ADVANCES IN SPECTRAL OBSERVERS WITH UNEVENLY SAMPLED DATA

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I. Introduction

The theory of spectral observers is a well-studied area, where the Fourier spectral analysis is performed in real-time, by ongoing recursive calculations [1]. The spectral observer structure is derived from the state-variable formulation and it is essentially a deadbeat state observer. Spectral observers can be used as parametric signal estimators when a measured signal is considered as an output signal of a system represented by state space model. Fig.1. shows the block diagram of the spectral observer structure. A computationally efficient realization is already developed for the regularly sampled case in [2]. The main advantage of this structure is its realizability in hard real-time embedded systems, as the coefficients can be computed offline and only modest amount of calculations need to be performed online.

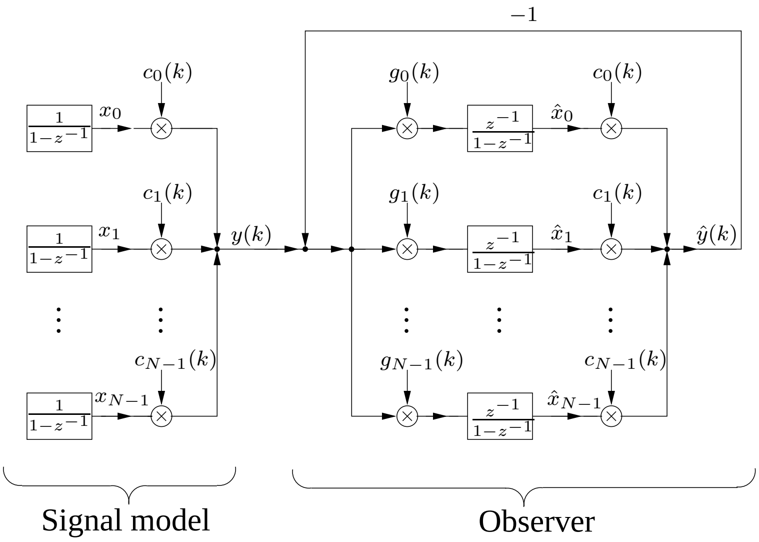

Figure 1: Spectral observer

Unfortunately, the efficient solution heavily rests on the assumption that the input signal is sampled regularly. However, in some cases this assumption does not hold as it is possible that due to implementation issues or resource constraints only uneven samples are available. This problem is critical in embedded and sensor network environments where the designer has to deal with serious resource constraints. First, the processor has to share its limited amount of CPU cycles among its tasks. Hence, it can not be guaranteed that sampling is performed precisely at a given time instant if it is not the highest priority task. Second, in case of wireless sensor networks (WSNs) the communication bandwidth is very limited and error prone, e.g. packet losses can result in missing samples if the signal samples are transmitted between two nodes of the network.

This paper presents the current state of the work in the area of spectral observers with unevenly sampled data. First, in section II. we summarize the foundations of our work. This is useful as the notations introduced here will be used in following sections. In section III. we examine the possibilities of improving an inefficient but theoretically sound solution to the general case of the uneven sampling problem. In section IV. we present an efficient solution to the special case of periodic uneven sampling. Finally, simulation results are shown and the conclusions are drawn.

II. Spectral Observers with Unevenly Sampled Data

This section presents the foundations of spectral observer theory in order to introduce the notations and the problem domain of our work. The block diagram of the conceptual signal model and the basic observer structure is shown in Fig.1. Let's denote $y(k)$ the sample of the input signal at sampling time instant t_k. Similarly, the values $c_m(k)$ and $g_m(k)$ denote the coefficients of the model at the time instant t_k. Note that in case of regular sampling $t_k = kT_s$ where T_s is the sampling interval.

Several different observers can be realized by the above structure, depending on the values $c_m(k)$. If $c_m(k) = e^{j(2\pi/N)mk}$, then a spectral observer is realized. Let's denote $\mathbf{c}(i) = [c_0(i)\ c_1(i)\ \ldots c_{N-1}(i)]^T$ and $\mathbf{g}(i) = [g_0(i)\ g_1(i)\ \ldots g_{N-1}(i)]^T$, $\mathbf{x}(i) = [x_0(i)\ x_1(i)\ \ldots x_{N-1}(i)]^T$ similarly.

The observer is called deadbeat if $\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1) = 0 \quad \forall k > N$. From the state equations it can be derived that:

$$\prod_{i=k-N+1}^{k} [\mathbf{I} - \mathbf{g}(i)\mathbf{c}^T(i)] = 0 \quad \Longrightarrow \quad \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1) = 0 \qquad \forall k > N \tag{1}$$

It is shown in [2], that if the signal is evenly sampled, then (1) is satisfied for every basis/reciprocal basis system. In case of the spectral observer $c_m(k) = e^{j(2\pi/N)mk}$ and $g_m(k) = \frac{1}{N}e^{-j(2\pi/N)mk}$.

