Then, we have $|L_1| + |L_2| = n_1 - 6$. By Corollary 5.14, the number of paths $P_4(u, y_1)$ is at most $|L_1| - 1$, and the number of paths $P_4(v, x_2)$ is at most $|L_2| - 1$.

The crossing C_5's for C_t are formed by a $P_4(u, y_1)$ and a vertex of $N_X(y_1) \setminus \{x_1\}$, or by a path $P_4(v, x_2)$ and a vertex of $N_Y(x_2) \setminus \{y_2\}$. Similarly to the previous case, we have $d_X(y_1) \leq (n_2-5)/2$ and $d_Y(x_2) \leq (n_2-5)/2$. Thus,

$$\begin{aligned}Cross(C_5) &\leq (|L_1|-1)(d_X(y_1)-1) + (|L_2|-1)(d_Y(x_2)-1)\\ &\leq (|L_1|-1)(\frac{n_2-5}{2}-1) + (|L_2|-1)(\frac{n_2-5}{2}-1)\\ &= \frac{(n_1-8)(n_2-7)}{2}\end{aligned}$$

Now, following the same computation as in 5.2, we obtain the desired result.

Next, assume that y_1 and x_2 have common neighbors in G_1, and let $B := \{b_1, \ldots, b_t\}$ be the set of their common neighbors. We may order them so that the interior region of C_t is divided into regions R, R' and R_i, for $1 \leq i \leq t-1$, bounded by cycles $ux_2b_1y_1x_1$, $vy_1b_tx_2y_2$, $y_1b_ix_2b_{i+1}$, respectively, such that none of these regions contain a common neighbor of y_1 and x_2 in their interior.

If any of these regions contain a vertex of G_1 in their interior, then the bounding cycle of the region forms a separating cycle in G. Similarly as in the previous case, we can define n', n'', G' and G'' and get the desired conclusion. Hence, we may assume that all these regions are empty, which means all the vertices in the interior of C_t are contained in B.

It is again easy to see that each vertex of B is contained in at most $d_X(y_1)+d_Y(x_2)$ 5-cycles in G. That is, the number of 5-cycles in G that contain the vertices in B is at most $|B|(d_X(y_1) + d_Y(x_2))$. If we delete them in the interior region of C_t and add them to Z, we can add the edge y_1x_2. Then, each of the vertices of B would be in at least $d_X(y_1) + d_Y(x_2) + 1$ 5-cycles, which means there are at least $|B|(d_X(y_1) + d_Y(x_2) + 1)$ 5-cycles of G that contain them. This shows that G has fewer 5-cycles than any member of $\mathcal{J}_n$, completing the proof. □

For the proof of Theorem 5.8 (and Theorem 5.9 in the next section), we use the concept of *triangular blocks* introduced in [53]. Let us first recall the definition and relevant results from them that we will rely on in the proofs.

Definition 5.1. [53] Let G be a plane graph. An edge $e \in E(G)$ is a triangular-block if it is not in any face of length 3 (this is called a **trivial triangular block**), otherwise we inductively build up the block; start with the subgraph $H := e$, keep adding to H all faces of length 3 (and their edges) that contain an edge of H until no such is left.

Let $\mathcal{T}$ be the set of all triangular blocks of G. It is easy to see that every edge of G is in exactly one triangular block. Then,

$$e(G) = \sum_{B \in \mathcal{T}} e(B). \tag{5.3}$$

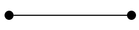
Obviously, every triangle is in at most one triangular block. However, there are possibly triangles in G that are not contained in any triangular block (as a subgraph), let T be the set of all such triangles. Let $\mathcal{T}_1$ be the set of triangular blocks of G that contain no edges of the triangles in T, and $\mathcal{T}_2 := \mathcal{T} \setminus \mathcal{T}_1$. Then

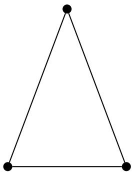
$$\mathcal{N}(C_3, G) = \sum_{B \in \mathcal{T}_1} \mathcal{N}(C_3, B) + \sum_{B \in \mathcal{T}_2} \mathcal{N}(C_3, B) + |T| \tag{5.4}$$

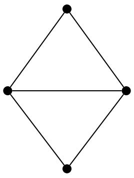
The first part of the following proposition was proven in [65] and the second part was done in [53].

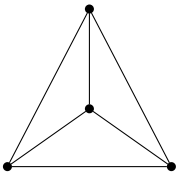
Proposition 5.15. *[53, 65] Let G be a plane graph and B be a triangular block of G. Then,*

1. *If G is C_5-free, then B has at most four vertices, and it is one of the following graphs:*

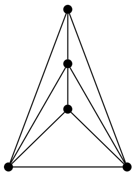

K_2

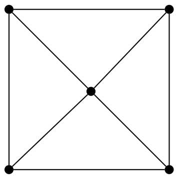

K_3

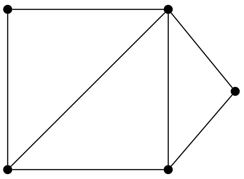

Θ_4

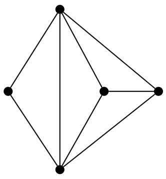

K_4

2. *If G is C_6-free, then B has at most five vertices, and it is either one of the four graphs mentioned above or one of the following:*


K_5^-


$B_{5,a}$


$B_{5,b}$


$B_{5,c}$

Proof of Theorem 5.8. Let G be a C_5-free planar graph. Let $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ and T denote the same sets as in (5.4). It is easy to see that $\mathcal{N}(C_3, B) \leq \frac{2}{3}e(B)$ for every triangular block B of G, with equality if and only if B is a K_4. This immediately, gives

$$\sum_{B \in \mathcal{T}_1} \mathcal{N}(C_3, B) \leq \sum_{B \in \mathcal{T}_1} \frac{2}{3}e(B). \tag{5.5}$$

Observe that no block in $\mathcal{T}_2$ can be a K_4, since it would form a C_5 in G. Also, for each triangle $t \in T$, either each of its edges are in different blocks or two of its edges are in the same block and the third edge is in a different block. In the former case, at least two out of the three blocks containing the edges of t must be trivial blocks, and t is the only triangle containing them. Thus, for each such $t \in T$, there are at least two trivial blocks in $\mathcal{T}_2$. In the latter case, the block containing the two edges of t must be a Θ_4 and the other block containing the third edge of t must be a trivial block, i.e. these two blocks form a K_4 in G (which is not a triangular block in G). Then, we obtain two such triangles in T, and two blocks K_2 and Θ_4 in $\mathcal{T}_2$, and no other triangles in T can contain the edges of these blocks. Therefore, computing the number of triangles and edges in T and $\mathcal{T}_2$, we obtain that

$$\sum_{B \in \mathcal{T}_2} \mathcal{N}(C_3, B) + |T| \leq \sum_{B \in \mathcal{T}_2} \frac{2}{3}e(B). \tag{5.6}$$

Substituting (5.5) and (5.6) in (5.4), and applying (5.3), we obtain

$$\mathcal{N}(C_3, G) \leq \sum_{B \in \mathcal{T}} \frac{2}{3}e(B) = \frac{2}{3} \sum_{B \in \mathcal{T}} e(B) = \frac{2}{3}e(G).$$

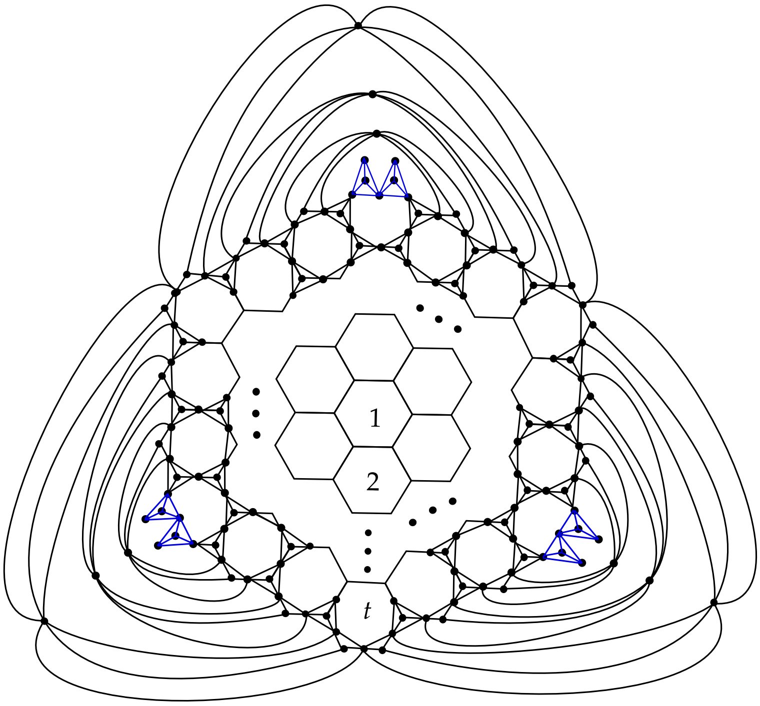


Figure 5.3: A construction that shows the sharpness in Theorem 5.8.

Also, it is proved in [65] (see also [19]) that for every $n \geq 11$, $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$, which gives:

$$\mathrm{ex}_{\mathcal{P}}(n, C_3, C_5) \leq \frac{2}{3}\left\lfloor \frac{12n-33}{5} \right\rfloor \leq \left\lfloor \frac{8n-22}{5} \right\rfloor$$

To prove the sharpness of the result, we take the graph Figure 5.3: we slightly modify the extremal construction for the number of edges (i.e. for $\mathrm{ex}_{\mathcal{P}}(n, C_5)$) given in [65]. In their construction, $n = 15t^2 - 6$, $e = 36t^2 - 21$, and all triangular blocks are K_4's except three of them, which are Θ_4's. We replace each Θ_4 by two copies of K_4 sharing a vertex. Thus, we add 9 vertices and 21 edges, so we get $n = 15t^2 + 3$ and $e = 36t^2 = \left\lfloor \frac{12n-33}{5} \right\rfloor$ (This means the new construction is still extremal for the number of edges, too). In the new graph, every triangular block is a K_4, and hence, it attains the stated bound on the number of triangles. □

5.4 Triangles and 6-cycles

Here, we prove Theorems 5.9 to 5.12. While the proof of Theorem 5.9 has a simple short proof, which is analogous to the proof of Theorem 5.7, the proof of Theorem 5.11 is more sophisticated.

Proof of Theorem 5.9. Let G be a C_6-free plane graph, let $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ and T denote the same sets as in (5.4). Every triangular-block B of G contains at most $7e(B)/9$

triangles, with equality if and only if B is a K_5^-. This implies

$$\sum_{B\in\mathcal{T}_1} \mathcal{N}(C_3, B) \leq \sum_{B\in\mathcal{T}_1} \frac{7}{9}e(B). \tag{5.7}$$

Notice that no triangular block in $\mathcal{T}_2$ can be a K_5^-, since this would form a C_6 in G.

Assume that each edge of t is in a different triangular block. For $i \in [3]$, let e_i be the edges of t, and $B_i \in \mathcal{T}_2$ be the blocks that contain them such that for each i, $e_i \in B_i$. Not that none of the blocks can be a $B_{5,a}$. If for some $i \in [3]$, $B_i \in \{K_4, B_{5,b}, B_{5,c}\}$ (see these blocks in Proposition 5.15), then the other two blocks must be trivial, and t is the only triangle that contains them. Thus, for each such triangle in T, there are two trivial blocks in $\mathcal{T}_2$, so the ratio of the number of triangles in these blocks and t to the number of edges in them is less than 7/9.

