Multidegrees of Singularities
and Nonreductive Quotients
Thesis booklet

Gergely Bérczi
Supervisor: András Szenes

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1 Background

The thesis lies in singularity theory, and uses methods of algebraic geometry and algebraic topology.

Algebraic geometry combines techniques of abstract algebra with the language and intuition of geometry. It occupies a central place in modern mathematics and has also multiple connections with physics. The central objects of algebraic geometry are polynomial equations in many variables: algebraic geometers attempt to understand the totality of the solutions of such a system of equations. The most elementary examples are well-known plane curves like the line, circle, parabola, hyperbola (such curves were already extensively studied by the Hellenistic Greeks such as Archimedes and Apollonius, who can be considered the founders of algebraic geometry). For instance, the two-dimensional sphere in three dimensional Euclidean space can be defined as the set of all points \((x, y, z)\) with

\[x^2 + y^2 + z^2 - 1 = 0.\]

In the 1950s and 1960s Jean-Pierre Serre and Alexander Grothendieck (aided by Artin, Mumford, and many others) recast the foundations making use of sheaf theory. This transformed the subject putting it on a radically new footing. As a result, algebraic geometry became very useful in other areas of mathematics, as well as physics. For instance, Deligne used it to prove a variant of one of the most famous unsolved problems of mathematics, the Riemann hypothesis, and Andrew Wiles’ celebrated proof of Fermat’s last theorem also used tools developed in algebraic geometry. These tools are also the building blocks – among other theories of mathematics – of one of the most prominent and ambitious theories in fundamental physics, string theory.

The history of algebraic geometry is closely related to that of topology, another fundamental branch of mathematics, which also plays a key rôle in the thesis, especially localisation methods in algebraic topology. The motivating insight behind topology is that the answers to many geometric problems depend not on the exact shape of the objects involved, but rather on a much looser concept of shape. For example the square and the circle are essentially the same from a topological point of view: they are both one-dimensional objects and both separate the plane into two parts, the part inside and the part outside. So such fundamental notions as length, angle and area do not exist in topology. Intuitively, two spaces are topologically equivalent if one can be deformed into the other without cutting or gluing. Combining the fine tools of algebraic geometry with topological approaches has resulted in many very important theorems.

The remaining crucial ingredient in the thesis is symmetry; that is, group actions. Objects in geometry often come with some symmetry; in formal
terms, we say that an object is symmetric with respect to a given mathematical operation if, when applied to the object, this operation does not change the object or its appearance. For example a square is symmetric with respect to rotation about the centre with angle some multiple of $\pi/2$, and is also symmetric with respect to reflection in its diagonals. Another simple example is the polynomial $p(x, y, z) = x + y + z$ in three variables: permuting the variables does not change the polynomial, and the permutation group $S_3$ acts as a group of symmetries. Symmetry is modelled mathematically via group actions $G \times X \to X$, where $G$ is a group acting on a space $X$ and the image of $(g, x)$ is written as $gx$. The orbit of $x$ in $X$ is $\{gx : g \in G\}$, which is the single point $x$ if $x$ is a fixed point for the action.

Symmetries are crucial throughout much of mathematics and physics, in particular in algebraic geometry and topology. The set of fixed points of a group action often stores significant information about the topology of a space – we can often read off topological invariants, such as the Euler characteristic, from the behaviour of a group action or a vector field near its fixed points. This is the basis for the localisation theorems [2, 6, 15] to be used in the thesis in order to study the topology of ‘quotient spaces’ in algebraic geometry. Such quotient spaces appear throughout geometry when we do not want to make a distinction between points which lie in the same orbits of a group action, so we try to construct a ‘space of orbits’ or quotient space, for the group action. This is often crucial in constructing and understanding moduli spaces (parameter spaces for families of geometric objects) which is one of the central problems of algebraic geometry, and is of great importance in related areas of geometry and of theoretical physics.

This thesis gives an iterated residue formula for the Thom polynomials of $A_d$ singularities, for all $d \leq 1$. Thom polynomials are important topological and algebraic invariants of singularities. The main difficulty of computation of these invariants is the highly nontrivial symmetry group, which is a diffeomorphism group. By now there are three known efficient methods for calculating them. The first, classical way is the method of resolutions – this is also effective at computations of other invariants of singular varieties. The second, is based on an idea of R. Rimányi and called the method of restriction equations. A standard reference of this is [30], and a nice summary of this method and its applications is [25].

The problem has a rich history. The case $d = 1$ is the classical formula of Giambelli-Thom-Porteous in algebraic geometry. The Thom polynomial in the $d = 2$ case was computed by Ronga in [31]. More recently, in [7], the authors proposed a formula for $d = 3$; P. Pragacz has given a sketch of a proof for this conjecture [28]. Finally, using his method of restriction equations, Rimányi was able to treat the zero-codimension case [29] (cf. [12] for the case $d = 4$. \vspace{3pt}
2 Purpose of the thesis

In the thesis we give a short formula for an algebraic invariant – called multidegree – of Morin singularities. These singularities – which are also called $A_d$ singularities – play a central role in global singularity theory.