In case of uneven sampling there is no constant sampling frequency, hence explicit time indexing has to be used. If we multiply the exponent by $f_sT_s = 1$ than f_s can be eliminated from the coefficients:

$$c_m(k) = e^{j(2\pi/N)mk} = e^{\frac{j2\pi f_s}{N}m(kT_s)} = e^{j\omega_1 m t_k}$$

where $\omega_1 = (2\pi f_s)/N$ is the circular frequency of the resonator with lowest nonzero circular frequency (the first spectral component).

It is important to note that observer theory assumes deterministic signals (i.e. signals without noise). The model does not address the problem of noise or measurement uncertainity in the theoretical level. Hence, theoretical results must be also examined from this point of view as these problems inevitably occur in practical applications.

III. General Case - Observer Gain Calculation Algorithm

In [3] Hostetter proposed an algorithm that ensures the deadbeat property of the spectral observer working with arbitrarly sampled data. This solution uses the same observer structure, the only difference is that the observer gain values $\mathbf{g}(k)$ are not known a priori, they are calculated online for every new time instant. The system equations of this structure are equivalent to the system depicted in Fig.1.

The proposed algorithm is proved to ensure a deadbeat observer (see [4] for the proof), but the amount of required online calculations makes this solution unrealizable in embedded or WSN environments. A promising goal of our future work is to develop an efficiently realizable variant of this algorithm.

IV. Special Case - Periodic Uneven Sampling

Although an efficient spectral observer is not known so far for the general case, we show that it does exist for a special case, namely if there are repeating patterns in the sampling time instants. If there is periodicity in the sampling process, then the sampling time instants follow each other with arbitrary difference, but the time difference between t_k and t_{k+N} is constant: $t_{(k+N)} - t_k = T_p \quad \forall k = 0..\infty$.

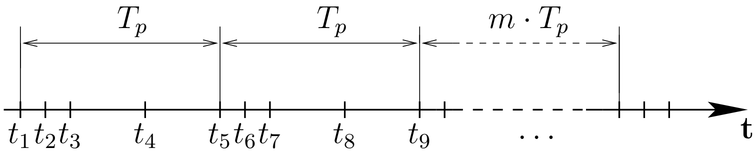

Figure 2: Periodic uneven sampling, N=4

Fig.2 shows an example of periodic uneven sampling with $N = 4$, where $t_1, t_2, ...t_i$ are the sampling time instants and T_p is the period. This situation, when a pattern N long is repeated periodically in the sampling time instants is realistic in certain applications, e.g. in case of burst sampling.

If the above property is true and the repeating pattern is known a priori, then the spectral observer can be utilized to reconstruct the measured signal in a deadbeat fashion i.e. with zero error. Denote $\mathbf{C}(k)$ the N x N matrix formed from the last N $\mathbf{c}(k)$ column vectors: $\mathbf{C} = [\mathbf{c}(k)\, \mathbf{c}(k-1)\, \mathbf{c}(k-2)\, \dots\, \mathbf{c}(k-N+1)]$ and similarly $\mathbf{G} = [\mathbf{g}(k)\, \mathbf{g}(k-1)\, \mathbf{g}(k-2)\, \dots\, \mathbf{g}(k-N+1)]$.

With these notations it can be easily shown that if we choose $\mathbf{G}(k)$ to statisfy $\mathbf{G}(k) = \mathbf{C}(k)^{-1}$, then Eq. 1 is satisfied $\forall k > N$. As $\mathbf{C}(k)$ is an N x N matrix formed from the last N $\mathbf{c}(i)$ column vectors ($i = k \dots k - N + 1$), therefore $\mathbf{C}(k+1)$ is formed from the column vectors $\mathbf{c}(i), i = k+1 \dots k-N$, i.e. it is a shifted version of the columns of $\mathbf{C}(k)$ except for the last column. If we ensure that the last column of $\mathbf{C}(k+1)$ is the same as the first column of $\mathbf{C}(k)$ for all k, than all $\mathbf{C}(i)$ matrices will consist of the same columns, but they are circularly shifted. In this case all of the corresponding inverse matrices $\mathbf{G}(i)$ will consist of the same rows which are shifted circularly. This means that the $\mathbf{G}(k)$ can be calculated offline and the actual $\mathbf{g}(k)$ values can be read out easily.

