Characterisation sets for the prenucleolus

PhD Dissertation

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Chapter 1

Introduction

Transferable utility games (TU-games) are cooperative games in which each subset of players, or coalition, is assigned a value representing the total value the coalition can generate and freely redistribute among its members. This assumption of full transferability of utility—typically modeled by expressing utility in a common, fungible unit such as money—makes TU-games a special and mathematically tractable subclass of cooperative games with cardinal utilities. In contrast, non-transferable utility (NTU) games allow for more general preference structures where utility cannot be freely transferred, leading to more complex solution concepts and less tractable analysis. TU-games can be viewed as a special case of NTU-games, where the possibility set of each coalition is constrained so that the total utility is at most the coalition's value in the corresponding TU-game, reflecting perfect transferability.

Compared to TU-games, solution concepts for NTU-games are significantly more complex. To illustrate the difference, consider a simple two-player bargaining situation. In a TU-game, if the total value from cooperation is 100 units (e.g., dollars), any division of this amount – such as $(60, 40)$ or $(50, 50)$ – is feasible and represents a potential outcome. The utility is transferable, and the set of core payoffs (assuming each player receives 0 in case of no cooperation) is the line segment connecting $(0, 100)$ and $(100, 0)$.

In contrast, in an NTU-game, suppose players care not only about money but also about how it is earned or the form it takes (e.g., leisure vs. labor). The feasible utility pairs might then form a curved set—say, a region $\{(x, y)\colon \exists t \in [\pi, \frac{3\pi}{2}],\ \text{such that } x \leq 100 + 100cos(t), y \leq 100 + 100sin(t)\}$ reflecting diminishing marginal utility or non-monetary preferences. In this case, a division like $(60, 40)$ may be infeasible if it requires one player to work excessively while the other does nothing. The set of core payoffs is no longer a simple line segment but a more complex shape – here, a quarter-circle arc – and utility cannot be freely shifted between players. This richer modeling comes at the cost of analytical complexity: many solution concepts that are well-behaved in TU-games become difficult to define or compute in NTU-games.

The goal in TU-games is to distribute the value of the grand coalition among the players in a "fair" way. Sets of such distributions are known as solutions and values, with a value referring to a single-valued solution. Numerous approaches exist to define such distributions, including solution concepts like the core (Shapley, 1955; Gillies, 1959), the kernel (Davis and Maschler, 1965), and the bargaining set (Aumann and Maschler, 1964), as well as value concepts like the Shapley value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969).

In TU-games, players are typically modeled as agents who can form coalitions and redistribute the coalition's value among themselves. Two TU-games are considered strategically equivalent if they induce the same set of feasible payoff vectors, up to affine transformations. This equivalence allows for normalization and simplification in analysis. For example, scaling the value of each coalition by a positive constant "preserves"[1] the structure of many solution concepts, such as the core, the Shapley value, or the (pre)nucleolus.

This thesis focuses on the prenucleolus of TU-games, along with its variants and generalizations. The prenucleolus is a solution concept designed to maximize the satisfaction of the most dissatisfied coalitions. Its practical relevance is evident in its wide-ranging applications to real-world cost allocation problems. A comprehensive review of such applications can be found in Inarra et al (2020) and Fiestras-Janeiro et al (2012). Here, we present a brief overview of selected applications to illustrate the importance and real-world applicability of TU-games and the (pre)nucleolus.

To begin with, Engevall et al (1998, 2004) applied the nucleolus and a variant thereof to a cost allocation problem arising in distribution planning at the Logistics Department of Norsk Hydro Olje AB in Stockholm, Sweden. In Engevall et al (1998), the problem is modeled as a travelling salesman game, while in Engevall et al (2004), it is framed as a vehicle routing game. Both studies compare Norsk Hydro's existing allocation principle – proportional to demand, which does not reflect the true origins of costs – with the nucleolus and demand nucleolus of the respective games. The results show that these game-theoretic solutions better incorporate the geographic location of customers. Notably, customers with low demand, even if located far from the depot, can benefit from forming coalitions with others beyond their immediate delivery route. Some models yield game classes where the nucleolus is easily computable, while others present computational challenges due to the size of the games. In such cases, alternative cost allocations are proposed that are

[1]covariance

computationally simpler and closely approximate the nucleolus.

Furthermore, Kuipers et al (2000) extended the applicability of the nucleolus by demonstrating that, for sequencing and routing games, it can be computed in polynomial time with respect to the number of players, more precisely in $\mathcal{O}(n^4)$ time for an n-player game.

In another application, Lejano and Davos (1995) employed the nucleolus and normalized nucleolus in their study of the State Water Project in Los Angeles. A six-year drought prompted the city to rely more heavily on reused wastewater – its most expensive water source. The Department of Public Works proposed a water reclamation program at the Terminal Island Treatment Plant (TITP), involving four agencies. Each agency's decision to participate depended on comparing the costs of joining the grand coalition, undertaking an independent project, or forming a subgroup coalition. The cost allocations derived from the nucleolus and normalized nucleolus satisfied all individual and group rationality conditions, thereby incentivizing all agencies not to leave the TITP project. In contrast, the Shapley value failed to meet these conditions under slight parameter changes.

Okada and Mikami (1992) used real-world data from North America and Canada to study conflicts in acid rain emission control. The challenge lies in assigning emission reduction targets to major sources located far from the affected receptors. Each receptor aims to minimize its own burden while maintaining a fixed total reduction across all receptors, leading to conflicting interests. The authors applied game-theoretic methods to derive fair and effective allocations of emission reductions. Both the nucleolus and the Shapley value exhibited similar patterns in this context.

Additionally, several studies have analyzed the nucleolus in power industry cost allocation problems, including Tsukamoto and Iyoda (1996), Stamtsis and Erlich (2004), and Bjorndal et al (2005). These involve allocating the fixed costs of electricity networks among users. Bjorndal et al (2005) also proposed a method akin to the nucleolus, combining various usage-based approaches to produce allocations that lie within, or as close as possible to, the core.

The diversity of applications and the demand for efficient algorithms to compute the (pre)nucleolus and its variants underscore the practical significance of this solution concept in addressing complex real-life problems.

Computing the prenucleolus is time-consuming. Although, it has been established that $2n-2$ coalitions are sufficient to characterize the prenucleolus of an n-player game (Brune, 1983; Reijnierse and Potters, 1998), identifying these $2n-2$ coalitions is not straightforward.

Despite significant progress in algorithm development over the past few decades, the runtime of these algorithms for computing the prenucleolus remains exponential in the number of players. The weighted-sum approach introduced by Kohlberg (1972) employs a single, but extremely large, linear program (LP) with $\mathcal{O}(2^n!)$ constraints for an n-player game. The wide range of coefficients in these constraints leads to severe numerical issues, even for games with as few as three or four players. An improved version was proposed by Owen (1974), who formulated a single LP equivalent to Kohlberg's, but more moderate in size, using $2^{n+1} + n$ variables and $4^n + 1$ constraints. However, the numerical instability due to the large range of coefficients persists.

Maschler et al (1979) introduced an alternative definition of the (pre)nucleolus, known as the lexicographic center of the game. This definition is based on an iterative process that reduces the set of feasible payoff vectors to a singleton. The resulting lexicographic center algorithm involves solving a sequence of LPs with $m+1$ variables and 2^n constraints, where all coefficients are drawn from the set $\{-1, 0, 1\}$, making it more practical for implementation.

About a decade later, Sankaran (1991) proposed an algorithm that solves at most 2^n slightly larger LPs but still using constraint coefficients from $\{-1, 0, 1\}$. In 1993, Solymosi (1993) introduced a faster algorithm that solves a sequence of at most $n-1$ LPs, each with $n + 1$ variables and initially $2^n + n - 1$ constraints. The number of constraints rapidly decreases to approximately n, and all coefficients remain within $\{-1, 0, 1\}$.

Potters and Reijnierse (1996) developed a prolonged simplex method, treating the sequence of at most $n - 1$ LPs – each with $2^n + n - 1$ variables and $2^n - 1$ constraints with coefficients from the set $\{-1, 0, 1\}$ – as a single prolonged simplex algorithm. This approach leverages the sparsity and structural similarity of the matrices involved. Also using the special properties of the LPs involved, Derks and Kuipers (1997) proposed a new simplex method that reduces the computational effort of each pivoting step by a factor of n compared to standard implementations. They also proved that this complexity cannot be improved further, as inspecting all constraints in each pivoting step is theoretically necessary.

In addition, there are ongoing efforts to enhance the algorithms for computing the nucleolus. Perea and Puerto (2013) introduced a novel single-LP formulation with $\mathcal{O}(4^n)$ constraints and variables, using only coefficients from $\{-1, 0, 1\}$. Nguyen and Thomas (2016) propose a nested-LPs-based approach for finding the nucleolus of large games, including some structured games with more than 50 players. Benedek et al (2021) presents an improved algorithm capable of handling games of moderately large size by utilizing a refined version of Kohlberg's criterion. This refinement reduces the verification process to at most $(n-1)$ collections of coalitions for balancedness, instead of an exponentially large number. Similarly, Koenemann and Toth (2023) introduces a framework for designing efficient nucleolus computation algorithms and demonstrates that, using this approach, the nucleolus of b-matching games on graphs with bounded treewidth can be computed in polynomial time.

It is highly unlikely that a general algorithm exists which computes the (pre)-nucleolus in polynomial time with respect to the number of players, as determining the nucleolus has been proven to be NP-hard for several classes of games, including weighted voting games Elkind et al (2007) and utility games with non-unit capacities Deng et al (2009).

On the other hand, for some classes of games, such as neighbour games (Hamers et al, 2003), permutation games under certain conditions (Solymosi et al, 2005), tree games (Maschler et al, 2010), and a large class of directed acyclic graph games (Sziklai et al, 2017), the nucleolus can be computed in polynomial time in the number of players. Additionally, some heuristics provide efficient algorithms to find an allocation close to the nucleolus (Perea and Puerto, 2019).

In this thesis, we consider the lexicographic center algorithm (Kopelowitz, 1967; Maschler et al, 1979), which calculates the nucleolus using a series of LPs with $\mathcal{O}(n)$ variables and $\mathcal{O}(2^n)$ constraints. The number of constraints can be reduced by identifying a smaller characterization set for the nucleolus than the one that includes all coalitions.

Huberman (1980) showed that the so-called essential coalitions provide a characterization set for the nucleolus of balanced games. In certain classes of games (e.g., matching games), the cardinality of essential coalitions is polynomial in the number of players, and they are also easy to find, thereby providing a way of computing the nucleolus in polynomial time.

Another characterization set for the (pre)nucleolus in balanced games was provided by Solymosi and Sziklai (2016), who showed that the family of dually essential coalitions forms a characterization set for the (pre)nucleolus. They also demonstrated that the intersection of essential and dually essential coalitions provides a characterization set for the (pre)nucleolus in balanced games if the grand coalition is vital.

Up to this point, we have discussed TU-games, where the values of all subsets of the grand coalition are known. However, this is not always the case. There are TU-games where not all coalitions are feasible, meaning their values are not considered at all. These games are called TU-games with restricted cooperation. Although this might seem like a slight difference, it significantly impacts the analysis of such games.

For example, in the case of TU-games with restricted cooperation, the prenucleolus is no longer a single-valued solution but a set of payoff vectors. This set can be multi-element or even empty, depending on the properties of the feasible coalitions. For the same reason, Huberman's theorem cannot be used in the same way, as the definition of essential coalitions relies on the fact that all singleton coalitions are feasible. Therefore, when analysing such games, we must consider that some well-known facts may no longer hold for TU-games with restricted cooperation. Understanding what happens in these cases provides deeper knowledge of TU-games and the logic behind them. In this thesis, we discuss TU-games with restricted cooperation.

In addition to dealing with games with restricted cooperation, we also aim to generalize the prenucleolus. The prenucleolus has numerous known variants and generalizations, such as the percapita prenucleolus (Grotte, 1970, 1972), the weighted prenucleolus (Derks and Haller, 1999) or q-nucleolus (Solymosi, 2019), – which two are the same concept, where the excesses are divided by coalition specific positive weights – the modified nucleolus (Sudhölter, 1996, 1997), and the general nucleolus (Potters and Tijs, 1992; Maschler et al, 1992). From the viewpoint of this research, the most generalized variant is the one by Potters and Tijs (1992) and Maschler et al (1992), called the general nucleolus. We introduce the so-called $\mathbf{u}$-prenucleolus (Dornai and Pintér, 2024, 2005), which is a special case of the general nucleolus and a common generalization of the per capita prenucleolus, the weighted prenucleolus, and the q-nucleolus.

All of these variants and generalizations commonly rely on modified versions of the excess function. In our framework, we define a function $\mathbf{u}$, referred to as a utility function, which transforms the excesses. By doing so, we generalize the prenucleolus to what we call the $\mathbf{u}$-prenucleolus.

It is important to clarify the use of the term utility function in this context to avoid confusion. In classical game theory and economics, utility typically refers to a player's preference or satisfaction level. However, in our setting, the utility function $\mathbf{u}$ is not meant to represent individual preferences directly. Instead, it serves as a transformation applied to the excesses of coalitions, reflecting how utility gains are perceived or weighted under different conditions.

The underlying idea is that the nominal value of utility may not be equally appreciated by the coalitions. For instance, a player may value a small increase in utility more when starting from a low baseline than when already enjoying a high level of utility. Similarly, a coalition might consider that the utility gain is distributed among its members – larger coalitions may dilute the benefit more than smaller ones. The term utility function is used here to emphasize that these transformations reinterpret the raw utility values in a way that better captures the perceived or effective value of coalition gains.

Using this **u** function, we also define a generalization of balanced games and the core (Shapley, 1955; Gillies, 1959): the **u**-balanced games and the **u**-core, respectively. Furthermore, we show that a game is **u**-balanced if and only if its **u**-core is nonempty, thus generalizing the Bondareva–Shapley theorem (Bondareva, 1963; Shapley, 1967; Faigle, 1989). Similarly, we generalize other important theorems in this field as well. As a result, we do not just generalize the prenucleolus but also extend TU-games to TU-games with utility functions.

At the core of this research is the definition of several characterization sets for the **u**-prenucleolus. We define the **u**-essential coalitions as a generalization of essential coalitions and prove that the **u**-essential coalitions provide a characterization set for the **u**-prenucleolus of **u**-balanced games. By choosing our **u** function accordingly, this generalization of Huberman's theorem can be applied to give a characterization set for the prenucleolus of non-balanced games Dornai and Pintér (2022) or a characterization set for the percapita prenucleolus of balanced games, among others.

In addition, we define another variant of essential coalitions, namely the dually-**u**-essential coalitions Dornai and Pintér (2025). We prove a variant of Theorem 7 on p. 420 in Huberman (1980), which claims that the dually-**u**-essential coalitions form a characterization set for the **u**-prenucleolus in **u**-balanced games. This result also generalizes Theorem 10 on p. 522 in Solymosi and Sziklai (2016), which states that dually essential coalitions form a characterization set for the (pre)nucleolus in balanced games.

Furthermore, we generalize Theorem 2.3 on p. 362 in Granot et al (1998) regarding characterization sets of the prenucleolus to TU-games with utilities (and with restricted cooperation). We also show that the intersection of the set of **u**-essential and dually-**u**-essential coalitions characterizes the **u**-least-core of **u**-balanced games if the **u**-least-core is a proper subset of the **u**-core. With the help of these results, we generalize Theorem 11 on p. 523 in Solymosi and Sziklai (2016), demonstrating that the intersection of **u**-essential and dually-**u**-essential coalitions forms a characterization set for the **u**-prenucleolus in **u**-balanced games if the **u**-least-core is a proper subset of the **u**-core.

The dissertation is structured as follows: In Chapter 2, we discuss the basic concepts and notions used in the thesis. In Section 2.1, we explore the different notions and definitions we use for TU-games, along with some solutions and values. Section 2.2 delves into the dual of TU-games and the solutions and values specific to dual games. In Section 2.3, we describe the lexicographic center algorithm, and in Section 2.4, we discuss Huberman's theorem regarding a characterization set for the prenucleolus.

In Chapter 3, we introduce and discuss TU-games with utility functions. In Section 3.1, we present the concepts of TU-games with utilities, **u**-excess, **u**-prenucleolus, and **u**-core. In Section 3.2, we define **u**-balanced games and demonstrate that a game is **u**-balanced if and only if its **u**-core is nonempty. In Section 3.3, we provide a generalization of the lexicographic center algorithm for calculating the **u**-prenucleolus. This algorithm is a special case of the lexicographic center algorithm for calculating the general nucleolus by Maschler et al (1992). In Section 3.4, we generalize Katsev and Yanovskaya (2013)'s theorem, giving a sufficient and necessary condition for the nonemptiness of the **u**-prenucleolus. Finally, in Section 3.5, we generalize another theorem by Katsev and Yanovskaya, providing a sufficient and necessary condition for the single-valuedness of the **u**-prenucleolus.

In Chapter 4, we define our first characterization set for the **u**-prenucleolus, namely, the **u**-essential coalitions. In Section 4.1, we define the so-called **u**-essential coalitions and generalize Huberman (1980)'s theorem by proving that the **u**-essential coalitions form a characterization set for the **u**-prenucleolus of **u**-balanced games. In Section 4.2, we describe the **u** functions for which the prenucleolus and the core coincide with the **u**-prenucleolus and the **u**-core, respectively. In Section 4.3, we show that in case of assignment games, using the reciprocal percapita utility function, we obtain a polynomial number of **u**-essential coalitions in the number of players.

In Chapter 5, we introduce the dual of TU-games with utility functions and define additional characterization sets with their help. In Section 5.1, we define the dual of TU-games with utility functions and present some of its basic properties. In Section 5.2, we demonstrate a modification of the lexicographic center algorithm described in Section 3.3, which can be used to calculate the **u**-anti-prenucleolus. In Section 5.3, we define the dually-**u**-essential coalitions and generalize a theorem by Solymosi and Sziklai (2016), proving that these coalitions form a characterization set for the **u**-prenucleolus of **u**-balanced games. In Section 5.4, we generalize a theorem by Granot et al (1998) concerning characterization sets. Using this theorem, we generalize another result by Solymosi and Sziklai (2016), demonstrating that the intersection of **u**-essential and dually-**u**-essential coalitions forms a characterization set for the **u**-prenucleolus of **u**-balanced games if the **u**-least-core is a proper subset of the **u**-core.

Finally, in Chapter 6, we provide a brief conclusion of our results and present a table that outlines the various notions considered in this thesis and their relationships to each other.

Chapter 2

Preliminaries

2.1 TU-games and solutions

Given a nonempty finite set of the players N and a function $v\colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$; then v is called a TU-*game* (henceforth game for short). Let $\mathcal{G}^N$ denote the class of games with player set N, moreover, let the set of coalitions be denoted by $\mathcal{P}(N) := \{S \subseteq N\}$ and the set of non-trivial coalitions be denoted by $\mathcal{P}^*(N) := \{S \subseteq N\colon S \neq \emptyset, S \neq N\}$. Let $\mathcal{D}_S$ denote the class of partitions of set $S \subseteq N$ except $\{S\}$.

Let $\mathcal{A} \subseteq \mathcal{P}(N)$ be such that $\emptyset, N \in \mathcal{A}$, then $\mathcal{A}$ is called a set of feasible coalitions. In this case, the function $v\colon \mathcal{A} \to \mathbb{R}$, $v(\emptyset) = 0$ is called a game with restricted cooperation. Let $\mathcal{G}^{N,\mathcal{A}}$ denote the class of games with restricted cooperation, where $\mathcal{A}$ is the set of feasible coalitions. If $\mathcal{A} = \mathcal{P}(N)$ then $\mathcal{G}^{N,\mathcal{A}} = \mathcal{G}^N$, therefore each introduced concept for games with restricted cooperation is a generalization of the related concept for classical games.

Let $\mathcal{A}^* := \mathcal{A} \setminus \{N, \emptyset\}$ denote the set of non-trivial feasible coalitions, and let $\mathcal{D}_S^{\mathcal{A}^*} := \{B \in \mathcal{D}_S\colon B \subseteq \mathcal{A}^*\}$ denote the non-trivial $\mathcal{A}^*$-partitions of set $S \in \mathcal{A}^*$.

A set of coalitions $\mathcal{S} \subseteq \mathcal{A}$ is a balanced set system if there exist $\lambda_S \in \mathbb{R}_+$, $S \in \mathcal{S}$, called balancing weight system, such that

$$\sum_{S \in \mathcal{S}} \lambda_S \, \chi_S = \chi_N ,$$

where $\chi_E \in \mathbb{R}^N$ is the characteristic vector of set E.

A solution is a set-valued mapping from a set of games with player set N to $\mathbb{R}^N$. For example, the core (Shapley, 1955; Gillies, 1959), the kernel (Davis and Maschler, 1965), and the bargaining set (Aumann and Maschler, 1964). A value is a singleton valued solution, for example the Shapley-value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969).

Let $I(v) := \{x \in \mathbb{R}^N\colon \sum_{i\in N} x_i = v(N) \text{ and } x_i \geq v(\{i\})\ \forall \{i\} \in \mathcal{A}\}$ and $I^*(v) := \{x \in \mathbb{R}^N\colon \sum_{i\in N} x_i = v(N)\}$ denote the set of imputations and preimputations of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ respectively.

Given a game with restricted cooperation $v \in \mathcal{G}^{N,\mathcal{A}}$, a coalition $S \in \mathcal{A}$ and a payoff a vector $x \in \mathbb{R}^N$, the *excess* of coalition S by the payoff vector x in the game v is $e(S,x) := v(S) - x(S)$, where $x(S) := \sum_{i\in S} x_i$.

The core of a game with restricted cooperation v is the set of preimputations for which the excess of every feasible coalition is non-positive:

$$\mathrm{core}(v) := \left\{x \in \mathbb{R}^N \colon x(N) = v(N) \text{ and } e_v(S,x) \leq 0,\ \forall S \in \mathcal{A}^*\right\}.$$

In case, the core is nonempty, we say the game is balanced.

The ε-core of the game is the set of preimputations, for which the maximal excess is at most ε, that is

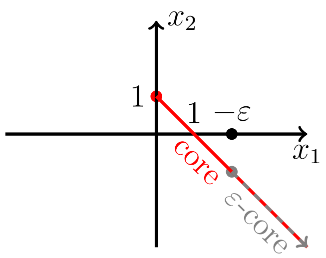
$$\mathrm{core}_\varepsilon(v) := \{x \in I^*(v)\colon \max_{S\in\mathcal{A}^*} e_v(S,x) \leq \varepsilon\}.$$

Moreover, the least-core of the game is its smallest nonempty ε-core, provided it exists. The least-core is always well-defined, when all coalitions are feasible. However, in games with restricted cooperation, it may happen that a smallest nonempty ε-core does not exist. The following example illustrates this phenomenon.

Example 1. Consider the following game: Let $N = \{1,2\}$, and let the set of non-trivial feasible coalitions be $\mathcal{A}^* = \{\{1\}\}$. The characteristic function is defined as $v(\{1\}) = 0$ and $v(N) = 1$. To determine the least-core, we examine the smallest nonempty ε-core:

$$\inf\{\varepsilon\colon x_1 + x_2 = 1, 0 - x_1 \leq \varepsilon, x_1, x_2 \in \mathbb{R}\} = -\infty.$$

This implies that no smallest nonempty ε-core exists, and therefore, the least-core is not well-defined for this game.



The following theorem provides a necessary and sufficient condition for the least-core to be well-defined. This result is well-established in the literature on game theory.

Theorem 2. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game. The least-core of the game is well-defined if and only if $\mathcal{A}^*$ contains a balanced set system.*

Proof. The least-core of the game is well-defined if and only if $\min_{\text{core}_\varepsilon \neq \emptyset} \varepsilon$ exists. That is, when the following LP has an optimal solution:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & e_v(S,x) \leq t, \qquad S \in \mathcal{A}^* \\ & x(N) = v(N) \\ & x \in \mathbb{R}^N, t \in \mathbb{R} \end{aligned} \tag{2.1}$$

It is easy to see, that (2.1) has a feasible solution.

The dual LP is the following:

$$\begin{aligned} & \textstyle\sum_{S \in \mathcal{A}^* \cup \{N\}} \lambda_S v(S) \to \max \\ \text{s.t.} \quad & \textstyle\sum_{S \in \mathcal{A}^* \cup \{N\}} \lambda_S \chi_S = \mathbf{0}, \\ & \textstyle\sum_{S \in \mathcal{A}^*} \lambda_S = 1 \\ & \lambda_S \geq 0 \qquad\qquad S \in \mathcal{A}^* \\ & \lambda_N \in \mathbb{R} \end{aligned} \tag{2.2}$$

$\sum_{S \in \mathcal{A}^* \cup \{N\}} \lambda_S \chi_S = \mathbf{0}$ is equivalent to $\sum_{S \in \mathcal{A}^*} \lambda_S \chi_S = -\lambda_N \chi_N$. Since λ_N is unrestricted and $\lambda_S \geq 0$ for all $S \in \mathcal{A}^*$: for any feasible solution $\lambda_N \leq 0$. $\lambda_N \neq 0$ because $\lambda_S \geq 0$ for all $S \in \mathcal{A}^*$ and $\sum_{S \in \mathcal{A}^*} \lambda_S = 1$; therefore, $\sum_{S \in \mathcal{A}^*} \lambda_S \chi_S \neq 0$. It follows, that we can divide by $\lambda'_N := -\lambda_N > 0$:

$$\sum_{S \in \mathcal{A}^*} \frac{\lambda_S}{\lambda'_N} \chi_S = \chi_N.$$

The nonemptiness of

$$\begin{aligned} \{\lambda \in \mathbb{R}^{\mathcal{A}^* \cup \{N\}} \colon \sum_{S \in \mathcal{A}^*} \frac{\lambda_S}{\lambda'_N} \chi_S = \chi_N, \\ \sum_{S \in \mathcal{A}^*} \lambda_S = 1, \\ \lambda_S \geq 0 \text{ for all } S \in \mathcal{A}^*, \\ \lambda'_N > 0\} \end{aligned} \tag{2.3}$$

is equivalent to the nonemptiness of

$$\begin{aligned} \{\lambda \in \mathbb{R}^{\mathcal{A}^*} \colon \sum_{S \in \mathcal{A}^*} \lambda_S \chi_S = \chi_N, \\ \lambda_S \geq 0 \text{ for all } S \in \mathcal{A}^* \\ \text{and there exists } S \text{ such that } \lambda_S > 0\}. \end{aligned} \tag{2.4}$$

(2.4) is nonempty if and only if $\mathcal{A}^*$ contains a balanced set system.

Therefore, the dual of (2.1) has a feasible solution if and only if $\mathcal{A}^*$ contains a balanced set system. Since (2.1) always admits a feasible solution, the strong duality theorem implies that it has an optimal solution if and only if $\mathcal{A}^*$ contains a balanced set system.

□

The idea behind this proof is closely related to the approach used in analyzing the relationship between exact games and balancedness, as discussed in Section 3 of Csóka et al (2011), in Schmeidler (1972) and in Bartl and Pintér (2024). As in the primal LP (2.1), we impose an equality constraint on a single coalition – namely, the grand coalition – while all other coalitions are subject to inequality constraints. This structure leads to a dual LP (2.2) in which the dual variable corresponding to the equality constraint is unrestricted in sign, whereas the dual variables associated with the inequality constraints are non-negative.

The vector $E_v(x) := [\ldots \geq e_v(S,x) \geq \ldots : S \in \mathcal{A}^*]$ is called excess vector. It consists of all the excesses in non-increasing order. The lexicographical ordering between $x, y \in \mathbb{R}^n$ is the following: $x \leq_L y$ if $x = y$ or if there exists k such that $x_k < y_k$ and for every $i < k$ it holds that $x_i = y_i$. The nucleolus is the set of imputations which lexicographically minimize the excess vectors over the set of imputations, that is,

$$N(v) := \{x \in I(v) \colon E_v(x) \leq_L E_v(y),\ \forall y \in I(v)\}\,.$$

Moreover, the prenucleolus is the set of preimputations which lexicographically minimize the excess vectors over the set of preimputations, that is,

$$N^*(v) := \{x \in I^*(v) \colon E_v(x) \leq_L E_v(y),\ \forall y \in I^*(v)\}\,.$$

We say that a set of coalitions $\mathcal{S} \subseteq \mathcal{A}^*$ forms a characterization set of the prenucleolus of the game $v \in \mathcal{G}^{N,\mathcal{A}}$, if for the game $v' = v|_{\mathcal{S} \cup \{N\}}$ it holds that $N^*(v) = N^*(v')$.

2.2 The dual of TU-games

Given a game $v \in \mathcal{G}^N$, then its dual is the game $v^* \colon 2^N \to \mathbb{R}$ such that $v^*(S) = v(N) - v(N \setminus S)$, for all $S \in \mathcal{P}(N)$.

Let $N \setminus \mathcal{A}$ denote the set of the complements of the sets from $\mathcal{A}$, that is, $N \setminus \mathcal{A} := \{N \setminus S \colon S \in \mathcal{A}\}$. Given a game with restricted cooperation $v \in \mathcal{G}^{N,\mathcal{A}}$, then its dual is the game with restricted cooperation $v^*\colon 2^{N\setminus\mathcal{A}} \to \mathbb{R}$, such that $v^*(S) = v(N) - v(N \setminus S)$, for all $S \in N \setminus \mathcal{A}$. Moreover, let anti- $I(v^*) := \{x \in I^*(v^*)\colon v^*(N \setminus \{i\}) \geq x(N \setminus \{i\}), \forall N \setminus \{i\} \in N \setminus \mathcal{A}\}$ denote the set of anti-imputations of the dual game v^*. Notice that $I(v) = \text{anti-}I(v^*)$, for all $v \in \mathcal{G}^{N,\mathcal{A}}$.

Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$, a coalition $S \in N \setminus \mathcal{A}$ and a payoff vector $x \in \mathbb{R}^N$. Then the satisfaction of coalition S by the payoff vector x in the dual game v^* is $f_{v^*}(S,x) := x(S) - v^*(S)$. Let $F_{v^*}(x) := [\ldots \geq f_{v^*}(S,x) \geq \ldots : S \in N \setminus \mathcal{A}^*]$ denote the satisfaction vector of the dual game v^*. It consists of all the satisfactions in non-increasing order. The anti-nucleolus of the dual game is the set of anti-imputations which lexicographically minimize the satisfaction vectors over the set of anti-imputations, that is,

$$\text{anti-}N(v^*) := \{x \in \text{anti-}I(v^*)\colon F_{v^*}(x) \leq_L F_{v^*}(y),\ \forall y \in \text{anti-}I(v^*)\}\,.$$

Moreover, the anti-prenucleolus of the dual game is the set of preimputations which lexicographically minimize the satisfaction vectors over the set of preimputations, that is,

$$\text{anti-}N^*(v^*) := \{x \in I^*(v^*)\colon F_{v^*}(x) \leq_L F_{v^*}(y),\ \forall y \in I^*(v^*)\}\,.$$

The anti-core of the dual of the game v is the set of preimputations for which the satisfaction of any coalition from the complementer set of the set of feasible coalitions is non-positive, that is,

$$\text{anti-core}(v^*) := \{x \in I^*(v^*)\colon f_{v^*}(S,x) \leq 0,\ \forall S \in N \setminus \mathcal{A}^*\}\,.$$

Similarly, the anti-ε-core of the dual game is the set of preimputations for which the maximal satisfaction of the coalitions from the complementer set of the set of feasible coalitions is not greater then ε, that is,

$$\text{anti-core}_\varepsilon(v^*) = \{x \in I^*(v^*)\colon \max_{S \in N \setminus \mathcal{A}^*} f_{v^*}(S,x) \leq \varepsilon\}.$$

In addition, the least-anti-core of the dual game is its smallest non-empty anti-ε-core, if it exists. Theorem 2 also applies here, meaning, that the least-anti-core of the dual game is well-defined if and only if $\mathcal{A}^*$ contains a balanced set-system.

Solutions of primal and dual games are related to each other as follows:

$$N(v) = \text{anti-}N(v^*)\,,$$
$$N^*(v) = \text{anti-}N^*(v^*)\,,$$
$$\text{core}(v) = \text{anti- core}(v^*)\,,$$
$$\text{least-core}(v) = \text{least-anti-core}(v^*)\,.$$

2.3 The lexicographic center algorithm

The lexicographic center algorithm Kopelowitz (1967); Maschler et al (1979) is one of the most well-known algorithms for computing the (pre)nucleolus. In this section, we will discuss the lexicographic center algorithm with the modifications by Huberman (1980).

Consider a game $v \in \mathcal{G}^N$ and the following problem:

$$\begin{aligned} & t \to \min \\ \text{s.t.}\;\; & e(S,x) \leq t, \quad S \in \mathcal{P}^*(N) \\ & x \in I^*(v) \\ & t \in \mathbb{R} \end{aligned} \tag{2.5}$$

It is easy to see, that (2.5) has an optimal solution. Let the optimal value of (2.5) be denoted by t_1 and

$$X_1 = \{x \in I^*(v) \colon e(S,x) \leq t_1,\ \forall S \in \mathcal{P}^*(N)\}.$$

Let W_1 denote the fix-set of (2.5), that is

$$W_1 = \{S \in \mathcal{P}^*(N) \colon \exists c_S \in \mathbb{R},\ \text{such that } e(S,x) = c_S,\ \forall x \in X_1\}.$$

For all $k \geq 2$ consider the following LP:

$$\begin{aligned} & t \to \min \\ \text{s.t.}\;\; & e(S,x) \leq t, \quad S \in \mathcal{P}^*(N) \setminus (\cup_{r=1}^{k-1} W_r) \\ & x \in X_{k-1} \\ & t \in \mathbb{R} \end{aligned} \tag{2.6}$$

It is easy to see, that (2.6) has an optimal solution. Let the optimal value of (2.6) be denoted by t_k and

$$X_k = \{x \in X_{k-1} \colon e(S,x) \leq t_k,\ \forall S \in \mathcal{P}^*(N) \setminus (\cup_{r=1}^{k-1} W_r)\}.$$

Let W_k denote the fix-set of (2.6), that is

$$W_k = \{S \in \mathcal{P}^*(N) \colon \exists c_S \in \mathbb{R}, \text{ such that } e(S, x) = c_S, \ \forall x \in X_k\}.$$

It is easy to see, that $t_k \geq t_{k+1}$, $X_k \supseteq X_{k+1}$ for all k and there exists a k^*, such that for all $l \geq k^*$ $X_l = X_{k^*}$.

Kopelowitz (1967); Maschler et al (1979) proved that the lexicographic center algorithm returns with the prenucleolus. The above described algorithm is a modification of the lexicographic center algorithm by Huberman (1980). These modifications do not change the result of the algorithm. Therefore, we can say, that Kopelowitz (1967) and Maschler et al (1979) proved the following theorem:

Theorem 3. *$N^*(v) = X_{k^*}$, for all $v \in \mathcal{G}^N$.*

2.4 Huberman's theorem

Huberman (1980) showed that the so-called essential coalitions give a characterization set for the nucleolus of balanced TU-games. Since in case of balanced games, the nucleolus and the prenucleolus coincide, the essential coalitions also give a characterization set for the prenucleolus. First, consider the definition of essential coalitions used by Huberman (1980).

Definition 4. *Let $v \in \mathcal{G}^N$ be a game. Then, a coalition $S \in \mathcal{P}^*(N)$ is essential, if either $|S| = 1$, or*

$$v(S) > \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v(T).$$

Let $\mathcal{E}_v$ denote the class of essential coalitions of the game v.

Here is Huberman (1980)'s theorem:

Theorem 5 (Huberman (1980))**.** *Let $v \in \mathcal{G}^N$ be a balanced game. Then $\mathcal{E}_v$ is a characterization set for the nucleolus, that is, the values $(v(S))_{S \in \mathcal{E}_v}$ determine the nucleolus of the game v.*

Huberman's theorem can be used to show that for certain classes of games the prenucleolus can be calculated in polynomial time (in the number of players), since there are only polynomial many essential coalitions.

For example, in case of matching games (see Example 6), it can be shown, that only the singletons and the two-element coalitions are essential; therefore, if the core is non-empty, the prenucleolus can be calculated in polynomial time.

Similarly, in case of assignment games (see Section 4.3), only the singletons and the pairs are essential. In addition, in case of assignment games, the core is always non-empty; therefore the prenucleolus can be calculated in polynomial time.

Example 6. In case of matching games, the value of singletons is 0, the values of pairs is given by $v(\{i,j\}) = a_{i,j}$ for all $i \neq j \in N$. For all other coalitions $S \in \mathcal{P}^*(N)$ $v(S) = \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{\{i,j\} \in \mathcal{B}} a_{i,j}$.

Therefore, for all $S \in \mathcal{P}^*$, $|S| > 2$ there exists a $\mathcal{B}^* \subseteq \mathcal{D}_S$ containing only singletons and pairs, such that:

$$v(S) = \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{\{i,j\} \in \mathcal{B}} a_{i,j} = \sum_{\{i,j\} \in \mathcal{B}^*} v(\{i,j\}) + \sum_{\{k\} \in \mathcal{B}^*} v(\{k\}),$$

hence S is not essential.

Therefore, by applying Huberman's theorem, if v is balanced, the pairs and the singletons are enough to calculate the prenucleolus of v.

In this thesis, we provide multiple generalizations and variations of Huberman's theorem, that can be applied in different settings (Theorems 43, 68 and 70).

Chapter 3

TU-games with utility functions

3.1 Games with utility functions

A well-known variant of the prenucleolus is the percapita prenucleolus (Grotte, 1970, 1972). The percapita prenucleolus differs from the preucleolus in a way that instead of using the excesses, it uses the so-called percapita excesses. The percapita excess of a coalition $S \in \mathcal{A}^*$ of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a payoff vector $x \in \mathbb{R}^N$ is $\frac{e_v(S,x)}{|S|}$. Similarly, the percapita excess vector is $E_v^{pc}(x) := (\frac{e_v(S,x)}{|S|})_{S \in \mathcal{A}^*} \in \mathbb{R}^{|\mathcal{A}^*|}$, where $E_v^{pc}(x)_i \geq E_v^{pc}(x)_j$ if $i \leq j$. Accordingly, the percapita prenucleolus is defined as follows: $N_{pc}^*(v) = \{x \in I^*(v) \colon E_v^{pc}(x) \leq_L E_v^{pc}(y),\ \forall y \in I^*(v)\}$.

Solymosi (2019) considers a further generalization of the percapita prenucleolus, where, instead of dividing the excess by the cardinality of S, it is divided by $q(S)$, where q is a positive real valued function over the feasible coalitions. Thereby, Solymosi (2019) introduced the notion of q-nucleolus N_q, which is defined as follows: $N_q^*(v) = \{x \in I^*(v) \colon E_v^q(x) \leq_L E_v^q(y),\ \forall y \in I^*(v)\}$, where the q-excess vector is defined as $E_v^q(x) := (\frac{e_v(S,x)}{q(S)})_{S \in \mathcal{A}^*}$, where $E_v^q(x)_i \geq E_v^q(x)_j$ if $i \leq j$.

Partially inspired by the above generalizations of the prenucleolus we generalize the prenucleolus further by introducing functions, called utility functions, applied to the excesses. Formally, see the following definition.

Definition 7. *A utility function* $\mathbf{u} \colon \mathcal{A}^* \times \mathbb{R} \to \mathbb{R}$ *is a family of functions* $(u_S)_{S \in \mathcal{A}^*}$ *such that* $u_S \colon \mathbb{R} \to \mathbb{R}$ *is strictly monotone increasing, continuous, and its domain is* $\mathbb{R}$*. Moreover, the ranges of* u_S *and* u_T *are the same for every* $S, T \in \mathcal{A}^*$*; let* $R_{\mathbf{u}}$ *denote this common range.*

Let the $\mathbf{u}$-excess of a coalition $S \in \mathcal{A}^*$ by the payoff vector $x \in \mathbb{R}^N$ in the game

v be as follows: $u_S \circ e_v(S,x) = u_S(v(S) - x(S))$. Moreover, let the **u**-excess vector be defined as $E_v(x) := (u_S(e_v(S,x)))_{S\in\mathcal{A}^*} \in \mathbb{R}^{|\mathcal{A}^*|}$, where $E_v(x)_i \geq E_v(x)_j$ if $i \leq j$.

We can now define the **u**-prenucleolus similarly to the percapita prenucleolus.

Definition 8. *The* **u***-prenucleolus is the set of preimputations, which lexicographically minimizes the* **u***-excess vectors over the set of preimputations. Formally,*

$$N^*_{\mathbf{u}}(v) := \{x \in I^*(v) \colon E^{\mathbf{u}}_v(x) \leq_L E^{\mathbf{u}}_v(y)\ \forall y \in I^*(v)\}.$$

Example 9. Some examples of utility functions:

- If **u** is the identity function, then the **u**-penucleolius is the prenucleolus.
- If **u** is defined for all $S \in \mathcal{A}^*$ as $u_S(t) = \frac{t}{|S|}$, then the **u**-prenucleolus is the percapita prenucleolus.
- We can also define **u** as a shift by a constant c. In this case $u_S(t) = t+c$, and for any game $v \in \mathcal{G}^{N,\mathcal{A}}$ the **u**-prenucleolus is the prenucleolus of the game v', where $v'(S) = v(S) + c$ for all $S \in \mathcal{A}^*$, and $v'(N) = v(N)$. Since the prenucleolus is invariant for shifting, in this case the prenucleolus and the **u**-prenucleolus of the game are the same.
- Note that **u** is not necessarily a family of linear functions. For example $u_S(t) = \arctan(t)$ for all $S \in \mathcal{A}^*$ can also be a utility function.

Certain properties of the **u**-prenucleolus may differ from those of the classical prenucleolus. This is illustrated in the following remark.

Remark 10. Unlike the classical prenucleolus, the **u**-prenucleolus is not necessarily covariant under strategic equivalence. Two games $v, w \in \mathcal{G}^{N,\mathcal{A}}$ are strategically equivalent if there exists $\alpha > 0$ and $\beta \in \mathbb{R}^N$, such that $w(S) = \alpha v(S) + \beta(S)$. A solution concept σ is said to be covariant under strategic equivalence if, for such games v and w, it holds that $\sigma(w) = \alpha\sigma(v) + \beta$. To determine whether the **u**-prenucleolus satisfies this property, we must check whether the excess vector $E^{\mathbf{u}}_w(\alpha x + \beta)$ is lexicographically minimal over $I^*(w)$ for all $x \in N^*_{\mathbf{u}}(v)$ and whether there is no other lexicographically minimizing payoff vector. For a coalition $S \in \mathcal{A}^*$, the **u**-excess in this excess vector is given by $u_S \circ (\alpha v(S) + \beta(S) - (\alpha x(S) + \beta(S))) = u_S \circ (\alpha(v(S) - x(S)))$. Thus, the **u**-prenucleolus is covariant under strategic equivalence if u_S is positive 1-homogeneous for every $S \in \mathcal{A}^*$, i.e., $u_S \circ (\alpha(v(S) - x(S))) = \alpha u_S \circ (v(S) - x(S))$.

Next, we introduce a generalization of the core (Shapley, 1955; Gillies, 1959):

Definition 11. *Given a utility function* $\mathbf{u}$*, the* $\mathbf{u}$-core *of a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *is defined as follows:*

$$\mathbf{u}\text{-core}(v) := \left\{ x \in \mathbb{R}^N : x(N) = v(N) \text{ and } u_S \circ e_v(S, x) \leq 0, \forall S \in \mathcal{A}^* \right\}.$$

Notice that, if $\mathcal{A} = \mathcal{P}(N)$ and $\mathbf{u}$ is the identity function, then the $\mathbf{u}$-core is the core.

To illustrate the above introduced definitions consider the following game.

Example 12. $N = \{1, 2, 3\}, \mathcal{A} = \mathcal{P}(N)$ and

$$v(S) = \begin{cases} 0, & \text{if } |S| = 1 \text{ or } S = \{1, 2\}, \\ 4, & \text{if } S = \{1, 3\}, \\ -1, & \text{if } S = \{2, 3\}, \\ 2, & \text{if } S = N. \end{cases}$$

Let the utility function be a shift by -1, that is $u_S(t) = t - 1$ for all $S \in \mathcal{A}^*$.

Then the $\mathbf{u}$-excess of coalition S by the payoff vector x is $v(S) - x(S) - 1$ and the $\mathbf{u}$-prenucleolus of the game is $(2, -1, 1)$, which coincides with the prenucleolus.

However, the $\mathbf{u}$-core of the game is the non-empty set: $\{x \in I^*(v) : x_i \geq -1, x_1 + x_2 \geq -1, x_1 + x_3 \geq 3, x_2 + x_3 \geq -2\}$, while the core of the game is empty.

Similarly, we introduce a generalization of the $\mathbf{u}$-ε-core, the following way: given a utility function $\mathbf{u}$, the $\mathbf{u}$-ε-core of a game $v \in \mathcal{G}^{N,\mathcal{A}}$ is

$$\mathbf{u}\text{-core}_\varepsilon(v) = \{x \in I^*(v) : \max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x) \leq \varepsilon\}.$$

In addition, the $\mathbf{u}$-least-core of the game is its smallest nonempty $\mathbf{u}$-ε-core, provided it exists. The following theorem provides a sufficient and necessary condition on the existence of the $\mathbf{u}$-least-core.

Theorem 13. *Let* $v \in \mathcal{G}^{N,\mathcal{A}}$ *be a game and* $\mathbf{u}$ *be a utility function. The* $\mathbf{u}$*-least-core of the game is well-defined if and only if* $\mathcal{A}^*$ *contains a balanced set system.*

Lemma 20, discussed in Section 3.3, implies that the proof of Theorem 2 is applicable in this context. Consequently, Theorem 13 follows directly from Lemma 20 and Theorem 2.

The assumptions on the utility function in Definition 7 are essential for the results presented in this dissertation. All of these assumptions – except for continuity – are crucial to ensure that the $\mathbf{u}$-prenucleolus behaves in a manner consistent with the expected properties of a generalized prenucleolus solution concept. Continuity, while not strictly necessary for certain properties of the **u**-prenucleolus, is needed in other results like the generalization of the Bondareva–Shapley theorem (Theorem 18).

If strict monotonicity of the utility functions fails, then Lemmata 20 and 21 no longer hold. As a consequence, the generalizations of the theorems by Katsev and Yanovskaya – namely, Theorems 26 and 27 – also do not hold. This highlights the critical role of strict monotonicity in preserving the structural properties and expected behavior of the **u**-prenucleolus.

Example 14. Consider the following game and utility function, where all properties of a utility function are satisfied except strict monotonicity: Let $N = \{1, 2\}$, the characteristic function is defined as $v(\{1\}) = v(\{2\}) = 1$, $v(N) = 0$ and the utility function is $u_{\{1\}}(t) = -t^2$, $u_{\{2\}}(t) = -(t-2)^2$. For a payoff vector $x \in \mathbb{R}^N$, the **u**-excesses for the singleton coalitions are $u_{\{1\}} \circ e_v(\{1\}, x) = -(1-x_1)^2$ and $u_{\{2\}} \circ e_v(\{2\}, x) = -(1-x_2-2)^2$. Since $v(N) = 0$, the efficiency condition implies $x_1 + x_2 = 0$, i.e., $x_1 = -x_2$. Substituting this into the second excess: $u_{\{2\}} \circ e_v(\{2\}, x) = -(1-x_1-2)^2 = -(x_1-1)^2 = -(1-x_1)^2 = u_{\{1\}} \circ e_v(\{1\}, x)$. Thus, for all $x \in I^*(v)$, the excess vector is $E_v^{\mathbf{u}}(x) = (-(1-x_1)^2, -(1-x_1)^2)$. This means the excesses equal for both coalitions and depend only on x_1. However, since x_1 can vary continuously and $-(1-x_1)^2$ can be made arbitrarily small (approaching $-\infty$), no lexicographic minimum exists. Therefore, the **u**-prenucleolus does not exist, even though all coalitions are feasible.

This example illustrates that strict monotonicity is essential for the existence of the **u**-prenucleolus, even in games without restricted cooperation.

The domain of u_S for all $S \in \mathcal{A}^*$ should be $\mathbb{R}$, since the range of the excesses is $\mathbb{R}$. To ensure that the utility transformation is applicable to any excess value, it is necessary that u_S is defined over the entire set of real numbers.

The ranges of u_S and u_T, for any $S, T \in \mathcal{A}^*$, should coincide to ensure the comparability of the **u**-excesses across coalitions. Since the lexicographic minimization of excess vectors relies on comparing transformed excess values, having consistent ranges is essential for maintaining the integrity of the ordering and ensuring meaningful comparisons.

Example 15. Consider the following game and function, which has all properties of a utility function, but the coincidence of the ranges: $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = v(\{1, 2\}) = 0$, $u_{\{1\}}(t) = \arctan(t)$, $u_{\{2\}}(t) = \pi + \arctan(t)$. Then for all $x \in I^*(v)$

$$u_{\{1\}} \circ e_v(\{1\}, x) < u_{\{2\}} \circ e_v(\{2\}, x).$$

Therefore, $E_v^{\mathbf{u}}(x) = (u_{\{2\}} \circ e_v(\{2\}, x), u_{\{1\}} \circ e_v(\{1\}, x))$. The **u**-prenucleolus does not exist, because for all $x \in I^*(v)$ there exists $x' \in I^*(v)$ – for example $x' =$

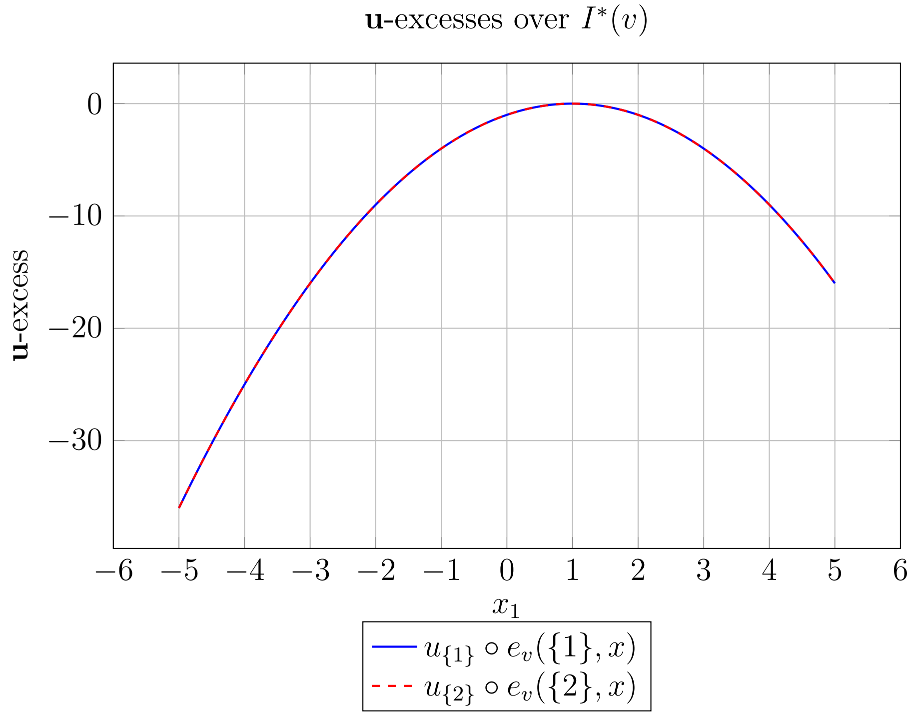


Figure 3.1: **u**-excesses for both coalitions, showing identical behavior and lack of lexicographic minimum.

$(x_1 - 1, x_2 + 1)$ – such that $E_v^{\mathbf{u}}(x') \leq_L E_v^{\mathbf{u}}(x)$.

If strict monotonicity and the range consistency of the utility function hold, then the lack of continuity does not affect the validity of Theorems 26 and 27. However, continuity becomes useful in the definition of **u**-balancedness (Definition 16) and in the generalization of the Bondareva–Shapley theorem (Theorem 18). Specifically, continuity is used to justify the assumption that $0 \in R_{\mathbf{u}}$, whenever $R_{\mathbf{u}} \cap \mathbb{R}_- \neq \emptyset$ and $R_{\mathbf{u}} \cap \mathbb{R}_+ \neq \emptyset$, ensuring that the image of a **u**-excess includes zero when both positive and negative values are taken.

It is worth noting that an alternative approach to applying utility functions is possible: rather than applying the utility function to the excesses, one could instead consider the excess of the utilities. This perspective is motivated by the idea that a utility function may represent a different currency or measurement scale, making it reasonable to transform the values before computing the excesses. In the case of additive utility functions – such as scalar multiplication – both approaches coincide. However, for more general transformations, the two approaches diverge.

In this thesis, we adopt the approach of applying the utility function to the excesses for several reasons. First, this formulation is widely used in the literature, including in the per capita prenucleolus (Grotte, 1970, 1972), the weighted prenucleolus (Derks and Haller, 1999) or q-nucleolus (Solymosi, 2019), and the general nucleolus (Potters and Tijs, 1992; Maschler et al, 1992). Second, if we were to apply the utility function before computing the excess – i.e., using excesses of utilities – the utility function would transform the prenucleolus but not the core. As a result, important results such as defining characterization sets for non-balanced games (see Example 12) would no longer hold. Third, from the perspective of decision theory, applying the utility function to the excesses aligns with the concept of reference dependence, where the zero utility excess serves as the reference point.

Nevertheless, adopting the perspective of excess of the utilities rather than the utility of excesses presents a novel and promising direction that warrants further exploration in future research.

3.2 u-balanced games

Let $\mathfrak{B}$ denote the class of balanced set systems of $\mathcal{A}$.

Definition 16. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$, the game v is $\mathbf{u}$-balanced, if either $R_{\mathbf{u}} \subseteq \mathbb{R}_- \setminus \{0\}$ or if $0 \in R_{\mathbf{u}}$ and*

$$\max_{\mathcal{B}\in\mathfrak{B}} \left(\lambda_N v(N) + \sum_{S\in\mathcal{B}\setminus\{N\}} \lambda_S \left(v(S) - \mathbf{u}_S^{-1}(0)\right) \right) \leq v(N), \tag{3.1}$$

where $(\lambda_S)_{S\in\mathcal{B}}$ is the balancing weight system of the balanced coalition system $\mathcal{B}$.

Notice that, if $\mathcal{A} = \mathcal{P}(N)$ and $\mathbf{u}$ is the identity function, then the $\mathbf{u}$-balancedness pins down to balancedness.

Example 17. Consider the game from Example 12 with feasible coalitions $\mathcal{A} = \{\emptyset, \{2\}, \{1,3\}, N\}$. Then, the only balanced set system of $\mathcal{A}^*$ is $\{\{2\}, \{1,3\}\}$ with balancing weights $\lambda_{\{2\}} = \lambda_{\{1,3\}} = 1$.

Then, considering the identity utility function, the game is non-balanced, since $\sum_{S\in\mathcal{B}\setminus\{N\}} \lambda_S \left(v(S) - \mathbf{u}_S^{-1}(0)\right) = \sum_{S\in\mathcal{B}\setminus\{N\}} \lambda_S\, v(S) = 4$, therefore the left side of (3.1) is strictly larger, than 2.

However, considering the utility function of Example 12, the game is balanced, since $\sum_{S\in\mathcal{B}\setminus\{N\}} \lambda_S \left(v(S) - \mathbf{u}_S^{-1}(0)\right) = 1(0-1) + 1(4-1) = 2$, therefore the left side of (3.1) equals 2.

The following theorem is a generalization of the Bondareva–Shapley theorem (Bondareva, 1963; Shapley, 1967; Faigle, 1989) to games with utilities.

Theorem 18. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$, the $\mathbf{u}$-core$(v) \neq \emptyset$ if and only if v is $\mathbf{u}$-balanced.*

Proof. We consider three cases:

Case 1: $R_{\mathbf{u}} \subseteq \mathbb{R}_- \setminus \{0\}$. In this case, the game is **u**-balanced by definition. In addition, in case of such a utility function, the **u**-core of a game is always non-empty. The reason for this is that $u_S \circ e_v(S, x) < 0$ for all $x \in I^*(v)$ and $S \in \mathcal{A}^*$. Therefore, the **u**-core of the game is not empty, even more, **u**-core$(v) = I^*(v)$.

Case 2: $R_{\mathbf{u}} \subseteq \mathbb{R}_+ \setminus \{0\}$. In this case, the game is not **u**-balanced by definition. In addition, in case of such a utility function the **u**-core of a game is always empty. The reason behind it is that $u_S \circ e_v(S, x) > 0$ for all $x \in I^*(v)$ and $S \in \mathcal{A}^*$; therefore, the **u**-core of the game is empty.

Case 3: Otherwise, that is, $0 \in R_{\mathbf{u}}$. First, consider the following problem

$$\begin{aligned} & x(N) && \to \min && \\ \text{s.t.} \quad & u_S \circ e_v(S, x) && \leq 0 && \forall S \in \mathcal{A}^* \\ & e_v(N, x) && \leq 0 && \\ & x && \in \mathbb{R}^N && \end{aligned} \tag{3.2}$$

The problem (3.2) is equivalent to the following LP (here we use that $0 \in R_u$)

$$\begin{aligned} & x(N) && \to \min && \\ \text{s.t.} \quad & e_v(S, x) && \leq u_S^{-1}(0) && \forall S \in \mathcal{A}^* \\ & e_v(N, x) && \leq 0 && \\ & x && \in \mathbb{R}^N && \end{aligned} \tag{3.3}$$

The LP (3.3) is equivalent to the following LP

$$\begin{aligned} & x(N) && \to \min && \\ \text{s.t.} \quad & x(S) && \geq v(S) - u_S^{-1}(0) && \forall S \in \mathcal{A}^* \\ & x(N) && \geq v(N) && \\ & x && \in \mathbb{R}^N && \end{aligned} \tag{3.4}$$

It is easy to see that LP (3.4) always has a feasible solution. Moreover, since its objective function is bounded from below ($x(N) \geq v(N)$) it always has an optimal solution. Let x^* denote the optimal solution of (3.4). Then the **u**-core is nonempty if and only if $x^*(N) = v(N)$.

The dual of (3.4) is the following:

$$\begin{aligned} & \lambda_N v(N) + \textstyle\sum_{S \in \mathcal{A}^*} \lambda_S (v(S) - u_S^{-1}(0)) && \to \max && \\ \text{s.t.} \quad & \textstyle\sum_{S \in \mathcal{A}^* \cup \{N\}} \lambda_S \chi_S && = \chi_N && \\ & \lambda_S && \geq 0 && \forall S \in \mathcal{A}^* \cup \{N\} \end{aligned}$$

By the strong duality theorem of LPs we know that the optimum of the primal LP equals the optimum of the dual LP.