If two of the blocks, say B_1 and B_2 are not the trivial block, then B_3 must be trivial and B_1 and B_2 are either a K_3 or a Θ_4. In this case, t is the only triangle that contains e_3, and if any of e_1 or e_2 is in another triangle $t' \in T$, then the other two edges of t' must be in trivial blocks. Again, the ratio of the number of triangles to the number of edges is less than 7/9. If one of B_1 or B_2, say B_2 is also trivial, but e_2 is in another triangle $t' \in T$, then the other two edges of t' must be in trivial blocks. If further B_1 is also trivial, and e_1 is contained in another triangle $t'' \in T$, then again the two other edges of t'' must be in trivial blocks. Thus, computing such triangles and edges in these blocks, we get to the same conclusion.

Now, assume that two edges of t, say e_1 and e_2 are in the same triangular block B and the third edge e_3 is in another block B'. Then, $B \in \{\Theta_4, B_{5,a}, B_{5,b}, B_{5,c}\}$. If B is a Θ_4, then B' is a K_2, K_3 or a Θ_4, and in each case if there is another triangle $t' \in T$ that contains e_3, then the other two edges of t' must be in trivial blocks. Then computing the number of triangles and the number of edges in each possibility we get a ratio of at most 7/9. Similarly, analyzing the other possibilities of $B \in \{B_{5,a}, B_{5,b}, B_{5,c}\}$, we get the same conclusion.

Therefore, we obtain

$$\sum_{B\in\mathcal{T}_2} \mathcal{N}(C_3, B) + |T| \leq \sum_{B\in\mathcal{T}_2} \frac{7}{9}e(B). \tag{5.8}$$

Substituting (5.7) and (5.8) in (5.4), and applying (5.3), we obtain

$$\mathcal{N}(C_3, G) \leq \sum_{B\in\mathcal{T}} \frac{7}{9}e(B) = \frac{7}{9}e(G)$$

It was proved in [53] that for every $n \geq 18$, $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq 5n/2 - 7$, which gives:

$$\mathrm{ex}_{\mathcal{P}}(n, C_3, C_6) \leq \frac{7}{9}\left(\frac{5}{2}n - 7\right) \leq \frac{35n - 98}{18}$$

Sharpness of the bound follows from the construction given in [53] for the number of edges, as every triangular block is a K_5^- that contains $7e(B)/9$ triangles (See Theorem 4 and the construction in Section 2 of [53]). $\square$

Proof of Theorem 5.10. Each path $P_5(u,v)$ uses a neighbor of u, a neighbor of v and a middle vertex joined to them. No neighbor of u (or v) can be the middle vertex, as otherwise it creates a triangle. Let X and Y be the sets of neighbors of u and v, respectively. We may assume that every vertex is on a $P_5(u,v)$, as otherwise deleting it does not reduce the number of such paths. Then, every vertex of $Z := V(G) \setminus (X \cup Y \cup \{u,v\})$ is the middle vertex of some $P_5(u,v)$, i.e. it has neighbors in both X and Y and each vertex of X and Y has neighbors in Z. Likewise, we may assume that every edge is in a $P_5(u,v)$. In particular, this means Z is an independent set and there are no edges from X to Y. Recall that, as G is triangle-free, X and Y are also independent sets. Let $Z = \{z_1, z_2, \ldots, z_l\}$. We consider the following cases.

Case 1. Assume that u and v have no common neighbors. That is, $|X \cap Y| = 0$.

Each path $P_5(u,v)$ corresponds to a path of length two of the form xzy, for some $x \in X$, $z \in Z$ and $y \in Y$. For each $i \in [l]$, let $\alpha_i := d_X(z_i)$ and $\beta_i := d_Y(z_i)$. Then,

$$\#P_5(u,v) = \sum_{i=1}^{l} \alpha_i \beta_i \tag{5.9}$$

If $l = 1$, then easily we get the upper bound $|X||Y| \leq (|X|+|Y|)^2/4 = (n-3)^2/4 < (n-1)^2/4 - 2$. So, we may assume that $l \geq 2$. Since each vertex of Z has neighbors in both X and Y, for each $i \in [l]$, $1 \leq \alpha_i \leq |X|$ and $1 \leq \beta_i \leq |Y|$. Moreover, $\sum_{i=1}^{l}(\alpha_i + \beta_i) = |E(Z, X \cup Y)|$.

Note that $F := G[X \cup Y \cup Z \cup \{u,v\}]$ is a bipartite graph with partite sets $X \cup Y$ and $Z \cup \{u,v\}$ in which there are $|X|$ edges incident to u and $|Y|$ edges incident to v. Since F is planar, it has at most $2|V(F)| - 4$ edges, and hence

$$\begin{aligned}|E(Z, X \cup Y)| &\leq 2(|X| + |Y| + |Z| + 2) - 4 - |X| - |Y| \\ &= |X| + |Y| + 2l\end{aligned}$$

Clearly, the right-hand side of Equation 5.9 is maximum if for some i, α_i and β_i are as large as possible and all others are as small as possible.

Without loss of generality, assume that $\alpha_1 = |X|$ and $\beta_1 = |Y|$, then $\sum_{i=2}^{l}(\alpha_i + \beta_i) \leq 2l$. So we have the maximum number of paths if further, again without loss, $\alpha_2 = 2 = \beta_2$, and for $i \in \{3, \ldots, l\}$, we have $\alpha_i = 1 = \beta_i$. This implies

$$\begin{aligned}\#P_5(u,v) &\leq |X| \cdot |Y| + 2 \cdot 2 + (l-2) = |X||Y| + l + 2 \\ &\leq \left(\frac{|X| + |Y|}{2}\right)^2 + l + 2 \\ &= \left(\frac{n-l-2}{2}\right)^2 + l + 2\end{aligned}$$

It is easy to see that the maximum is attained at $l = 2$ (note that $2 \leq l = |Z| \leq n-4$). Therefore,

$$\#P_5(u,v) \leq \frac{(n-4)^2}{4} + 4 < \frac{(n-1)^2}{4} - 2,$$

when $n \geq 8$. If n is at most 7 and $l \geq 2$, then at least one of X and Y contains one vertex, and hence, the $2 \cdot 2$ term in computing $\#P_5(u, v)$ above would be $1 \cdot 2$, which gives the upper bound of $(n-3)^2/4 + 1 < (n-1)^2/4 - 2$.

Case 2. Assume that u and v have exactly one common neighbor, that is, $|X \cap Y| = 1$.

Let $X \cap Y = \{w\}$, $X_0 = X \setminus \{w\}$ and $Y_0 = Y \setminus \{w\}$. If $l = 1$, then from the previous case, without using w, the number of paths from u to v is at most $((n-1)-3)^2/4$. And clearly w is in $|X_0| + |Y_0| = n - 4$ paths from u to v. Thus, we have at most

$$\frac{(n-4)^2}{4} + (n-4) < \frac{(n-1)^2}{4} - 2$$

Hence, we assume that $l \geq 2$. Again, from the previous case, without using w, the number of paths from u to v is at most $|X_0||Y_0| + l + 2$.

Now, let us count the paths $P_5(u, v)$ that use w. Each such path either uses w as a neighbor of u and a path $P_4(w, v)$ using a vertex from Z and a vertex from Y_0, or uses w as a neighbor of v and a path $P_4(w, u)$ using a vertex from Z and another from X_0. That is the umber of paths $P_5(u, v)$ that use w is equal to $\#P_4(w, u) + \#P_4(w, v)$. Let $Z_1 \subseteq Z$ be the set of vertices that lie on a path $P_4(w, u)$ and $Z_2 \subseteq Z$ be those on paths $P_4(w, v)$. By Corollary 5.14,

$$\#P_4(w, u) \leq |Z_1| + |X_0| - 1 \quad \text{and} \quad \#P_4(w, v) \leq |Z_2| + |Y_0| - 1$$

We claim that $|Z_1 \cap Z_2| \leq 2$. To see this, let $z_1, z_2, z_3 \in Z_1 \cap Z_2$. Then, each of them is adjacent to w, has a neighbor in X_0, and has a neighbor in Y_0. We can then find a subdivision of $K_{3,3}$ as follows. The vertices z_1, z_2 and z_3 form one partite set of it. The other partite set contains w, and two other vertices: If each of z_1, z_2 and z_3 has a distinct neighbor in X_0, then choose u, otherwise choose $x \in X$ that is adjacent to at least two of the z_i's (the other z_i is either adjacent to x or connected to x by a path through another $x_1 \in X$ and u), and similarly we either choose v or a vertex $y \in Y$. This contradicts the planarity of G.

Thus, $|Z_1| + |Z_2| \leq l + 2$, and hence the number of paths $P_5(u, v)$ that contain w is at most $|X_0| + |Y_0| + l = n - 3$. Therefore, in total we have

$$\begin{aligned}
\#P_5(u, v) &\leq |X_0||Y_0| + l + 2 + |X_0| + |Y_0| + l \\
&\leq \left(\frac{|X_0| + |Y_0|}{2}\right)^2 + n + l - 1 \\
&\leq \left(\frac{n - l - 3}{2}\right)^2 + n + l - 1 \\
&\leq \left(\frac{n-5}{2}\right)^2 + n + 1 = \left(\frac{n-1}{2}\right)^2 - 2 - (n - 9).
\end{aligned}$$

Thus, we are done, since if $n \leq 8$, then at least one of X_0 and Y_0 contains at most one vertex, and in this case we would have fewer paths, i.e. in the above computations we have over counted enough paths to compensate.

Case 3. Assume that u and v have at least two common neighbors.

Let $W := \{w_1, w_2, \ldots, w_k\}$ be the set of common neighbors of u and v. Then, $G[W \cup \{u, v\}]$ is an induced $K_{2,k}$ subgraph of G. When drawing G, this subgraph divides the plane into k regions R_i each bounded by the cycle uw_ivw_{i+1}, where the addition is modulo k, see Figure 5.4. We also have that each R_i contains only two common neighbors of u and v, namely w_i and w_{i+1}. Each path $P_5(u, v)$ lies in only one of these regions (may use boundary vertices of the region as well), otherwise it forms a triangle. So we first restrict ourselves to one of the regions. Take a region R_i bounded by the cycle uw_ivw_{i+1} (when $k = 2$, then there are two regions bounded by the same cycle, it should not be confusing that we take one of them). Let M be the set of vertices in the interior of R_i, with $|M| = m$. Let X, Y and Z be defined as in the beginning restricted to the vertices in the interior of R_i.

Let $P_5(u, v)_{R_i}$ be a $P_5(u, v)$ that uses only vertices in the region R_i (including the boundary vertices of the region).