In the following, we show that the columns $\mathbf{c}(i+N)$ and $\mathbf{c}(i)$ are equal if and only if the time difference between the sampling time instants t_i and $t_i - N$ is a multiplier of T_1, the period of the lowest frequency resonator.

$$
\begin{aligned}
\mathbf{c}(i+N) = \mathbf{c}(i) \iff c_m(i+N) &= c_m(i) \qquad \forall m \in 1 \dots N-1 \\
e^{jm\omega_1 t_{i+N}} &= e^{jm\omega_1 t_i} \\
jm\omega_1 t_{i+N} &= jm\omega_1 t_i + k2\pi \qquad k \in \mathbf{N} \\
t_{i+N} &= t_i + \frac{k2\pi}{m\omega_1} \quad = \quad t_i + \frac{kT_1}{m} \\
t_i - t_{i-N} &= k\frac{T_1}{m} \qquad \forall m \in 1 \dots N-1 \qquad k \in \mathbf{N}
\end{aligned}
$$

This has to be true for all $m \in 1 \dots N-1$. But as k can be arbitrary, if the property is true for $m = 1$ then it is true for all other m, as an appropriate k multplier can always be found to satisfy the equation. We can summarize this as follows:

$$
c(i+N) = c(i) \iff t_i - t_{i-N} = T_p = kT_1. \tag{2}
$$

An attractive property of the spectral observer is that the frequencies of the resonators can be tuned easily by using the Adaptive Fourier Analyzer (AFA) algorithm proposed in [5]. Combining the AFA with the above proposed observer gain calculation algorithm the property (2) can be ensured by tuning the frequency of the first resonator to $\frac{1}{T_p}$.

Another condition that falls out from the above deduction is that exactly N arbitrary samples are needed from the T_p period. If there are more samples available, N of them has to be chosen in a way that the N different $c(k)$ vectors are linearly independent of each other to ensure that the $C(k)$ matrix is invertible.

It is important to point out a possible numerical problem of the solution. In case the $\mathbf{C}(k)$ matrix is ill-conditioned i.e. its condition number is so large that its reciprocal is close to the numerical precision of the computing machine, then the calculated $\mathbf{G}(k)$ values can be very unprecise. Therefore, if there are more than N samples are available from the T_p period, those samples have to be chosen which provide the minimum condition number for $\mathbf{C}(k)$. In case this minimum condition number is still too large, then the spectral observation is not possible by the proposed solution.

V. Simulation Results

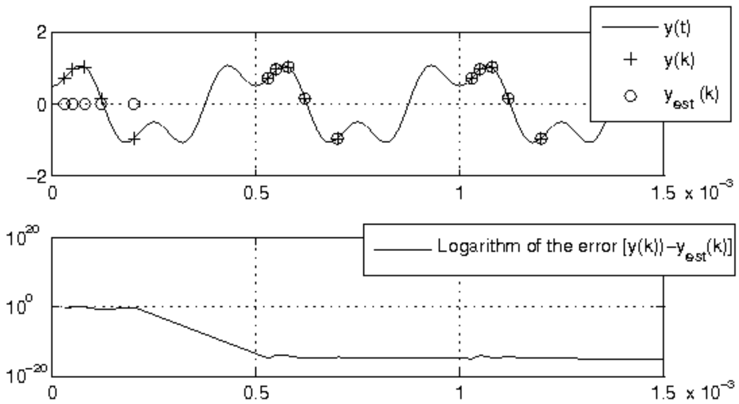

Figure 3: The estimated signal samples and the error of the observer $[y(k) - \hat{y}(k)]$

In Fig. 3 we present an example with an input signal sampled periodically and unevenly ($N = 5$). The input signal is a simple periodic signal formed from two sinusoid components, a base harmonic and the third upper harmonic. The original analog signal $y(t)$ is plotted with a solid line, the sampled signal $y(k)$ is represented by the "+" signs, and the output of the observer $\hat{y}(k)$ is shown by the "o" signs. The observer error $[e(k) = y(k) - \hat{y}(k)]$ is plotted with a logarithmic scale. It is clear that the observer error is reduced to zero (within numeric precision) after N steps, proving the deadbeat property.

VI. Conclusion and Future Work

The well-known spectral observer structure can be modified to work with unevenly sampled data easily. The algorithm mentioned in section III. is an inefficient but formally correct solution to realize a spectral observer with arbitrary signal samples. The main reason of the inefficiency is the online calculation of the observer gain values. Section IV. shows an efficient solution to this problem for the special case of periodic uneven sampling, where the observer gain values can be calculated offline.

In the future, first we will examine the mentioned numerical limitation of the solution proposed in section IV. and work out a formal bound on the usability of the solution. Second, we will try to develop an efficient and realizable algorithm based on Hostetter's algorithm for the general case.

References

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