Suppose that x^* and λ^* are optimal solutions of the primal and the dual LPs, respectively. Notice, that $\lambda^*_N v(N) + \sum_{S\in\mathcal{A}^*} \lambda^*_S (v(S) - u_S^{-1}(0))$ equals the left-hand side of (3.1). Due to the strong duality theorem, it is less or equal than $v(N)$ if and only if $x^*(N)$ is less than or equal to $v(N)$, which is equivalent to the **u**-core being nonempty. □

Example 19. Consider the game from Example 12 with feasible coalitions $\mathcal{A} = \{\emptyset, \{2\}, \{1,3\}, N\}$. We have seen in Example 17 that if the utility function is the identity function, then the game is not balanced, however if the utility function is the shifting by -1, then the game is balanced. Now, consider the core in these two cases: $\text{core}(v) = \{x \in I^*(v) \colon v(\{2\}) \geq 0, v(\{1,3\}) \geq 4\} = \emptyset$; however $\mathbf{u}\text{-core}(v) = \{x \in I^*(v) \colon v(\{2\}) \geq -1, v(\{1,3\}) \geq 3\} \neq \emptyset$, as Theorem 18 says.

3.3 A lexicographic center approach for the u-prenucleolus

In this section, we introduce a modification of the lexicographic center algorithm (Kopelowitz, 1967; Maschler et al, 1979) for the **u**-prenucleolus. More precisely, we show how the idea behind the lexicographic center algorithm can be applied for the **u**-prenucleolus.

The lexicographic center algorithm works by solving a series of LPs. Note that in our case the optimization problems are not necessarily linear. We do not provide any algorithm to solve the non-linear problems, but we introduce a condition to decide whether the problems have optimums or not. The following lemma provides the considered condition:

Lemma 20. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game,* **u** *be a utility function and $X \subseteq I^*(v)$. Then*

$$\begin{aligned} & k \to \min \\ \text{s.t.} \quad & e_v(S,x) \leq k \quad S \in \mathcal{A}^* \\ & x \in X \end{aligned} \tag{3.5}$$

has an optimal solution, if and only if

$$\begin{aligned} & k \to \min \\ \text{s.t.} \quad & u_S \circ e_v(S,x) \leq k \quad S \in \mathcal{A}^* \\ & x \in X \end{aligned} \tag{3.6}$$

has an optimal solution.

Proof. If $X = \emptyset$, then neither problem (3.5) nor problem (3.6) has an optimal solution. Therefore, w.l.o.g. we can assume that $X \neq \emptyset$.

Since X is nonempty, if problem (3.5) does not have an optimal solution, then for every $k \in \mathbb{R}$ there exists an $x_k \in X$ such that $\max_{S \in \mathcal{A}^*} e_v(S, x_k) \leq k$.

Let us define the following sequence: let $k_1 \in \mathbb{R}$ be an arbitrary number and $x_1 \in X$ a payoff vector such that $\max_{S \in \mathcal{A}^*} e_v(S, x_1) \leq k_1$.

Let $k_2 := \min_{S \in \mathcal{A}^*} e_v(S, x_1) - 1$, and x_2 be such that $\max_{S \in \mathcal{A}^*} e_v(S, x_2) \leq k_2$.

For $i > 2$ let $k_i := \min_{S \in \mathcal{A}^*} e_v(S, x_{i-1}) - 1$, and let $x_i \in X$ be such that $\max_{S \in \mathcal{A}^*} e_v(S, x_i) \leq k_i$.

Then for every $n \in \mathbb{N}^+$ $\min_{S \in \mathcal{A}^*} e_v(S, x_n) > \max_{S \in \mathcal{A}^*} e_v(S, x_{n+1})$, therefore, for all $S \in \mathcal{A}^*$ we have that $e_v(S, x_n) > e_v(S, x_{n+1})$. Since u_S is strictly monotone increasing for every $S \in \mathcal{A}^*$, it follows that for every $S \in \mathcal{A}^*$ it holds that $u_S \circ e_v(S, x_n) > u_S \circ e_v(S, x_{n+1})$. Therefore, $\max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_n) > \max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_{n+1})$.

So for every $x \in X$ there exists an $x' \in X$ such that

$$\max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x) > \max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x').$$

Therefore, (3.6) does not have an optimal solution.

Now suppose that (3.6) does not have an optimal solution. Since X is nonempty and $D_{u_S} = \mathbb{R}$ (where D_f is the domain of f) for every $S \in \mathcal{A}^*$ we have that problem (3.6) has a feasible solution. Let $R_{\mathbf{u}} = (a, b)$ denote the range of $\mathbf{u}$, where $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$.

If problem (3.6) does not have an optimal solution, then $\inf\{t\colon u_S \circ e_v(S, x) \leq t \, \forall S \in \mathcal{A}^*, x \in X\} = a$. It means that for every $k \in (a, b) \cap \mathbb{R}$ there exists $x_k \in X$ such that $\max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_k) \leq k$. Furthermore, it is clear that $\min_{S \in \mathcal{A}^*} u_S \circ e_v(S, x) \in (a, b) \cap \mathbb{R}$ for every $x \in X$.

Let us define the following sequence: let $k_1 \in (a, b) \cap \mathbb{R}$ be an arbitrary number and $x_1 \in X$ be such that $\max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_1) \leq k_1$.

Since $\min_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_1) \in (a, b)$, there exists $\varepsilon > 0$ such that $\min_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_1) - \varepsilon > a$. Let $k_2 := \min_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_1) - \varepsilon$ and $x_2 \in X$ be such that $\max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_2) \leq k_2$.

For any $n > 2$: let $\varepsilon_n > 0$ be such that $\min_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_{n-1}) - \varepsilon_n > a$. Let $k_n := \min_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_{n-1}) - \varepsilon_n$ and $x_n \in X$ be such that $\max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_n) \leq k_n$.

Then, for every $n \in \mathbb{N}^+$ $\min_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_n) > \max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x_{n+1})$, furthermore, for each $S \in \mathcal{A}^*$ it holds that $u_S \circ e_v(S, x_n) > u_S \circ e_v(S, x_{n+1})$. Since u_S is strictly monotone increasing, it holds that $e_v(S, x_n) > e_v(S, x_{n+1})$, for all $S \in \mathcal{A}^*$. Meaning that $\max_{S \in \mathcal{A}^*} e_v(S, x_n) > \max_{S \in \mathcal{A}^*} e_v(S, x_{n+1})$.

So, for every $x \in X$ there exists an $x' \in X$ such that

$$\max_{S \in \mathcal{A}^*} e_v(S,x) > \max_{S \in \mathcal{A}^*} e_v(S,x') \,.$$

Therefore, problem (3.5) does not have an optimal solution either. □

The following lemma is a direct corollary of Lemma 20.

Lemma 21. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game, and $\mathbf{u^1}$, $\mathbf{u^2}$ be utility functions. Let $X \subseteq I^*(v)$, then*

$$\min_{x \in X} \max_{S \in \mathcal{A}^*} u^1_S \circ (v(S) - x(S))$$

exists if and only if

$$\min_{x \in X} \max_{S \in \mathcal{A}^*} u^2_S \circ (v(S) - x(S))$$

exists.

Next, we introduce a variant of the lexicographic center algorithm (Kopelowitz, 1967; Maschler et al, 1979), which can be used for calculating the **u**-prenucleolus of a game.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and $\mathbf{u}$ be a utility function. Consider the following problem:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_S \circ e_v(S,x) \leq t, \quad S \in \mathcal{A}^* \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{3.7}$$

If problem (3.7) has an optimal solution, let t_1 denote the optimum of (3.7).

Let X_1 be defined as follows:

$$X_1 = \{x \in I^*(v) \colon u_S \circ e_v(S,x) \leq t_1, \ \forall S \in \mathcal{A}^*\}.$$

Furthermore, let

$$W_1 = \{S \in \mathcal{A}^* \colon \exists c_S \in \mathbb{R}, \text{ such that } u_S \circ e_v(S,x) = c_S, \ \forall x \in X_1\}.$$

Let $k \geq 2$, and let us consider the following problem:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_S \circ e_v(S,x) \leq t, \quad S \in \mathcal{A}^* \setminus (\cup_{r=1}^{k-1} W_r) \\ & x \in X_{k-1} \\ & t \in \mathbb{R} \end{aligned} \tag{3.8}$$

If (3.8) has an optimal solution, let t_k denote the optimum of (3.8).

Let X_k be defined as follows

$$X_k = \{x \in X_{k-1} \colon u_S \circ e_v(S,x) \leq t_k,\ \forall S \in \mathcal{A}^* \setminus (\cup_{r=1}^{k-1} W_r)\}.$$

Furthermore, let

$$W_k = \{S \in \mathcal{A}^* \colon \exists c_S \in \mathbb{R},\ \text{such that } u_S \circ e_v(S,x) = c_S,\ \forall x \in X_k\}.$$

It is easy to see that $t_k \geq t_{k+1}$ and $X_k \supseteq X_{k+1}$ for all $k \in \mathbb{N}_+$, and there exists k^* such that for all $l \geq k^*$ it holds that $X_l = X_{k^*}$, and $X_{k^*} \neq \emptyset$.

In Algorithm 2.4 on page 90 of Maschler et al (1992), Maschler et al proved that a more general version of the above algorithm - the lexicographical center algorithm for finding the general prenucleolus - returns with the general prenucleolus. The **u**-prenucleolus is a special case of the general prenucleolus, hence the result by Maschler et al (1992) implies the following theorem:

Theorem 22. *For every game $v \in \mathcal{G}^{N,\mathcal{A}}$ and utility function $\mathbf{u}$ it holds that*

$$N^*_{\mathbf{u}}(v) = X_{k^*}.$$

Example 23. Consider the game from Example 12 with the feasible coalitions $\mathcal{A} = \{N, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$ and utility function $u_S(t) = t - 1$. Then the first step of the modified lexicographic center algorithm is:

$$\begin{aligned}
& t \to \min \\
\text{s.t.}\quad & 0 - x_2 - 1 \leq t, \\
& 0 - x_3 - 1 \leq t, \\
& 0 - x_1 - x_2 - 1 \leq t, \\
& 4 - x_1 - x_3 - 1 \leq t, \\
& x_1 + x_2 + x_3 = 2 \\
& t \in \mathbb{R},
\end{aligned} \tag{3.9}$$

where the optimum is $t_1 = 0$, $X_1 = \{x \in I^*(v) \colon 0 - x_2 - 1 \leq 0, 0 - x_3 - 1 \leq 0, 0 - x_1 - x_2 - 1 \leq 0, 4 - x_1 - x_3 - 1 \leq 0\}$ and $W_1 = \{\{2\}, \{1,3\}\}$.

Therefore, the second step of the modified lexicographic center algorithm is:

$$
\begin{aligned}
& t \to \min \\
\text{s.t.}\quad & 0 - x_2 - 1 = 0, \\
& 0 - x_3 - 1 \leq t, \\
& 0 - x_1 - x_2 - 1 \leq t, \\
& 4 - x_1 - x_3 - 1 = 0, \\
& x_1 + x_2 + x_3 = 2 \\
& t \in \mathbb{R},
\end{aligned}
\tag{3.10}
$$

where the optimum is $t_2 = -2$, $X_2 = \{x \in I^*(v) \colon 0 - x_2 - 1 = 0, 0 - x_3 - 1 \leq -2, 0 - x_1 - x_2 - 1 \leq -2, 4 - x_1 - x_3 - 1 = 0\}$ and $W_2 = \{\{3\}, \{1, 2\}\}$.

Then, all coalitions are fixed, so we reached the end of the algorithm and the **u**-prenucleolus is: $(2, -1, 1)$

3.4 The nonemptyness of the u-prenucleolus

In case of classical TU-games ($\mathcal{A} = \mathcal{P}(N)$ and u_S is the identity function for all $S \in \mathcal{A}^*$), the prenucleolus always consists of exactly one point (payoff vector) (Schmeidler, 1967). However, this does not hold in case of TU-games with restricted cooperation, where the prenucleolus is a set – not necessarily a singleton set – of payoff vectors. Katsev and Yanovskaya (2013) showed that the prenucleolus of a game is nonempty if and only if the set of feasible coalitions is a balanced set of coalitions. In this section we are going to prove that this statement holds for the **u**-prenucleolus of a game as well. The proof relies on the generalization of Kohlberg's theorem (Theorem 25) and on Lemma 21.

Kohlberg's theorem (Kohlberg (1971)) for classical TU-games is as follows:

Theorem 24 (Kohlberg's theorem)**.** *Given a game $v \in \mathcal{G}^N$ and a payoff vector $x \in I^*(v)$, x is the prenucleous if and only if for every $\alpha \in \mathbb{R}$ it holds that either $\mathcal{D}(\alpha, x) := \{S \in \mathcal{P}^*(N) \colon e_v(S, x) \geq \alpha\} = \emptyset$, or $\mathcal{D}(\alpha, x)$ is a balanced set of coalitions.*

Maschler et al (1992) proved a generalisation of Kohlberg's theorem for the general prencucleolus in Theorem 7.2 on pages 102-104 of Maschler et al (1992). The **u**-prenucleolus is a special case of the general prenucleolus, therefore the following generalization of Kohlberg's theorem is a corollary of the result by Maschler et al (1992):

Theorem 25 (Generalization of Kohlberg's theorem)**.** *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, a utility function $\mathbf{u}$, and $x \in I^*(v)$: x is an element of the $\mathbf{u}$-prenucleolus ($x \in N^*_{\mathbf{u}}(v)$) if and only if $\mathcal{D}_{\mathbf{u}}(\alpha, x)$ is a balanced set of coalitions for every α such that $\mathcal{D}_{\mathbf{u}}(\alpha, x) \neq \emptyset$, where $\mathcal{D}_{\mathbf{u}}(\alpha, x^*) := \{S \in \mathcal{A}^* \colon u_S \circ e_v(S, x^*) \geq \alpha\}$.*

The following proposition, which generalizes Theorem 2. on page 58 of Katsev and Yanovskaya (2013), is a corollary of Theorem 25.

Proposition 26. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and $\mathbf{u}$ be a utility function. If the $\mathbf{u}$-prenucleolus of the game is nonempty, then $\mathcal{A}^*$ is a balanced set of coalitions.*

Proof. Since the $\mathbf{u}$-prenucleolus is nonempty, there exists $x \in N^*_{\mathbf{u}}(v)$. Then by Theorem 25 it holds that $\mathcal{D}_{\mathbf{u}}(\alpha, x)$ is a balanced set of coalitions for every α such that $\mathcal{D}_{\mathbf{u}}(\alpha, x) \neq \emptyset$.

Take α^* such that it is the smallest element of $E^{\mathbf{u}}_v(x)$, that is, let $\alpha^* := E^{\mathbf{u}}_v(x)_{|\mathcal{A}^*|}$. Then, $\mathcal{D}_{\mathbf{u}}(\alpha^*, x) = \mathcal{A}^*$, hence, by Theorem 25, $\mathcal{A}^*$ is a balanced set of coalitions. □

Proposition 27. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and $\mathbf{u}$ be a utility function. If $\mathcal{A}^*$ is a balanced set of coalitions, then the $\mathbf{u}$-prenucleolus of the game is nonempty.*

Proof. The $\mathbf{u}$-prenucleolus of the game is nonempty, if the considered optimization problem admits an optimal solution in every iteration of the generalized lexicographic center approach discussed in Section 3.3. We are going to show that if $\mathcal{A}^*$ is a balanced set of coalitions, then in every iteration of the generalized lexicographic center approach the considered optimization problem attains an optimal solution.

Take an arbitrary step in the generalized lexicographic center approach (see Section 3.3). Then, we have to solve the following minimization problem:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_S \circ e_v(S, x) \leq t, && S \in \mathcal{A}^* \setminus W \\ & u_S \circ e_v(S, x) = c_S, && S \in W \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}}, \end{aligned} \tag{3.11}$$

where $W = \cup_{i=1}^k W_i^u$ for some k.

By Lemma 21, problem (3.11) has an optimal solution if and only if the following LP has:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & v(S) - x(S) \leq t, && S \in \mathcal{A}^* \setminus W \\ & v(S) - x(S) = u_S^{-1}(c_S), && S \in W \\ & x(N) = v(N) \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{3.12}$$

Since the LP (3.12) has a feasible solution (every solution of the optimization problem in the previous iteration is a feasible solution here, or, if we are at the first iteration, that is $W = \emptyset$, then it is easy to see that LP (3.12) has a feasible solution), it is enough to prove that the objective function in (3.12) is bounded from below.

Let $\{\lambda_S\}_{S\in\mathcal{A}^*}$ be a balancing weight system, then

$$\begin{aligned}\sum_{S\in\mathcal{A}^*\setminus W}\lambda_S(v(S)-t)+\sum_{S\in W}\lambda_S(v(S)-u_S^{-1}(c_S)) &\leq \sum_{S\in\mathcal{A}^*}\lambda_S\chi_S^\top x = v(N)\\ \sum_{S\in\mathcal{A}^*\setminus W}\lambda_S v(S)+\sum_{S\in W}\lambda_S(v(S)-u_S^{-1}(c_S))-v(N) &\leq \sum_{S\in\mathcal{A}^*\setminus W}\lambda_S t,\end{aligned}$$

where χ_S denotes the characteristic vector of a set $S \subseteq N$.

Since $0 < \lambda_S$, $\forall S \in \mathcal{A}^*$ it holds that $\sum_{S\in\mathcal{A}^*\setminus W}\lambda_S > 0$, hence

$$\frac{\sum_{S\in\mathcal{A}^*\setminus W}\lambda_S v(S)+\sum_{S\in W}\lambda_S(v(S)-u_S^{-1}(c_S))-v(N)}{\sum_{S\in\mathcal{A}^*\setminus W}\lambda_S}\leq t$$

The left-hand side of the inequality is a constant, hence it gives a lower bound for the right-hand side, which is the objective function of problem (3.12). Therefore, it is bounded from below meaning that problem (3.12) has an optimal solution. □

The following theorem, which generalizes Theorem 2. on page 58 of Katsev and Yanovskaya (2013) comes from Propositions 26 and 27:

Theorem 28. *Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and $\mathbf{u}$ be a utility function. Then $\mathcal{A}^*$ is a balanced set of coalitions, if and only if the $\mathbf{u}$-prenucleolus of the game is nonempty.*

Example 29. Consider the game from Example 12 with the utility function $u_S(t) = t-1$.

We have already seen in Example 23, that if the feasible coalitions are $\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1,2\}, \{1,3\}, N\}$, then the prenucleolus of the game is non-empty. Notice, that in this case $\mathcal{A}^*$ is a balanced set of coalitions.

Now, let us consider a non-balanced coalition set: $\{\{1,2\},\{1,3\}\}$, then

$$\begin{aligned} & t \to \min\\ \text{s.t.}\quad & 0-x_1-x_2-1\leq t,\\ & 4-x_1-x_3-1\leq t,\\ & x_1+x_2+x_3=2\\ & t,x_1,x_2,x_3\in\mathbb{R}\end{aligned}\tag{3.13}$$

does not have an optimal solution, since for any $K \in \mathbb{R}$ $x_1 = K, x_2 = -\frac{K}{2}-1, x_3 = -\frac{K}{2}+3, t = -\frac{K}{2}$ is a feasible solution of (3.13); therefore, the $\mathbf{u}$-prenucleolus is empty.

3.5 The cardinality of the u-prenucleolus

In this section, we consider the size of the **u**-prenucleolus. First, we introduce a notation. For a family of coalitions $\mathcal{A} \subseteq \mathcal{P}(N)$ let $X(\mathcal{A})$ denote the $|\mathcal{A}| \times |N|$ dimensional matrix, where its row vectors are the characteristic vectors of the sets from $\mathcal{A}$.

The following theorem is a generalization of Theorem 3. on page 59 of Katsev and Yanovskaya (2013).

Theorem 30. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, where $\mathcal{A}^*$ is a balanced set of coalitions, and a utility function $\mathbf{u}$, the $\mathbf{u}$-prenucleolus of the game v is a singleton if and only if rank$(X(\mathcal{A})) = |N|$.*

Proof. Only if: Let $x \in N^*_{\mathbf{u}}(v)$ be the only element of the **u**-prenucleolus. Suppose for contradiction that $\text{rank}(X(\mathcal{A})) < |N|$. Consider the following system of linear equations:

$$\begin{cases} y(S) = x(S), \ \forall S \in \mathcal{A}^* \\ y(N) = v(N) \end{cases} \tag{3.14}$$

Then (3.14) can be rewritten as follows:

$$X(\mathcal{A})y = e(x),$$

where $e(x) = (0, \ldots, x(S), \ldots, v(N))^\top$, $S \in \mathcal{A}$.

Since $\text{rank}(X(\mathcal{A})) < |N|$, the system (3.14) has an infinite many solutions and all of those belong to the **u**-prenucleolus, which is a contradiction.

If: Suppose for contradiction that $\text{rank}(X(\mathcal{A})) = |N|$ and there exist $x, y \in N^*_{\mathbf{u}}(v)$ such that $x \neq y$. This means that $E^{\mathbf{u}}_v(x) = E^{\mathbf{u}}_v(y)$.

Notice that for all $S \in \mathcal{A}^*$, such that $x(S) = y(S)$ we have the following:

$$u_S \circ \left(v(S) - \frac{x+y}{2}(S)\right) = u_S \circ (v(S) - x(S)) = u_S \circ (v(S) - y(S)).$$

If there exists a coalition $S \in \mathcal{A}^*$ such that $x(S) \neq y(S)$ (without loss of generality we can suppose that $x(S) > y(S)$), then

$$u_S \circ (v(S) - x(S)) < u_S \circ \left(v(S) - \frac{x+y}{2}(S)\right) < u_S \circ (v(S) - y(S)).$$

Then let T_i be the first coalition according to the order by vector $E_v^{\mathbf{u}}(\frac{x+y}{2})$ for which $x(T_i) \neq y(T_i)$. Then either $u_{T_i} \circ (v(T_i) - \frac{x+y}{2}(T_i)) < u_{T_i} \circ (v(T_i) - y(T_i))$ or $u_{T_i} \circ (v(T_i) - \frac{x+y}{2}(T_i)) < u_{T_i} \circ (v(T_i) - x(T_i))$.

Since the **u**-excesses are in non-increasing order in $E_v^{\mathbf{u}}$ we have that $E_v^{\mathbf{u}}(\frac{x+y}{2}) <_L E_v^{\mathbf{u}}(x) = E_v^{\mathbf{u}}(y)$. Therefore, $x(S) = y(S)$ for all $S \in \mathcal{A}$, and y is a solution of the following system of linear equations:

$$\begin{cases} z(S) = x(S), \forall S \in \mathcal{A}^* \\ z(N) = v(N) \end{cases} \tag{3.15}$$

which can be rewritten as

$$X(\mathcal{A})z = e(x).$$

Since $rank(X(\mathcal{A})) = |N|$, this system has a unique solution $y = x$, which is a contradiction. □

Example 31. Consider the game from Example 12 with the utility function $u_S(t) = t - 1$.

We have already seen in Example 23, that if the feasible coalitions are $\{\emptyset, \{2\}, \{3\}, \{1,2\}, \{1,3\}, N\}$, then the **u**-prenucleolus of the game is single-valued and indeed

$$\text{rank}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = 3.$$

Consider a balanced coalition set $\{N, \{2\}, \{1,3\}\}$, where

$$\text{rank}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = 2 \neq 3.$$

In this case, the **u**-prenucleolus is not empty, but not single valued either.

$$\begin{aligned} & t \to \min \\ \text{s.t.}\quad & 0 - x_2 - 1 \leq t, \\ & 4 - x_1 - x_3 - 1 \leq t, \\ & x_1 + x_2 + x_3 = 2 \\ & t \in \mathbb{R} \end{aligned} \tag{3.16}$$

has an optimum $t_1 = 0$. $W_1 = \{\{2\}, \{1,3\}\}$, hence both non-trivial feasible coalitions are fixed in the first iteration; however $N_{\mathbf{u}}^*(v) = \{x \in I^*(v) \colon x_2 = -1, x_1 + x_3 = 3\}$ is not single-valued. We can easily check, that this set really is the **u**-prenucleolus, since for all $x \in N^*_{\mathbf{u}}(v)$ $E^{\mathbf{u}}_v(x) = (0,0)$. Due to 0 being the optimum of (3.16), we know, that for any $y \in I^*(v) \setminus N^*_{\mathbf{u}}(v)$: $E^{\mathbf{u}}_v(x) <_L E^{\mathbf{u}}_v(y)$.

Chapter 4

Characterization sets for the u-prenucleolus

4.1 The u-essential coalitions

When generalizing Huberman's theorem, we need to "redefine" the essential coalitions (Definition 4).

Definition 32. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system and utility function $\mathbf{u}$, a coalition $S \in \mathcal{A}^*$ is $\mathbf{u}$-essential, if either $\mathcal{D}_S^{\mathcal{A}^*} = \emptyset$ or if $\exists x \in \mathbf{u}$-least-core(v) such that*

$$u_S \circ e_v(S,x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}} \sum_{T \in \mathcal{B}} u_T \circ e_v(T,x).$$

Let $\mathcal{E}_v^{\mathbf{u}}$ denote the class of $\mathbf{u}$-essential coalitions of the game v.

Note that Definition 32 uses x, while Definition 4 seemingly does not. However, if $\mathbf{u}$ is the identity function, x is cancelled out from the inequality. Moreover, the idea of the proof of Huberman's theorem is that the excesses of the essential coalitions exceed the excesses of the other coalitions.

Moreover, notice that if $\mathbf{u}$ is the identity function and $\mathcal{A} = \mathcal{P}(N)$, then the $\mathbf{u}$-essential coalitions are the essential coalitions, hence the $\mathbf{u}$-essential coalitions are generalizations of the essential coalitions.

In addition, one could consider using '$\forall x \in \mathbf{u}$-least-core(v)' instead of '$\exists x \in \mathbf{u}$-least-core(v)' in Definition 32, as both lead to reasonable generalizations of essential coalitions. By using the alternative definition ('$\forall x \in \mathbf{u}$-least-core(v)'), one would obtain a subset of our class of $\mathbf{u}$-essential coalitions. However, our proofs would not hold in that case, since Lemma 34 would only establish that there exists $x \in \mathbf{u}$-least-core(v) for which the lemma's statements apply. In the proof of Theorem 43, when Lemma 34 is applied, we use that the lemma holds for all $x \in \mathbf{u}$-least-core(v) to show that the statements of Lemma 34 hold for y^* as well. Consequently, using the alternative definition, the proof of Theorem 43 would not work.

We should also note that the inclusion of x in Definition 32 raises the question which x must be considered. A straightforward option would be that every element of the preimputations, as it is stated in the definition of the **u**-prenucleolus, have to be considered. However, it turns out that not all preimputations must be considered, in Definition 32 we choose the elements of the **u**-least-core. Of course, any set that contains the **u**-least-core could work.

Example 33. Consider the following game, which is a modification of the game examined in Example 3 on page 1099 in Solymosi (2019): $v(N) = 12, v(\{1,2\}) = v(\{3,4\}) = v(\{2,3,4\}) = 6, v(\{1,4\}) = 4, v(\{4\}) = 3, v(\{1,2,3\}) = 9$ and for every other coalition $S \in \mathcal{P}(N)$ let $v(S) = 0$.

Let the utility function be the percapita-utility function, that is $u_S(t) = \frac{t}{|S|}$ for all $S \in \mathcal{P}^*(N)$.

The percapita-core coincides with the core and $t_1 = 0$, so

$$\mathbf{u}\text{-core}(v) = \mathbf{u}\text{-least-core}(v) = \{(t, 6-t, 3, 3) \colon 1 \leq t \leq 6\}.$$

The percapita-prenucleolus of v is $(3,3,3,3)$. The first iteration in the lexicographic center algorithm (see (3.7)) gives $t_1 = 0$ and

$$W_1 = \{\{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}\}.$$

The second iteration (see (3.8)) gives $t_2 = -1$ and

$$W_2 = \{\{1\}, \{2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

Therefore, the **u**-prenucleolus is $(3,3,3,3)$.

However, if we consider only the essential coalitions in the calculation of the percapita-prenucleolus of the game, we get $(4+\frac{1}{3}, 1+\frac{2}{3}, 3, 3)$. The essential coalitions of v are:

$$\mathcal{E}_v = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,4\}, \{1,2,3\}\}.$$

The first iteration gives $t_1 = 0$ and

$$W_1 = \{\{3\},\{4\},\{1,2\},\{3,4\},\{1,2,3\}\}.$$

The second iteration gives: $t_2 = -(1+\frac{2}{3})$ and

$$W_2 = \{\{1\},\{2\},\{1,4\}\}.$$

Therefore, the **u**-prenucleolus of the game with restricted coalitions $\mathcal{E}_v$ is $(4+\frac{1}{3}, 1+\frac{2}{3}, 3, 3)$.

However, if we consider the **u**-essential coalitions in calculating the percapita-prenucleolus, we get $(3,3,3,3)$, which coincides with the percapita-prenucleolus. Indeed, the **u**-essential coalitions are

$$\mathcal{E}_v^{\mathbf{u}}(v) = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\\ \{1,2,3\},\{1,3,4\},\{2,3,4\}\}.$$

Notice that not all coalitions are **u**-essential. Coalition $\{1,2,4\}$ is not **u**-essential, since for all $x \in \mathbf{u}\text{-core}(v)$ $u_{\{1,2,4\}} \circ e_v(\{1,2,4\},x) = \frac{0-9}{3} = -3$ and $u_{\{1,2\}} \circ e_v(\{1,2\},x) + u_{\{4\}} \circ e_v(\{4\},x) = \frac{6-6}{2} + 3 - 3 = 0$, hence, $u_{\{1,2,4\}} \circ e_v(\{1,2,4\},x) < u_{\{1,2\}} \circ e_v(\{1,2\},x) + u_{\{4\}} \circ e_v(\{4\},x)$.

