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Modelling and measuring of viscoelastic material behaviour under cyclic loads

Booklet of theses

by

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1 Subject of the work

Polymers are very popular structural materials nowadays. Therefore, description of their cyclic mechanical behaviour is an important problem. My PhD thesis deals with some subjects related to this field.

The first part of the work presents the choice of material model parameters for the finite element description of the cyclic mechanical behaviour of a polypropylene in the 10–100 Hz range. A large number of previous works and material models is available in the literature. However, only a few of these material models are incorporated to finite element codes. Therefore, it seems logical to choose the well-known and relatively simple generalized Maxwell (or generalized standard solid[1]) model. Contrarily, material parameters should be determined by measurements.

In the second part of the work, a finite element description of linear viscoelastic beams is examined. The beam element based on the anelastic displacement field formulation (Lesieutre and Lee, 1996) is more suitable for some problems in vibration analysis of linear viscoelastic beam structures than the formulation usually included in commercial software. However, the original paper leaves some questions open, which makes room for several variants of the element.

Finally, the third part deals with the solution of a fractionally damped single-degree-of-freedom vibration equation. Fractional calculus is used by some researchers to model material behaviour, and is therefore of interest for the present work. Several authors have proposed solution methods (Suarez and Shokooh, 1997; Yuan and Agrawal, 2002; Saha Ray et al., 2005), but neither of them has been widely adopted. Therefore, it seems useful to compare some of the existing procedures. Also, new solution methods can be proposed for the accurate solution of the fractionally damped vibration equation.

2 Goal of the work

In the above, some outlines of the context of the work have been presented. Based on them, the goals of the work are:

1. To find a material model for the finite element description of the vibration behaviour of polypropylene in the 10–100 Hz range.

2. To examine the finite element proposed by Lesieutre and Lee, and propose a modification, if necessary.

[1] In this work, the term 'generalized Maxwell model' is used, conforming with most finite element codes.

3. To compare literature solution methods for the fractionally damped vibration equation.

4. To find a more efficient technique to obtain high-precision solutions of the fractionally damped vibration equation.

3 Theses

1. **I have shown that the cyclic mechanical behaviour of polypropylene in the observed frequency range can be efficiently modelled by the generalized Maxwell model with the parameters given in the dissertation.**

   To find material model parameters for polypropylene in the range of 10–100 Hz, forced and free vibration measurements have been performed on specimens of the material, and two parameter sets have been proposed for the generalized Maxwell model. To verify them, forced vibration of a machine part made of the same material has been studied by measurement and finite element calculation. Results show a fair agreement, with the second set of material model parameters performing better.

   These results are detailed in Part I of the dissertation.
   Related publications: [1, 2, 6, 8–10].

2. **I have proven that the beam finite element based on the anelastic displacement field description can be used more efficiently, if the boundary conditions are applied also on the internal degrees of freedom.**

   Finite element modelling of linear viscoelasticity using anelastic displacement fields (ADF) has been studied earlier by other researchers. In the present work, two new variants of the Euler–Bernoulli beam element developed based on the ADF method are analysed. One of them adds boundary conditions to the viscoelastic degrees of freedom (RADF: Restricted ADF; the idea has been mentioned in the literature), while the other introduces an internal node for these degrees of freedom (EADF: Element ADF). I have shown that these variants lead to the same physical vibration modes as the original one. Also, I have proven that numerical conditioning of the original ADF element is inferior to the new variants. Furthermore, a model by the RADF variant has at most the same number of degrees of freedom as a model by EADF.

   These results are detailed in Part II of the dissertation.
   Related publication: [4].

3. **Comparing different methods on sample problems of fractionally damped vibrations, I have shown the followings:**

   i. For the solution of a fractionally damped vibration equation with zero initial conditions, the method of Yuan and Agrawal is usually efficient.

   ii. A direct method using Caputo's derivative can sometimes be even more efficient (depending on parameters).

   iii. For non-zero initial conditions, either a method based on Adomian decomposition or a direct method based on the Grünwald–Letnikov definition can be advised.

   The solution of a fractionally damped vibration equation is a field in development. I compared some literature methods on sample problems. Based on the results, it can be stated that:

   (a) For a derivative of order 1/2, the method of Suarez and Shokooh is efficient enough in terms of error and required calculation effort. It provides an analytical solution for a step function excitation, but is not the most accurate among the examined methods for the test case with a harmonic excitation.

   (b) The method based on Adomian decomposition provides the same accuracy as the method of Suarez and Shokooh, but is one-two magnitudes slower for a long simulated time interval. For a shorter simulated time, the required CPU time is favourable.

   (c) The method of Yuan and Agrawal is usually very efficient, as it leads to an engineering precision in a reasonable time for all studied problems, being the fastest among the tested methods in several cases.

   (d) The tested direct methods (based on the Grünwald–Letnikov and Caputo formulations) are much less accurate than either the solution of Suarez and Shokooh or the Adomian decomposition. However, their calculation time was acceptable for the test problems. In some test cases for a low–medium precision, they are faster than the method of Yuan and Agrawal.