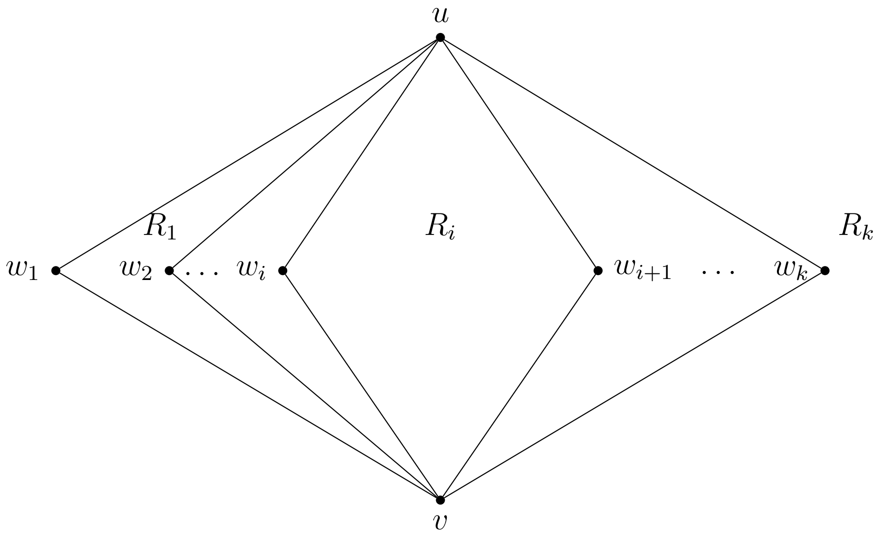


Figure 5.4: The common neighbors of u and v form k regions

Claim 5.3. *In a region R_i, $\#P_5(u, v)_{R_i} \le (\frac{m+3}{2})^2 - 2$.*

Proof. If $|Z| = 1$, then we can easily compute as in the previous cases: there are at most $(m-1)^2/4$ paths not containing w_i and w_{i+1}, each of w_i and w_{i+1} is contained in at most $m - 1$ paths, and there can be two paths that contain both w_i and w_{i+1}. Thus, we get

$$\#P_5(u, v)_{R_i} \le \frac{(m-1)^2}{4} + 2(m-2) + 2 = \frac{(m+3)^2}{4} - 2.$$

Now, assume that $|Z| \ge 2$. First, suppose that w_i and w_{i+1} have at most one common neighbor in M. By the first case, there are at most $|X||Y| + |Z| + 2$ paths $P_5(u, v)$ without using w_i and w_{i+1}.

A path $P_5(u, v)$ that uses w_i must use w_i as a neighbor of u and a path $P_4(w_i, v)$ using a vertex from Z and a neighbor of v, or use w_i as a neighbor of v and a path

$P_4(w_i, u)$ using a vertex from Z and a neighbor of u. Let Z_i' and Z_i'' be those vertices in Z that are on paths $P_4(w, u)$ and paths $P_4(w, v)$, respectively. Define sets Z_{i+1}' and Z_{i+1}'' with respect to w_{i+1} in the same way.

Then, any vertex in $(Z_i' \cup Z_i'') \cap (Z_{i+1}' \cup Z_{i+1}'')$ is a common neighbor of w_i and w_{i+1} in M, and hence this intersection contains at most one vertex. Moreover, any common neighbor of w_i and w_{i+1} is in all the four sets Z_i', Z_i'', Z_{i+1}' and Z_{i+1}''.

We claim that $|Z_i' \cap Z_i''| \leq 1$. Let $z \in Z_i' \cap Z_i''$, then z is adjacent to w_i and has a neighbor $x \in X$ and a neighbor $y \in Y$. If any of x or y is w_{i+1}, then the region R_i is divided into the regions R and R' bounded by the cycles $uw_izw_i + 1$ and vw_izw_{i+1}, respectively. Obviously, any vertex of X lies in R and any vertex of Y lies in R'. So, if any other vertex $z' \in Z$ is in R, then it cannot be adjacent to any vertex of Y, since it can only be adjacent to w_{i+1} and hence be another common neighbor of w_i and w_{i+1}, a contradiction. Thus, any vertex of Z that lies in R cannot be in Z_i''. Similarly, the vertices of Z that are in R' cannot be in Z_i'.

If none of x and y is w_{i+2}, we still have the region R_i divided into three regions bounded by cycles uw_izx, vw_izy and $uxzyvw_{i+1}$. Again, they separate the vertices of Z, and we can similarly obtain $|Z_i' \cap Z_i''| \leq 1$. Likewise, $|Z_{i+1}' \cap Z_{i+1}''| \leq 1$.

Therefore, If w_i and w_{i+1} have no common neighbors, then $|(Z_i' \cup Z_i'') \cap (Z_{i+1}' \cup Z_{i+1}'')| = 0$, $|Z_i' \cap Z_i''| \leq 1$ and $|Z_{i+1}' \cap Z_{i+1}''| \leq 1$. Thus, $|Z_i'| + |Z_i''| + |Z_{i+1}'| + |Z_{i+1}''| \leq |Z| + 2$.

Also, if w_i and w_{i+1} have one common neighbor, then The only vertex in all the four sets is the this common neighbor, and hence, $|Z_i'| + |Z_i''| + |Z_{i+1}'| + |Z_{i+1}''| \leq |Z| + 3$.

By Corollary 5.14, there are at most $(|Z_i'| + |X| - 1) + (|Z_i''| + |Y| - 1)$ paths that use w_i (not using w_{i+1}) and there are at most $(|Z_{i+1}'| + |X| - 1) + (|Z_{i+1}''| + |Y| - 1)$ paths that use w_{i+1} (not using w_i). Also, if w_i and w_{i+1} have a common neighbor, then there are two other paths $P_5(u, v)$ that use both w_i and w_{i+1}. Thus the number of paths containing w_i or w_{i+1} is at most

$$\begin{aligned}
&(|Z_i'| + |X| - 1) + (|Z_i''| + |Y| - 1) + (|Z_{i+1}'| + |X| - 1) + (|Z_{i+1}''| + |Y| - 1) + 2 \\
&= |Z_i'| + |Z_i''| + |Z_{i+1}'| + |Z_{i+1}''| + 2|X| + 2|Y| - 2 \\
&= |Z| + 2|X| + 2|Y| + 1.
\end{aligned}$$

Hence, for the total number of paths in the region R_i, we get

$$\begin{aligned}
\#P_5(u, v)_{R_i} &\leq |X||Y| + |Z| + 2 + |Z| + 2|X| + 2|Y| + 1 \\
&= |X||Y| + 2(|X| + |Y| + |Z|) + 3 \\
&\leq \left(\frac{|X| + |Y|}{2}\right)^2 + 2(|X| + |Y| + |Z|) + 3 \\
&= \left(\frac{m - |Z|}{2}\right)^2 + 2m + 3 \leq \left(\frac{m - 2}{2}\right)^2 + 2m + 3 < \frac{(m+3)^2}{4} - 2.
\end{aligned}$$

Note that the last inequality is true for all $m \geq 4$. If $m \leq 3$, then as $l \geq 2$, we one of X and Y is empty and the other one contains at most one vertex. We can easily check that the claim it true in this case.

Now, Suppose that w_i and w_{i+1} have at least two common neighbors in M. let $A := \{a_1, a_2, \ldots, a_t\}$ be the set of their common neighbors, where $t \geq 2$. Then the region R_i is divided into regions Q, Q' and Q_i, for $1 \leq i \leq t-1$, bounded by $uw_ia_1w_{i+1}$, $vw_ia_tw_{i+1}$, and $w_ia_iw_{i+1}a_{i+1}$, for $1 \leq i \leq t-1$, respectively. Then, every vertex of X lies in Q, and every vertex of Y lies in Q'. No vertex of $Z \setminus A$ can lie in Q_i, for any $1 \leq i \leq t-1$, otherwise it must be adjacent to both w_i and w_{i+1}, but this means it is in A. So, vertices of $Z \setminus A$ lie either in Q or Q'. Let $Z' \subseteq Z$ be the vertices in Q (including a_1), and $Z'' \subseteq Z$ be the vertices in Q' (including a_t). Each path $P_5(u,v)_{R_i}$ that use the vertices of X must use $\{w_i, w_{i+1}\}$ as neighbors of v. Thus, it either uses w_i and a path $P_4(w_i, u)$ with a vertex of Z' and a vertex of X, or uses w_{i+1} and a path $P_4(w_{i+1,u})$ with a vertex of Z' and a vertex of X. Let Z'_i and Z'_{i+1} be the sets of vertices in Z' that are on paths $P_4(w_i, u)$ and paths $P_4(w_{i+1}, u)$, respectively. Since only a_1 in Z' is a common neighbor of w_i and w_{i+1}, then $Z'_i \cap Z'_{i+1} = \{a_1\}$. Then, $|Z'_i| + |Z'_{i+1}| = |Z'| + 1$ By Corollary 5.14, we have $|Z'_i| + |X| - 1$ paths $P_4(w_i, u)$ and $|Z'_{i+1}| + |X| - 1$ paths $P_4(w_{i+1}, u)$. Thus, the number of paths $P_5(u,v)_{R_i}$ that contain vertices of X is at most

$$(|Z'_i| + |X| - 1) + (|Z'_{i+1}| + |X| - 1) = |Z'| + 2|X| - 1$$

Similarly, we can show that the number of paths $P_5(u,v)_{R_i}$ that contain vertices of Y is at most $|Z''| + 2|Y| - 1$.

For each $2 \leq i \leq t-1$, each vertex a_i is in two paths $P_5(u,v)_{R_i}$ (they are adjacent to only w_i and w_{i+1} as neighbors of u and v). So, they give $2(t-2) = 2(l - |Z'| - |Z''|)$ paths.

Thus, in total, we obtain

$$\begin{aligned}\#P_5(u,v)_{R_i} &\leq |Z'| + 2|X| - 1 + |Z'| + 2|Y| - 1 + 2(l - |Z'| - |Z''|) \\ &= 2|X| + 2|Y| + 2l - |Z'| - |Z''| - 2 \\ &\leq 2m - 4 < \frac{(m+3)^2}{4} - 2.\end{aligned}$$

□

Now, suppose that two regions R_i and R_j both contain paths of length four from u to v, so they are not empty, say they contain m_i and m_j vertices, respectively. Then, applying the claim above, in the two regions together the number of paths is at most

$$\begin{aligned}\left(\frac{m_i+3}{2}\right)^2 + \left(\frac{m_j+3}{2}\right)^2 - 4 &= \frac{m_i^2 + m_j^2 + 6(m_i+m_j) + 18}{4} - 4 \\ &= \frac{(m_i+m_j)^2 - 2m_im_j + 6(m_i+m_j) + 18}{4} - 4 \\ &= \left(\frac{m_i+m_j+3}{2}\right)^2 - \frac{7 + 2m_im_j}{4} < \left(\frac{m_i+m_j+3}{2}\right)^2 - 2.\end{aligned}$$

That is, we obtain more paths if all the $m_i + m_j$ vertices are in one region and the other is empty. Thus, easily by induction, we will have more paths if all the regions except one of them are empty, in which case by Claim 5.3 we will have

$$\#\ P_5(u,v) \leq \left(\frac{n-(k+2)+3}{2}\right)^2 - 2 \leq \left(\frac{n-1}{2}\right)^2 - 2.$$

□

Now, we prove three lemmas to prepare the proof of Theorem 5.11. In fact, we prove that for sufficiently large n any extremal graph on n vertices has to be isomorphic to H_n. Clearly, H_n is a triangle-free planar graph on n-vertices.