Considering only the **u**-essential coalitions the first iteration gives $t_1 = 0$ and

$$W_1 = \{\{3\},\{4\},\{1,2\},\{3,4\},\{1,2,3\}\}.$$

The second iteration gives $t_2 = -1$ and

$$W_2 = \{\{1\},\{2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,3,4\},\{2,3,4\}\}.$$

Therefore, the per-capita prenucleolus of the game with restricted coalitions $\mathcal{E}_v^{\mathbf{u}}$ is $(3,3,3,3)$, which coincides with the percapita-prenucleolus of the original game.

In the following, we show that the **u**-essential coalitions form a characterization set for the **u**-prenucleolus in case of **u**-balanced games. First, consider the following two optimization problems:

$$\begin{aligned} & t \to \min \\ \text{s.t.}\quad & u_S \circ e(S,x) \le t, \quad S \in \mathcal{A}^* \\ & x \in I^*(v) \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{4.1}$$

and

$$
\begin{aligned}
& t \to \min \\
\text{s.t.} \quad & u_S \circ e_v(S,x) \leq t, \quad S \in \mathcal{E}_v^{\mathbf{u}} \\
& x \in I^*(v) \\
& t \in R_{\mathbf{u}}
\end{aligned}
\tag{4.2}
$$

Let t_1 denote the optimum of problem (4.1) and X_1 be the set of optimal solutions of problem (4.1) except from t, that is, $X_1 = \{x \in I^*(v) \colon u_S \circ e_v(S,x) \leq t_1 \; \forall S \in \mathcal{A}^*\}$. Similarly, let t_1' be the optimum of problem (4.2) and X_1' be the set of optimal solutions of problem (4.2) except t, that is, $X_1' = \{x \in I^*(v) \colon u_S \circ e_v(S,x) \leq t_1' \; \forall S \in \mathcal{E}_v^{\mathbf{u}}\}$.

Next, we consider some lemmata which are needed for showing that $X_1 = X_1'$ (Proposition 41). Then, by these results we show that the **u**-essential coalitions characterize the **u**-prenucleolus of **u**-balanced games (Theorem 43).

In order to help the reader in following the interdependence of the upcoming results, we have constructed the following graph:

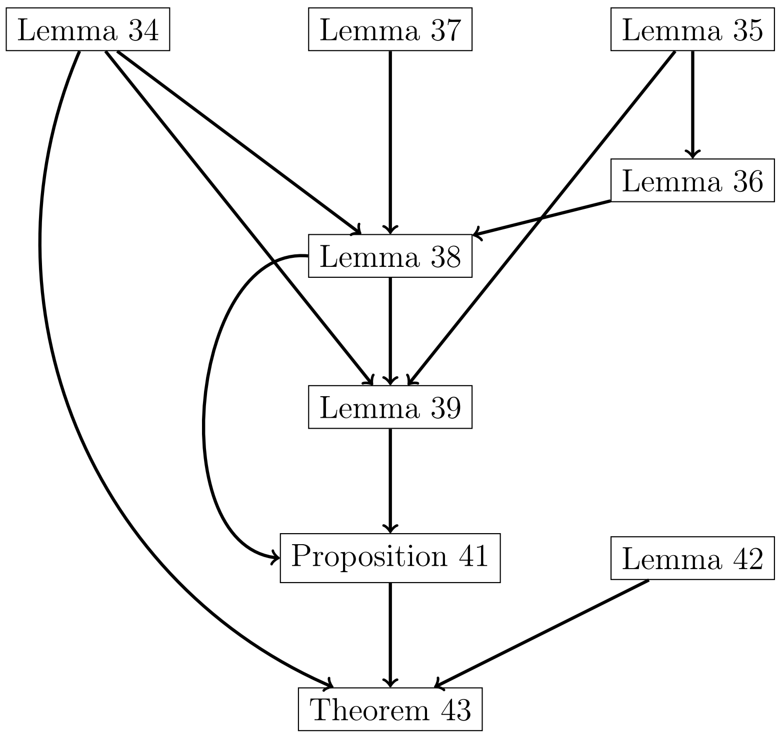


Figure 4.1: The relationships of the results of Section 4.1

Although Lemma 34 is technical, it is a core observation for many other lemmata and theorems. It shows that we can use the **u**-excesses of only the **u**-essential coalitions to give an upper estimate of the **u**-excess of a not **u**-essential coalition.

Lemma 34. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, such that* $\mathcal{A}^*$ *contains a balanced set system. Let* $S \in \mathcal{A}^* \setminus \mathcal{E}_v^{\mathbf{u}}$*. Then for every* $x \in \mathbf{u}$-least-core(v) *there exists* $\mathcal{B}^* \in \mathcal{D}_S^{\mathcal{A}^*}$ *such that* $u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e_v(T,x)$ *and* $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}}$*.*

Proof. Since S is not $\mathbf{u}$-essential, $\forall x \in \mathbf{u}$-least-core(v) there exists $\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}$ such that $u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{B}} u_T \circ e_v(T,x)$ by definition. For all $x \in \mathbf{u}$-least-core(v) let $\mathbf{B}(x) := \{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*} : u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{B}} u_T \circ e_v(T,x)\}$ and let $\mathcal{B}^* \in \mathbf{B}(x)$ be such that for every partition $\mathcal{B} \in \mathbf{B}(x)$ it holds that $|\mathcal{B}^*| \geq |\mathcal{B}|$.

Assuming for the purpose of deriving a contradiction, that a coalition $T^* \in \mathcal{B}^*$ is not $\mathbf{u}$-essential. Since T^* is not $\mathbf{u}$-essential by Definition 32 $\exists \mathcal{B}' \in \mathcal{D}_{T^*}^{\mathcal{A}^*}$ such that $u_{T^*} \circ e_v(T^*,x) \leq \sum_{T' \in \mathcal{B}'} u_{T'} \circ e_v(T',x)$, therefore,

$$u_S \circ e_v(S,x) \leq \sum_{T' \in (\mathcal{B}^* \setminus \{T^*\}) \cup \mathcal{B}'} u_{T'} \circ e_v(T',x),$$

and $|(\mathcal{B}^* \setminus \{T^*\}) \cup \mathcal{B}'| > |\mathcal{B}^*|$, which is a contradiction. □

Next, we introduce the following notion: for a class of coalitions $\mathcal{S} \subseteq \mathcal{A}^*$ and $t \in \mathbb{R}$ let $X(\mathcal{S},t) := \{x \in I^*(v) \colon u_S \circ e_v(S,x) \leq t, \forall S \in \mathcal{S}\}$.

Through the following lemmata, we aim to show that $X(\mathcal{A}^*,t_1) = X(\mathcal{E}_v^{\mathbf{u}},t_1')$, that is, $X_1 = X_1'$. Since, we do not just switch from $\mathcal{A}^*$ to $\mathcal{E}_v^{\mathbf{u}}$, but also from t_1 to t_1', we need to check first, what happens, if we leave the set of coalitions unchanged, and only manipulate t.

Lemma 35. *Given a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system and* $t', t'' \in \mathbb{R}$ *such that* $t' \leq t''$*, then* $X(\mathcal{S},t') \subseteq X(\mathcal{S},t'')$*.*

Proof. For any $x \in X(\mathcal{S},t')$ it holds that $x \in I^*(v)$, and for all $S \in \mathcal{S}$:

$$u_S \circ e_v(S,x) \leq t' \leq t'' \ .$$

Therefore, $x \in X(\mathcal{S},t'')$. □

Lemmata 36 and 37 show that some very important geometrical properties of the sets $X(\mathcal{A}^*,t)$ and $X(\mathcal{E}_v^{\mathbf{u}},t)$ remain unchanged by the introduction of utility functions (compared to using the identity utility function).

Lemma 36. *Given a utility function* $\mathbf{u}$ *and a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system, for every* $t_1 \leq t$ *it holds that both* $X(\mathcal{A}^*,t)$ *and* $X(\mathcal{E}_v^{\mathbf{u}},t)$ *are nonempty, convex and closed.*

Proof. By Lemma 35, we have $X(\mathcal{A}^*,t) \supseteq X(\mathcal{A}^*,t_1) = \mathbf{u}$-least-core$(v)$. Since $\mathbf{u}$-least-core$(v) \neq \emptyset$, and noting that $X(\mathcal{A}^*,t) \subseteq X(\mathcal{E}_v^{\mathbf{u}},t)$, because $\mathcal{E}_v^{\mathbf{u}} \subseteq \mathcal{A}^*$, it follows that $X(\mathcal{E}_v^{\mathbf{u}},t) \neq \emptyset$.

Let $H_S := \{x \in \mathbb{R}^N \colon u_S \circ e_v(S,x) \le t\}$ for all $S \in \mathcal{A}^*$. Then $H_S = \{x \in \mathbb{R}^N \colon e_v(S,x) \le u_S^{-1}(t)\}$, hence H_S is a closed half-space, therefore, it is convex and closed. $X_0 = I^*(v) = \{x \in \mathbb{R}^N \colon x(N) = v(N)\}$ is a hyperplane, therefore it is convex and closed. Finally,

$$X(\mathcal{A}^*,t) = \{x \in I^*(v) \colon u_S \circ e_v(S,x) \le t \ \forall S \in \mathcal{A}^*\} = \cap_{S \in \mathcal{A}^*} H_S \cap X_0 \ ,$$

hence, $X(\mathcal{A}^*,t)$ is an intersection of convex, closed sets; therefore, it is convex and closed.

Similarly, $X(\mathcal{E}_v^{\mathbf{u}},t) = \cap_{S \in \mathcal{E}_v^{\mathbf{u}}} H_S \cap X_0$ is an intersection of convex, closed sets, hence, it is convex and closed. □

Lemma 37. *Given a utility function* $\mathbf{u}$ *and a game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, take* $x_1, x_2 \in \mathbb{R}^N$ *and* $S \in \mathcal{A}$*. If* $u_S \circ e_v(S,x_1) < u_S \circ e_v(S,x_2)$*, then for every* $\lambda \in (0,1)$ *it holds that*

$$u_S \circ e_v(S,x_1) < u_S \circ e_v(S,\lambda x_1 + (1-\lambda)x_2) < u_S \circ e_v(S,x_2).$$

Proof. Let $\lambda \in (0,1)$, then

$$(\lambda x_1 + (1-\lambda)x_2)(S) = \lambda x_1(S) + (1-\lambda)x_2(S).$$

Therefore,

$$\begin{aligned} e_v(S,\lambda x_1 + (1-\lambda)x_2) &= v(S) - (\lambda x_1 + (1-\lambda)x_2)(S) \\ &= \lambda v(S) + (1-\lambda)v(S) - (\lambda x_1(S) + (1-\lambda)x_2(S)) \\ &= \lambda e_v(S,x_1) + (1-\lambda)e_v(S,x_2). \end{aligned}$$

Since u_S is strictly monotone increasing we have that

$$u_S \circ e_v(S,x_1) < u_S \circ e_v(S,\lambda x_1 + (1-\lambda)x_2) < u_S \circ e_v(S,x_2).$$

□

Lemma 38 shows that if we switch from $\mathcal{A}^*$ to $\mathcal{E}_v^{\mathbf{u}}$, but not from t_1 to t_1', then we have the equality $X(\mathcal{A}^*,t_1) = X(\mathcal{E}_v^{\mathbf{u}},t_1)$. This lemma provides the main step in showing that $X_1 = X_1'$. The proof of Lemma 38 sheds light on the additional theoretical difficulty of using $\mathbf{u}$-least-core(v) instead of $I^*(v)$ in Definition 32.

If we used $I^*(v)$ in Definition 32, then showing for a $\mathbf{u}$-balanced game v that $X(\mathcal{A}^*,t_1) = X(\mathcal{E}_v^{\mathbf{u}},t_1)$ would simplify to the following: for all $x \in X(\mathcal{E}_v^{\mathbf{u}},t_1)$ and $S \in \mathcal{E}_v^{\mathbf{u}}$ we have $u_S \circ e_v(S,x) \le t_1$. Due to this modification, we know that for every $S' \in \mathcal{A}^* \setminus \mathcal{E}_v^{\mathbf{u}}$ there exists $B \in \mathcal{D}_{S'}^{\mathcal{A}*}$, $\mathcal{B} \subseteq \mathcal{E}_v^{\mathbf{u}}$ (due to Lemma 34) such that

$u_{S'} \circ e_v(S', x) \leq \sum_{T \in \mathcal{B}} u_T \circ e_v(T, x) \leq |\mathcal{B}| t_1 \leq t_1$, where the last inequality comes from $t_1 \leq 0$.

However, since we use $\mathbf{u}$-least-core(v) in Definition 32, we only obtain the inequality $u_{S'} \circ e_v(S', x) \leq \sum_{T \in \mathcal{B}} u_T \circ e_v(T, x)$ for those payoff vectors that belong to $\mathbf{u}$-least-core(v). Therefore, if there exists a payoff vector $x' \in X(\mathcal{E}_v^{\mathbf{u}}, t_1)$ such that $x' \notin \mathbf{u}$-least-core$(v) = X(\mathcal{A}^*, t_1)$, then the above reasoning cannot be applied.

Lemma 38. *Given a utility function* $\mathbf{u}$ *and a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system, if* v *is* $\mathbf{u}$*-balanced, then* $X(\mathcal{A}^*, t_1) = X(\mathcal{E}_v^{\mathbf{u}}, t_1)$.

Proof. Since $\mathcal{E}_v^{\mathbf{u}} \subseteq \mathcal{A}^*$, it holds that $X(\mathcal{A}^*, t_1) \subseteq X(\mathcal{E}_v^{\mathbf{u}}, t_1)$.

Indirectly assume that $\exists x^1 \in X(\mathcal{E}_v^{\mathbf{u}}, t_1) \setminus X(\mathcal{A}^*, t_1)$. This means that $\exists S \in \mathcal{A}^*$ such that

$$u_S \circ e_v(S, x^1) > t_1. \tag{4.3}$$

Let $\mathcal{S}_{x^1} = \{S \in \mathcal{A}^* \colon u_S \circ e_v(S, x^1) > t_1\}$. Then, for all $S \in \mathcal{S}_{x^1}$ it holds that $S \notin \mathcal{E}_v^{\mathbf{u}}$.

Let $x^* \in X(\mathcal{A}^*, t_1)$ be the closest point of set $X(\mathcal{A}^*, t_1)$ to point x^1. It is clear that such x^* exists, since $X(\mathcal{A}^*, t_1)$ is nonempty and closed by Lemma 36.

Since $X(\mathcal{E}_v^{\mathbf{u}}, t_1)$ is a convex set, for every $\lambda \in [0, 1]$ it holds that $\lambda x^* + (1-\lambda)x^1 \in X(\mathcal{E}_v^{\mathbf{u}}, t_1)$.

By Lemma 34 we have that for each $S \in \mathcal{S}_{x^1}$ $\exists \mathcal{B}_S^* \subseteq \mathcal{E}_v^{\mathbf{u}}$, $\mathcal{B}_S^* \in \mathcal{D}_{S^1}^{\mathcal{A}^*}$ such that

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*). \tag{4.4}$$

Since $t_1 \leq 0$ and for all $x \in X(\mathcal{E}_v^{\mathbf{u}}, t_1)$ and $T \in \mathcal{E}_v^{\mathbf{u}}$ it holds that $u_T \circ e_v(T, x) \leq t_1$, therefore for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*) \leq t_1 \ . \tag{4.5}$$

By (4.3), (4.5) and the continuity of u_S, for every $S \in \mathcal{S}_{x^1}$ there exists $\lambda_S \in [0, 1]$ such that

$$u_S \circ e_v(S, \lambda_S x^* + (1-\lambda_S)x^1) = t_1 \ .$$

Let $S^1 \in \operatorname{argmin}_{S \in \mathcal{S}_{x^1}} ||x^* - (\lambda_S x^* + (1-\lambda_S)x^1))||$, and let $x^2 = \lambda_{S^1} x^* + (1-\lambda_{S^1})x^1$. Then,

$$u_{S^1} \circ e_v(S^1, x^2) \geq \sum_{T \in \mathcal{B}_{S^1}^*} u_T \circ e_v(T, x^2),$$

because $t_1 \leq 0$ and $u_T \circ e_v(T, x^2) \leq t_1$ for all $T \in \mathcal{B}^*_{S^1}$.

Then, there are two cases:

Case 1:

$$u_{S^1} \circ e_v(S^1, x^2) = \sum_{T \in \mathcal{B}^*_{S^1}} u_T \circ e_v(T, x^2).$$

In this case, $t_1 = u_{S^1} \circ e_v(S^1, x^2) = \sum_{T \in \mathcal{B}^*_{S^1}} u_T \circ e_v(T, x^2) \leq |\mathcal{B}^*_{S^1}| t_1 \leq t_1$; therefore, $t_1 = 0$.

Then, $\sum_{T \in \mathcal{B}^*_{S^1}} u_T \circ e_v(T, x^2) = 0$, hence, for all $T \in \mathcal{B}^*_{S^1}$ it holds that $u_T \circ e_v(T, x^2) = 0$.

We know that $u_{S^1} \circ e_v(S^1, x^1) > t_1 \geq \sum_{T \in \mathcal{B}^*_{S^1}} u_T \circ e_v(T, x^1)$ and that $x^2(S^1) = \sum_{T \in \mathcal{B}^*_{S^1}} x^2(T)$ and $x^1(S^1) = \sum_{T \in \mathcal{B}^*_{S^1}} x^1(T)$. Then $x^1(S^1) < x^2(S^1)$, but then $\exists T' \in \mathcal{B}^*_{S^1}$ such that $x^1(T') < x^2(T')$. However, $u_{T'}$ is a strictly monotone increasing function, hence we have that $u_{T'} \circ e_v(T', x^1) > t_1$, which is a contradiction, because $x^1 \in X(\mathcal{E}^{\mathbf{u}}_v, t_1)$.

Case 2:

$$u_{S^1} \circ e_v(S^1, x^2) > \sum_{T \in \mathcal{B}^*_{S^1}} u_T \circ e_v(T, x^2) \ . \tag{4.6}$$

By the choice of x^2 and Lemma 37, for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S, x^2) \leq t_1 \ .$$

Since for all $S \in \mathcal{A}^* \setminus \mathcal{S}_{x_1}$ it holds that $u_S \circ e_v(S, x^1) \leq t_1$; by Lemma 37

$$u_S \circ e_v(S, x^2) \leq t_1 \ .$$

This means that for all $S \in \mathcal{A}^*$ it holds that $u_S \circ e_v(S, x^2) \leq t_1$, hence $x^2 \in X(\mathcal{A}^*, t_1)$.

Since $x^* \in X(\mathcal{A}^*, t_1)$ is the closest point of set $X(\mathcal{A}^*, t_1)$ to point x^1, we have that $x^2 = x^*$. However, then (4.6) contradicts (4.4). □

We proved that $X(\mathcal{A}^*, t_1) = X(\mathcal{E}^{\mathbf{u}}_v, t_1)$. To show that $X(\mathcal{A}^*, t_1) = X(\mathcal{E}^{\mathbf{u}}_v, t'_1)$ we need $t_1 = t'_1$, for which we must prove that for all $t < t_1$ it holds that $X(\mathcal{E}^{\mathbf{u}}_v, t) = \emptyset$.

Lemma 39. *Given a utility function $\mathbf{u}$, a $\mathbf{u}$-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system and $t \in \mathbb{R}$, if $t < t_1$, then $X(\mathcal{A}^*, t) = X(\mathcal{E}^{\mathbf{u}}_v, t) = \emptyset$.*

Proof. By the definition of t_1 we have that $X(\mathcal{A}^*, t) = \emptyset$, and we know that $X(\mathcal{A}^*, t) \subseteq X(\mathcal{E}^{\mathbf{u}}_v, t)$.

By Lemma 35, since $t < t_1$, we have that $X(\mathcal{E}^{\mathbf{u}}_v, t) \subseteq X(\mathcal{E}^{\mathbf{u}}_v, t_1)$. Furthermore, by Lemma 38 it holds that $X(\mathcal{E}^{\mathbf{u}}_v, t_1) = X(\mathcal{A}^*, t_1) = \mathbf{u}$-least-core$(v)$. Therefore,

$$X(\mathcal{E}^{\mathbf{u}}_v, t) \subseteq X(\mathcal{E}^{\mathbf{u}}_v, t_1) = X(\mathcal{A}^*, t_1) = \mathbf{u}\text{-least-core}(v) \ .$$

Then, for every $x \in X(\mathcal{E}_v^{\mathbf{u}}, t)$ and for every $S \in \mathcal{A}^* \setminus \mathcal{E}_v^{\mathbf{u}}$ by Lemma 34 and by the non-positivity of t it holds that $\exists \mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}$, $\mathcal{B} \subseteq \mathcal{E}_v^{\mathbf{u}}$ such that

$$u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}} u_T \circ e_v(T, x) \leq t \ .$$

Then, $x \in X(\mathcal{A}^*, t)$, that is, $X(\mathcal{E}_v^{\mathbf{u}}, t) \subseteq X(\mathcal{A}^*, t)$. Summing up, we can conclude that $X(\mathcal{E}_v^{\mathbf{u}}, t) = X(\mathcal{A}^*, t) = \emptyset$. □

Remark 40. Notice, that all the results we have discussed so far hold if one defines **u**-essentiality (Definition 32) with any set containing the **u**-least-core – for example the **u**-core – instead of the **u**-least-core.

The following proposition is a consequence of Lemmata 38 and 39. Here, we finally conclude that $X(\mathcal{A}^*, t_1) = X(\mathcal{E}_v^{\mathbf{u}}, t_1')$, if the game is **u**-balanced.

Proposition 41. *Given a utility function* $\mathbf{u}$*, a* **u***-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system, the following holds:* $t_1 = t_1'$ *and* $X_1 = X_1'$*.*

Proof. By definition $t_1' = \min\{t \colon X(\mathcal{E}_v^{\mathbf{u}}, t) \neq \emptyset\}$. We know, that $t_1' \leq t_1$; therefore, by Lemmata 38 and 39 we have that $X_1' = X(\mathcal{E}_v^{\mathbf{u}}, t_1') = X(\mathcal{A}^*, t_1')$. However, $X(\mathcal{A}^*, t_1') \neq \emptyset$ if and only if $t_1' \geq t_1$, hence $t_1 = t_1'$ and $X_1 = X_1'$. □

Lemma 42 is a technical result, that we will use in the proof of Theorem 43, that is our first generalization of Huberman's theorem (Theorem 5).

Lemma 42. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$*, such that* $\mathcal{A}^*$ *is a balanced set system. Let* k *be a positive integer,* $S \in \mathcal{A}^* \setminus \cup_{r=1}^{k-1} W_r$ *be such that* $\mathcal{D}_S^{\mathcal{A}^*} \neq \emptyset$*, and* $\mathcal{B}^* \in \mathcal{D}_S^{\mathcal{A}^*}$*. Then, there exists a coalition* $T^* \in \mathcal{B}^*$ *such that* $T^* \notin \cup_{r=1}^{k-1} W_r$*.*

Proof. If $k = 1$, then $\cup_{r=1}^{k-1} W_r = \emptyset$, hence, $T^* \notin \cup_{r=1}^{k-1} W_r$.
If $k \geq 2$, then indirectly assume that $\mathcal{B}^* \subseteq \cup_{r=1}^{k-1} W_r$. Then for every $x \in X_{k-1}$

$$\begin{aligned} u_S \circ e_v(S, x) = u_S(v(S) - x(S)) &= u_S \left(v(S) - \sum_{T \in \mathcal{B}^*} x(T) \right) \\ &= u_S \left(v(S) - \sum_{T \in \mathcal{B}^*} (v(T) - u_T^{-1}(c_T)) \right) \ . \end{aligned}$$

Therefore, for each $x, x' \in X_{k-1}$ it holds that $u_S \circ e_v(S, x) = u_S \circ e_v(S, x')$, meaning that $S \in \cup_{r=0}^{k-1} W_r$, which is a contradiction. □

The following theorem generalizes Huberman (1980)'s theorem (Theorem 7 on page 420 of Huberman (1980)):

Theorem 43. *Consider a* **u***-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system, and let*

$$Y_1 = \{x \in I^*(v) \colon u_S \circ e_v(S, x) \leq t_1, \ \forall S \in \mathcal{E}_v^{\mathbf{u}}\},$$

and for all $k \geq 2$ *let* Y_k *be defined as follows:*

$$Y_k = \{x \in X_{k-1} \colon u_S \circ e_v(S, x) \leq t_k, \ \forall S \in \mathcal{E}_v^{\mathbf{u}} \setminus (\cup_{r=1}^{k-1} W_r)\},$$

where t_k *is the optimum of* (3.8)*, if it exists and* $-\infty$ *otherwise. Then,* $X_k = Y_k$ *for all* $k \geq 1$*.*

In other words, Theorem 43 claims that the **u**-essential coalitions give a characterization set for the **u**-prenucleolus of **u**-balanced games.

Proof. First, notice that $X_k \subseteq Y_k$ holds for all k by definition.

By Proposition 41, we have that $X_1 = Y_1$; therefore, $X_1 = Y_1 =$ **u**-least-core(v).

Suppose for contradiction that there exists $k > 1$ such that $X_k \not\supseteq Y_k$, which means that there exist $y^* \in Y_k$ and $S \in \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cup (\cup_{r=1}^{k-1} W_r))$ such that $u_S \circ e_v(S, y^*) > t_k$.

By Lemma 34 for each $x \in X_1$ there exists $\mathcal{B}_x \in \mathcal{D}_S^{\mathcal{A}^*} \cap \mathcal{E}_v^{\mathbf{u}}$ such that $u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}_x} u_T \circ e_v(T, x)$.

Then

$$u_S \circ e_v(S, y^*) \leq \sum_{T \in \mathcal{B}_{y^*}} u_T \circ e_v(T, y^*) \ .$$

Moreover, by definition $u_S \circ e_v(S, x) \leq 0$ for all $S \in \mathcal{A}^*, x \in X_{k-1}$. Therefore, for every coalition $T \in \mathcal{B}_{y^*}$ we have that $u_S \circ e_v(S, y^*) \leq u_T \circ e_v(T, y^*)$. By Lemma 42, there exists $T^* \in \mathcal{B}_{y^*}$ such that $T^* \notin \cup_{r=1}^{k-1} W_r$. Therefore, $u_S \circ e_v(S, y^*) \leq u_{T^*} \circ e_v(T^*, y^*) \leq t_k$, which is a contradiction. □

In words, Theorem 43 gives a characterization set for the **u**-prenucleolus in case of **u**-balanced games. As a direct corollary of this theorem, we can find a characterization set for the percapita prenucleolus in case of balanced games (notice, that the core and the percapita core coincide), or we can shift the values of the non-trivial coalitions uniformly so that the **u**-core (**u** here is the shift) of the game becomes nonempty, while the **u**-prenucleolus coincides with the prenucleolus. We discuss these applications in more detail in Section 4.2.

4.2 Two invariance results

There are **u** functions such that the **u**-prenucleolus of a game is the same as its prenucleolus, while the **u**-core of the game is different from its core. These utility functions can be useful in finding a characterization set for the prenucleolus of non-balanced games.

In this section, we characterize the classes of **u**-functions under which the **u**-prenucleolus and the **u**-core are the same as the prenucleolus and the core, respectively.

Lemma 44. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, and utility functions $\mathbf{u}^1$ and $\mathbf{u}^2$, the $\mathbf{u}^1$-prenucleolus coincides with the $\mathbf{u}^2$-prenucleolus if for every $S, T \in \mathcal{A}^*$ and $x \in I^*(v)$*

$$u_S^1 \circ e_v(S, x) \leq u_T^1 \circ e_v(T, x) \tag{4.7}$$

if and only if

$$u_S^2 \circ e_v(S, x) \leq u_T^2 \circ e_v(T, x). \tag{4.8}$$

Proof. Notice, that w.l.o.g. we can assume that $\mathbf{u}^1 = \mathbf{id}$. Then let $\mathbf{u}$ denote $\mathbf{u}^2$.

Let $x, y \in I^*(v)$ be such that $E_v(x) \leq_L E_v(y)$, where

$$E_v(x) := [e_v(S_1, x), e_v(S_2, x), \ldots, e_v(S_{|\mathcal{A}^*|}, x)]$$

$$E_v(y) := [e_v(T_1, y), e_v(T_2, y), \ldots, e_v(T_{|\mathcal{A}^*|}, y)].$$

Case 1: $E_v(x) =_L E_v(y)$: By (4.7) and (4.8) $e_v(S_n, x) = e_v(T_n, y)$ for all $1 \leq n \leq |\mathcal{A}^*|$ is equivalent with $u_{S_n} \circ e_v(S_n, x) = u_{T_n} \circ e_v(T_n, y)$ for all $1 \leq n \leq |\mathcal{A}^*|$. Meaning $E_v^{\mathbf{u}}(x) =_L E_v^{\mathbf{u}}(y)$.