   These results are detailed in Part III of the dissertation (Section 8.1). Related publications: [3, 7].

4. **I have developed a new method for the solution of a fractionally damped, single-degree-of-freedom vibration equation, combining Taylor and Adomian series. I have proven that a very**

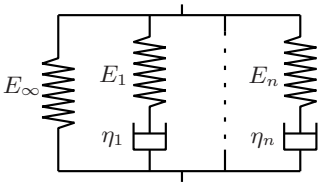

Figure 1: Generalized Maxwell model with parameters.

accurate solution can be obtained magnitudes faster by it than by a direct method.

The solution of a fractionally damped, single-degree-of-freedom vibration equation can be obtained quite quickly with an almost arbitrary precision using the new method, if the excitation function has a convergent Taylor series. The method allows non-zero initial conditions. Its main drawback is the fast increase of calculation time with the length of the simulated time interval.

These results are detailed in Part III of the dissertation (Section 8.2). Related publication: [5].

4 Appendices to theses

4.1 Appendix to Thesis 1

The goal was to find material model parameters for polypropylene in the range of 10–100 Hz. The workflow was the following:

1. Forced vibration of a polypropylene machine part has been studied experimentally. The frequency–amplification function was calculated.

2. Similar forced vibration experiments have been carried out on a simple-geometry specimen to study material behaviour. The same specimen has also been studied in free vibration.

3. Based on the simple-geometry measurements, material parameters for the generalized Maxwell (or generalized standard solid) model have been chosen.

4. The forced vibration of the machine part, measured in step 1, has been simulated by finite elements using the material model from step 3.

The results in the last step have shown a fair agreement. The material model parameters were:

$$E_0 = 1650 \text{ MPa}, \; E_1 = 175 \text{ MPa}, \; \tau_1 = 7.958 \cdot 10^{-3} \text{ s},$$
$$E_2 = 175 \text{ MPa and } \tau_2 = 1.326 \cdot 10^{-3} \text{ s}.$$

4.2 Appendix to Thesis 2

The Anelastic Displacement Field description, used by Lesieutre and Lee to develop a viscoelastic beam finite element, uses internal strains to describe the behaviour of the generalized Maxwell model. To these internal strains, displacement fields are attribued, which are treated the same way as physical displacements. Therefore, in a finite element model, additional degrees of freedom describe these displacement fields.

The questions studied in thesis 2 are:

1. Should boundary conditions be applied to the degrees of freedom added to nodes (RADF variant)? *(Answer: yes, applying them leads to better numerical conditions.)*

2. Should the additional degrees of freedom be added at each node, or should they be added to each element (EADF variant)? *(Answer: both lead to the same physical result, with the degrees of freedom on elements leading to a larger model.)*

Based on the experiences, the RADF variant is proposed for future use.

4.3 Appendix to Thesis 3

The fractionally damped vibration equation is written as

$$m\mathbf{D}^2 x(t) + c\mathbf{D}^\alpha x(t) + kx(t) = F(t)\,, \tag{1}$$

where m and k are the mass and stiffness of the system, as usual, $F(t)$ is the excitational force and c and α describe the damping, $\mathbf{D}$ being the derivative operator. Another common form for the equation is

$$\mathbf{D}^2 x(t) + 2\zeta\omega_\mathrm{n}^{2-\alpha}\mathbf{D}^\alpha x(t) + \omega_\mathrm{n}^2 x(t) = f(t)\,,$$

with

$$2\zeta\omega_\mathrm{n}^{2-\alpha} = \frac{c}{m}, \; \omega_\mathrm{n}^2 = \frac{k}{m} \text{ and } f(t) = \frac{F(t)}{m}.$$

The above-mentioned methods (Suarez and Shokooh, 1997; Yuan and Agrawal, 2002; Saha Ray et al., 2005; direct method based on the Grünwald–Letnikov definition; direct method based on the Caputo definition) have been compared on some sample problems with zero initial conditions. The problems are:

1. Harmonic excitation, with $\alpha = 1/2$, $\omega_\mathrm{n} = 10$, $\zeta = 0.5$ and $f(t) = \sin(4\pi t)$, for $t = 0 \ldots 10$. For this problem, no analytical solution was not known.

2. Unit step excitation, with $\alpha = 1/2$, $\omega_\mathrm{n} = 10$ or 5, $\zeta = 0.5$ or 0.1 (all four combinations have been investigated) for $t = 0 \ldots 4$. One case ($\omega_\mathrm{n} = 10$, $\zeta = 0.5$) has also been calculated for $t = 0 \ldots 10$. The analytical solution was known here, given by Suarez and Shokooh in 1997.