Let us count the number of 6-cycles in H_n. There are $|A||B||C|$ 6-cycles on $V(H_n) \setminus \{z_1, z_2\}$, and each of z_1 and z_2 is in $|A||B| + |A||C| + |B||C|$ 6-cycles that does not contain the other. Finally, we count those 6-cycles that contain both z_1 and z_2, any such cycle must contain u_1, u_2 and u_3, and hence the last vertex comes from one of A, B or C, but each of them twice depending on which edges we use. Thus, there are $2(|A| + |B| + |C|)$ such 6-cycles. Hence, all together, the number of 6-cycles in G is at most

$$|A||B||C| + 2(|A||B| + |A||C| + |B||C|) + 2(|A| + |B| + |C|).$$

Note that if $n \equiv 2 \pmod 3$, then each of A, B and C contains $(n-5)/3$ vertices, if $n \equiv 1 \pmod 3$, then one of them contains $(n-7)/3$ vertices and the others contain $(n-4)/3$ each, and if $n \equiv 0 \pmod 3$, then one of them contains $(n-3)/3$ vertices and the other two contain $(n-6)/3$ each. Calculating as above, H_n contains $h(n)$ 6-cycles. Note that among such graphs with various sizes of A, B and C, this is the maximum number of 6-cycles.

Also, through Theorem 5.10, it is easy to see that the vertices of H_n that are in the largest set among A, B and C, are in the fewest number of 6-cycles. Simply computing that, we see that each vertex of H_n is contained in at least $h_1(n)$ 6-cycles (and there are vertices that are contained in exactly $h_1(n)$ 6-cycles), where $h_1(n)$ is defined as follows.

$$h_1(n) = \begin{cases} \left(\frac{n}{3}\right)^2 - 2 = \frac{n^2}{9} - 2, & \text{if } n \equiv 0 \ (mod\ 3) \\ \left(\frac{n-1}{3}\right)\left(\frac{n+2}{3}\right) - 2 = \frac{n^2}{9} + \frac{n}{9} - \frac{20}{9}, & \text{if } n \equiv 1 \ (mod\ 3) \\ \left(\frac{n+1}{3}\right)^2 - 2 = \frac{n^2}{9} + \frac{2n}{9} - \frac{17}{9}, & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

Note that a vertex of degree one is not contained in any 6-cycle, so we may assume that our graphs have minimum degree at least 2. We fist prove that for n large enough, if an n-vertex triangle-free planar graph has the property that each of its vertices is in at least $h_1(n)$ 6-cycles, then it has to be isomorphic to H_n. This is done through the following lemmas, following the approach in [86] and [52].

Lemma 5.16. *Let $\gamma > 0$ be a small constant and n be sufficiently large. Assume that G is a triangle-free planar graph on n vertices such that every vertex of G is contained in at least $n^2/10$ 6-cycles. Then, G contains three vertices u_1, u_2 and u_3 such that each of $|N(u_1, u_2)|, |N(u_1, u_3)|$ and $|N(u_2, u_3)|$ is at least γn.*

Proof. Suppose G is a plane graph on n vertices satisfying all the conditions of the lemma.

Claim 5.4. *If G contains two vertices x and y with $|N(x, y)| \geq \alpha n$, where $0 < \alpha < 1$ is a constant, then G contains a vertex of degree 2, whose only two neighbors are x and y.*

Proof. Let $N(x, y) = \{w_1, w_2, \ldots, w_k\}$, then $k \geq \alpha n$. As in the proof of Lemma 5.10, they form k regions. For each $1 \leq i \leq k$, the region R_i is bounded by the 4-cycle xw_iyw_{i+1}, where addition in the subscript is modulo k. Since G is triangle-free, $N(x, y)$ is an independent set, and hence, for any $1 \leq i \leq k$, the vertex w_i can only be adjacent to x, y and vertices inside R_{i-1} or R_i. Thus, it suffices to show that there are at least two consecutive empty regions.

We show that if a region is not empty, then it must contain more than $n^{1/3}$ vertices. On the contrary, assume that a region R_i bounded by xw_iyw_{i+1} is not empty and contains at most $n^{1/3}$ vertices. Let z be a vertex inside R_i. We count all the 6-cycles that contain z. If such a 6-cycle uses vertices only in R_i (including the boundary vertices), then owing to Theorem 5.1, there are at most $(n^{1/3}+4)^3/9+o((n^{1/3}+4)^2) < n$, as n is large.

If a 6-cycle containing z is a crossing one, i.e. uses at least one vertex outside of R_i, then it uses at least two vertices on the boundary of R_i, and hence it can contain at most three vertices inside R_i.

Assume that the 6-cycle contains only z inside R_i. Then, z must be adjacent to at least two of the boundary vertices of R_i. As z is not a common neighbor of x and y, and since G is triangle-free, the two boundary vertices must be w_i and w_{i+1}. Now, without using any of x and y we cannot have a 6-cycle containing w_izw_{i+1}, and if one of x and y are used, we need two vertices outside R_i, that form a path of length three from x to w_i or from x to w_{i+1}, if x is used (and for y it is similar). By Corollary 5.14, there are fewer than n such paths. Also, if both x and y are used, to complete the 6-cycle we need a common neighbor of x and y, for which there are $k-2$ choices. Thus, all together, there are at most $5n$ such 6-cycles.

Assume that the 6-cycle contains z and another vertex, say z', inside R_i. There are at most $n^{1/3}$ choices for z'. If only two boundary vertices are used, then we need two more vertices outside of R_i that form a path of length three between the two boundary vertices. Again, by Corollary 5.14, there are fewer than n such paths for each pair of the boundary vertices. Thus, there are at most $6nn^{1/3} = 6n^{4/3}$ choices for such a cycle. If the 6-cycle uses three vertices on the boundary of R_i, then together with z and z', it is already 5 vertices, so we need one vertex outside, for which there are at most n choices. Therefore, there are at most $4nn^{1/3} = 4n^{4/3}$, as there are 4 possible ways to choose three boundary vertices. Note that it may not be the case that any two or three of the boundary vertices could be chosen, but even this loose upper bound works for us now. In this case, there are at most $10n^{4/3}$ such 6-cycles.

Finally, assume that the 6-cycle contains two other vertices z_1 and z_2 besides z inside the region R_i. As it uses at least a vertex outside of R_i, it must use exactly two vertices on the boundary of R_i and the outside vertex is a common neighbor of them. Thus, the two boundary vertices cannot be adjacent as this would form a triangle, and hence, they are either x and y or w_i and w_{i+1}. There are no common neighbors of w_i and w_{i+1} outside the region R_i because of the other common neighbors of x and y and planarity of G. Thus, the two boundary vertices must be x and y for which there are $k-2$ possible common neighbors outside R_i. Then, x, y, z, z_1 and z_2 form a path of length four from x to y in the region R_i (including boundary vertices), applying Theorem 5.10, there are at most $((n^{1/3}+3)/2)^2-2$ such paths, which is less than $n^{5/6}$ as n is large. Even if z is in all those paths, there is at most $kn^{5/6} < n^{11/6}$ 6-cycles in this case.

Therefore, in total, z is contained in at most $n + 5n + 10n^{4/3} + n^{11/6} < n^2/10$ 6-cycles, since n is large, obtaining the desired contradiction. Thus, if there are t non-empty regions, they contain more than $tn^{1/3}$ vertices, and hence $tn^{1/3} < n$, which means $t < n^{2/3}$. Since there are $k \geq \alpha n$ regions and n is sufficiently large, there exist consecutive empty regions, which completes the proof. ☐

Claim 5.5. *Let $\beta > 0$ be a small constant. If a vertex v has degree at most 5, then it has two neighbors w_1 and w_2 for which there is a vertex w such that each of $|N(w, w_1)|$ and $|N(w, w_2)|$ is at least βn.*

Proof. Every 6-cycle containing v must contain xvy for some pair of neighbors x and y of v. There are at most 10 such pairs and v is in at least $n^2/10$ 6-cycles. Thus, there is a pair w_1 and w_2 such that w_1vw_2 is in at least $n^2/100$ 6-cycles. Each such cycle is determined by a path of length four from w_1 to w_2, hence there are $n^2/100$ paths $P_5(w_1, w_2)$. Let P be the set of vertices that are on such paths from w_1 to w_2. Let $X := N(w_1) \cap P$, $Y := N(w_2) \cap P$ and $Z := P \setminus X \cup Y$. Each path uses an edge $yz \in E(Y, Z)$ and a neighbor of z in X, and each edge $yz \in E(Y, Z)$ is in at most $|N_X(z)|$ such paths. Since the graph is planar, the bipartite graph between Y and Z has fewer that $2n$ edges, i.e. $|E(Y, Z)| < 2n$, and there are at most $\sum_{yz \in E(Y,Z)} |N_X(z)|$ paths of length four from w_1 to w_2. Let $L := \{yz \in E(Y, Z) : \ |N_X(z)| \geq n/300\}$. We then have $n^2/100 \leq \#P_5(w_1, w_2) \leq |L|n + (2n - |L|)n/300$, which implies that $|L| \geq n/299$.

Since we also have $E(X, Z) < 2n$, at most 600 vertices in Z can be the endpoints of the edges in L, and hence, there is a vertex $w \in Z$, which is the end point of at least $n/179400$ edges in L. This means $N(w, w_1) \geq n/300$ and $N(w, w_2) \geq n/179400$. As β is small enough, this completes the proof of the claim. ☐

Now, we can complete the proof of the lemma. Since G is planar, it contains a vertex of degree at most 5. Then By Claim 5.5, there are vertices u_1 and u_2 such that $|N(u_1, u_2)|$ is at least γn. Hence, by Claim 5.4, there is a vertex v_1 of degree 2, whose only two neighbors are u_1 and u_2. Applying Claim 5.5 on v_1, there is a vertex u_3 such that each of $|N(u_1, u_3)|$ and $|N(u_2, u_3)|$ is at least γn. ☐

For a set of vertices M, we denote by $P_5(x, y)_M$ a path of length four from x to y that contains some vertices from M.

Lemma 5.17. *Let G be a triangle-free planar graph, and $u_1v_1u_2v_2u_3v_3$ be an induced 6-cycle in G. Suppose M is the set of vertices in the interior of the region bounded by the 6-cycle, with $|M| = m$, such that each vertex in M is adjacent to at most one of u_1, u_2 and u_3. Then,*

$$\sum_{i=1}^{3} \#P_5(u_i, u_{i+1})_M \leq 3\left(\frac{m+5}{3}\right)^2 - 3,$$

where $i + 1$ is taken modulo 3.

Proof. First observe that no path $P_5(u_i, u_{i+1})$, for each $i \in [3]$, can use vertices from both the interior and the exterior of the bounding 6-cycle, since it would create a triangle in G. Thus, any path $P_5(u_i, u_{i+1})_M$ uses only the vertices of M and those on the 6-cycle. For each $i \in [3]$, let X_i be the set of neighbors of u_i in M together with it's neighbors v_i and v_{i+2} on the 6-cycle. Since each vertex in M is adjacent to at most one of u_i's, we have $X_i \cap X_{i+1} = \{v_i\}$. Each $P_5(u_i, u_{i+1})$ contains a vertex from X_i, a vertex from X_{i+1} and a common neighbor of them, which we call the *middle vertex* of the path. Let $Z = M \setminus \cup_{i=1}^{3} X_i$. Then, for each $i \in [3]$, the middle vertex of any $P_5(u_i, u_{i+1})$ is in $Z \cup X_{i+2}$. Accordingly, we distinguish the following cases.