Case 2: $E_v(x) \neq_L E_v(y)$: Then, there exists k such that

$$\begin{aligned} e_v(S_n, x) = e_v(T_n, y) \ \forall n < k, \\ e_v(S_k, x) < e_v(T_k, y). \end{aligned} \tag{4.9}$$

By (4.7) and (4.8) we have that (4.9) is equivalent with

$$\begin{aligned} u_{S_n} \circ e_v(S_n, x) = u_{T_n} \circ e_v(T_n, y) \ \forall n < k, \\ u_{S_k} \circ e_v(S_k, x) < u_{T_k} \circ e_v(T_k, y). \end{aligned} \tag{4.10}$$

This proves that for each $x, y \in I^*(v)$ $E_v(x) \leq_L E_v(y)$ if and only if $E_v^{\mathbf{u}}(x) \leq_L E_v^{\mathbf{u}}(y)$. Therefore, $x \in N^*(v)$ if and only if $x \in N_{\mathbf{u}}^*(v)$. □

For example, if $\mathbf{u}$ is such that $u_S = u_T$ for all $S, T \in \mathcal{A}^*$, then the $\mathbf{u}$-prenucleolus coincides with the prenucleolus.

Example 45. In this example, we show how to find a characterization set for the prenucleolus of non-balanced games using Lemma 44.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game such that $\mathcal{A}^*$ contains a balanced set system and let $\mathbf{u}$ be the utility function, such that $u_S \circ e_v(S, x) := e_v(S, x) - \varepsilon^*$ for all $S \in \mathcal{A}^*$, where ε^* is the optimum of the following LP:

$$\begin{aligned} & \varepsilon \to \min \\ \text{s.t.} \quad & e_v(S, x) \leq \varepsilon, \quad S \in \mathcal{A}^* \\ & x \in I^*(v) \\ & \varepsilon \in \mathbb{R}. \end{aligned} \tag{4.11}$$

In other words, ε^* is such that the least core is the ε^*-core.

Then, by Lemma 44 the $\mathbf{u}$-prenucleolus of the game coincides with the prenucleolus. In addition, the game is $\mathbf{u}$-balanced because the least core is never empty.

Therefore, by Theorem 43, in the case of this $\mathbf{u}$ function ($u_S(t) = t - \varepsilon^*$, $S \in \mathcal{A}^*$), the $\mathbf{u}$-essential coalitions form a characterization set for the prenucleolus of v even if v is not balanced.

To illustrate the application of $\mathbf{u}$-essential coalitions described in Example 45, consider the following example:

Example 46. Consider the game from Example 12, where $\mathcal{A} = \mathcal{P}(N)$.

The core of this game is empty, as we have seen in Example 12; therefore, we cannot apply Huberman's theorem (Theorem 5).

The essential coalitions (Definition 4) are $\mathcal{E}_v = \{\{1\}, \{2\}, \{3\}, \{1, 3\}\}$ and the prenucleolus of the game is $(2, -1, 1)$. However, if we only consider the essential coalitions when calculating the prenucleolus, we would get $(\frac{3}{2}, -1, \frac{3}{2})$.

On the other hand, if we take the utility function $u(t) = t - 1$, where $t_1 = -1$, the $\mathbf{u}$-core is non-empty as we have seen in Example 12; therefore, we can apply Theorem 43. The $\mathbf{u}$-essential coalitions (Definition 32) are $\mathcal{E}_v^{\mathbf{u}} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and the $\mathbf{u}$-prenucleolus is $(2, -1, 1)$.

Due to Lemma 44, we know that in case of this $\mathbf{u}$ function, the $\mathbf{u}$-prenucleolus coincides with the prenucleolus, so $\mathcal{E}_v^{\mathbf{u}}$ characterize the prenucleolus as well.

In the following, we consider the equivalence of the core and the $\mathbf{u}$-core.

Lemma 47. *Given a game $v \in \mathcal{G}^{N,\mathcal{A}}$, and utility functions $\mathbf{u}^1$ and $\mathbf{u}^2$, the $\mathbf{u}^1$-core coincides with the $\mathbf{u}^2$-core if for every $S \in \mathcal{A}^*$ and $x \in I^*(v)$*

$$u_S^1 \circ e_v(S,x) \leq 0 \tag{4.12}$$

if and only if

$$u_S^2 \circ e_v(S,x) \leq 0. \tag{4.13}$$

Proof. $\mathbf{u}^1$-core$(v) = \{x \in I^*(v) \colon u_S^1 \circ e_v(S,x) \leq 0 \ \forall S \in \mathcal{A}^*\}$. Due to the equivalence of (4.12) and (4.13) this equals $\{x \in I^*(v) \colon u_S^2 \circ e_v(S,x) \leq 0 \ \forall S \in \mathcal{A}^*\} = \mathbf{u}^2$-core$(v)$. □

For example, if $\mathbf{u}$ is such that $u_S(0) = 0$ for all $S \in \mathcal{A}^*$, then the $\mathbf{u}$-core coincides with the core; therefore a game is balanced if and only if it is $\mathbf{u}$-balanced.

Example 48. In this example, we show how to find a characterization set for the percapita prenucleolus of balanced games.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game and let $\mathbf{u}$ be the percapita function, that is, $u_S \circ e_v(S,x) = \frac{e_v(S,x)}{|S|}$ for all $S \in \mathcal{A}^*$.

By Lemma 47, the $\mathbf{u}$-core coincides with the core.

Then, a coalition $S \in \mathcal{A}^*$ is $\mathbf{u}$-essential (see Definition 32), if either $\mathcal{D}_S^{\mathcal{A}^*} = \emptyset$, (if $\mathcal{A} = \mathcal{P}(N)$, these coalitions are the singletons) or if there exists $x \in \mathbf{u}$-least-core(v) such that

$$\frac{e_v(S,x)}{|S|} > \max_{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}} \sum_{T \in \mathcal{B}} \frac{e_v(T,x)}{|T|}.$$

By Theorem 43, the $\mathbf{u}$-essential (percapita-essential) coalitions form a characterization set for the percapita prenucleolus in case of balanced games.

4.3 An example

There are certain classes of games and utility functions for which there are only polynomial many $\mathbf{u}$-essential coalitions in the number of players. For example, consider the class of assignment games with the reciprocal percapita utility function $\mathbf{u}$. Our definition of the the reciprocal percapita utility function $\mathbf{u}$ is that for all $v \in \mathcal{G}^N, S \in \mathcal{P}^*(N)$ we have that $u_S \circ e_v(S,x) = |S| e_v(S,x)$. The reciprocal percapita utility function can be "interpreted" in a way that the value of a coalition is a public value for the members of the coalition. Therefore, each player's utility is the excess of the coalition; hence the total utility of the excess of the coalition is the excess of the coalition multiplied by the size of the coalition.

In case of an assignment game, there are sellers (M') and buyers (M). Each seller $j \in M'$ has a reservation value of $c_j \geq 0$ and each buyer $i \in M$ values the object of seller j to $h_{i,j} \geq 0$. If a buyer and a seller trade, they make a joint profit of $a_{i,j} = \max\{0, h_{i,j} - c_j\}$. These joint profits can be displayed in an assignment matrix A:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m'} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,m'} \end{bmatrix}$$

A matching μ' is a subset of $M \times M'$, where each agent appears in at most one pair. Let $\mathcal{M}(M, M')$ be the set of matchings.

An assignment game has the set of players $M \cup M'$ and the characteristic function w_A defined as: for all $S \subseteq M, T \subseteq M'$

$$w_A(S \cup T) = \max \left\{ \sum_{(i,j) \in \mu} a_{i,j} \colon \mu \in \mathcal{M}(S, T) \right\}.$$

The core can be described using only the matchings and the singletons in the following way:

$$\text{core}(w_A) = \Big\{ (x, y) \in \mathbb{R}^M \times \mathbb{R}^{M'} \colon \sum_{i \in M} x_i + \sum_{j \in M'} y_j = w_A(M \cup M'),$$
$$x_i + y_j \geq a_{i,j} \ \forall (i, j) \in M \times M', x_i \geq 0 \ \forall i \in M, y_j \geq 0 \ \forall j \in M' \Big\}.$$

Moreover, the core of an assignment game is non-empty (Shapley and Shubik, 1972). By Lemma 47, the core of an assignment game coincides with the **u**-core of the game in case of the reciprocal percapita utility function.

Let us find the **u**-essential coalitions in case of assignment games. The singletons are **u**-essential by definition. The mixed-pairs are **u**-essential, if $a_{i,j}$ is positive, but the other pairs are not **u**-essential. Indeed, let $i, j \in N$, $\{i, j\} \notin \mathcal{M}(M, M')$, then for each $x \in$ **u**-least-core$(w_A) \subseteq$ **u**-core(w_A) we have that $2(w_A(\{i, j\}) - x_i - x_j) \leq -x_i - x_j$, because $0 \leq x_i + x_j$.

Consider a coalition $S \in \mathcal{P}^*(N)$ with cardinality larger than two. Let $x \in$ **u**-core(w_A) and μ^* be an optimal matching for S. The left-hand side of the inequality in Definition 32 is:

$$u_S \circ e_v(S,x) = |S|(w_A(S) - x(S)) = |S| \left(\sum_{(i,j)\in\mu^*} (a_{i,j} - x_i - x_j) - \sum_{\substack{k\in N,\\ (k,\cdot)\notin\mu^*,\\ (\cdot,k)\notin\mu^*}} x_k \right) .$$

Moreover, for $\mathcal{B}^* = \mu^* \cup \{\{k\}\}_{k\in N,(k,\cdot)\notin\mu^*,(\cdot,k)\notin\mu^*}$ the right-hand side of the inequality in Definition 32 is

$$\sum_{T\in\mathcal{B}^*} u_T \circ u(T,x) = 2 \left(\sum_{(i,j)\in\mu^*} (a_{i,j} - x_i - x_j) \right) - \sum_{\substack{k\in N,\\ (k,\cdot)\notin\mu^*,\\ (\cdot,k)\notin\mu^*}} x_k \,. \tag{4.14}$$

Subtract $2(\sum_{(i,j)\in\mu^*}(a_{i,j} - x_i - x_j)) - |S| \sum_{\substack{k\in N,\\ (k,\cdot)\notin\mu^*,\\ (\cdot,k)\notin\mu^*}} x_k$ from both sides. Then we get

$$(|S| - 2) \sum_{(i,j)\in\mu^*} a_{i,j}$$

on the left-hand side, and

$$(|S| - 2) \sum_{(i,j)\in\mu^*} (x_i + x_j) + (|S| - 1) \sum_{\substack{k\in N,\\ (k,\cdot)\notin\mu^*,\\ (\cdot,k)\notin\mu^*}} x_k$$

on the right-hand side. Since $x \in$ **u**-core(w_A), it holds that $\sum_{(i,j)\in\mu^*}(x_i + x_j) \geq \sum_{(i,j)\in\mu^*} a_{i,j}$ and $\sum_{\substack{k\in N,\\ (k,\cdot)\notin\mu^*,\\ (\cdot,k)\notin\mu^*}} x_k \geq 0$. Then, it follows that

$$u_S \circ e_v(S,x) \leq \sum_{T\in\mathcal{B}^*} u_T \circ u(T,x) \,.$$

It means that there is no $x \in$ **u**-core(w_A) such that the left-hand side would be strictly larger than the right-hand side, hence S is not **u**-essential.

In conclusion, only the singletons and the positive-valued mixed-pair coalitions are **u**-essential in case of assignment games, hence, there are only quadratic many **u**-essential coalitions in the number of players.

Chapter 5

Dual games

5.1 The dual of games with utility functions

The idea of applying utility functions to games can be extended to dual games. Since the feasible coalitions of the dual game are the complementer set of the feasible coalitions of the primal game, we define the **u**-satisfaction of the dual game using the u_S functions corresponding to their complementers.

Definition 49. *The dual of a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with utility function* $\mathbf{u}$ *is the game* $v^* \in \mathcal{G}^{N,N\setminus\mathcal{A}}$ *with utility function* $\mathbf{u}^* := [u_{N\setminus S}]_{S\in N\setminus\mathcal{A}^*}$.

Notice that the dual of the dual game with a utility function is the primal game with the same utility function. This follows from the identities $v^{**} = v$ and $\mathbf{u}^{**} = ([u_{N\setminus S}]_{S\in N\setminus\mathcal{A}^*})^* = [u_{N\setminus(N\setminus S)}]_{S\in N\setminus(N\setminus\mathcal{A}^*)} = [u_S]_{S\in\mathcal{A}^*}$. Thus, applying the dual operation twice restores both the original game and the original utility function.

Example 50. Consider the following game with the percapita utility function: $N = \{1,2,3\}$,

$$v(S) = \begin{cases} 1, & \text{if } |S| = 1, \\ 2, & \text{if } |S| = 2, \\ 6, & \text{if } S = N \end{cases}$$

and $\mathcal{A}^* = \{\{1\},\{2\},\{1,2\},\{1,3\}\}$.

Then, $N \setminus \mathcal{A}^* = \{\{2\},\{3\},\{1,3\},\{2,3\}\}$, $v^*(\{2\}) = v^*(\{3\}) = 6 - 2 = 4$, $v^*(\{1,3\}) = v^*(\{2,3\}) = 6 - 1 = 5$ and $v^*(N) = 6 - 0 = 6$.

Let $\mathbf{u}$ be the percapita utility function, that is $u_S(t) = \frac{t}{|S|}$ for all $S \in \mathcal{A}^*$. Therefore, $u^*_{\{3\}}(t) = u^*_{\{2\}}(t) = \frac{t}{2}$. Similarly, $u^*_{\{1,3\}}(t) = u^*_{\{2,3\}}(t) = \frac{t}{1}$.

Definition 51. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$. *Then the* $\mathbf{u}^*$*-satisfaction of a coalition* $S \in N \setminus \mathcal{A}$ *by the payoff vector* $x \in \mathbb{R}^N$ *in the dual game* v^* *is* $u_{N\setminus S} \circ f_{v^*}(S,x)$, *and the* $\mathbf{u}^*$*-satisfaction vector is* $F_{v^*}^{\mathbf{u}^*}(x) := [\ldots \geq u_{N\setminus S} \circ f_{v^*}(S,x) \geq \ldots : S \in N \setminus \mathcal{A}^*]$.

The $\mathbf{u}^*$-anti-nucleolus and $\mathbf{u}^*$-anti-prenucleolus is defined with the $\mathbf{u}^*$-satisfaction vectors:

Definition 52. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$. *Then the* $\mathbf{u}^*$*-anti-nucleolus of* v^* *is the set of* $\mathbf{u}^*$*-anti-imputations, which lexicographically minimize the* $\mathbf{u}^*$*-satisfaction vectors over the set of* $\mathbf{u}^*$*-anti-imputations, that is,*

$$\text{anti-}N_{\mathbf{u}^*}(v^*) = \left\{x \in \mathbf{u}^*\text{-anti-}I(v^*) \colon F_{v^*}^{\mathbf{u}^*}(x) \leq_L F_{v^*}^{\mathbf{u}^*}(y),\ \forall y \in \mathbf{u}^*\text{-anti-}I(v^*)\right\},$$

where $\mathbf{u}^*\text{-anti-}I(v^*) := \{x \in I^*(v^*) \colon u_{\{i\}} \circ f_{v^*}(N \setminus \{i\}, x) \leq 0,\ \forall N \setminus \{i\} \in N \setminus \mathcal{A}^*\}$.

Moreover, the $\mathbf{u}^*$*-anti-prenucleolus of* v^* *is the set of preimputations which lexicographically minimize the* $\mathbf{u}^*$*-satisfaction vectors over the set of preimputations, that is,*

$$\text{anti-}N_{\mathbf{u}^*}^*(v^*) = \left\{x \in I^*(v^*) \colon F_{v^*}^{\mathbf{u}^*}(x) \leq_L F_{v^*}^{\mathbf{u}^*}(y),\ \forall y \in I^*(v^*)\right\}.$$

Similarly, the $\mathbf{u}^*$-anti-core of the dual game is also defined with the $\mathbf{u}^*$-satisfactions:

Definition 53. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *and a utility function* $\mathbf{u}$. *Then the* $\mathbf{u}^*$*-anti-core of the dual game* v^* *is the set of preimputations for which the* $\mathbf{u}^*$*-satisfactions are non-positive:*

$$\mathbf{u}^*\text{-anti-core}(v^*) = \left\{x \in I^*(v^*) \colon u_{N\setminus S} \circ f_{v^*}(S,x) \leq 0,\ \forall S \in N \setminus \mathcal{A}^*\right\}.$$

If the $\mathbf{u}^*$-anti-core of a game is not empty then we say that the game is $\mathbf{u}^*$-anti-balanced.

Example 54. Consider the game described in Example 50. $\mathbf{u}^*\text{-anti-core}(v^*) = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6, \frac{x_2-4}{2} \leq 0, \frac{x_3-4}{2} \leq 0, \frac{x_2+x_3-5}{1} \leq 0, \frac{x_1+x_3-5}{1} \leq 0\} = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6, x_2 \leq 4, x_3 \leq 4, x_2 + x_3 \leq 5, x_1 + x_3 \leq 5\}$.

Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$. The $\mathbf{u}^*$-anti-ε-core of the dual game is

$$\mathbf{u}^*\text{-anti-core}(v^*)_\varepsilon = \{x \in I^*(v^*) \colon \max_{S \in N\setminus\mathcal{A}^*} u_{N\setminus S} \circ f_{v^*}(S,x) \leq \varepsilon\}.$$

In addition, the $\mathbf{u}^*$-anti-least-core of the dual game is its smallest nonempty $\mathbf{u}^*$-anti-ε-core, provided it exists.

Similar relations hold for solutions of TU-games with utility functions as for TU-games:

Theorem 55. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$. Then the following hold:*

1. $N_{\mathbf{u}}(v) = \text{anti-}N_{\mathbf{u}^*}(v^*)$,
2. $N^*_{\mathbf{u}}(v) = \text{anti-}N^*_{\mathbf{u}^*}(v^*)$,
3. $\mathbf{u}\text{-core}(v) = \mathbf{u}^*\text{-anti-core}(v^*)$,
4. $\mathbf{u}\text{-least-core}(v) = \mathbf{u}^*\text{-least-anti-core}(v^*)$.

Proof. For every $S \in \mathcal{A}, x \in I^*(v)$ it holds that

$$\begin{aligned} u_S \circ f_{v^*}(N \setminus S, x) &= u_S \circ (x(N \setminus S) - v^*(N \setminus S)) \\ &= u_S \circ (x(N) - x(S) - (v(N) - v(S))) \\ &= u_S \circ (v(S) - x(S)) = u_S \circ e_v(S, x). \end{aligned}$$

3: Then it follows

$$\begin{aligned} \mathbf{u}\text{-core}(v) &= \{x \in I^*(v) \colon u_S \circ e_v(S, x) \leq 0,\ \forall S \in \mathcal{A}^*\} \\ &= \{x \in I^*(v^*) \colon u_{N \setminus S} \circ f_{v^*}(S, x) \leq 0,\ \forall S \in N \setminus \mathcal{A}^*\} \\ &= \mathbf{u}^*\text{-anti-core}(v^*)\,. \end{aligned}$$

4: Similarly, for any $\alpha \in \mathbb{R}$

$$\begin{aligned} \mathbf{u}\text{-core}_\varepsilon(v) &= \{x \in I^*(v) \colon \max_{S \in \mathcal{A}^*} u_S \circ e_v(S, x) \leq \varepsilon\} \\ &= \{x \in I^*(v) \colon \max_{S \in \mathcal{A}^*} u_S \circ f_{v^*}(N \setminus S, x) \leq \varepsilon\} \\ = \{x \in I^*(v) \colon &\max_{S \in N \setminus \mathcal{A}^*} u_{N \setminus S} \circ f_{v^*}(S, x) \leq \varepsilon\} = \mathbf{u}^*\text{-anti-core}_\varepsilon(v^*)\,. \end{aligned}$$

Therefore, $\mathbf{u}\text{-least-core}(v) = \mathbf{u}^*\text{-least-anti-core}(v^*)$.

1-2: $v^*(N) = v(N) - v(\emptyset) = v(N)$, therefore $I^*(v) = I^*(v^*)$ and $\mathbf{u}\text{-}I(v) = \mathbf{u}^*\text{-anti-}I(v^*)$.

Since $u_S \circ f_{v^*}(N \setminus S, x) = u_S \circ e_v(S, x)$ for all $S \in \mathcal{A}, x \in I^*(v)$ we have that $E^{\mathbf{u}}_v(x) = F^{\mathbf{u}^*}_{v^*}(x)$ for all $x \in I^*(v)$. Therefore, $N_{\mathbf{u}}(v) = \text{anti-}N_{\mathbf{u}^*}(v^*)$ and $N^*_{\mathbf{u}}(v) = \text{anti-}N^*_{\mathbf{u}^*}(v^*)$. □

The following theorem is a consequence of Theorems 13 and 55.

Theorem 56. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ and a utility function $\mathbf{u}$. The $\mathbf{u}^*$-anti-least-core of the dual game is well-defined if and only if $\mathcal{A}^*$ contains a balanced set system.*

Example 57. Consider the game described in Example 50. $\mathbf{u}$-core$(v) = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6, \frac{1-x_1}{1} \leq 0, \frac{1-x_2}{1} \leq 0, \frac{2-x_1-x_3}{2} \leq 0, \frac{2-x_1-x_2}{2} \leq 0\} = \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = 6, x_1 \geq 1, x_2 \geq 1, x_1 + x_3 \geq 2, x_1 + x_2 \geq 2\}$, which equals $\mathbf{u}^*$-anti-core(v^*) calculated in Example 54.

5.2 The lexicographic center method for the u-anti-prenucleolus

Later we will use the modified version of the lexicographic center method (Kopelowitz, 1967; Maschler et al, 1979) for the $\mathbf{u}^*$-anti-prenucleolus, hence we discuss it shortly.

Let $v \in \mathcal{G}^{N,\mathcal{A}}$ be a game with a utility function $\mathbf{u}$. The modified lexicographic center algorithm for calculating the $\mathbf{u}$-prenucleolus of game v is described in Section 3.3.

Similarly, the modified lexicographic center algorithm for calculating the $\mathbf{u}^*$-anti-prenucleolus of the dual game v^* solves the following optimization problems iteratively:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_{N \setminus S} \circ f_{v^*}(S, x) \leq t, \quad S \in (N \setminus \mathcal{A}^*) \setminus (\cup_{r=0}^{k-1} W_r^d) \\ & x \in X_{k-1}^d \\ & t \in R_{\mathbf{u}}, \end{aligned} \tag{5.1}$$

where $X_0^d := I^*(v^*)$ and $W_0^d := \emptyset$ and for $k \geq 1$ t_k^d denotes the optimum of (5.1) if it exists, and

$$X_k^d := \{x \in X_{k-1}^d \colon u_{N \setminus S} \circ f_{v^*}(S, x) \leq t_k^d, \ \forall S \in (N \setminus \mathcal{A}^*) \setminus (\cup_{r=0}^{k-1} W_r^d)\},$$
$$W_k^d := \{S \in N \setminus \mathcal{A}^* \colon \exists c_S \in \mathbb{R}, \text{ such that } u_{N \setminus S} \circ f_{v^*}(S, x) = c_S, \ \forall x \in X_k^d\}.$$

It is easy to see that $t_k^d \geq t_{k+1}^d$ and $X_k^d \supseteq X_{k+1}^d$ for all $k \in \mathbb{N}_+$, and there exists k^* such that for all $l \geq k^*$ it holds that $X_{k^*}^d = X_l^d$. Since for every $x \in I^*(v), S \in \mathcal{A}^*$ $u_S \circ e_v(S, x) = u_S \circ f_{v^*}(N \setminus S, x)$, we have that $t_i^d = t_i$ and $X_i^d = X_i$ for all $i \in \mathbb{N}_+$. In addition, we learned in Theorem 55, that the $\mathbf{u}$-prenucleolus of the primal game equals the $\mathbf{u}^*$-anti-prenucleolus of the dual game. Therefore, we can apply the result of Maschler et al (1992), saying, that $X_{k^*}^d = \text{anti-} N_{\mathbf{u}^*}^*(v^*)$.

5.3 The $\mathbf{u}^*$-anti-essential and dually-u-essential coalitions

In this section, we discuss the notion of $\mathbf{u}^*$-anti-essential coalitions and show that those form a characterization set of the $\mathbf{u}^*$-anti-prenucleolus for games having non-empty $\mathbf{u}^*$-anti-core. Finally, using the results for $\mathbf{u}^*$-anti-essential coalitions, we show that dually-$\mathbf{u}$-essential coalitions form a characterization set for the $\mathbf{u}$-prenucleolus of $\mathbf{u}$-balanced games.

In order to help the reader in following the interdependence of the upcoming results, we have constructed a graph in Figure 5.1.

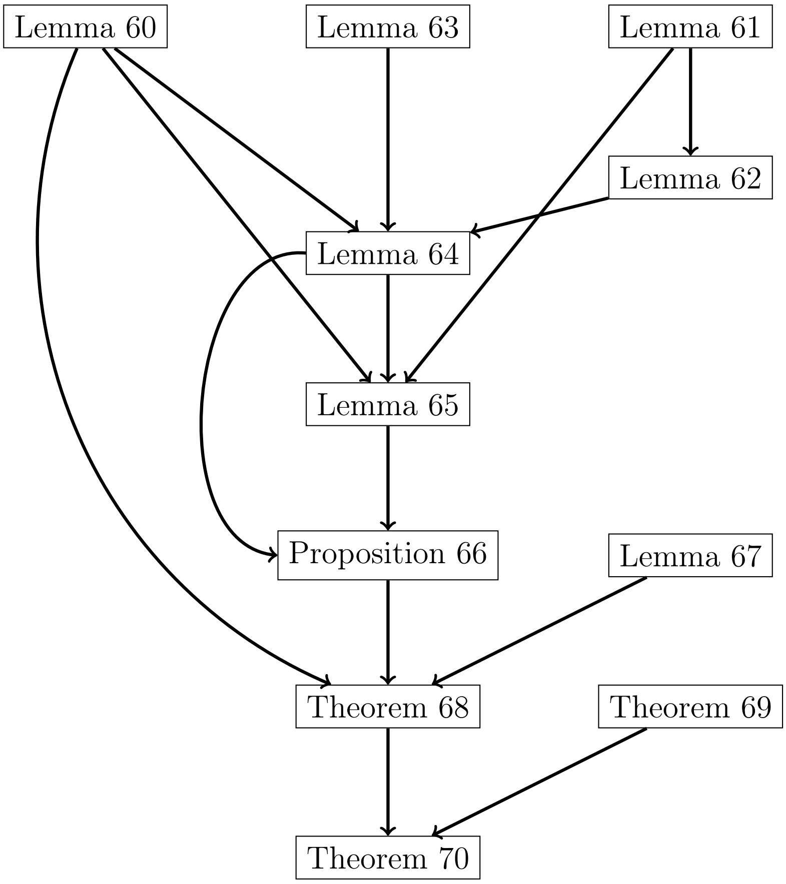


Figure 5.1: The relationships of the results of Section 5.3

Definition 58. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system with a utility function $\mathbf{u}$. Then a coalition $S \in N \setminus \mathcal{A}^*$ is $\mathbf{u}^*$-anti-essential if $\mathcal{D}_S^{N\setminus\mathcal{A}^*} = \emptyset$ or if there exists $x \in \mathbf{u}^*\text{-least-anti-core}(v^*)$ such that*

$$u_{N\setminus S} \circ f_{v^*}(S,x) > \max_{\mathcal{B}\in\mathcal{D}_S^{N\setminus\mathcal{A}^*}} \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)\,.$$

The complementer set of the class of $\mathbf{u}^*$*-anti-essential coalitions (of game* v^**) is called the class of dually-*$\mathbf{u}$*-essential coalitions (of game* v*). Let* $\mathcal{E}_{v^*}^{a-\mathbf{u}^*}$ *denote the class of* $\mathbf{u}^*$*-anti-essential coalitions (of* v^**) and* $\mathcal{E}_v^{d-\mathbf{u}}$ *denote the class of dually-*$\mathbf{u}$*-essential coalitions (of* v*).*

In words, the dually-$\mathbf{u}$-essential coalitions of a game are the complements of the $\mathbf{u}^*$-anti-essential coalitions of the dual of the game.

Example 59. Consider an additive game with the pecapita utility function $\mathbf{u}$, where $N = \{1, 2, 3\}$, $\mathcal{A}^* = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$ and $v(\{i\}) = 1$ for all $\{i\} \in \mathcal{A}^*$.

Then, $\mathbf{u}$-core$(v) = \mathbf{u}^*$-anti-core$(v^*) = \{(1, 1, 1)\}$ and $N \setminus \mathcal{A}^* = \{\{2\}, \{3\}, \{2, 3\}, \{1, 3\}\}$. Therefore, the $\mathbf{u}^*$-anti-essential coalitions are $\mathcal{E}_{v^*}^{a-\mathbf{u}^*} = \{\{2\}, \{3\}, \{1, 3\}\}$, since for all three of these coalitions $\mathcal{D}_S^{N\setminus\mathcal{A}^*} = \emptyset$. $\{2, 3\}$ is not $\mathbf{u}^*$-anti-essential, because for all $x \in \mathbf{u}^*$-anti-core(v^*) $\frac{x_2+x_3-v^*(\{2,3\})}{1} = \frac{x_2+x_3-2}{1} = 0$ and $\mathcal{D}_{\{2,3\}}^{N\setminus\mathcal{A}^*} = \{\{\{2\}, \{3\}\}\}$; therefore, $\max_{\mathcal{B}\in\mathcal{D}_{\{2,3\}}^{N\setminus\mathcal{A}^*}} \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T, x) = \frac{x_2-1}{2} + \frac{x_3-1}{2} = 0$.

The dually-$\mathbf{u}$-essential coalitions of v are the complements of the anti-$\mathbf{u}^*$-essential coalitions of v^*; therefore, $\mathcal{E}_v^{d-\mathbf{u}} = \{\{2\}, \{1, 3\}, \{1, 2\}\}$.