To make the purpose clear, we give a short description of the subject in a little more detail. The main objects of study in global singularity theory are maps between manifolds. Manifolds are objects which locally look like Euclidean spaces, though globally they can have very varied ‘shapes’: a sphere at every point looks locally like the plane, as does a torus, but globally they are very different. Maps between manifolds can be described by choosing local coordinates: near any point $p$ a smooth map $f : M \to N$ from an $m$-dimensional manifold $M$ to an $n$-dimensional manifold $N$ with local coordinates $x_1, \ldots, x_m$ on $M$ near $p$ and $y_1, \ldots, y_n$ on $N$ near $f(p)$ is given by a smooth map from $\mathbb{R}^m$ to $\mathbb{R}^n$:

$$(x_1, x_2, \ldots, x_m) \mapsto (f_1(x_1, x_2, \ldots, x_m), \ldots, f_n(x_1, x_2, \ldots, x_m)).$$

(It is often important to work with complex manifolds, replacing the real numbers $\mathbb{R}$ with the complex numbers $\mathbb{C}$). There is of course a great deal of freedom in choosing the local coordinates; different choices are related by local diffeomorphisms of the neighbourhoods. In singularity theory, in order to understand global maps, we study local maps between Euclidean spaces, but we need to handle the change of coordinates, so we need to understand the local diffeomorphism groups, which are highly complicated, infinite dimensional, non-reductive groups, and to take appropriate quotients by their actions.

For a given map $f : M \to N$ between manifolds of dimensions $m$ and $n$, one can classify the points of the source manifold $M$ according to the local behaviour of the map at the point. There are singular points in the source space, where infinitesimally $f$ has smaller dimensional image than the dimension of $M$, and singular points of different types form ‘cycles’ in $M$. Here, the classification of the singular points is based on the local algebra of the map germ at the given point, defined as follows. Taking the local form (1) as above, the local algebra at this point is defined as

$$A_f(p) = \mathbb{C}[x_1, \ldots, x_n]/(f_1, f_2, \ldots, f_k)$$

This is invariant up to isomorphism under local reparametrisation of the source and target spaces. Given a local algebra $A$, the points of the source manifold where the map germ has local algebra isomorphic to $A$ provide us with a cycle $M(A)$, with topological invariants given by intersection numbers. The most fundamental intersection number is the Poincaré dual, or multidegree, of $M(A)$.

In this thesis we investigate the multidegree of $M(A)$ for a very special class of local algebras $A$, called Morin singularities. Using an algebraic model
of Gaffney [12], we give an iterated residue formula for the multidegrees of Morin singularities. The heart of the thesis is to handle the problem of quotienting by the diffeomorphism group.

The standard construction of quotient spaces in algebraic geometry uses Mumford’s Geometric Invariant Theory (GIT) [27]. This theory, however, applies for reductive quotients, where we quotient by a reductive group. The purpose of this thesis is to make the first steps for nonreductive quotients, and we give a proper compactification of such quotients in special cases, for nonreductive diffeomorphism groups.

3 Methods

The first observation is that the germs (1) with local algebra $A$ (this set is denoted by $\Sigma(A)$) fibres equivariantly over a base space with respect to the action of a sub-torus of the diffeomorphism group. The base of this fibration is a non-compact quotient $X//H$ of a vector space $X$ by a linear action of a non-reductive finite-dimensional subgroup $H$ of the diffeomorphism group of $\mathbb{C}$. The main ingredient of the work is a subtle compactification of this non-reductive quotient, which makes it possible to use a two-step localisation process:

1. First we observe that $X//H$ fibres over a partial flag variety $X//B$ (where $B$ is a Borel containing $H$), and an iterated residue formula on the flag variety is given, allowing attention to be restricted to the fibres over those flags which are fixed by the torus action.

2. After compactification of the fibres over the fixed flags, we use the abelian localisation theorem [2, 6] on the fibres.

The result is a residue formula which is a sum of terms indexed by the fixed points in the compactified fibre. However something unexpected happens: all but one of these terms contribute zero to the result, so that all the relevant information is stored at one fixed point.

The resulting formula involves an extra parameter: a polynomial, which is the multidegree of a Borel orbit in a vector space. The computation of multidegrees for toric varieties is a well-understood task (see [26]), but much less is known for Borel orbits. In the last section 5.3 we use Gröbner degeneration and representation theoretic methods to provide a relation between the multidegrees of the Borel orbit and its toric part.

4 Results of the paper

- **Thesis 1**
  Using a model of Gaffney [12], we give an iterated residue formula
for multidegrees of an important one-parameter class of singularities, called Morin singularities. This formula is a deep generalisation of the classical Giambelli-Thom-Porteous formula [11, ?, 13]. This result is the first half of the thesis, including Chapters 1-4, and Section 5.1. This result is published in [4].

- Thesis 2
  The main idea of the formula is a proper compactification of a nonreductive quotient. We do localisation on this quotient. The compactification is stated in Proposition 4.2.8. This was also published in [4].

- Thesis 3 In Section 5.1 we compute the above mentioned unknown parameter of the residue formula, which is the multidegree of a Borel orbit in a complex vector space. The results for \( d = 2, 3, 4, 5 \) are 5.1.1, 5.1.2, 5.1.7, 5.1.10, respectively. These examples are also published in [4].

- Thesis 4
  The iterated residue formula of Thesis 1 involves an extra parameter: a polynomial, which is the multidegree of a Borel orbit in a complex vector space. Each Borel-orbit contains a toric-orbit, namely, the orbit of the maximal torus contained in the Borel subgroup. In the last Section 5.2 we use Gröbner degeneration and representation theoretic methods to provide a relation between the multidegrees of the Borel orbit and its toric part.

  This result is going to be published in the forthcoming paper [5].

- Thesis 5 In Section 5.3 we give an important application of our main theorem mentioned in Thesis 1. We prove a conjecture of R. Rimnyi for \( d = 3 \). This says that the coefficients of the Thom polynomial are positive for any \( d \). This result is also published in [4].

5 Publications connected to the dissertation


References