Case 1. For each $i \in [3]$, the middle vertex of any $P_5(u_i, u_{i+1})$ is in Z.

For each $i \in [3]$, let $Z_i \subseteq Z$ be the set of middle vertices of all the paths $P_5(u_i, u_{i+1})$. It is easy to see, due to the properties of G being planar and triangle-free, that $|\cap_{i=1}^{3} Z_i| \leq 1$. Analogous to the proof of Theorem 5.10, we have the most number of paths between u_i and u_{i+1} if for each $i \in [3]$, we have Z_i is a singleton. Furthermore, a simple computation gives that we have more paths if $Z_1 = Z_2 = Z_3 = Z = \{z\}$, see Figure 5.5. Then, $\sum_{i=1}^{3} |X_i| \leq m + 5$ (since we exclude z from M and each v_i is counted twice), and for each $i \in [3]$, we have $|X_i||X_{i+1}| - 1$ paths $P_5(u_i, u_{i+1})$. The term -1 is because choosing v_i in both of X_i and X_{i+1} does not give a path of length four from u_i to u_{i+1}. This gives

$$\sum_{i=1}^{3} \#P_5(u_i, u_{i+1})_M = \sum_{i=1}^{3} (|X_i||X_{i+1}| - 1) \leq 3 \left(\frac{m+5}{3}\right)^2 - 3$$

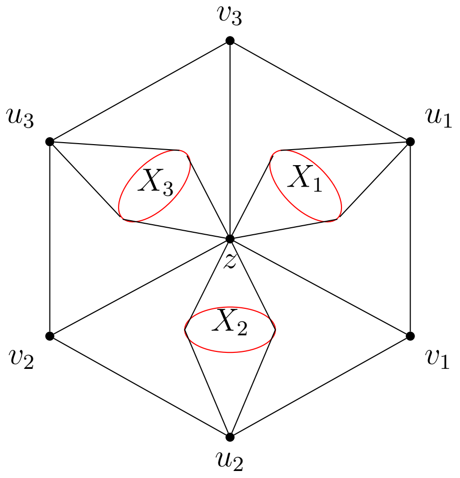


Figure 5.5: The best configuration that achieves the bound in Lemma 5.17.

Case 2. For some $i \in [3]$, there is a vertex in X_{i+2}, which is the middle vertex of a $P_5(u_i, u_{i+1})$.

To ease the notation, without loss of generality, assume there is a vertex $x_3 \in X_3$, which is the middle vertex of a path $P_5(u_1, u_2)$. Since x_3 is not adjacent to any of u_1 and u_2, it must have neighbors in both of X_1 and X_2. Let $N_{X_1}(x_3) = Y_1$

and $N_{X_2}(x_3) = Y_2$. Take $x_1 \in Y_1$ such that the region R_3 bounded by the cycles $u_1x_1x_3u_3v_3$ does not contain a common neighbor of u_1 and x_3. Similarly, choose $x_2 \in Y_2$ such that the region R_2 bounded by the cycle $u_2x_2x_3u_3v_2$ does not contain a common neighbor of u_2 and x_3. Let R_1 be the region bounded by the cycle $u_1v_1u_2x_2x_3x_1$. For each $i \in [3]$, let m_i be the number of vertices in the interior of R_i, so $m_1 + m_2 + m_3 = m - 3$. See Figure 5.6.

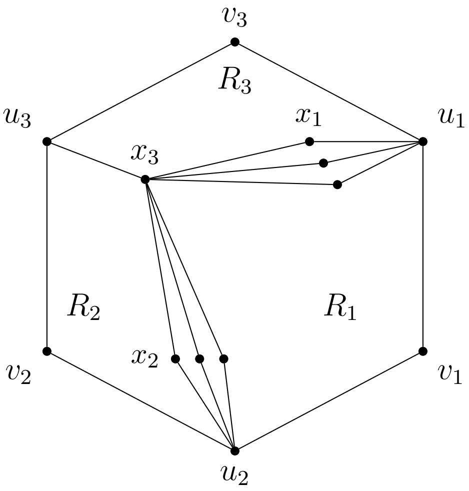


Figure 5.6: $x_3 \in X_3$ is the a middle vertex of some $P_5(u_1, u_2)$.

Note that, since G is planar and triangle-free, none of the x_i's can be any of the v_i's, and no other vertices of X_3 can be the middle vertex of a $P_5(u_1, u_2)$. Also, for the same reasons, if Y_1 contains only x_1, then it is possible that x_1 is the middle vertex of a path $P_5(u_2, u_3)$, but then no vertex in X_2 can be the middle vertex of a path $P_5(u_1, u_3)$. That is, in each X_i at most one vertex can be the middle vertex, and at most two of the X_i's can contain a middle vertex. We now have the following possibilities:

Subcase 2.1 Each of Y_1 and Y_2 contains at least two vertices. Thus, none of x_1 and x_2 can be the middle vertex of the considered paths. Let us count the paths according the regions R_1, R_2 and R_3.

No vertex inside R_3 can be on a path $P_5(u_1, u_2)$ or $P_5(u_2, u_3)$, and by Theorem 5.10 there are at most $\left(\frac{m_3+4}{2}\right)^2 - 2$ paths $P_5(u_1, u_3)$ (note that there are m_3 vertices inside R_3 and 5 vertices on its boundary). Similarly, vertices inside R_2 cannot be on paths $P_5(u_1, u_2)$ and $P_5(u_1, u_3)$, and there are at most $\left(\frac{m_2+4}{2}\right)^2 - 2$ paths $P_5(u_1, u_3)$.

Let us consider R_1. Again there are at most $\left(\frac{m_1+5}{2}\right)^2 - 2$ paths $P_5(u_1, u_2)$. If a path $P_5(u_1, u_3)$ contains vertices inside R_1, then it must contain the edge x_3u_3 and a path $P_4(u_1, x_3)$, i.e. a path of length three from u_1 to x_3. Likewise, a $P_5(u_2, u_3)$ that uses vertices inside R_1 is determined by a $P_4(u_2, x_3)$. Let L_1 and L_2 be the sets of vertices inside R_1 that are on paths $P_4(u_1, x_3)$ and $P_4(u_2, x_3)$. We claim that $|L_1 \cap L_2| \leq 1$. To see this, assume a vertex $x' \in X_1$ (a neighbor of u_1) is on a path $P_4(u_2, x_3)$. Then x' must be adjacent to x_3 and must have a neighbor with u_2, since

x' is not adjacent to u_2. Thus, x' is a common neighbor of u_1 and x_3, and hence, it cannot be on a $P_4(u_1, x_3)$, as otherwise this would form a triangle. Therefore, no vertex in X_1 (inside R_1) can be in $L_1 \cap L_2$. Similarly, no vertex in X_2 inside R_1 can be in $L_1 \cap L_2$. Now, suppose a vertex w inside R_1 is in $L_1 \cap L_2$. Then w is not adjacent to any of u_1 and u_2 and it is on a $P_4(u_1, x_3)$ and a $P_4(u_2, x_3)$. Then, it must be adjacent to x_3 and have neighbors with both u_1 and u_2, which then separates the region R_1 into three other regions that can easily be seen that no other vertices can be common to both of L_1 and L_2. By Corollary 5.14, there are at most $|L_1| - 1$ paths $P_4(u_1, x_3)$, and at most $|L_2| - 1$ paths $P_4(u_2, x_3)$. Also, on the boundary, v_1 can be counted in $L_1 \cap L_2$. Thus, there are at most $m_1 + 4$ paths of length three from u_1 or u_2 to x_3, which means there are at most $m_1 + 4$ paths $P_5(u_1, u_3)$ or $P_5(u_2, u_3)$. Thus, we obtain

$$\begin{aligned}\sum_{i=1}^{3} \#P_5(u_i, u_{i+1})_M &= \left(\frac{m_3+4}{2}\right)^2 - 2 + \left(\frac{m_2+4}{2}\right)^2 - 2 + \left(\frac{m_1+5}{2}\right)^2 - 2 + m_1 + 4 \\ &= \left(\frac{m_3+4}{2}\right)^2 + \left(\frac{m_2+4}{2}\right)^2 + \left(\frac{m_1+5}{2}\right)^2 + m_1 - 2 \\ &< 3\left(\frac{m+5}{3}\right)^2 - 3.\end{aligned}$$

Subcase 2.2 At least one of Y_1 or Y_2 contains only one vertex. Then, it is possible for x_1 or x_2 to be the middle vertex of a considered path. Without loss of generality, assume x_1 is such, which means $Y_1 = \{x_1\}$. Then, x_1 has neighbors in X_2, which are not in Y_2 (as otherwise with x_3, they form a triangle) and may have other neighbors besides x_3 in X_3, which must be in the region R_3.

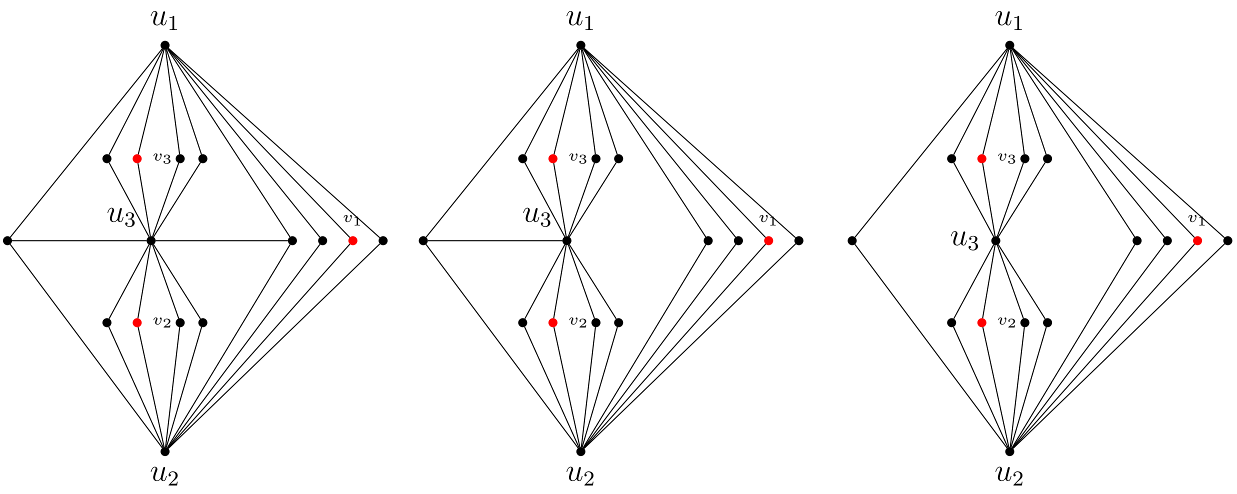
If x_1 has no other neighbors besides x_3 in X_3, then this situation is already covered in Subcase 2.1. And if x_3 has other neighbors besides x_3 in X_3, then the situation is the same as the previous subcase with the role of x_1 and x_3 being swapped. □

Lemma 5.18. *Let G be an n-vertex triangle-free planar graph. If every vertex of G is contained in at least $h_1(n)$ 6-cycles, then G is isomorphic to H_n, if n is sufficiently large.*

Proof. Let G be an n-vertex triangle-free planar graph, where n is sufficiently large. Assume every vertex of G is contained in at least $h_1(n)$ 6-cycles. By Lemma 5.16, there are three vertices u_1, u_2 and u_3 such that for every $i \in [3]$, $|N(u_i, u_{i+1})| \geq \alpha n$, where $0 < \alpha < 1$ is a constant. The addition in the subscript, here and in the rest of the proof, is taken modulo 3. Let $|N(u_i, u_{i+1})| = k_i$, for each $i \in [3]$. Let G_0 be the induced subgraph of G on $\{u_1, u_2, u_3\} \cup N(u_1, u_2) \cup N(u_2, u_3) \cup N(u_1, u_3)$.