Note, that the dually-$\mathbf{u}$-essential coalitions do not coincide with the $\mathbf{u}$-essential coalitions (Definition 32), which are $\mathcal{E}_v^{\mathbf{u}} = \{\{1\}, \{2\}, \{1, 3\}\}$.

At the end of this section, we prove that dually-$\mathbf{u}$-essential coalitions form a characterization set for the $\mathbf{u}$-prenucleolus of $\mathbf{u}$-balanced games. For this result, first we need the following lemmata.

The following lemmata are the dual equivalents of Lemmata 34-42. The ideas of the proofs are similar to those in Section 4.1; however, we need to check, whether these lemmata also hold in the dual scenario, since the previous proofs are not enough in themselves to prove these lemmata.

In particular, Lemmata 60, 61, 62, 63, 64, 65, 67, Proposition 66 and Theorem 68 are the dual equivalents of Lemmata 34, 35, 36, 37, 38, 39, 42, Proposition 41 and Theorem 43 respectively.

These Lemmata, Propositions and Theorems show the similarities between the two settings (the $\mathbf{u}$-prenucleolus and $\mathbf{u}$-essential coalitions of the primal game and the $\mathbf{u}^*$-anti-prenucleolus and $\mathbf{u}^*$-anti-essential coalitions of the dual game). By using these notions, the swift from primal to dual games is quite straightforward; even though, the $\mathbf{u}$-essential coalitions do not coincide with the $\mathbf{u}^*$-anti-essential coalitions nor with their complementers, the dually-$\mathbf{u}$-essential coalitions, as shown in Example 59.

Lemma 60. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system with a utility function* $\mathbf{u}$*. Let* $S \in N \setminus \mathcal{A}^*$ *be a non* $\mathbf{u}^*$*-anti-essential coalition (of*

v^*). Then for every $x \in \mathbf{u}^*$-least-anti-core(v^*) there exists $\mathcal{B}^* \in \mathcal{D}_S^{N\setminus\mathcal{A}^*}$ such that $u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T\in\mathcal{B}^*} u_{N\setminus T} \circ f_{v^*}(T,x)$ and $\mathcal{B}^* \subseteq \mathcal{E}_{v^*}^{a-\mathbf{u}^*}$.

Proof. Since S is not $\mathbf{u}^*$-anti-essential, by definition for each $x \in \mathbf{u}^*$-least-anti- core(v*) there exists $\mathcal{B} \in \mathcal{D}_S^{N\setminus\mathcal{A}^*}$ such that $u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)$. For each $x \in \mathbf{u}^*$-least-anti-core(v^*) let $\mathbf{B}(x) := \{\mathcal{B} \in \mathcal{D}_S^{N\setminus\mathcal{A}^*} : u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)\}$, and let $\mathcal{B}^* \in \mathbf{B}(x)$ be such that for every partition $\mathcal{B} \in \mathbf{B}(x)$ it holds that $|\mathcal{B}^*| \geq |\mathcal{B}|$.

Indirectly assume that there exists a coalition $T^* \in \mathcal{B}^*$ such that it is not $\mathbf{u}^*$-anti-essential. Since T^* is not $\mathbf{u}^*$-anti-essential, by Definition 58 $\exists \mathcal{B}' \in \mathcal{D}_{T^*}^{N\setminus\mathcal{A}^*}$ such that $u_{N\setminus T^*} \circ f_{v^*}(T^*,x) \leq \sum_{T'\in\mathcal{B}^*} u_{N\setminus T'} \circ f_{v^*}(T',x)$. Therefore,

$$u_{N\setminus S} \circ f_{v^*}(S,x) \leq \sum_{T'\in(\mathcal{B}^*\setminus\{T^*\})\cup\mathcal{B}'} u_{N\setminus T'} \circ f_{v^*}(T',x),$$

and $|(\mathcal{B}^* \setminus \{T\}) \cup \mathcal{B}'| > |\mathcal{B}^*|$, which is a contradiction. □

Consider the following notation: for a class of coalitions $\mathcal{S} \subseteq \mathcal{A}^*$ and $t \in \mathbb{R}$ let $X^d(\mathcal{S},t) := \{x \in I^*(v) \colon u_{N\setminus S} \circ f_{v^*}(S,x) \leq t, \forall S \in \mathcal{S}\}$.

Before proving the main theorem of this section, we need to prove that the $\mathbf{u}^*$-anti-essential coalitions characterize the $\mathbf{u}^*$-least-anti-core. For this result we need Lemmata 61, 62 and 63. The ideas behind these lemmata are similar to the ideas behind Lemmata 35, 36 and 37.

Lemma 61. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system with a utility function $\mathbf{u}$, and $t', t'' \in \mathbb{R}$ such that $t' \leq t''$. Then $X^d(\mathcal{S},t') \subseteq X^d(\mathcal{S},t'')$.*

Proof. For every $x \in X^d(\mathcal{S},t')$ it holds that $x \in I^*(v^*)$, and for all $S \in \mathcal{S}$:

$$u_{N\setminus S} \circ f_{v^*}(S,x) \leq t' \leq t'' \ .$$

Therefore, $x \in X^d(\mathcal{S},t'')$. □

Lemma 62. *Consider a utility function $\mathbf{u}$ and $\mathbf{u}$-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system. Then for every $t_1^d \leq t$ it holds that both $X^d(N\setminus\mathcal{A}^*,t)$ and $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*},t)$ are nonempty, convex and closed, where t_1^d is from Problem 5.1.*

Proof. By Lemma 61 $X^d(N\setminus\mathcal{A}^*,t) \supseteq X^d(N\setminus\mathcal{A}^*,t_1^d) = \mathbf{u}^*$-least-anti-core$(v^*)$. We know that $\mathbf{u}^*$-least-anti-core$(v^*) \neq \emptyset$, hence $X^d(N\setminus\mathcal{A}^*,t) \neq \emptyset$, and $X^d(N\setminus\mathcal{A}^*,t) \subseteq X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*},t)$, because $\mathcal{E}_{v^*}^{a-\mathbf{u}^*} \subseteq N\setminus\mathcal{A}^*$. Therefore, $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*},t) \neq \emptyset$ as well.

Let $H_S := \{x \in \mathbb{R}^N \colon u_{N\setminus S} \circ f_{v*}(S,x) \leq t\}$, for all $S \in N\setminus\mathcal{A}^*$. Then $H_S = \{x \in \mathbb{R}^N \colon f_{v^*}(S,x) \leq u_{N\setminus S}^{-1}(t)\}$, hence H_S is a closed half-space, therefore, it is convex and closed. $X_0^d = I^*(v) = \{x \in \mathbb{R}^N : x(N) = v^*(N)\}$ is a hyperplane, therefore it is convex and closed. Finally,

$$X^d(N \setminus \mathcal{A}^*, t) = \{x \in I^*(v^*) : u_{N\setminus S} \circ f_{v^*}(S, x) \leq t \ \forall S \in N \setminus \mathcal{A}^*\}$$
$$= \left(\cap_{S \in N\setminus \mathcal{A}^*} H_S\right) \cap X_0^d \ ,$$

hence $X^d(N \setminus \mathcal{A}^*, t)$ is the intersection of convex closed sets, therefore, it is convex and closed.

Similarly, $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t) = \left(\cap_{S \in \mathcal{E}_v^{\mathbf{u}}} H_S\right) \cap X_0^d$ is the intersection of convex closed sets, hence it is convex and closed. □

Lemma 63. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$, and take $x_1, x_2 \in \mathbb{R}^N$, $S \in N \setminus \mathcal{A}$. If $u_{N\setminus S} \circ f_{v^*}(S, x_1) < u_{N\setminus S} \circ f_{v^*}(S, x_2)$ then for every $\lambda \in (0,1)$ it holds that*

$$u_{N\setminus S} \circ f_{v^*}(S, x_1) < u_{N\setminus S} \circ f_{v^*}(S, \lambda x_1 + (1-\lambda)x_2) < u_{N\setminus S} \circ f_{v^*}(S, x_2)\,.$$

Proof. Let $\lambda \in (0,1)$, then

$$(\lambda x_1 + (1-\lambda)x_2)(S) = \lambda x_1(S) + (1-\lambda)x_2(S).$$

Therefore,

$$f_{v^*}(S, \lambda x_1 + (1-\lambda)x_2) = (\lambda x_1 + (1-\lambda)x_2)(S) - v^*(S)$$
$$= \lambda x_1(S) + (1-\lambda)x_2(S) - \lambda v^*(S) - (1-\lambda)v^*(S)$$
$$= \lambda f_{v^*}(S, x_1) + (1-\lambda) f_{v^*}(S, x_2)\,.$$

Since $u_{N\setminus S}$ is strictly monotone increasing we have that

$$u_{N\setminus S} \circ f_{v^*}(S, x_1) < u_{N\setminus S} \circ f_{v^*}(S, \lambda x_1 + (1-\lambda)x_2) < u_{N\setminus S} \circ f_{v^*}(S, x_2)\,.$$

□

The following lemma provides the main step in proving that the $\mathbf{u}^*$-anti-essential coalitions characterize the $\mathbf{u}^*$-least-anti-core.

Lemma 64. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system with a utility function $\mathbf{u}$. If v is $\mathbf{u}$-balanced, then $X^d(N \setminus \mathcal{A}^*, t_1^d) = X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t_1^d)$.*

Proof. Since $\mathcal{E}_{v^*}^{a-\mathbf{u}^*} \subseteq N \setminus \mathcal{A}^*$, it holds that $X^d(N \setminus \mathcal{A}^*, t_1^d) \subseteq X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t_1^d)$.

Indirectly assume that $\exists x^1 \in X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t_1^d) \setminus X^d(N \setminus \mathcal{A}^*, t_1^d)$. This means that $\exists S \in N \setminus \mathcal{A}^*$ such that

$$u_{N\setminus S} \circ f_{v^*}(S, x^1) > t_1^d \,. \tag{5.2}$$

Let $\mathcal{S}_{x^1} := \{S \in N \setminus \mathcal{A}^* \colon u_{N\setminus S} \circ f_{v^*}(S, x^1) > t_1^d\}$. Then for all $S \in \mathcal{S}_{x^1}$ it holds that $S \notin \mathcal{E}_{v^*}^{a-\mathbf{u}^*}$.

Let $x^* \in X^d(N \setminus \mathcal{A}^*, t_1^d)$ be the closest point of set $X^d(N \setminus \mathcal{A}^*, t_1^d)$ to the point x^1. It is clear that such x^* exists, since $X^d(N \setminus \mathcal{A}^*, t_1^d)$ is nonempty and closed by Lemma 62.

Since $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t_1^d)$ is a convex set, for every $\lambda \in [0,1]$ it holds that $\lambda x^* + (1-\lambda)x^1 \in X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t_1^d)$.

By Lemma 60 we have that for each $S \in \mathcal{S}_{x^1}$ there exists $\mathcal{B}_S^* \subseteq \mathcal{E}_{v^*}^{a-\mathbf{u}^*}$, $\mathcal{B}_S^* \in \mathcal{D}_S^{N\setminus\mathcal{A}^*}$ such that

$$u_{N\setminus S} \circ f_{v^*}(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_{N\setminus T} \circ f_{v^*}(T, x^*) \,. \tag{5.3}$$

Since $t_1^d \leq 0$ and for all $x \in X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*}, t_1^d)$ and $T \in \mathcal{E}_{v^*}^{a-\mathbf{u}^*}$ it holds that $u_{N\setminus T} \circ f_{v^*}(T, x) \leq t_1^d$, therefore for all $S \in \mathcal{S}_{x^1}$

$$u_{N\setminus S} \circ f_{v^*}(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_{N\setminus T} \circ f_{v^*}(T, x^*) \leq t_1^d \,. \tag{5.4}$$

By (5.2), (5.4) and the continuity of $u_{N\setminus S}$, for every $S \in \mathcal{S}_{x^1}$ there exists $\lambda_S \in [0,1]$ such that

$$u_{N\setminus S} \circ f_{v^*}(S, \lambda_S x^* + (1-\lambda_S)x^1) = t_1^d \,.$$

Let $S^1 \in \operatorname{argmin}_{S \in \mathcal{S}_{x^1}} ||x^* - (\lambda_S x^* + (1-\lambda_S)x^1))||$, and let $x^2 = \lambda_{S^1} x^* + (1-\lambda_{S^1})x^1$. Then

$$u_{N\setminus S^1} \circ f_{v^*}(S^1, x^2) \geq \sum_{T \in \mathcal{B}_{S^1}^*} u_{N\setminus T} \circ f_{v^*}(T, x^2) \,,$$

because $t_1^d \leq 0$ and $u_{N\setminus T} \circ f_{v^*}(T, x^2) \leq t_1^d$, for all $T \in \mathcal{B}_{S^1}^*$.

Then there are two cases:

Case 1:

$$u_{N\setminus S^1} \circ f_{v^*}(S^1, x^2) = \sum_{T \in \mathcal{B}_{S^1}^*} u_{N\setminus T} \circ f_{v^*}(T, x^2) \,.$$

In this case, $t_1^d = u_{N\setminus S^1} \circ f_{v^*}(S^1, x^2) = \sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T} \circ f_{v^*}(T, x^2) \leq |\mathcal{B}^*_{S^1}| t_1^d \leq t_1^d$; therefore $t_1^d = 0$.

Then $\sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T} \circ f_{v^*}(T, x^2) = 0$, hence for all $T \in \mathcal{B}^*_{S^1}$ it holds that $u_{N\setminus T} \circ f_{v^*}(T, x^2) = 0$.

We know that $u_{N\setminus S^1} \circ f_{v^*}(S^1, x^1) > t_1^d \geq \sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T} \circ f_{v^*}(T, x^1)$ and that $x^2(S^1) = \sum_{T\in\mathcal{B}^*_{S^1}} x^2(T)$ and $x^1(S^1) = \sum_{T\in\mathcal{B}^*_{S^1}} x^1(T)$. Then $x^1(S^1) > x^2(S^1)$ and $\exists T' \in \mathcal{B}^*_{S^1}$ such that $x^1(T') > x^2(T')$. However, $u_{N\setminus T'}$ is a strictly monotone increasing function, hence we have that $u_{N\setminus T'} \circ f_{v^*}(T', x^1) > t_1^d$, which is a contradiction because $x^1 \in X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t_1^d)$.

Case 2:

$$u_{N\setminus S^1} \circ f_{v^*}(S^1, x^2) > \sum_{T\in\mathcal{B}^*_{S^1}} u_{N\setminus T} \circ f_{v^*}(T, x^2)\,. \tag{5.5}$$

Since for all $S \in (N \setminus \mathcal{A}^*) \setminus \mathcal{S}_{x_1}$ it holds that $u_{N\setminus S} \circ f_{v^*}(S, x^1) \leq t_1^d$, by Lemma 63

$$u_{N\setminus S} \circ f_{v^*}(S, x^2) \leq t_1^d\,.$$

This means that for all $S \in N \setminus \mathcal{A}^*$ it holds that $u_{N\setminus S} \circ f_{v^*}(S, x^2) \leq t_1^d$, hence $x^2 \in X^d(N \setminus \mathcal{A}^*, t_1^d)$.

Since $x^* \in X^d(N \setminus \mathcal{A}^*, t_1^d)$ is the closest point of set $X^d(N \setminus \mathcal{A}^*, t_1^d)$ to the point x^1, we have that $x^2 = x^*$. However, then (5.5) contradicts (5.3). □

We need the following lemma to show that not only $X^d(N\setminus\mathcal{A}^*, t_1^d) = X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t_1^d)$ (see Lemma 64), but also $t_1^d = \min\{t\colon X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t) \neq \emptyset\}$.

Lemma 65. *Consider a utility function* $\mathbf{u}$*, a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system and* $t \in \mathbb{R}$*. If* $t < t_1^d$ *then* $X^d(N \setminus \mathcal{A}^*, t) = X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t) = \emptyset$*.*

Proof. By the definition of t_1^d (see Problem 5.1), we have that $X^d(N \setminus \mathcal{A}^*, t) = \emptyset$ and we know that $X^d(N \setminus \mathcal{A}^*, t) \subseteq X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t)$.

By Lemma 61, since $t < t_1^d$, it follows that $X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t) \subseteq X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t_1^d)$. Furthermore, by Lemma 64, we have $X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t_1^d) = X^d(N\setminus\mathcal{A}^*, t_1^d) = \mathbf{u}^*$-least-anti-core$(v^*)$. Therefore,

$$X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t) \subseteq X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t_1^d) = X^d(N \setminus \mathcal{A}^*, t_1^d) = \mathbf{u}^*\text{-least-anti-core}(v^*)\,.$$

Then for every $x \in X^d(\mathcal{E}^{a-\mathbf{u}^*}_{v^*}, t)$ and for every $S \in (N \setminus \mathcal{A}^*) \setminus \mathcal{E}^{a-\mathbf{u}^*}_{v^*}$ by Lemma 60 and by the non-positivity of t, it holds that $\exists \mathcal{B} \in \mathcal{D}^{N\setminus\mathcal{A}^*}_S$, $\mathcal{B} \subseteq \mathcal{E}^{a-\mathbf{u}^*}_{v^*}$ such that

$$u_{N\setminus S}\circ f_{v^*}(S,x)\leq\sum_{T\in\mathcal{B}}u_{N\setminus T}\circ f_{v^*}(T,x)\leq t\,.$$

Then $x\in X^d(N\setminus\mathcal{A}^*,t)$, that is, $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*},t)\subseteq X^d(N\setminus\mathcal{A}^*,t)$. Summing up, we can conclude that $X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*},t)=X^d(N\setminus\mathcal{A}^*,t)=\emptyset$. □

The following proposition is a corollary of Lemmata 64 and 65.

Proposition 66. *Consider a utility function* $\mathbf{u}$*, and a* $\mathbf{u}$*-balanced game* $v\in\mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system. Let* $t_1^{d'}:=\min\{t\colon X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*},t)\neq\emptyset\}$*, and* $X_1^{d'}:=X^d(N\setminus\mathcal{A}^*,t_1^{d'})$*. Then the following hold:* $t_1^d=t_1^{d'}$ *and* $X_1^d=X_1^{d'}$*.*

Proof. We know that $t_1^{d'}\leq t_1^d$. Therefore, by Lemmata 64 and 65 we have that $X_1^{d'}=X^d(\mathcal{E}_{v^*}^{a-\mathbf{u}^*},t_1^{d'})=X^d(N\setminus\mathcal{A}^*,t_1^{d\prime})$. However, $X^d(N\setminus\mathcal{A}^*,t_1^{d'})\neq\emptyset$ if and only if $t_1^{d'}\geq t_1^d$, hence $t_1^d=t_1^{d'}$ and $X_1^d=X_1^{d'}$. □

Before proving Theorem 68, we need one more lemma.

Lemma 67. *Consider a utility function* $\mathbf{u}$*, and a* $\mathbf{u}$*-balanced game* $v\in\mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system. Consider Problem* (5.1)*, and let* k *be an arbitrary positive integer. Then for every* $S\in(N\setminus\mathcal{A}^*)\setminus(\cup_{r=0}^{k-1}W_r^d)$ *such that* $\mathcal{D}_S^{N\setminus\mathcal{A}^*}\neq\emptyset$*, and for every* $\mathcal{B}^*\in\mathcal{D}_S^{N\setminus\mathcal{A}^*}$ *there exists* $T^*\in\mathcal{B}^*$ *such that* $T^*\notin\cup_{r=0}^{k-1}W_r^d$*.*

Proof. $k=1$: Then $\cup_{r=0}^{k-1}W_r^d=\emptyset$, hence for every $\mathcal{B}^*\in\mathcal{D}_S^{N\setminus\mathcal{A}^*}$ it holds that $T^*\notin\cup_{r=0}^{k-1}W_r^d$, for all $T^*\in\mathcal{B}^*$.

$k\geq 2$: Indirectly assume that for every $\mathcal{B}^*\in\mathcal{D}_S^{N\setminus\mathcal{A}^*}$ it holds that $\mathcal{B}^*\subseteq\cup_{r=0}^{k-1}W_r^d$. Then for every $\mathcal{B}^*\in\mathcal{D}_S^{N\setminus\mathcal{A}^*}$ and $x\in X_{k-1}^d$

$$\begin{aligned}u_{N\setminus S}\circ f_{v^*}(S,x)&=u_{N\setminus S}\circ(x(S)-v^*(S))=u_{N\setminus S}\circ\left(\sum_{T\in\mathcal{B}^*}x(T)-v^*(S)\right)\\&=u_{N\setminus S}\circ\left(\sum_{T\in\mathcal{B}^*}(u_{N\setminus T}^{-1}(c_T)+v^*(T))-v^*(S)\right),\end{aligned}$$

where c_T is a constant such that $u_{N\setminus T}\circ f_{v^*}(T,x)=c_T$, for all $x\in X_l^d$, where $T\in W_l^d$.

Therefore, for each $x,x'\in X_{k-1}^d$ it holds that $u_{N\setminus S}\circ f_{v^*}(S,x)=u_{N\setminus S}\circ f_{v^*}(S,x')$ meaning that $S\in\cup_{r=0}^{k-1}W_r^d$, which is a contradiction. □

The following theorem states that the $\mathbf{u}^*$-anti-essential coalitions form a characterization set for the $\mathbf{u}^*$-anti-prenucleolus of v^*, if v is $\mathbf{u}$-balanced.

Theorem 68. *Consider a utility function* $\mathbf{u}$*, and a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system; moreover, for* $k \geq 1$ *let*

$$Y_k^d := \{x \in X_{k-1}^d \colon u_{N \setminus S} \circ f_{v^*}(S,x) \leq t_k^d \ \forall S \in \mathcal{E}_{v^*}^{a-\mathbf{u}^*}\},$$

where t_k^d *is the optimum of* (5.1)*, if it exists and* $-\infty$ *otherwise. Then* $X_k^d = Y_k^d$ *for all* $k \geq 1$*. In other words, the* $\mathbf{u}^*$*-anti-essential coalitions (of* v^**) form a characterization set for the* $\mathbf{u}^*$*-anti-prenucleolus of* v^**.*

Proof. $X_k^d \subseteq Y_k^d$ for all $k \geq 1$ by definition.

By Proposition 66, we have that $X_1^d = Y_1^d$. Therefore, $Y_1^d = \mathbf{u}^*$-least-anti-core(v^*).

Indirectly assume that there exists $k \geq 2$ such that $X_k^d \not\supseteq Y_k^d$, that is, there exists $y^* \in Y_k^d$ and $S \in (N \setminus \mathcal{A}^*) \setminus (\mathcal{E}_{v^*}^{a-\mathbf{u}^*} \cup (\cup_{r=0}^{k-1} W_r^d))$ such that $u_{N \setminus S} \circ f_{v^*}(S, y^*) > t_k^d$.

It follows from Lemma 60 that for all $x \in \mathbf{u}^*$-least-anti-core(v^*) there exists $\mathcal{B}_x \in \mathcal{D}_S^{N \setminus \mathcal{A}^*}$, $\mathcal{B}_x \subseteq \mathcal{E}_{v^*}^{a-\mathbf{u}^*}$, such that $u_{N \setminus S} \circ f_{v^*}(S,x) \leq \sum_{T \in \mathcal{B}_x} u_{N \setminus T} \circ f_{v^*}(T,x)$. In particular, since $Y_k^d \subseteq \mathbf{u}^*$-least-anti-core$(v^*)$

$$u_{N \setminus S} \circ f_{v^*}(S, y^*) \leq \sum_{T \in \mathcal{B}_{y^*}} u_{N \setminus T} \circ f_{v^*}(T, y^*)\,.$$

Since $\mathbf{u}^*$-anti-core$(v^*) \neq \emptyset$, we have that $u_{N \setminus S} \circ f_{v^*}(S,x) \leq 0$, for all $S \in N \setminus \mathcal{A}^*$, $x \in X_{k-1}^d \subseteq \mathbf{u}^*$-least-anti-core$(v^*)$. Therefore, for any $T \in \mathcal{B}_{y^*}$ it holds that $u_{N \setminus S} \circ f_{v^*}(S, y^*) \leq u_{N \setminus T} \circ f_{v^*}(T, y^*)$. By Lemma 67, there exists $T^* \in \mathcal{B}_{y^*}$ such that $T^* \notin \cup_{r=0}^{k-1} W_r^d$. Therefore, $u_{N \setminus S} \circ f_{v^*}(S, y^*) \leq u_{N \setminus T^*} \circ f_{v^*}(T^*, y^*) \leq t_k^d$, which is a contradiction. □

The following theorem establishes the connection between characterization sets for the $\mathbf{u}$-prenucleolus of the primal game and characterization sets for the $\mathbf{u}^*$-anti-prenucleolus of the dual game.

Theorem 69. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *with a utility function* $\mathbf{u}$*. Then a set system is a characterization set for the* $\mathbf{u}$*-prenucleolus of* v *if and only if its complementer set system is a characterisation set for the* $\mathbf{u}^*$*-anti-prenucleolus of* v^**.*

Proof. Let $\mathcal{B} \subseteq \mathcal{A}$ be an arbitrary set system. In the first step of the generalized lexicographic center algorithm (see Section 5.2) for calculating the $\mathbf{u}$-prenucleolus of v with respect to $\mathcal{B}$, we have to solve the following optimization problem:

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_S \circ e_v(S,x) \leq t, \quad S \in \mathcal{B} \\ & x(N) = v(N) \\ & x \in \mathbb{R}^N \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{5.6}$$

Since $e_v(S,x) = v(S) - (x(N) - x(N \setminus S)) = x(N \setminus S) - (v(N) - v(S)) = x(N \setminus S) - v^*(N \setminus S))$ Problem (5.6) is equivalent with the following problem

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_S \circ (x(N \setminus S) - v^*(N \setminus S)) \leq t, \quad S \in \mathcal{B} \\ & v(N) = x(N) \\ & x \in \mathbb{R}^N \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{5.7}$$

From the dual perspective Problem 5.7 can be reformulated as follows

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_{N \setminus S} \circ f_{v^*}(S,x) \leq t, \quad S \in N \setminus \mathcal{B} \\ & v^*(N) = x(N) \\ & x \in \mathbb{R}^N \\ & t \in R_{\mathbf{u}} \end{aligned} \tag{5.8}$$

Problem (5.8) is the first step of the generalized lexicographic center algorithm (see Section 5.2) for calculating the $\mathbf{u}^*$-anti-prenucleolus of v^* with respect to $N \setminus \mathcal{B}$. Therefore, if its optimum exists then $t_1 = t_1^d$, $X_1 = X_1^d$ and $W_1 = W_1^d$.

Applying the steps in the lexicographic center algorithm for each $k \geq 2$ we have $t_{k-1} = t_{k-1}^d$ (if the optimum exists), $X_{k-1} = X_{k-1}^d$ and $W_{k-1} = W_{k-1}^d$, and we have to solve the following problem as the kth step of the algorithm

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_S \circ e_v(S,x) \leq t, \quad S \in \mathcal{B} \setminus \cup_{r=0}^{k-1} W_r \\ & x \in X_{k-1} \\ & t \in R_{\mathbf{u}} \end{aligned}$$

or equivalently

$$\begin{aligned} & t \to \min \\ \text{s.t.} \quad & u_{N \setminus S} \circ f_{v^*}(S,x) \leq t, \quad S \in (N \setminus \mathcal{B}) \setminus \cup_{r=0}^{k-1} W_r^d \\ & x \in X_{k-1}^d \\ & t \in R_{\mathbf{u}}. \end{aligned}$$

Therefore, if the optimum exists $t_k = t_k^d$, $X_k = X_k^d$ and $W_k = W_k^d$.

Summing up, for all $k \geq 1$ $X_k = X_k^d$, hence the **u**-prenucleolus of v with respect to $\mathcal{B}$ coincides with the $\mathbf{u}^*$-anti-prenucleolus of v^* with respect to $N \setminus \mathcal{B}$. Moreover, $\mathcal{B}$ is a characterization set of the **u**-prenucelolus of v if and only if $N \setminus \mathcal{B}$ is a characterizaton set of the $\mathbf{u}^*$-anti-prenculeolus of v^*. □

The following theorem is a corollary of Theorems 68 and 69.

Theorem 70. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system with a utility function* **u***, such that v is* **u***-balanced. Then the dually-***u***-essential coalitions form a characterization set for the* **u***-prenucleolus of v.*

5.4 The intersection of the two characterization sets

Solymosi and Sziklai (2016) showed that the intersection of the set of essential coalitions and the set of dually-essential coalitions is a characterization set for the nucleolus for games without restricted cooperation, where the least-core is a proper subset of the core (Theorem 11 on p. 523 in Solymosi and Sziklai (2016)).

In this section, we show that the intersection of the set of **u**-essential coalitions and the set of dually-**u**-essential coalitions is a characterization set for the **u**-prenucleolus for games where the **u**-least-core is a proper subset of the **u**-core. Therefore we generalize Solymosi and Sziklai (2016) in two directions: we consider games with restricted cooperation and we consider games with utility functions.

The graph in Figure 5.2 shows the relationships of the results of this section.