The calculation time (in Maple) and the average absolute error

$$\delta = \frac{1}{N} \sum_{i=1}^{N} \left| x_i^{\mathrm{num}} - x_i^{\mathrm{ref}} \right|$$

have been used to judge the efficiency of a method. Based on the tests, the followings can be stated:

1. For a step function excitation, the method of Suarez and Shokooh leads to an exact solution in a reasonable time (21–24 seconds for the 'short' test cases). For the harmonic case, the calculation time was 78 seconds, and the discretization of the excitation apparently caused a larger error than the error of the Yuan–Agrawal scheme.

2. The method based on Adomian decomposition provides the same accuracy as the method of Suarez and Shokooh, but is one-two magnitudes slower for a long simulated time interval. For a shorter simulated time, the required CPU time is favourable (47–59 seconds), if the goal is to deal with any order of derivative (taking into account that the method of Suarez and Shokooh is only usable for $\alpha = 1/2$).

3. The method of Yuan and Agrawal is usually very efficient, as it leads to an engineering precision (when the error is 1% of the steady-state value) in 4–10 seconds in all unit-step cases. Moreover, it is the fastest to do so among the tested methods, when damping is low.

4. The tested direct methods (based on the Grünwald–Letnikov and Caputo formulations) are much less accurate than either the solution of Suarez and Shokooh or the Adomian decomposition. However, their calculation time was acceptable for the test problems. In the highly damped test cases, calculation to an engineering precision was faster than the method of Yuan and Agrawal, requiring less than a single second. Contrarily, the same precision for the cases with a low damping needed 30–50 seconds.

Based on the above, the method of Yuan and Agrawal can be suggested for general use in case of a harmonic excitation, taking into account that it is restricted to zero initial conditions. When there are non-zero initial conditions, the method based on the Grünwald–Letnikov definition seems to be the best solution for general purpose. The Adomian decomposition is very useful whenever a high precision is required, but it has a higher computational cost.

4.4 Appendix to Thesis 4

The solution of the fractionally damped, single-degree-of-freedom vibration equation can be obtained with a relatively low computational cost, if the excitation function $F(t)$ can be written as the Taylor series

$$F(t) = \sum_{i=0}^{\infty} T_i t^i.$$

In this case, the solution of Equation (1) can be calculated as

$$x(t) = \sum_{n=0}^{\infty} \left[x_n^{\mathrm{IC}}(t) + \sum_{i=0}^{\infty} x_{n,i}^{f}(t) \right], \tag{2}$$

with

$$x_n^{\mathrm{IC}}(t) = \frac{(-1)^n}{m^n} t^{(2-\alpha)n} \sum_{j=0}^{n} \left[\binom{n}{j} c^{n-j} k^j t^{j\alpha} \times \left(\frac{X_0}{\Gamma(2n+1-(n-j)\,\alpha)} + \frac{V_0 t}{\Gamma(2n+2-(n-j)\,\alpha)} \right) \right] \tag{3}$$

and

$$x_{n,i}^{f}(t) = T_i \frac{(-1)^n}{m^n} \Gamma(i+1)\, t^{i+(2-\alpha)n+2} \times \sum_{j=0}^{n} \binom{n}{j} \frac{c^{n-j} k^j}{\Gamma(i+3+2n-(n-j)\,\alpha)} t^{j\alpha}, \tag{4}$$

where $X_0 = x(0)$ and $V_0 = \dot{x}(0)$ are the initial conditions, and $\Gamma(x)$ is the Gamma function.

As is immediate to see, the right-hand sides of Equations (3) and (4) are 'polynomials' with non-integer exponents, and these exponents are growing rather fast. Moreover, experience shows that they require the calculation of differences of large numbers close to each other, which means that conventional

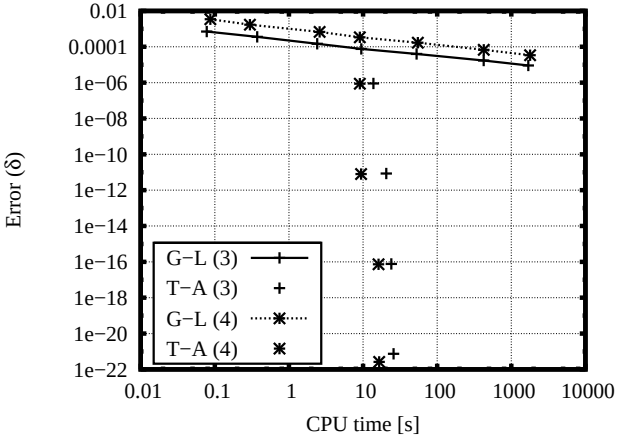

Figure 2: Comparison of methods for sample problems 3 and 4. G–L: direct method based on the Grünwald–Letnikov definition, T–A: new procedure, called Taylor–Adomian method.

double-precision arithmetics is not sufficient. However, the method is still several magnitudes faster than a direct method, when the required calculation precision is high, as illustrated by Figure 2, where the results of two sample problems with harmonic excitation are shown. The main drawback of the method is the fast increase of calculation time with the length of the simulated time interval.

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