Consider $G[\{u_1, u_2\} \cup N(u_1, u_2)]$. This divides the plane into k_1 regions, each bounded by a 4-cycle $u_1 x u_2 y$, for some $x, y \in N(u_1, u_2)$. Then, u_3 and its common neighbors with u_1 and with u_2 are in one of these regions, without loss of generality, say in the “first" region, as shown in Figure 5.7

Note that $|N(u_1, u_2, u_3)| \leq 2$, as otherwise G would contain a $K_{3,3}$. If $|N(u_1, u_2, u_3)| = 2$, then all the faces of G_0 are bounded by 4-cycles, if $|N(u_1, u_2, u_3)| = 1$, then there is one face bounded by a 6-cycle, if $|N(u_1, u_2, u_3)| = 0$, there are two faces bounded by 6-cycles. See Figure 5.7. We first show that $V(G) \setminus V(G_0) = \emptyset$.



(a) All faces are C_4
$|N(u_1, u_2, u_3)| = 2$

(b) One C_6-face
$|N(u_1, u_2, u_3)| = 1$

(c) Two C_6-faces
$|N(u_1, u_2, u_3)| = 0$

Figure 5.7: The possible cases of the subgrapb G_0.

From Claim 5.4 in the proof of Lemma 5.16, we obtain that for each $i \in [3]$, there is a vertex $v_i \in N(u_i, u_{i+1})$ such that the only neighbors of v_i are u_i and u_{i+1}. For each $i \in [3]$, we count all the 6-cycles that contain v_i. Then, by averaging, there is a vertex $v \in \{v_1, v_2, v_3\}$ that is contained in at most $\frac{1}{3}\sum_{i=1}^{3} \#C_6(v_i)$ (recall that $C_6(v_i)$ denotes a 6-cycle that contains the vertex v_i). If a 6-cycle contains v_i, then it must contain the path $u_i v_i u_{i+1}$ and a $P_5(u_i, u_{i+1})$. Also, any $P_5(u_i, u_{i+1})$ gives a 6-cycle containing v_i. Hence, $\#C_6(v_i) = \#P_5(u_i, u_{i+1})$, which means

$$\sum_{i=1}^{3} \#C_6(v_i) = \sum_{i=1}^{3} \#P_5(u_i, u_{i+1})$$

Assume a face of G_0 that is bounded by a 4-cycle $u_i x u_{i+1} y$, where $x, y \in N(u_i, u_{i+1})$, for some $i \in [3]$, is not empty in G (i.e. there are vertices of G in its interior). Let T be the set of vertices in its interior with $|T| = t$. Then as in the proof of Claim 5.4, we must have $t \geq n^{1/3}$. Observe that if none of x and y is adjacent to u_{i+2}, then no vertex of T can be on a $P_5(u_i, u_{i+2})$ or a $P_5(u_{i+1}, u_{i+2})$. Also, at most one of x and y can be adjacent to u_{i+2} (and this is possible only for two such faces), say x is adjacent to u_{i+2}. Then, any $P_5(u_i, u_{i+2})$ that uses vertices from T must use a path of length three from u_i to x and the edge xu_{i+2}. Then there are at most t such paths. Similarly there are at most t paths $P_5(u_{i+1}, u_{i+2})$ that uses vertices from T. By Theorem 5.10, T contributes to at most $(t+3)^2/4 - 2 = t^2/4 + 3t/2 + 1/4$ paths $P_5(u_i, u_{i+1})$. Thus, by moving $t/2$ of the vertices to each of $N(u_i, u_{i+2})$ and $N(u_{i+1}, u_{i+2})$, we lose at most $2t$ paths of length four from u_i to u_{i+2} and from u_{i+1}

to u_{i+2}, and at most $t^2/4 + 3t/2 + 1/4$ paths of length four between u_i and u_{i+1}, while we gain $t^2/4 + (k_{i+1} + k_{i+2})t/2$ paths between u_i and u_{i+1}, $k_i t/2$ paths between u_i and u_{i+2} and $k_i t/2$ paths between u_{i+1} and u_{i+2}.

Thus, the total increase of $\sum_{i=1}^{3} \#P_5(u_i, u_{i+1})$ is at least

$$t^2/4 + (k_{i+1} + k_{i+2})t/2 + k_i t - \big(t^2/4 + 3t/2 + 1/4 + 2t\big) \geq (k_{i+1} + k_{i+2})t/2 \geq \alpha n^{4/3}.$$

Therefore, we may assume that all the regions that are bounded by 4-cycles of G_0 are empty. We now show that the possible 6-faces of G_0 (i.e are bounded by 6-cycles of G_0) are empty in G, too. Recall that we can have at most two such faces, so suppose M and L are the sets of vertices in their interiors, with $|M| = m$ and $|L| = l$.

For each $i \in [3]$, a $P_5(u_i, u_{i+1})$ can not contain vertices from both of M and L, and hence it either contains no vertex from $M \cup L$ or it contains some vertices from M or it contains some vertices from L. Then, owing to Lemma 5.17, and assuming $k_1 + k_2 + k_3 = k$, we obtain

$$\begin{aligned}
\sum_{i=1}^{3} \#C_6(v_i) &= \sum_{i=1}^{3} (k_i k_{i+1} + \#P_5(u_i, u_{i+1})_M + \#P_5(u_i, u_{i+1})_L) \\
&\leq k_1 k_2 + k_2 k_3 + k_1 k_3 + 3\left(\frac{m+5}{3}\right)^2 - 3 + 3\left(\frac{l+5}{3}\right)^2 - 3 \\
&\leq 3\left(\frac{k}{3}\right)^2 + 3\left(\frac{m+5}{3}\right)^2 - 3 + 3\left(\frac{l+5}{3}\right)^2 - 3
\end{aligned}$$

Then, by averaging, there is a vertex $v \in \{v_1, v_2, v_3\}$ such that

$$\begin{aligned}
\#C_6(v) \leq 1/3 \sum_{i=1}^{3} \#C_6(v_i) &= \left(\frac{k}{3}\right)^2 + \left(\frac{m+5}{3}\right)^2 + \left(\frac{l+5}{3}\right)^2 - 2 \\
&= \frac{k^2}{9} + \frac{m^2 + 10m + 25}{9} + \frac{l^2 + 10l + 25}{9} - 2 \\
&= \frac{k^2 + m^2 + l^2}{9} + \frac{10m + 25}{9} + \frac{10l + 25}{9} - 2 \\
&= \frac{(k + m + l)^2 - 2(km + kl + ml)}{9} + \frac{10m + 25}{9} + \frac{10l + 25}{9} - 2 \\
&< \frac{(n-1)^2}{9} - 2 < h_1(n),
\end{aligned}$$

where the last inequality is because $k + m + l \leq n - 1$. Indeed, if both of m and l are non-zero, then $k + m + l = n - 3$, as they count all the vertices of G except the u_i's. Also, if one of them, say l is zero, then we can have a vertex in $N(u_1, u_2, u_3)$, which will be counted three times in k_i's, but still $k + m = n - 1$, and we reach the same conclusion. This contradiction proves that both of M and L must be empty, which gives $V(G_0) = V(G)$ as desired.

Now, to show that G is isomorphic to H_n, we must have all k_i's are as equal as possible and $|N(u_1, u_2, u_3)| = 2$. Since all faces are empty, then we can add appropriate chords in the two faces that are bounded by 6-cycles, and make $|N(u_1, u_2, u_3)| = 2$, increasing the number of paths of length four between each pair of the u_i's. Therefore, we may assume that $|N(u_1, u_2, u_3)| = 2$, and hence $k_1 + k_2 + k_3 = n + 1$. Then, for each $i \in [3]$, we have the number of $P_5(u_i, u_{i+1})$, and hence the number of 6-cycles that contain v_i, is exactly $k_{i+1}k_{i+2} - 2$.

Assume $n \equiv 0 \pmod 3$, which means $n + 1 \equiv 1 \pmod 3$. If for some $i \in [3]$, we have $k_i \geq n/3 + 1$, then $k_{i+1} + k_{i+2} \leq 2n/3$. Then, $\#C_6(v_i) = \#P_5(u_i, u_{i+1}) = k_{i+1}k_{i+2} - 2 \leq (n/3)^2 - 2 = h_1(n)$, and equality holds only when each of k_{i+1} and k_{i+2} is $n/3$, which means we must also have equality in $k_i \geq n/3 + 1$, which implies that G is isomorphic to H_n.

Assume $n \equiv 1 \pmod 3$, which means $n + 1 \equiv 2 \pmod 3$. If for some $i \in [3]$, we have $k_i \geq (n + 2)/3$, then $k_{i+1} + k_{i+2} \leq (2n + 1)/3$. Note that, since $n \equiv 1 \pmod 3$, we have $(2n + 1)/3$ is an odd integer, we can then have $\#C_6(v_i) = \#P_5(u_i, u_{i+1}) = k_{i+1}k_{i+2} - 2 \leq ((n+2)/3) \cdot ((n-1)/3) - 2 = h_1(n)$. Again, equality holds only when one of k_{i+1} and k_{i+2} is $(n + 2)/3$, and the other is $(n - 1)/3$, which means we must also have equality in $k_i \geq (n + 2)/3$, implying that that G is isomorphic to H_n.

Finally, if $n \equiv 2 \pmod 3$, we can similarly get the same conclusion. □

We are now ready to prove Theorem 5.11.

Proof of Theorem 5.11. Let n be sufficiently large, and G_n be an extremal graph on n-vertices. By Lemma 5.18, we have either G_n contains a vertex in fewer than $h_1(n)$ 6-cycles or it is isomorphic to H_n. So in any case G_n contains a vertex v in at most $h_1(n)$ 6-cycles. Then, by deleting v we obtain a graph G_{n-1} on $n - 1$ vertices that contains at least $\mathrm{ex}_{\mathcal{P}}(n, C_6, C_3) - h_1(n)$ 6-cycles, which implies $\mathrm{ex}_{\mathcal{P}}(n - 1, C_6, C_3) \geq \mathrm{ex}_{\mathcal{P}}(n, C_6, C_3) - h_1(n)$, i.e.

$$\mathrm{ex}_{\mathcal{P}}(n, C_6, C_3) - \mathrm{ex}_{\mathcal{P}}(n - 1, C_6, C_3) \leq h_1(n).$$

On the other hand, we have $h(n) - h(n - 1) = h_1(n)$. Therefore, if for if for infinitely many values of n an extremal graph G_n contains a vertex in strictly fewer than $h_1(n)$ 6-cycles, we obtain $\mathrm{ex}_{\mathcal{P}}(n, C_6, C_3) < h(n)$, which is a contradiction, since H_n is a triangle-free planar graph containing $h(n)$ 6-cycles. Thus, there is an n_0 such that if G is an extremal graph on $n > n_0$ vertices, then every vertex of G is contained in at least $h_1(n)$ 6-cycles. Then, by Lemma 5.18, they are isomorphic to H_n. □

Finally we present the proof of Theorem 5.12. We will be rather sketchy, as many of the arguments have already been used in the previous proofs. Again, the addition that appears in the subscripts is taken modulo 3.