First, we prove a generalization of Theorem 2.3 on p. 362 in Granot et al (1998). In case of games with restricted cooperation, the original conditions do not imply the equality of the considered **u**-prenucleoli, but only less:

Theorem 71. *Consider a* **u***-balanced game $v \in \mathcal{G}^{N,\mathcal{A}}$ with the utility function* **u***, and a set of coalitions $\mathcal{F} \subseteq \mathcal{A}^*$, $\mathcal{F} \neq \emptyset$. Let $v' \in \mathcal{G}^{N,\mathcal{F}\cup\{N,\emptyset\}}$ be $v' = v|_{\mathcal{F}\cup\{N,\emptyset\}}$. If $x \in N^*_{\mathbf{u}}(v')$, and for every $S \in \mathcal{A}^* \setminus \mathcal{F}$ there exists $\mathcal{F}_{S,x} \subseteq \mathcal{F}$, $\mathcal{F}_{S,x} \neq \emptyset$ such that*

1. *$u_S \circ e_v(S, x) \leq u_T \circ e_v(T, x)$ for all $T \in \mathcal{F}_{S,x}$,*
2. *$\chi_S \in \mathrm{Lin}\{\chi_T : T \in \mathcal{F}_{S,x} \cup \{N\}\}$,*

then $x \in N^*_{\mathbf{u}}(v)$.

To prove this theorem we need the following lemma and a generalization of Kohlberg's theorem (Kohlberg, 1971).

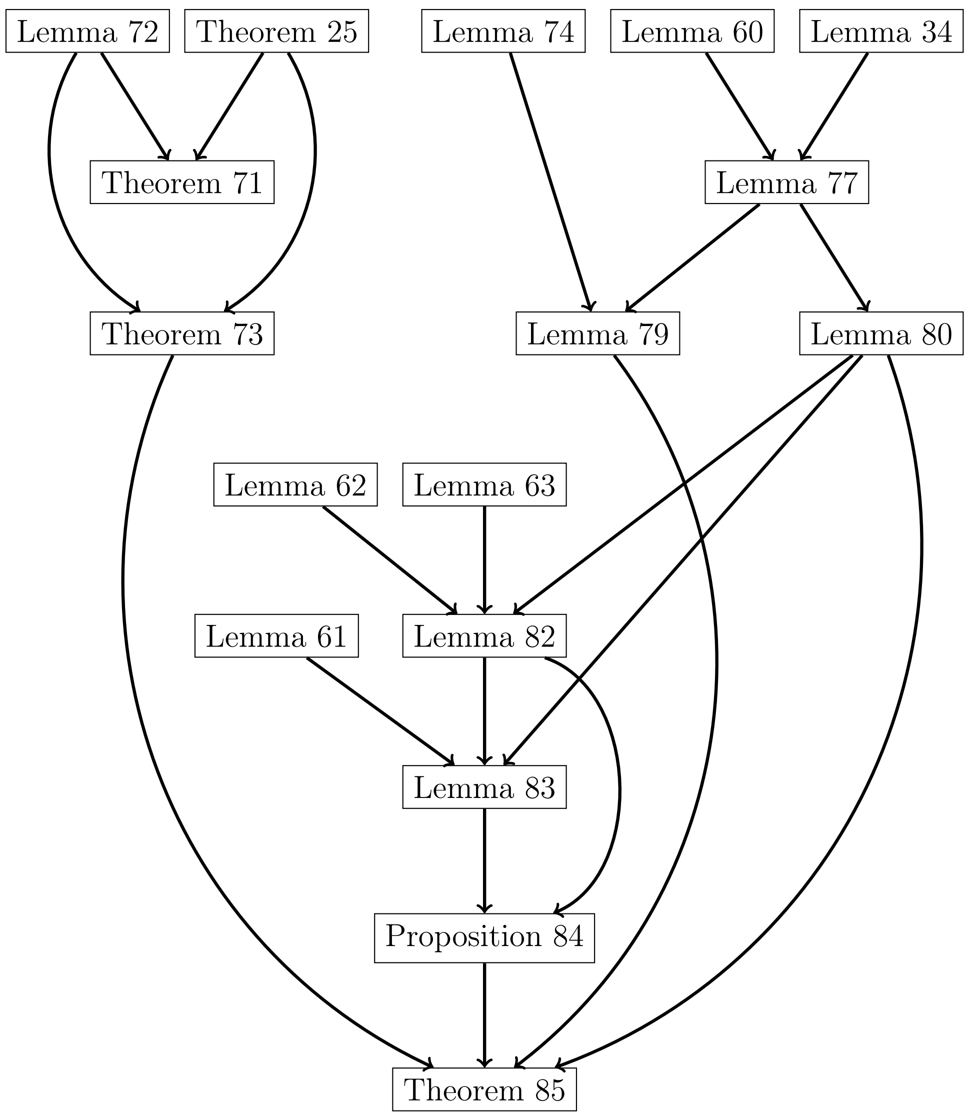


Figure 5.2: The relationships of the results of Section 5.4

Lemma 72. *Let $\mathcal{F} \subseteq 2^N$ be a balanced set of coalitions and $S \notin \mathcal{F}$ be such that $\chi_S \in \operatorname{Lin}\{\chi_T \colon T \in \mathcal{F} \cup \{N\}\}$. Then $\mathcal{F} \cup \{S\}$ is a balanced set of coalitions.*

Proof. $\mathcal{F}$ is balanced, hence there exist a weight system $\lambda_T > 0, T \in \mathcal{F}$ such that

$$\sum_{T \in \mathcal{F}} \lambda_T \chi_T = \chi_N \,.$$

$\chi_S \in \operatorname{Lin}\{\chi_T \colon T \in \mathcal{F} \cup \{N\}\}$, hence there exist $k_T \in \mathbb{R}, T \in \mathcal{F}$ and $k_N \in \mathbb{R}$ such that

$$k_N \chi_N + \sum_{T \in \mathcal{F}} k_T \chi_T = \chi_S \,.$$

Then there exists $\varepsilon > 0$ such that $\lambda_T - \varepsilon k_T > 0$ for all $T \in \mathcal{F}$ and $1 + \varepsilon k_N > 0$.

$$\chi_N = \sum_{T\in\mathcal{F}} \lambda_T \chi_T + \varepsilon\chi_S - \varepsilon\chi_S$$
$$= \sum_{T\in\mathcal{F}} (\lambda_T - \varepsilon k_T)\chi_T - \varepsilon k_N \chi_N + \varepsilon\chi_S\,.$$

Therefore,

$$\begin{aligned}(1+\varepsilon k_N)\chi_N &= \textstyle\sum_{T\in\mathcal{F}}(\lambda_T - \varepsilon k_T)\chi_T + \varepsilon\chi_S\\ \chi_N &= \textstyle\sum_{i\in\mathcal{F}} \frac{\lambda_T-\varepsilon k_T}{1+\varepsilon k_N}\chi_T + \frac{\varepsilon}{1+\varepsilon k_N}\chi_S,\end{aligned}$$

where $\frac{\lambda_T-\varepsilon k_T}{1+\varepsilon k_N} > 0$ and $\frac{\varepsilon}{1+\varepsilon k_N} > 0$. Therefore, the set $\mathcal{F}\cup\{S\}$ is balanced. □

Now we are ready to prove Theorem 71.

The proof of Theorem 71. Since $\mathcal{F}\subseteq\mathcal{A}^*$ we have that $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha,x)\subseteq\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha,x)$. Therefore, if $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha,x)\neq\emptyset$ then $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha,x)\neq\emptyset$. By Point *1.* if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha,x)\neq\emptyset$ then $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha,x)\neq\emptyset$.

By Theorem 25, if $x\in N^*_{\mathbf{u}}(v')$ then for every $\alpha\in\mathbb{R}$ it holds that $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha,x)$ is either empty or balanced. If $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha,x)=\emptyset$ then by Point *1.* $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha,x)=\emptyset$. If $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha,x)$ is balanced then by Lemma 72 and Point *2.* $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha,x)$ is balanced as well. Therefore, by Theorem 25 $x\in N^*_{\mathbf{u}}(v)$. □

Notice that by Theorem 71 we have that $N^*_{\mathbf{u}}(v')\subseteq N^*_{\mathbf{u}}(v)$. Therefore, if $N^*_{\mathbf{u}}(v)$ is a singleton and $N^*_{\mathbf{u}}(v')$ has at least one element, then we know, that the two are equal. However, we have already represented in Example 31, that in case of games with restricted cooperation, the $\mathbf{u}$-prenucleolus is not necessarily a singleton. Therefore, in case of games with restricted cooperation, we need an extra condition to get $N^*_{\mathbf{u}}(v') = N^*_{\mathbf{u}}(v)$.

Theorem 73. *Consider a game $v\in\mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$ such that v is $\mathbf{u}$-balanced, and a set sytem $\mathcal{F}\subseteq\mathcal{A}^*$, $\mathcal{F}\neq\emptyset$. Let $v'\in\mathcal{G}^{N,\mathcal{F}\cup\{N,\emptyset\}}$ be such that $v' = v|_{\mathcal{F}\cup\{N,\emptyset\}}$ and $X\subseteq I^*(v)$ be such that $N^*_{\mathbf{u}}(v), N^*_{\mathbf{u}}(v')\subseteq X$. If for every $x\in X$, $S\in\mathcal{A}^*\setminus\mathcal{F}$ there exists $\mathcal{F}_{S,x}\subseteq\mathcal{F}$, $\mathcal{F}_{S,x}\neq\emptyset$ such that*

1. *$u_S\circ e_v(S,x)\leq u_T\circ e_v(T,x)$ for all $T\in\mathcal{F}_{S,x}$,*
2. *$\chi_S\in\mathrm{Lin}\{\chi_T\colon T\in\mathcal{F}_{S,x}\cup\{N\}\}$,*
3. *for all $\alpha\in\mathbb{R}$ if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha)$ is balanced then $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(x,\alpha)$ is balanced,*

then $\mathcal{F}$ is a characterization set for $N^*_{\mathbf{u}}(v)$.

Proof. Take $x \in X \subseteq I^*(v)$. Since $\mathcal{F} \subseteq \mathcal{A}^*$ we have that $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x) \subseteq \mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha, x)$. Therefore, if $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x) \neq \emptyset$ then $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha, x) \neq \emptyset$. By Point *1.* if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha, x) \neq \emptyset$ then $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x) \neq \emptyset$.

By Lemma 72 if $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x)$ is balanced then $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha, x)$ is balanced. By Point *3.* if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha, x)$ is balanced then $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x)$ is balanced.

By Theorem 25 $x \in N^*_{\mathbf{u}}(v')$ if and only if for every $\alpha \in \mathbb{R}$ it holds that $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x)$ is either empty or balanced. In the first paragraph we concluded that $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x)$ is empty if and only if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha, x)$ is empty. In the second paragraph we concluded that $\mathcal{D}^{\mathcal{F}}_{\mathbf{u}}(\alpha, x)$ is balanced if and only if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(\alpha, x)$ is balanced. Therefore, by Theorem 25 $x \in N^*_{\mathbf{u}}(v)$ if and only if $x \in N^*_{\mathbf{u}}(v')$.

Since, for every $x \in X$ it holds that $x \in N^*_{\mathbf{u}}(v)$ if and only if $x \in N^*_{\mathbf{u}}(v')$ and $N^*_{\mathbf{u}}(v), N^*_{\mathbf{u}}(v') \subseteq X$, we conclude that $\mathcal{F}$ is a characterisation set for $N^*_{\mathbf{u}}(v)$. □

We need the following lemma to show the balancedness of certain set systems.

Lemma 74. *Consider the set systems $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(N)$ such that $\mathcal{T} \subseteq \mathcal{S}$, and a sequence of set systems $\mathcal{S} = \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ such that*

1. *if $\mathcal{F}_n$ is balanced then $\mathcal{F}_{n+1}$ is balanced, for every $n \in \mathbb{N}$,*
2. *$|\mathcal{T} \cap \mathcal{F}_n| \leq |\mathcal{T} \cap \mathcal{F}_{n+1}|$, for every $n \in \mathbb{N}$,*
3. *if $\mathcal{T} \subsetneq \mathcal{F}_n$ then $|\mathcal{F}_n \cap (\mathcal{S} \setminus \mathcal{T})| > |\mathcal{F}_{n+1} \cap (\mathcal{S} \setminus \mathcal{T})|$.*

Then there exists $n^* \in \mathbb{N}$ such that $\mathcal{F}_{n^*} = \mathcal{T}$, and if $\mathcal{S}$ is balanced then $\mathcal{T}$ is balanced.

Proof. By Point *3.* there exist $n^* \leq |\mathcal{S} \setminus \mathcal{T}|$ such that $|\mathcal{F}_{n^*} \cap (\mathcal{S} \setminus \mathcal{T})| = 0$, that is, $\mathcal{F}_{n^*} \subseteq \mathcal{T}$.

By $\mathcal{T} \subseteq \mathcal{S} = \mathcal{F}_0$ and Point *2.* it holds that $\mathcal{T} \subseteq \mathcal{F}_n$, for all $n \in \mathbb{N}$, in particular $\mathcal{T} \subseteq \mathcal{F}_{n^*}$.

Therefore, $\mathcal{T} = \mathcal{F}_{n^*}$, and by Point *1.* if $\mathcal{S}$ is balanced then for all $n \in \mathbb{N}$ it holds that $\mathcal{F}_n$ is balanced. In particular $\mathcal{F}_{n^*} = \mathcal{T}$ is balanced. □

Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ with a utility function $\mathbf{u}$ such that v is $\mathbf{u}$-balanced. Then for every $S \in \mathcal{A}^* \setminus (\mathcal{E}^{d-\mathbf{u}}_v \cap \mathcal{E}^{\mathbf{u}}_v)$ it holds that either S is not $\mathbf{u}$-essential or it is not dually-$\mathbf{u}$-essential.

Case 1: If S is not $\mathbf{u}$-essential then by Lemma 34 for every $x \in \mathbf{u}$-least-core(v) there exists $\mathcal{B}_S \in \mathcal{D}^{\mathcal{A}^*}_S$, $\mathcal{B}_S \subseteq \mathcal{E}^{\mathbf{u}}_v$ such that

$$u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}_S} u_T \circ e_v(T, x)\,.$$

Case 2: If S is $\mathbf{u}$-essential, that is, it is not dually-$\mathbf{u}$-essential then for every $x \in \mathbf{u}$-least-core(v)

$$\begin{aligned} u_S \circ e_v(S,x) &= u_S \circ (v(S) - x(S)) \\ &= u_S \circ (v(N) - v^*(N \setminus S) - (x(N) - x(N \setminus S))) \\ &= u_S \circ (x(N \setminus S) - v^*(N \setminus S)) = u_S \circ f_{v^*}(N \setminus S, x)\,, \end{aligned} \tag{5.9}$$

and by Lemma 60 there exists $\mathcal{B} \in \mathcal{D}_{N\setminus S}^{N\setminus\mathcal{A}^*}$, $\mathcal{B} \subseteq \mathcal{E}_v^{a-\mathbf{u}^*}$ such that $u_S \circ f_{v^*}(N \setminus S, x) \leq \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x)$. Then

$$\begin{aligned} u_S \circ f_{v^*}(N \setminus S, x) &\leq \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ f_{v^*}(T,x) \\ &= \sum_{T\in\mathcal{B}} u_{N\setminus T} \circ e_v(N \setminus T, x) = \sum_{T\in N\setminus\mathcal{B}} u_T \circ e_v(T,x). \end{aligned} \tag{5.10}$$

Let $\mathcal{B}_S := N \setminus \mathcal{B} \subseteq \mathcal{E}_v^{d-\mathbf{u}}$. Then by Eq. (5.9) and (5.10) we have that

$$u_S \circ e_v(S,x) \leq \sum_{T\in\mathcal{B}_S} u_T \circ e_v(T,x)\,.$$

For an $x \in \mathbf{u}$-least-core(v), for all $S \in \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ fix one such $\mathcal{B}_S$ partition (Case 1) or anti-partition (Case 2). Let a directed graph $\Gamma_{v,\mathbf{u}}(x)$ be defined as follows: the nodes are the coalitions of $\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, and there is a directed edge from S to S', $S, S' \in \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, if S' appears in the partition or in the anti-partition of S, that is, if $S' \in \mathcal{B}_S \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$.

Remark 75. Note that since the considered partitions and anti-partitions can be arbitrarily chosen, the graph $\Gamma_{v,\mathbf{u}}(x)$ can be defined multiple ways. However, this does not make any problem in our analysis, since we need only an instance of such graphs, we do not need to consider all of them.

Example 76. Consider the following game: $N = \{1,2,3,4\}$, $\mathcal{A}^* = \{\{1\},\{2\},\{3\},\{4\}, \{1,2\},\{1,3\},\{2,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$, $v(\{1\}) = v(\{2\}) = 1.5$, $v(\{3\}) = v(\{4\}) = 1, v(\{1,2\}) = 3.5, v(\{1,3\}) = v(\{2,4\}) = 2, v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 4, v(N) = 6$. Let $\mathbf{u}$ be the identity function.

The essential coalitions are: $\mathcal{E}_v = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3,4\},\{2,3,4\}\}$.

The dual game is the following: $v^*(\{1\}) = v^*(\{2\}) = v^*(\{3\}) = v^*(\{4\}) = 2, v^*(\{1,3\}) = v^*(\{2,4\}) = 4, v^*(\{3,4\}) = 2.5, v^*(\{1,2,3\}) = v^*(\{1,2,4\}) = 5, v^*(\{1,3,4\}) = v^*(\{2,3,4\}) = 4.5, v^*(N) = 6$.

The anti-essential coalitions of the dual game are: $\mathcal{E}^a_{v^*} = \{\{1\},\{2\},\{3\},\{4\},\{3,4\},\{1,2,3\},\{1,2,4\}\}$; therefore the dually-essential coalitions of the original game are: $\mathcal{E}^d_v = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2\},\{3\},\{4\}\}$.

Therefore, $\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v = \{\{3\},\{4\},\{1,2\},\{1,3,4\},\{2,3,4\}\}$ and the nodes of the graph are $\mathcal{A}^* \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v) = \{\{1\},\{2\},\{1,3\},\{2,4\},\{1,2,3\},\{1,2,4\}\}$.

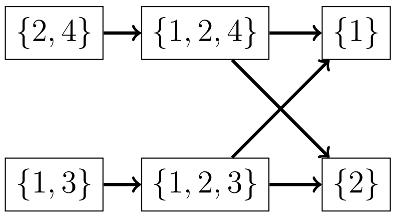
For all $S \in \mathcal{A}^* \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)$ there exists a $\mathcal{B}_S \in \mathcal{E}^{\mathbf{u}}_v$ partition (Case 1) or a $\mathcal{B}_S \in \mathcal{E}^{d-\mathbf{u}}_v$ anti-partition (Case 2) as described above. To define the graph, fix one of these coalition sets for each coalition.

$\mathcal{B}_{\{1,2,4\}} = \{\{1\},\{2\},\{4\}\}$, because $v(\{1,2,4\}) \leq v(\{1\}) + v(\{2\}) + v(\{4\})$ and $\{\{1\},\{2\},\{4\}\} \subseteq \mathcal{E}^{\mathbf{u}}_v$.

$\mathcal{B}_{\{2,4\}} = \{\{2,3,4\},\{1,2,4\}\}$, because $-v^*(\{1,3\}) \leq -v^*(\{1\}) - v^*(\{3\})$ and $\{\{2,3,4\},\{1,2,4\}\} \subseteq \mathcal{E}^{d-\mathbf{u}}_v$.

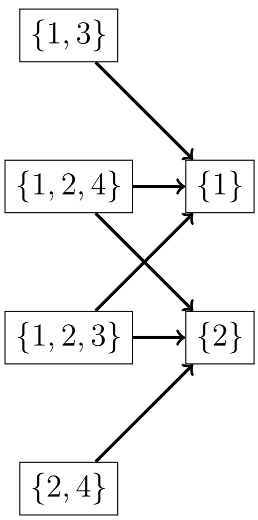
Similarly, let $\mathcal{B}_{\{1,2,3\}} = \{\{1\},\{2\},\{3\}\}$, $\mathcal{B}_{\{1,3\}} = \{\{1,2,3\},\{1,3,4\}\}$, $\mathcal{B}_{\{1\}} = \{\{1,2\},\{1,3,4\}\}$, $\mathcal{B}_{\{2\}} = \{\{1,2\},\{2,3,4\}\}$

For an arbitrary $x \in \mathbf{u}$-least-core(v), $\Gamma_{v,\mathbf{u}}(x)$ is the following:



There is an edge from $\{2,4\}$ to $\{1,2,4\}$, because $\{1,2,4\} \in \mathcal{B}_{\{2,4\}}$ and there is an edge from $\{1,2,4\}$ to $\{1\}$, because $\{1\} \in \mathcal{B}_{\{1,2,4\}}$, etc.

As mentioned in Remark 75, $\Gamma_{v,\mathbf{u}}(x)$ can be defined in multiple ways depending on the choice of the $\mathcal{B}_S$ partitions and anti-partitions. For example, $\mathcal{B}_{\{2,4\}}$ can be selected as $\{\{2\},\{4\}\}$, because $v(\{2,4\}) \leq v(\{2\}) + v(\{4\})$ and $\{\{2\},\{4\}\} \subseteq \mathcal{E}^{\mathbf{u}}_v$ and $\mathcal{B}_{\{1,3\}}$ can be selected as $\{\{1\},\{3\}\}$. Then the graph would be:



The following lemma shows that the above defined graph is acyclic, if $\mathbf{u}$-least-core$(v) \neq$ $\mathbf{u}$-core(v).

Lemma 77. *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system with the utility function* $\mathbf{u}$*, and* $x \in \mathbf{u}$-least-core(v)*. If* $\mathbf{u}$-least-core$(v) \neq$ $\mathbf{u}$-core(v) *then* $\Gamma_{v,\mathbf{u}}(x)$ *is acyclic.*

Proof. Indirectly assume that the graph is not acyclic. Then there is a cycle $T_1, T_2, \ldots, T_r \subseteq \mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ such that

$$\begin{aligned}
u_{T_1} \circ e_v(T_1, x) &\leq u_{T_2} \circ e_v(T_2, x) + \sum_{j=1}^{k_1} u_{S_j^1} \circ e_v(S_j^1, x) \\
u_{T_2} \circ e_v(T_2, x) &\leq u_{T_3} \circ e_v(T_3, x) + \sum_{j=1}^{k_2} u_{S_j^2} \circ e_v(S_j^2, x) \\
&\vdots \\
u_{T_r} \circ e_v(T_r, x) &\leq u_{T_1} \circ e_v(T_1, x) + \sum_{j=1}^{k_r} u_{S_j^r} \circ e_v(S_j^r, x),
\end{aligned}$$

where $\mathcal{B}_{T_i} = \{T_{i+1}\} \cup \{S_j^i\}_{j=1}^{k_i}$, $i = 1, \ldots, r-1$ and $\mathcal{B}_{T_r} = \{T_1\} \cup \{S_j^r\}_{j=1}^{k_r}$ are the corresponding partitions or anti-partitions.

Summing up the inequalities above, we have that

$$0 \leq \sum_{i=1}^{r} \sum_{j=1}^{k_r} u_{S_j^i} \circ e_v(S_j^i, x) \,. \tag{5.11}$$

By Eq. (5.11), we have that $u_{S_j^i} \circ e_v(S_j^i, x) = 0$, for all $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, k_r$. However, since $x \in \mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v), we have that $u_{S_j^i} \circ e_v(S_j^i, x) < 0$, for all $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, k_r$, which is a contradiction. □

In the following example, we show that if $\mathbf{u}$-least-core$(v) = \mathbf{u}$-core(v), then $\Gamma_{v,\mathbf{u}}(x)$ can be cyclic.

Example 78. Consider the game from Example 59. We have already seen, that $\mathcal{E}_v^{\mathbf{u}} = \{\{1\}, \{2\}, \{1,3\}\}$, $\mathcal{E}_v^{d-\mathbf{u}} = \{\{2\}, \{1,3\}, \{1,2\}\}$ and $\mathbf{u}$-core$(v) = \{(1,1,1)\}$.

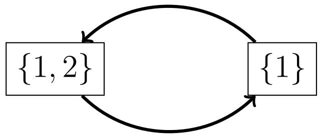
Since $\mathbf{u}$-least-core$(v) = \{(1,1,1)\}$, it equals the $\mathbf{u}$-core.

$\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}} = \{\{2\}, \{1,3\}\}$; therefore, $\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) = \{\{1\}, \{1,2\}\}$.

Let $x = (1,1,1) \in \mathbf{u}$-core$(v)$.

The nodes of $\Gamma_{v,\mathbf{u}}(x)$ are $\{1\}, \{1,2\}$. There is an edge from $\{1,2\}$ to $\{1\}$, because $\frac{v(\{1,2\}) - x_1 - x_2}{2} \leq \frac{v(\{1\} - x_1)}{1} + \frac{v(\{2\} - x_2)}{1}$; therefore, $\{1\}$ appears in the partition of $\{1,2\}$. In addition, there is an edge from $\{1\}$ to $\{1,2\}$, because $\frac{x_2 + x_3 - v^*(\{2,3\})}{1} \leq \frac{x_2 - v^*(\{2\})}{2} + \frac{x_3 - v^*(\{3\})}{2}$; therefore, $\{1,2\}$ appears in the anti-partition of $\{1\}$.

$\Gamma_{v,\mathbf{u}}(x)$ is not acyclic:



We need the following lemma to show that Point *3.* of Theorem 73 holds for the intersection of the set of **u**-essential and the set of dually-**u**-essential coalitions, if **u**-least-core(v) is a proper subset of **u**-core(v).

Lemma 79. *Consider a* **u***-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system with a utility function* **u***, and* $x \in$ **u**-least-core(v)*. If* **u**-least-core(v) $\neq$ **u**-core(v) *then the following holds: if* $\mathcal{A}^*$ *is balanced then* $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ *is balanced. In addition, for all* $\alpha \in \mathbb{R}$ *it holds that if* $\mathcal{D}_{\mathbf{u}}^{\mathcal{A}^*}(\alpha, x)$ *is balanced then* $\mathcal{D}_{\mathbf{u}}^{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}(\alpha, x)$ *is balanced.*

Proof. By Lemma 77, we have that $\Gamma_{v,\mathbf{u}}(x)$ is acyclic, hence there is a topological ordering on its nodes $T_1, T_2, \ldots, T_{|\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|}$. Then we can define the following sequence of coalitions:

$$
\begin{aligned}
\mathcal{F}_0 &= \mathcal{A}^*\,, \\
\mathcal{F}_n &= \mathcal{F}_{n-1} \setminus \{T_n\} \cup \mathcal{B}_{T_n}\,, \qquad n = 1, 2, \ldots, |\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|\,, \\
\mathcal{F}_n &= \mathcal{F}_{n-1}\,, n \in \mathbb{N}\,, \qquad\qquad n > |\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|\,.
\end{aligned}
$$

Let $\mathcal{B}_{T_n}$ denote the partition or anti-partition of T_n used in defining $\Gamma_{v,\mathbf{u}}(x)$. Next we check the three conditions of Lemma 74:

Point *1.*: If $\mathcal{F}_n$ is balanced then $\mathcal{F}_{n+1}$ is balanced, for every $n \in \mathbb{N}$.

If $n \geq |\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|$ then $\mathcal{F}_n = \mathcal{F}_{n+1}$, hence Point *1.* holds.

Otherwise, if $\mathcal{F}_n$ is balanced then there exists $(\lambda_S)_{S \in \mathcal{F}_n} \geq 0$ such that

$$
\sum_{S \in \mathcal{F}_n} \lambda_S \chi_S = \chi_N\,. \tag{5.12}
$$

If $\mathcal{B}_{T_{n+1}}$ is the corresponding partition of T_{n+1}, then T_{n+1} is not **u**-essential and

$$
\chi_{T_{n+1}} = \sum_{T \in \mathcal{B}_{T_{n+1}}} \chi_T\,.
$$

Therefore, Eq. (5.12) can be rewritten as

$$\sum_{S\in\mathcal{F}_n\setminus\{T_{n+1}\}} \lambda_S\chi_S + \sum_{T\in\mathcal{B}_{T_{n+1}}} \lambda_{T_{n+1}}\chi_T = \chi_N,$$

hence $\mathcal{F}_{n+1} = (\mathcal{F}_i \setminus \{T_{n+1}\}) \cup \mathcal{B}_{T_{n+1}}$ is balanced.

If $\mathcal{B}_{T_{n+1}}$ is the corresponding anti-partition of T_{n+1}, then T_{n+1} is not dually-**u**-essential and

$$\chi_{T_{n+1}} = \sum_{T\in\mathcal{B}_{T_{n+1}}} \chi_T - (|\mathcal{B}_{T_{n+1}}| - 1)\chi_N\,.$$

Therefore, Eq. (5.12) can be rewritten as

$$\sum_{S\in\mathcal{F}_n\setminus\{T_{n+1}\}} \lambda_S\chi_S + \sum_{T\in\mathcal{B}_{T_{n+1}}} \lambda_{T_{n+1}}\chi_T - \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1)\chi_N = \chi_N$$

$$\sum_{S\in\mathcal{F}_n\setminus\{T_{n+1}\}} \lambda_S\chi_S + \sum_{T\in\mathcal{B}_{T_{n+1}}} \lambda_{T_{n+1}}\chi_T = (1 + \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1))\chi_N$$

$$\sum_{S\in\mathcal{F}_n\setminus\{T_{n+1}\}} \frac{\lambda_S}{1 + \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1)}\chi_S + \sum_{T\in\mathcal{B}_{T_{n+1}}} \frac{\lambda_{T_{n+1}}}{1 + \lambda_{T_{n+1}}(|\mathcal{B}_{T_{n+1}}| - 1)}\chi_T = \chi_N,$$

hence, $\mathcal{F}_{n+1} = (\mathcal{F}_n \setminus \{T_{n+1}\}) \cup \mathcal{B}_{T_{n+1}}$ is balanced.

Point *2.*: $|(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_n| \le |(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_{n+1}|$, for every $n \in \mathbb{N}$.

If $n \ge |\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|$ then $\mathcal{F}_n = \mathcal{F}_{n+1}$, hence $|(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_n| = |(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_{n+1}|$.

Otherwise, since $\mathcal{F}_{n+1} = (\mathcal{F}_n \setminus \{T_{n+1}\}) \cup \mathcal{B}_{T_{n+1}}$, where $T_{n+1} \notin (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, the number of coalitions in the intersection never decreases, that is, $|(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_n| \le |(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \cap \mathcal{F}_{n+1}|$.

Point *3.*: If $(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}) \subsetneq \mathcal{F}_n$, then $|\mathcal{F}_n \cap (\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))| > |\mathcal{F}_{n+1} \cap (\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))|$.

By the topological ordering of the nodes (coalitions), for all $i, j \in \mathbb{N}_+, i < j \le |\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|$ it holds that $T_i \notin \mathcal{B}_{T_j}$. Therefore, $T_i \notin \mathcal{F}_j$. Since $T_i \in \mathcal{F}_{i-1}$ and if a coalition – not from the intersection – is removed then it never comes back, we have that $|\mathcal{F}_i \cap (\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))| > |\mathcal{F}_{i+1} \cap (\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}))|$.

In conclusion, by Lemma 74 we have that $\mathcal{F}_{|\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})|} = (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$, and if $\mathcal{A}^*$ is balanced, then $(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ is balanced.

To prove the second part of the theorem, for each $\alpha \in \mathbb{R}$ consider the induced subgraph of $\Gamma_{v,\mathbf{u}}(x)$ with nodes from $(\mathcal{A}^* \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})) \cap \mathcal{D}_{\mathbf{u}}^{\mathcal{A}^*}(x, \alpha)$. Let $\Gamma_{v,\mathbf{u}}^{\alpha}(x)$ denote this induced subgraph. It is easy to check that $\Gamma^{\alpha}_{v,\mathbf{u}}(x)$ is a directed acyclic graph, hence there is a topological ordering on its nodes: $T'_1, T'_2, \dots T'_{|(\mathcal{A}^* \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)) \cap \mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha)|}$.

Consider the following sequence of sets:

$$\begin{aligned}
\mathcal{F}'_0 &= \mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha)\,, \\
\mathcal{F}'_n &= (\mathcal{F}'_{n-1} \setminus \{T'_n\}) \cup \mathcal{B}_{T'_n}\,, \quad n = 1,2,\dots,|(\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha) \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v))|\,, \\
\mathcal{F}'_n &= \mathcal{F}'_{n-1}\,, \qquad\qquad\qquad\quad n \in \mathbb{N},\ n > |(\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha) \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v))|\,.
\end{aligned}$$

The proof of that Points *1.* and *2.* of Lemma 74 hold for the sequence of sets $(\mathcal{F}'_n)$ goes as it goes for the sequence of sets $(\mathcal{F}_n)$, hence we omit its proof. For Point *3.* of Lemma 74 consider the following:

For every $n = 1,2,\dots,|(\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha) \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v))|$ and $S \in \mathcal{B}_{T'_n} \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)$ it holds that S is a node in $\Gamma^{\alpha}_{v,\mathbf{u}}(x)$. Indeed, since $x \in$ **u**-least-core(v), **u**-least-core$(v) \neq$ **u**-core(v) and $u_{T'_n} \circ e_v(T'_n, x) \leq \sum_{S \in \mathcal{B}_{T'_n}} u_S \circ e_v(S,x)$, we have that for every $S \in \mathcal{B}_{T'_n}$ it holds that $u_S \circ e_v(S,x) \geq u_{T'_n} \circ e_v(T'_n, x) \geq \alpha$.

Therefore, we can apply Lemma 74 and get that

$$\mathcal{F}'_{|(\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha) \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v))|} = \mathcal{D}^{(\mathcal{E}^{\mathbf{u}}_v(x,\alpha) \cap \mathcal{E}^{d-\mathbf{u}}_v))}_{\mathbf{u}}(x,\alpha)$$

and that, if $\mathcal{D}^{\mathcal{A}^*}_{\mathbf{u}}(x,\alpha)$ is balanced then $\mathcal{D}^{(\mathcal{E}^{\mathbf{u}}_v(x,\alpha) \cap \mathcal{E}^{d-\mathbf{u}}_v))}_{\mathbf{u}}(x,\alpha)$ is balanced. □

The following lemma shows that Points *1.* and *2.* of Theorem 73 hold for the intersection of the set of **u**-essential and the set of dually-**u**-essential coalitions, if **u**-least-core(v) is a proper subset of **u**-core(v).

Lemma 80. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system with a utility function* **u** *such that the game is* **u**-*balanced, and a coalition $S \in \mathcal{A}^* \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)$. If* **u**-least-core$(v) \neq$ **u**-core(v) *then for every $x \in$* **u**-least-core(v) *there exists $\mathcal{B}^* \subseteq \mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v$ such that $u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e_v(T,x)$ and $\chi_S \in \mathrm{Lin}\{\chi_T \colon \mathcal{B}^* \cup \{N\}\}$.*

Proof. If **u**-least-core$(v) \neq$ **u**-core(v) then by Lemma 77 $\Gamma_{v,\mathbf{u}}(x)$ is acyclic. Take the induced subgraph of $\Gamma_{v,\mathbf{u}}(x)$ consisting only of the nodes $\{S\} \cup \{T \in \mathcal{A}^* \setminus (\mathcal{E}^{\mathbf{u}}_v \cap \mathcal{E}^{d-\mathbf{u}}_v)\colon$ there is a directed path in $\Gamma_{v,\mathbf{u}}(x)$ from S to $T\}$. Let $\Gamma'_{v,\mathbf{u}}(x)$ denote this induced subgraph. Then $\Gamma'_{v,\mathbf{u}}(x)$ is also acyclic, hence there is a topological ordering on its nodes: $T_1, T_2, \dots T_r$, where $T_1 = S$.

Consider the following sequence of sets:

$$\begin{aligned}
\mathcal{T}_0 &= \{S\}\,, \\
\mathcal{T}_n &= (\mathcal{T}_{n-1} \setminus \{T_n\}) \cup \mathcal{B}_{T_n}\,, \quad n = 1,\dots,r\,.
\end{aligned}$$

By the definition of $\mathcal{B}_{T_n}$, $n = 1, \ldots, r$

$$u_S \circ e_v(S,x) \;\; \leq \;\; \sum_{T \in \mathcal{T}_n} u_T \circ e_v(T,x)\,,$$

in particular

$$u_S \circ e_v(S,x) \;\; \leq \;\; \sum_{T \in \mathcal{T}_r} u_T \circ e_v(T,x)\,.$$

Moreover, since for all $1 \leq i \leq j \leq r$ it holds that $T_i \notin \mathcal{B}_{T_j}$, and for all $T \in \mathcal{B}_{T_i} \setminus (\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}})$ we have that T is a node of the graph $\Gamma'_{v,\mathbf{u}}(x)$, we have that $\mathcal{T}_r \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$.

Since for each $n = 1, \ldots, r$ it holds that $\mathcal{B}_{T_n}$ is either a partition or an anti-partition of T_n, we have that $\chi_{T_n} \in \mathrm{Lin}\{\chi_T \colon T \in \mathcal{B}_{T_n} \cup \{N\}\}$. If $\mathcal{B}_{T_n}$ is a partition of T_n then

$$\chi_{T_n} = \sum_{T \in \mathcal{B}_{T_n}} \chi_T\,.$$

If $\mathcal{B}_{T_n}$ is an anti-partition of T_n then

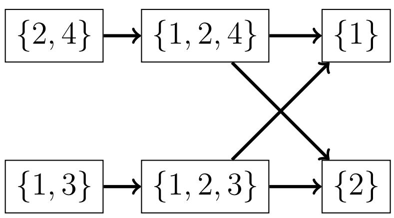
$$\chi_{T_n} = \sum_{T \in \mathcal{B}_{T_n}} \chi_T - (|\mathcal{B}_{T_n}| - 1)\chi_N\,.$$

Therefore, for every $n = 1, \ldots, r$ it holds that $\chi_S \in \mathrm{Lin}\{\chi_T \colon T \in \mathcal{T}_n \cup \{N\}\}$. In particular $\chi_S \in \mathrm{Lin}\{\chi_T \colon T \in \mathcal{T}_r \cup \{N\}\}$.

In conclusion, let $\mathcal{B}^* := \mathcal{T}_r$. Then $\mathcal{B}^* \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, $u_S \circ e_v(S,x) \leq \sum_{T \in \mathcal{B}^*} u_T \circ e_v(T,x)$, and $\chi_S \in \mathrm{Lin}\{\chi_T \colon T \in \mathcal{B}^* \cup \{N\}\}$. □

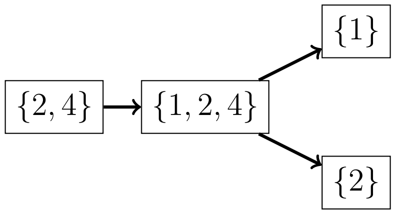
In the following example, we illustrate the idea behind the proof of Lemma 80.

Example 81. Consider the game of Example 76 with the identity utility function, $x \in \mathbf{u}$-least-core(v) and $\Gamma_{v,\mathbf{u}}(x)$:



In this case, the game is $\mathbf{u}$-balanced and $\mathbf{u}$-least-core$(v) \neq \mathbf{u}$-core(v).

Consider the coalition $S = \{2, 4\}$. Then, $\Gamma'_{v,\mathbf{u}}$ is:



Therefore, a topological ordering on the nodes of $\Gamma'_{v,\mathbf{u}}$ is $T_1 = \{2,4\}, T_2 = \{1,2,4\}, T_3 = \{1\}, T_4 = \{2\}$. Then, $\mathcal{T}_0 = \{\{2,4\}\}$, $\mathcal{T}_1 = \{\{2,3,4\},\{1,2,4\}\}$ and

$$v(\{2,4\}) \leq v(\{2,3,4\}) + v(\{1,2,4\}). \tag{5.13}$$

Here, $\{2,3,4\} \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, but $\{1,2,4\} \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$. $\mathcal{T}_2 = \{\{2,3,4\},\{1\},\{2\},\{4\}\}$ and $v(\{1,2,4\}) \leq v(\{1\}) + v(\{2\}) + v(\{4\})$; therefore,

$$v(\{2,4\}) \leq v(\{2,3,4\}) + v(\{1\}) + v(\{2\}) + v(\{4\}). \tag{5.14}$$

Here $\{2,3,4\},\{4\} \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, but $\{1\},\{2\} \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$. $\mathcal{T}_3 = \{\{2,3,4\},\{1,2\},\{1,3,4\},\{2\},\{4\}\}$ and $v(\{1\}) \leq v(\{1,2\}) + v(\{1,3,4\})$; therefore,

$$v(\{2,4\}) \leq v(\{2,3,4\}) + v(\{1,2\}) + v(\{1,3,4\}) + v(\{2\}) + v(\{4\}). \tag{5.15}$$

Here $\{2,3,4\},\{1,2\},\{1,3,4\},\{4\} \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, but $\{2\} \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$. $\mathcal{T}_4 = \{\{2,3,4\},\{1,2\},\{1,3,4\},\{4\}\}$ and $v(\{2\}) \leq v(\{1,2\}) + v(\{2,3,4\})$; therefore,

$$v(\{2,4\}) \leq 2v(\{2,3,4\}) + 2v(\{1,2\}) + v(\{1,3,4\}) + v(\{4\}), \tag{5.16}$$

where $T_4 \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$.

In order to apply Theorem 73 in the proof of that $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ is a characterization set for the $\mathbf{u}$-prenucleolus we must find a set $X \subseteq I^*(v)$ such that $N_{\mathbf{u}}^*(v), N_{\mathbf{u}}^*(v') \subseteq X$, where $v' \in \mathcal{G}^{N,\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$ is such that $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$.

We know that $N_{\mathbf{u}}^*(v) \subseteq \mathbf{u}\text{-least-core}(v)$ and $N_{\mathbf{u}}^*(v') \subseteq \mathbf{u}\text{-least-core}(v')$. The following two Lemmata state that $\mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-least-core}(v')$, hence we can apply Theorem 73 with $X = \mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-least-core}(v')$.

As introduced in Section 4.1, for a class of coalitions $\mathcal{S} \subseteq \mathcal{A}^*$ and $t \in \mathbb{R}$ let $X(\mathcal{S},t) := \{x \in I^*(v) \colon u_S \circ e_v(S,x) \leq t, \forall S \in \mathcal{S}\}$.

Lemma 82. *Consider a game $v \in \mathcal{G}^{N,\mathcal{A}}$ such that $\mathcal{A}^*$ contains a balanced set system with a utility function $\mathbf{u}$ such that v is $\mathbf{u}$-balanced. If $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ then $X(\mathcal{A}^*, t_1) = X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$.*

Proof. Since $\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}} \subseteq \mathcal{A}^*$ it holds that $X(\mathcal{A}^*, t_1) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$, where t_1 is from Section 5.2.

Indirectly assume that $\exists x^1 \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1) \setminus X(\mathcal{A}^*, t_1)$. This means that $\exists S \in \mathcal{A}^*$ such that

$$u_S \circ e_v(S, x^1) > t_1 \,. \tag{5.17}$$

Let $\mathcal{S}_{x^1} := \{S \in \mathcal{A}^* \colon u_S \circ e_v(S, x^1) > t_1\}$. Then for all $S \in \mathcal{S}_{x^1}$ it holds that $S \notin \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$.

Let $x^* \in X(\mathcal{A}^*, t_1)$ be the closest point of set $X(\mathcal{A}^*, t_1)$ to the point x^1. It is clear that such x^* exists, since $X(\mathcal{A}^*, t_1)$ is nonempty and closed by Lemma 36.

Since $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$ is a convex set, for every $\lambda \in [0, 1]$ it holds that $\lambda x^* + (1 - \lambda)x^1 \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$.

By Lemma 80 we know that for each $S \in \mathcal{S}_{x^1}$ there exists $\mathcal{B}_S^* \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ such that

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*) \,. \tag{5.18}$$

Since $t_1 < 0$, for every $x \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$ and $T \in \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ it holds that $u_T \circ e_v(T, x) \leq t_1$. Therefore, for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S, x^*) \leq \sum_{T \in \mathcal{B}_S^*} u_T \circ e_v(T, x^*) \leq t_1 \,. \tag{5.19}$$

By Eqs. (5.17), (5.19) and the continuity of u_S, we have that for every $S \in \mathcal{S}_{x^1}$ there exists $\lambda_S \in [0, 1]$ such that

$$u_S \circ e_v(S, \lambda_S x^* + (1 - \lambda_S)x^1) = t_1 \,.$$

Let $S^1 \in \operatorname{argmin}_{S \in \mathcal{S}_{x^1}} ||x^* - (\lambda_S x^* + (1 - \lambda_S)x^1))||$, and let $x^2 := \lambda_{S^1} x^* + (1 - \lambda_{S^1})x^1$. Since $t_1 < 0$ and $u_T \circ e_v(T, x^2) \leq t_1$, for all $T \in \mathcal{B}_{S^1}^*$ we have that

$$u_{S^1} \circ e_v(S^1, x^2) > \sum_{T \in \mathcal{B}_{S^1}^*} u_T \circ e_v(T, x^2) \,. \tag{5.20}$$

By the choice of x^2, Lemma 37, for all $S \in \mathcal{S}_{x^1}$

$$u_S \circ e_v(S, x^2) \leq t_1 \,\,.$$

Since for all $S \in \mathcal{A}^* \setminus \mathcal{S}_{x_1}$ it holds that $u_S \circ e_v(S, x^1) \leq t_1$. By Lemma 37

$$u_S \circ e_v(S, x^2) \leq t_1 \,.$$

This means that for each $S \in \mathcal{A}^*$ it holds that $u_S \circ e_v(S, x^2) \leq t_1$, hence $x^2 \in X(\mathcal{A}^*, t_1)$.

Since $x^* \in X(\mathcal{A}^*, t_1)$ is the closest point of set $X(\mathcal{A}^*, t_1)$ to the point x^1, we have that $x^2 = x^*$. However, then (5.20) contradicts (5.18). $\square$

Lemma 83. *Consider a utility function* $\mathbf{u}$*, a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system and* $t \in \mathbb{R}$*. If* $t < t_1$ *then* $X(\mathcal{A}^*, t) = X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) = \emptyset$*.*

Proof. By the definition of t_1 (see Section 5.2), we have that $X(\mathcal{A}^*, t) = \emptyset$. Moreover, we know that $X(\mathcal{A}^*, t_1) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$.

By Lemma 35 and that $t < t_1$ we have that $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1)$. Furthermore, by Lemma 82 it holds that $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1) = X(\mathcal{A}^*, t_1) = \mathbf{u}\text{-least-core}(v)$. Therefore,

$$X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) \subseteq X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t_1) = X(\mathcal{A}^*, t_1) = \mathbf{u}\text{-least-core}(v)\,.$$

Then by Lemma 80 and by the non-positivity of t for every $x \in X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t)$ and $S \in \mathcal{A}^* \setminus \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$ it holds that $\exists \mathcal{B} \subseteq \mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}$, such that

$$u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{B}} u_T \circ e_v(T, x) \leq t\,.$$

Then $x \in X(\mathcal{A}^*, t)$, that is, $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) \subseteq X(\mathcal{A}^*, t)$. Summing up, we can conclude that $X(\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}, t) = X(\mathcal{A}^*, t) = \emptyset$. $\square$

The following proposition is a direct corollary of Lemmata 82 and 83, hence we omit its proof.

Proposition 84. *Consider a* $\mathbf{u}$*-balanced game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system with the utility function* $\mathbf{u}$*, and* $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$*. If* $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ *then* $\mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-least-core}(v')$*.*

The following theorem is the main result of this section.

Theorem 85. *Consider a game* $v \in \mathcal{G}^{N,\mathcal{A}}$ *such that* $\mathcal{A}^*$ *contains a balanced set system with a utility function* $\mathbf{u}$ *such that* v *is* $\mathbf{u}$*-balanced. If* $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ *then* $\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ *is a characterization set for the* $\mathbf{u}$*-prenucleolus.*

Proof. If $\mathbf{u}\text{-least-core}(v) \neq \mathbf{u}\text{-core}(v)$ then by Proposition 84 we have that $\mathbf{u}\text{-least-core}(v) = \mathbf{u}\text{-least-core}(v')$, where $v' = v|_{\mathcal{E}_v^{\mathbf{u}} \cap \mathcal{E}_v^{d-\mathbf{u}}}$. Therefore, $N_{\mathbf{u}}^*(v)$, $N_{\mathbf{u}}^*(v') \subseteq \mathbf{u}\text{-least-core}(v)$.

By Lemma 80 for all $S \in \mathcal{A}^* \setminus (\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}})$, $x \in \mathbf{u}\text{-least-core}(v)$, there exists $\mathcal{F}_{S,x} \subseteq \mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ such that $u_S \circ e_v(S, x) \leq \sum_{T \in \mathcal{F}_{S,x}} u_T \circ e_v(T, x)$ and $\chi_S \in \operatorname{Lin}\{\chi_T \colon T \in$

$\mathcal{F}_{S,x} \cup \{N\}\}$. Since for all $T \in \mathcal{F}_{S,x}$ it holds that $u_T \circ e_v(T,x) < 0$ we have that $u_S \circ e_v(S,x) \leq u_T \circ e_v(T,x)$, for all $T \in \mathcal{F}_{S,x}$.

Moreover, by Lemma 79 for all $\alpha \in \mathbb{R}$ it holds that if $\mathcal{D}_{\mathbf{u}}^{\mathcal{A}^*}(x,\alpha)$ is balanced then $\mathcal{D}_{\mathbf{u}}^{\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}}(x,\alpha)$ is balanced.

Therefore, we can apply Theorem 73 on $X = \mathbf{u}$-least-core$(v), \mathcal{F} = \mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ and conclude that $\mathcal{E}_v^{d-\mathbf{u}} \cap \mathcal{E}_v^{\mathbf{u}}$ is a characterization set for $N_{\mathbf{u}}^*(v)$. □

Chapter 6

Summary

We introduce a generalization of the prenucleolus using utility functions, namely the **u**-prenucleolus. This generalization encompasses both the per capita prenucleolus (Grotte, 1970, 1972) and the q-nucleolus (Solymosi, 2019). Additionally, the **u**-prenucleolus is a special case of the general prenucleolus (Potters and Tijs, 1992; Maschler et al, 1992).

We consider TU-games with restricted cooperation. For such games, some of the original properties of the prenucleolus change; for example, the prenucleolus is no longer a single-valued solution. Katsev and Yanovskaya (2013) provided necessary and sufficient conditions for the prenucleolus to be non-empty and single-valued. We generalize these results to the **u**-prenucleolus.

Using the concept of utility functions, we introduce generalizations of the core, least core, balanced games, and essential coalitions: the **u**-core, **u**-least-core, **u**-balanced games, and **u**-essential coalitions, respectively. We generalize the Bondareva–Shapley theorem (Bondareva, 1963; Shapley, 1967; Faigle, 1989) by showing that a game is **u**-balanced if and only if its **u**-core is nonempty. Additionally, we generalize Huberman's theorem (Huberman, 1980), demonstrating that **u**-essential coalitions form a characterization set for the **u**-prenucleolus in **u**-balanced games.

We provide sufficient conditions on the utility functions for the **u**-prenucleolus and the **u**-core to be invariant. Additionally, we discuss a class of games and a utility function in which a game has a polynomial number of **u**-essential coalitions.

In the last chapter, we define a variant of essential coalitions, namely the class of dually-**u**-essential coalitions, which generalizes the concept of dually essential coalitions by Solymosi and Sziklai (2016) to TU-games with utility functions (Dornai and Pintér, 2024). Using this notion, we prove a variant of Huberman (1980)'s theorem, which also generalizes a theorem by Solymosi and Sziklai (2016), demonstrating that dually-**u**-essential coalitions form a characterization set for the **u**-prenucleolus of $\mathbf{u}$-balanced games.

Furthermore, we explore the dual of TU-games with utility functions and define the $\mathbf{u}^*$-anti-nucleolus, $\mathbf{u}^*$-anti-prenucleolus, $\mathbf{u}^*$-anti-core, $\mathbf{u}^*$-least-anti-core, and dually-$\mathbf{u}$-essential coalitions.

In addition, we generalize a result by Granot et al (1998) regarding characterization sets of the prenucleolus to TU-games with utility functions. Leveraging this theorem, we generalize a result by Solymosi and Sziklai (2016), demonstrating that the intersection of the set of $\mathbf{u}$-essential coalitions and the set of dually-$\mathbf{u}$-essential coalitions forms a characterization set for the $\mathbf{u}$-prenucleolus, provided the $\mathbf{u}$-least-core is a proper subset of the $\mathbf{u}$-core. Furthermore, we show that this intersection also characterizes the $\mathbf{u}$-least-core under these conditions.

Primal game	**Dual game**	**Coincide**
Game: $v(S)$	Dual game: $v^*(S) = v(N) - v(N \setminus S)$	✗
Feasible coalitions: $\mathcal{A} \subseteq \mathcal{P}(N)$, where $\emptyset, N \in \mathcal{A}$	Complementer set of feasible coalitions: $N \setminus \mathcal{A} := \{N \setminus S \colon S \in \mathcal{A}\}$	✗
excess: $e_v(S, x) = v(S) - x(S)$	satisfaction: $f_{v^*}(S, x) = x(S) - v^*(S)$	✗
$E_v(x) = (\ldots \geq e_v(S, x) \geq \ldots : S \in \mathcal{A}^*)$	$F_{v^*}(x) = (\ldots \geq f_{v^*}(S, x) \geq \ldots : S \in N \setminus \mathcal{A}^*)$	✗
$\text{core}(v) = \{x \in I^*(v) \colon x(S) \geq v(S), \forall S \in \mathcal{A}^*\}$	$\text{anti-core}(v^*) = \{x \in I^*(v^*) \colon x(S) \leq v^*(S), \forall S \in N \setminus \mathcal{A}^*\}$	✓
ε-core: $\{x \in I^*(v) \colon \max_{S \in \mathcal{A}^*} e_v(S, x) \leq \varepsilon\}$	ε-anti-core: $\{x \in I^*(v^*) \colon \max_{S \in N \setminus \mathcal{A}^*} f_{v^*}(S, x) \leq \varepsilon\}$	✓
least-core: ε^*-core, where $\varepsilon^* = \min_{\text{core}_\varepsilon(v) \neq \emptyset} \varepsilon$	least-anti-core: ε^*-anti-core, where $\varepsilon^* = \min_{\text{anti-core}_\varepsilon(v) \neq \emptyset} \varepsilon$	✓
nucleolus $N(v) = \{x \in I(v) \colon E_v(x) \leq_L E_v(y), \forall y \in I(v)\}$	anti-nucleolus: anti- $N(v^*) =$ $\{x \in \text{anti-}\, I(v^*) \colon F_{v^*}(x) \leq_L F_{v^*}(y), \forall y \in \text{anti-}\, I(v^*)\}$	✓
prenucleolus: $N^*(v) = \{x \in I^*(v) \colon E_v(x) \leq_L E_v(y), \forall y \in I^*(v)\}$	anti-prenucleolus: anti- $N^*(v^*) = \{x \in I^*(v^*) \colon F_{v^*}(x) \leq_L F_{v^*}(y), \forall y \in I^*(v^*)\}$	✓
essential coalitions ($\mathcal{A} = \mathcal{P}(N)$): $S \in \mathcal{P}^*(N)$, if $\lvert S \rvert = 1$ or $v(S) > \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v(T)$	anti-essential coalitions ($\mathcal{A} = \mathcal{P}(N)$): $S \in \mathcal{P}^*(N)$, if $\lvert S \rvert = 1$ or $v^*(S) < \min_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v^*(T)$	✗
essential coalitions: $S \in \mathcal{A}^*$, if $\mathcal{D}_S^{\mathcal{A}^*} = \emptyset$ or $v(S) > \max_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v(T)$	anti-essential coalitions: $S \in N \setminus \mathcal{A}^*$, if $\mathcal{D}_S^{N \setminus \mathcal{A}^*} = \emptyset$ or $v^*(S) < \min_{\mathcal{B} \in \mathcal{D}_S} \sum_{T \in \mathcal{B}} v^*(T)$	✗
$\mathbf{u}$-excess: $u_S \circ e_v(S, x)$	$\mathbf{u}^*$-satisfaction: $u_{N \setminus S} \circ f_{v^*}(S, x)$	✗
$E_v^{\mathbf{u}}(x) = (\cdots \geq u_S \circ e_v(S, x) \geq \ldots : S \in \mathcal{A}^*)$	$F_{v^*}^{\mathbf{u}^*}(x) = (\cdots \geq u_{N \setminus S} \circ f_{v^*}(S, x) \geq \ldots : S \in N \setminus \mathcal{A}^*)$	✗
$\mathbf{u}$-core$(v) = \{x \in I^*(v) \colon u_S \circ e_v(S, x) \leq 0, \forall S \in \mathcal{A}^*\}$	$\mathbf{u}^*$-anti-core$(v^*) = \{x \in I^*(v^*) \colon u_{N \setminus S} \circ f_{v^*}(S, x) \leq 0, \forall S \in N \setminus \mathcal{A}^*\}$	✓
$\mathbf{u}$-nucleolus: $N_{\mathbf{u}}(v) =$ $\{x \in \mathbf{u}\text{-}I(v) \colon E_v^{\mathbf{u}}(x) \leq_L E_v^{\mathbf{u}}(y), \forall y \in \mathbf{u}\text{-}I(v)\}$	$\mathbf{u}^*$-anti-nucleolus: anti- $N_{\mathbf{u}^*}(v^*) =$ $\{x \in \mathbf{u}^*\text{- anti-}\, I(v^*) \colon F_{v^*}^{\mathbf{u}^*}(x) \leq_L F_{v^*}^{\mathbf{u}^*}(y), \forall y \in \mathbf{u}^*\text{- anti-}\, I(v^*)\}$	✓
$\mathbf{u}$-prenucleolus $N_{\mathbf{u}}^*(v) = \{x \in I^*(v) \colon E_v^{\mathbf{u}}(x) \leq_L E_v^{\mathbf{u}}(y), \forall y \in I^*(v)\}$	$\mathbf{u}^*$-anti-prenucleolus anti- $N_{\mathbf{u}^*}^*(v^*) = \{x \in I^*(v^*) \colon F_{v^*}^{\mathbf{u}^*}(x) \leq_L F_{v^*}^{\mathbf{u}^*}(y), \forall y \in I^*(v^*)\}$	✓
$\mathbf{u}$-essential coalitions: $S \in \mathcal{A}^*$ if $\mathcal{D}_S^{\mathcal{A}^*} = \emptyset$ or if there exists $x \in \mathbf{u}$-least-core(v), such that $u_S \circ e_v(S, x) > \max_{\mathcal{B} \in \mathcal{D}_S^{\mathcal{A}^*}} \sum_{T \in \mathcal{B}} u_T \circ e_v(T, x)$	$\mathbf{u}^*$-anti-essential coalitions : $S \in N \setminus \mathcal{A}^*$ if $\mathcal{D}_S^{N \setminus \mathcal{A}^*} = \emptyset$ or if there exists $x \in \mathbf{u}^*$- least-anti-core(v^*), such that $u_{N \setminus S} \circ f_{v^*}(S, x) > \max_{\mathcal{B} \in \mathcal{D}_S^{N \setminus \mathcal{A}^*}} \sum_{T \in \mathcal{B}} u_{N \setminus T} \circ f_{v^*}(T, x)$	✗

Table 6.1: Summary

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