Proof of Theorem 5.12. Let u_1, u_2 and u_3 be three distinct vertices of G. Let $|N(u_i, u_{i+1})| = k_i$, for each $1 \leq i \leq 3$, and let $k = k_1 + k_2 + k_3$. First, assume that each k_i is non-zero, in particular this means $\{u_1, u_2, u_3\}$ is independent, since G is triangle-free. Consider $G_0 := G[\cup_{i=1}^{3} N(u_i, u_{i+1}) \cup \{u_1, u_2, u_3\}]$. As G is planar, $|N(u_1, u_2, u_3)| \leq 2$. If $|N(u_1, u_2, u_3)| = 2$, then we have all the faces of G_0 are bounded by 4-cycles, and similar to the proof of Lemma 5.18, we obtain more paths in total if all the faces are empty and $G = G_0$. Consequently, we have

$$\sum_{i=1}^{3} \#P_5(u_i, u_{i+1}) = \sum_{i=1}^{3} (k_i k_{i+1} - 2) \leq 3\left(\frac{n+1}{3}\right)^2 - 6$$

where the inequality is because $k_1 + k_2 + k_3 = n + 1$ (since they count all the vertices except the u_i's and the two common neighbors are each counted 2 times). The equality holds if each k_i is exactly $\frac{n+1}{3}$, which happens when $n \equiv 2 \pmod 3$. (Note that in this case $G \cong H_n$.)

If $|N(u_1, u_2, u_3)| = 1$, then G_0 has a face bounded by a 6-cycle. If $k_i \geq 2$, for all $i \in [3]$, then this 6-cycle is an induced one. Assume M is the set of vertices in its interior with $|M| = m$. Applying Lemma 5.17, we have

$$\begin{aligned}\sum_{i=1}^{3} \#P_5(u_i, u_{i+1}) &= \sum_{i=1}^{3} (k_i k_{i+1} - 1) + \sum_{i=1}^{3} \#P_5(u_i, u_{i+1})_M \\ &\leq \sum_{i=1}^{3} (k_i k_{i+1} - 1) + 3\left(\frac{m+5}{3}\right)^2 - 3 \\ &\leq 3\left(\frac{k}{3}\right)^2 - 3 + 3\left(\frac{m+5}{3}\right)^2 - 3\end{aligned}$$

Clearly, this is maximum if k or m is the smallest possible. Note that $k \geq 6$ (and counts the vertex in $N(u_1, u_2, u_3)$ three times), and hence, m can be as large as $n-7$. A simple computation shows that we have the maximum possible value if $m = 0$ (and $k = n - 1$), which is then still much smaller than the stated bound in the theorem.

Assume for some $i \in [3]$, we have $k_i = 1$. Without loss of generality assume $k_2 = 1$, then $N(u_2, u_3) = N(u_1, u_2, u_3) := \{v\}$. The C_6-face of G_0 is not an induced one then, and the edge $u_1 v$ can be in paths of length four from u_1 to u_2 or to u_3. Any such path must use a path of length three from v to u_2 or to u_3. Similar to the Subcase 2.2 in the proof of Lemma 5.17 we can show that there are at most $m + 1$ such paths. Thus, there are at most $m + 1$ paths of length four from u_1 to u_2 or u_3 that contain the edge $u_1 v$. Therefore we have

$$\begin{aligned}\sum_{i=1}^{3} \#P_5(u_i, u_{i+1}) &\leq \sum_{i=1}^{3} (k_i k_{i+1} - 1) + \sum_{i=1}^{3} \#P_5(u_i, u_{i+1})_M \\ &\leq \sum_{i=1}^{3} (k_i k_{i+1} - 1) + 3\left(\frac{m+5}{3}\right)^2 - 3 + m + 1 \\ &= k_1 + k_3 + k_2 k_3 - 3 + 3\left(\frac{m+5}{3}\right)^2 - 3 + m + 1 \quad [\text{since } k_2 = 1] \\ &\leq (k-1) + \left(\frac{k-1}{2}\right)^2 - 3 + 3\left(\frac{m+5}{3}\right)^2 - 3 + m + 1 \\ &\leq \frac{n^2+n+1}{3} - 8,\end{aligned}$$

where the last inequality holds because we have the maximum value in case $k = 3$ (the smallest possible) and $m = n - 6$.

If $|N(u_1, u_2, u_3)| = 0$, Then G_0 has two such C_6-faces, each of which is an induced 6-cycle. Again, applying Lemma 5.17 to the interior of the C_6-faces and counting all the paths as before, we can deduce that there are strictly fewer paths than the stated bound in the theorem.

Finally, Suppose for some $i \in [3]$, we have $k_i = 0$. Then, again for each $i \in [3]$, a $P_5(u_i, u_{i+1})$ uses a vertex in $N(u_i)$, a vertex in $N(u_{i+1})$ and a middle vertex. One can repeat the argument in the proof of Lemma 5.17 to see that the number of all paths is maximum if there is only one middle vertex and the rest of the vertices are evenly distributed into the neighbors of u_1, u_2 and u_3. Then going through the computations, one sees that there are strictly fewer paths P_5 joining all the pairs of u_i's than the stated bound in the theorem.

□

5.5 Conclusion

The maximum number of an even cycle in triangle-free planar graphs is asymptotically the same as in planar graphs in general, which is given in Theorem 5.1. This is clearly true for C_4 and C_6 as we have shown. For any even cycle C_{2k}, the following construction attains the best asymptotic value.

Take an even cycle C_{2k}, and blow up every other vertex in a balanced way such that all blow-up subsets intersect in two vertices, call such graphs $\mathcal{G}_{\text{even}}$ (Note that H_n is $\mathcal{G}_{\text{even}}$ for $k = 3$). We conjecture that $\mathcal{G}_{\text{even}}$ is the unique extremal graph for $\text{ex}_{\mathcal{P}}(n, C_{2k}, C_3)$.

However, in case of the odd cycles, we have already seen that $\text{ex}_{\mathcal{P}}(n, C_5, C_3)$ is not asymptotically the same as $\text{ex}_{\mathcal{P}}(n, C_5, \emptyset)$. In this case, we conjecture the following (similar to $\mathcal{J}_n$) construction $\mathcal{G}_{\text{odd}}$ to be the extremal graphs. Take a cycle $C_{2k+1} = v_1 v_2 v_3 \dots v_{2k} v_{2k+1}$ and join two vertices z_1 and z_2 (one from inside and the other form the outside of C_{2k+1}) to each vertex v_{2i+1} for $i = 1, 2, \dots, k-1$. Then, blow-up each vertex v_{2i}, for $i = 1, 2, \dots, k-1$, to independent sets of vertices, and replace the edge $v_{2k} v_{2k+1}$ by a tree with color classes A and B, such that the size of each of the blown-up sets and $|A \cup B| - 1$ are as equal as possible.

Conjecture 5.19. *Let n be sufficiently large and $k \geq 2$. Then,*

1. $\text{ex}_{\mathcal{P}}(n, C_{2k}, C_3) = \mathcal{N}(C_{2k}, \mathcal{G}_{\text{even}})$*, and* $\mathcal{G}_{\text{even}}$ *is the unique extremal graph.*
2. $\text{ex}_{\mathcal{P}}(n, C_{2k+1}, C_3) = \mathcal{N}(C_{2k+1}, G_n)$*, for a* $G_n \in \mathcal{G}_{\text{odd}}$*, and* $\mathcal{G}_{\text{odd}}$ *is the set of all extremal graphs.*

Note that the Theorems 5.2 and 5.11 show that the first part of this conjecture holds for $k = 2, 3$ and Theorem 5.7 yields part 2 for $k = 2$.

Bibliography

[1] N. Alon and Y. Caro. On the number of subgraphs of prescribed type of planar graphs with a given number of vertices. In *North-Holland Mathematics Studies*, volume 87, pages 25–36. Elsevier, 1984.

[2] N. Alon and M. Krivelevich. Extremal and probabilistic combinatorics. In T. Gowers, editor, *The Princeton Companion to Mathematics*, pages 562–575. Princeton University Press, 2008.

[3] N. Alon and C. Shikhelman. Many T copies in H-free graphs. *J. Combin. Theory Ser. B*, 121:146–172, 2016.

[4] B. Andrásfai, P. Erdős, and V. Sós. On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.*, 8(3):205–218, 1974.

[5] C. Berge. Sur le couplage maximum d'un graphe. *C.R. Acad. Sci. Paris*, 247:258–259, 1958.

[6] G. Blekherman, A. Raymond, A. Razborov, and F. Wei. On domination exponents for pairs of graphs. *arXiv preprint, arXiv:2506.12151*, 2025.

[7] B. Bollobás. *Extremal graph theory.* Academic Press, 1978.

[8] B. Bollobás. *Modern graph theory*, volume 184. Springer, 1998.

[9] B. Bollobás and E. Győri. Pentagons vs. triangles. *Discrete Math.*, 308:4332–4336, 2008.

[10] J. A. Bondy and M. Simonovits. Cycles of even length in graphs. *J. Combin. Theory Ser. B*, 16(2):97–105, 1974.

[11] S. Cambie, R. de Joannis de Verclos, and R. J. Kang. Regular Turán numbers and some Gan–Loh–Sudakov-type problems. *J. Graph Theory*, 102(1):67–85, 2023.

[12] Y. Caro and Z. Tuza. Regular Turán numbers. *Australas. J. Combin.*, 78:133–144, 2020.

[13] Z. Chase. The maximum number of triangles in a graph of given maximum degree. *Adv. Com.*, 2020. paper No. 10, 5 pp.

[14] F. Chung. Unavoidable stars in 3-graph. *J. Comb. Theory Ser. A*, 3:252–262, 1983.

[15] F. Chung and P. Erdős. On unavoidable graphs. *Combinatorica*, 3:167–176, 1983.

[16] F. Chung and P. Frankl. The maximum number of edges in a 3-graph not containing a given star. *Graphs and Combinatorics*, 3:111–126, 1987.

[17] L. N. Coregliano and A. Razborov. Semantic limits of dense combinatorial objects. *Usp. Mat. Nauk*, 75(4):45–152, 2020.

[18] C. Cox and R. R. Martin. Counting paths, cycles, and blow-ups in planar graphs. *Journal of Graph Theory*, 101(3):521–558, 2022.

[19] C. Dowden. Extremal C_4-free/C_5-free planar graphs. *Journal of Graph Theory*, 83(3):213–230, 2016.

[20] R. Duke and P. Erdős. Systems of finite sets having a common intersection. In *Proceedings, 8th S-E Conf. Combinatorics, Graph Theory and Computing*, pages 247–252, 1977.

[21] P. Erdős. On sequences of integers no one of which divides the product of two others and on some related problems. *Isvestia Nauchno-Issl. Inst. Mat. i Meh. Tomsk*, 2:74–82, 1938.

[22] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Magy. Tud. Akad. Mat. Kut. Intéz. Közl.*, 7:459–474, 1962.

[23] P. Erdős. Extremal problems in graph theory. In *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, pages 29–36, 1964.

[24] P. Erdős. Some recent results on extremal problems in graph theory. In *Theory of Graphs (Internl. Symp. Rome)*, pages 118–123, 1966.

[25] P. Erdős. On some new inequalities concerning extremal properties of graphs. In P. Erdős and G. Katona, editors, *Theory of Graphs*, pages 77–81. Academic Press, New York, 1968.

[26] P. Erdős. On some problems in graph theory, combinatorial analysis and combinatorial number theory. In *Graph theory and combinatorics (Cambridge, 1983)*, pages 1–17. Academic Press, London, 1984.

[27] P. Erdős, Z. Füredi, R. Gould, and D. Gunderson. Extremal graphs for intersecting triangles. *J. Comb. Theory Ser. B*, 64:89–100, 1995.

[28] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.*, 10:337–356, 1959.

[29] P. Erdős and H. Sachs. Reguläre graphen gegebener taillenweite mit minimaler knotenzahl. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 12:251–257, 1963. In German.

[30] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Scientiarum Mathematicarum Hungarica*, 1:51–57, 1966.

[31] P. Erdős and A. Stone. On the structure of linear graphs. *Bulletin of the American Mathematical Society*, 52:1087–1091, 1946.

[32] Z. Füredi. A proof of the stability of extremal graphs, Simonovits' stability from Szemerédi's regularity. *J. Combin. Theory Ser. B*, 115:66–71, 2015.

[33] Z. Füredi and M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős Centennial*, pages 169–264. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[34] D. Gerbner. On Turán-good graphs. *Discrete Math.*, 344(8):112445, 2021.

[35] D. Gerbner. A note on the uniformity threshold for Berge hypergraphs. *European J. Combin.*, 105:103561, 2022.

[36] D. Gerbner. On weakly Turán-good graphs. *Discuss. Math. Graph Theory*, 44:1539–1550, 2022.

[37] D. Gerbner. Generalized Turán problems for $\mathrm{K}_{2,\mathrm{t}}$. *Electron. J. Comb.*, 30(1), 2023.

[38] D. Gerbner. Some exact results for non-degenerate generalized Turán problems. *Electron. J. Comb.*, 30(4), 2023.

[39] D. Gerbner. Some stability and exact results in generalized Turán problems. *Studia Scientiarum Mathematicarum Hungarica*, 60(1):16–26, 2023.

[40] D. Gerbner. The Turán number of Berge book hypergraphs. *SIAM Journal on Discrete Mathematics*, 38(4):2896–2912, 2024.

[41] D. Gerbner and H. Hama Karim. Generalized regular Turán numbers. *Australasian Journal of Combinatorics*, 90(3):326–340, 2024.

[42] D. Gerbner and H. Hama Karim. Stability from graph symmetrization arguments in generalized Turán problems. *Journal of Graph Theory*, 107(4):681–692, 2024.

[43] D. Gerbner, H. Hama Karim, and G. Kucheriya. On Turán-type problems and the abstract chromatic number. *arXiv preprint, arXiv:2410.09617*, 2024.

[44] D. Gerbner, A. Methuku, T. Mészáros, and C. Palmer. Generalized rainbow Turán numbers. *The Electronic Journal of Combinatorics*, 29, 2022.

[45] D. Gerbner, A. Methuku, D. T. Nagy, D. Pálvölgyi, G. Tardos, and M. Vizer. Turán problems for edge-ordered graphs. *J. of Combin. Theory, Ser. B*, 160:66–113, 2023.

[46] D. Gerbner and C. Palmer. Counting copies of a fixed subgraph of F-free graphs. *European J. Combin.*, 82:103001, 2019.

[47] D. Gerbner and C. Palmer. Some exact results for generalized Turán problems. *European J. of Combin.*, 103:103519, 2022.

[48] D. Gerbner and C. Palmer. Survey of generalized Turán problems–counting subgraphs. *arXiv preprint, arXiv:2506.03418*, 2025.

[49] D. Gerbner and B. Patkós. Generalized Turán results for intersecting cliques. *Discrete Mathematics*, 347(1):113710, 2024.

[50] D. Gerbner, B. Patkós, Z. Tuza, and M. Vizer. Some exact results for regular Turán problems for all large orders. *European J. Combin.*, 117:103828, 2024.

[51] D. Gerbner, B. Patkós, Z. Tuza, and M. Vizer. Singular Turán numbers and worm-colorings. *Discuss. Math. Graph Theory*, 42(4):1061–1074, 2022.

[52] D. Ghosh, E. Győri, O. Janzer, A. Paulos, N. Salia, and O. Zamora. The maximum number of induced C_5's in a planar graph. *Journal of Graph Theory*, 99(3):378–398, 2022.

[53] D. Ghosh, E. Győri, R. Martin, A. Paulos, and C. Xiao. Planar Turán number of the 6-cycle. *SIAM J. Discret. Math.*, 36(3):2028–2050, 2022.

[54] L. Gishboliner and A. Shapira. A generalized Turán problem and its applications. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 760–772, 2018.

[55] A. W. Goodman. On sets of acquaintances and strangers at any party. *Amer. Math. Monthly*, 66(9):778–783, 1959.

[56] A. Grzesik. On the maximum number of five-cycles in a triangle-free graph. *J. Combin. Theory Ser. B*, 102(5):1061–1066, 2012.

[57] A. Grzesik, E. Győri, N. Salia, and C. Tompkins. Subgraph densities in K_r-free graphs. *Electron. J. Comb.*, 30(1), 2023.

[58] E. Győri and H. Hama Karim. Generalized planar Turán numbers related to short cycles. *arXiv preprint arXiv:2405.08162*, 2024.

[59] E. Győri, K. Varga, and X. Zhu. A new construction for the planar Turán number of cycles. *Graphs and Combinatorics*, 40(6):124, 2024.

[60] E. Győri. On the number of C_5's in a triangle-free graph. *Combinatorica*, 9(1):101–102, 1989.

[61] E. Győri, J. Pach, and M. Simonovits. On the maximal number of certain subgraphs in K_r-free graphs. *Graphs and Combinatorics*, 7(1):31–37, 1991.

[62] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora. Generalized planar Turán numbers. *Electron. J. Comb.*, 28(4), 2021.

[63] E. Győri, A. Paulos, N. Salia, C. Tompkins, and O. Zamora. The maximum number of pentagons in a planar graph. *Journal of Graph Theory*, 108(2):229–256, 2025.

[64] E. Győri, N. Salia, C. Tompkins, and O. Zamora. The maximum number of P_l copies in P_k-free graphs. *Discrete Math. Theor. Comput. Sci.*, 21, 2019. #14.

[65] E. Győri, X. Wang, and Z. Zheng. Extremal planar graphs with no cycles of particular lengths. *arXiv preprint*, 2022.

[66] S. L. Hakimi and E. F. Schmeichel. On the number of cycles of length k in a maximal planar graph. *Journal of Graph Theory*, 3(1):69–86, 1979.

[67] H. Hatami, J. Hladký, D. Kráľ, S. Norine, and A. Razborov. On the number of pentagons in triangle-free graphs. *J. Combin. Theory Ser. A*, 120(3):722–732, 2013.

[68] D. Hei and X. Hou. The cycle of length four is strictly F-Turán-good. *Bulletin of the Malaysian Mathematical Sciences Society*, 47(1):5, 2024.

[69] S. Jukna. *Extremal combinatorics: with applications in computer science*, volume 571. Springer, 2011.

[70] G. Katona, T. Nemetz, and M. Simonovits. On a problem of Turán in the theory of graphs. *Mat. Lapok*, 15:228–238, 1964.

[71] P. Keevash, D. Mubayi, B. Sudakov, and J. Verstraëte. Rainbow Turán problems. *Combinatorics, Probability and Computing*, 16(1):109–126, 2007.

[72] T. Kovári, V. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloq. math.*, 3:50–57, 1954.

[73] Y. Lan, Y. Shi, and Z.-X. Song. Extremal H-free planar graphs. *Electron. J. Combin.*, 26(2):P2.11, 2019.

[74] Y. Lan, Y. Shi, and Z.-X. Song. Extremal theta-free planar graphs. *Discrete Math.*, 342(12):111610, 2019.

[75] Y.-X. Lan and Y.-T. Shi. Extremal P_8-free/P_9-free planar graphs. *Journal of the Operations Research Society of China*, 11(3):451–457, 2023.

[76] E. Liu and J. Wang. The generalized Turán problem of two intersecting cliques. *Discussiones Mathematicae Graph Theory*, 45:565–594, 2025.

[77] H. Liu, O. Pikhurko, M. Sharifzadeh, and K. Staden. Stability from graph symmetrisation arguments with applications to inducibility. *Journal of the London Mathematical Society*, 108(3):1121–1162, 2023.

[78] L. Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[79] R. Luo. The maximum number of cliques in graphs without long cycles. *J. Combin. Theory Ser. B*, 128:219–226, 2017.

[80] Z. Lv, E. Győri, Z. He, N. Salia, C. Tompkins, and X. Zhu. The maximum number of copies of an even cycle in a planar graph. *J. Combin. Theory, Ser. B*, 167:15–22, 2024.

[81] J. Ma and Y. Qiu. Some sharp results on the generalized Turán numbers. *European J. Combin.*, 84:103026, 2020. 16 pages.

[82] M. Mantel. Problem 28. *Wiskundige Opgaven*, 10:60–61, 1907.

[83] N. Morrison, J. Nir, S. Norin, P. Rzążewski, and A. Wesolek. Every graph is eventually Turán-good. *J. Combin. Theory Ser. B*, 162:231–243, 2023.

[84] J. Pach and G. Tardos. Forbidden paths and cycles in ordered graphs and matrices. *Israel Journal of Mathematics*, 155(1):359–380, 2006.

[85] I. Ruzsa and E. Szemerédi. Triple systems with no six points carrying three triangles, combinatorics (keszthely, 1976), 18 (1978), vol. ii., 939-945.

[86] M. Savery. Planar graphs with the maximum number of induced 6-cycles. *Electron. J. Comb.*, 30(4), 2023.

[87] M. Savery. Planar graphs with the maximum number of induced 4-cycles or 5-cycles. *Graphs and Combinatorics*, 40(3):46, 2024.

[88] A. Shapira and R. Yuster. On the density of a graph and its blowup. *Journal of Combinatorial Theory, Series B*, 100(6):704–719, 2010.

[89] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319, 1968.

[90] M. Simonovits. Extremal graph problems with symmetrical extremal graphs. additional chromatic conditions. *Discrete Math.*, 7:349–376, 1974.

[91] V. Sós. Remarks on the connection of graph theory, finite geometry and block designs. *Teorie Combinatorie*, 2:223–233, 1976.

[92] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[93] X. Zhu and Y. Chen. Generalized Turán number for linear forests. *Discrete Mathematics*, 345(10):112997, 2022.

[94] X. Zhu, Y. Chen, D. Gerbner, E. Győri, and H. Hama Karim. The maximum number of triangles in F_k-free graphs. *European J. Combin.*, 114:103793, 2023.

[95] A. A. Zykov. On some properties of linear complexes. *Matematicheskii Sbornik*, 66(2):163–188, 1949.