New constructions in classical invariant theory
Summary of PhD dissertation

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1 Orthogonal vector invariants in characteristic 2

1.1 Introduction

1.1.1 Summary of results

Chapter 1 of the dissertation is based on the two papers [DF1, DF2], which are joint with Mátyás Domokos. Strictly speaking, only this chapter of the dissertation is about invariant theory, i.e., about the structure of the ring of invariant polynomials under a certain group action. Namely, it is shown that the invariant theory of the orthogonal group acting on the direct sum of several copies of the standard vector representation differs drastically over fields of characteristic 2 from the well-known theory in all other characteristics. As a result, we encounter non-classical behaviour also over the ring of integers.

More precisely, we investigate the ring of those polynomials depending on the entries of a generic $n \times m$ matrix that are invariant under left multiplication by elements of the group O_n, resp. SO_n. Over fields of characteristic zero, the structure of these rings (in terms of generators and relations) has been known for about a century, due mainly to Hermann Weyl [W]. In the 1970's, De Concini and Procesi [CP] showed that formally the same results hold over fields of odd characteristic. Elementary proofs of their results were given by Richman [R] in 1989. The ring generators in the case of O_n are of degree 2; for SO_n they are of degree 2 and of degree n.

In Theorems 1.4.2, 1.4.5 and 1.4.8 of the dissertation, we show that the case of characteristic 2 is drastically different: for fixed $n \geq 3$ and m large, indecomposable invariants of arbitrarily high degree appear.

We encounter similar phenomena over $\mathbb{Z}$. The orthogonal group is defined with respect to a non-degenerate quadratic form with integer coefficients. If we start with the usual sum of squares as quadratic form, then the results of De Concini and Procesi [CP] are valid (low degree generators). However, if we start with the quadratic form

$$x_1y_1 + \cdots + x_ry_r\{+z^2\},$$

then our result (existence of high degree indecomposable invariants) is valid. This is the content of Corollaries 1.4.3, 1.4.6 and 1.4.9 in the dissertation.

On the other hand, we show that invariant rational functions in characteristic 2 behave basically the same way as in all other characteristics, and so do the invariant polynomials if $m \leq n$. We show for all n and m that the algebra of invariant polynomials is a finitely generated module over the subalgebra generated by the quadratic invariants.

1.1.2 The orthogonal group

Let $\mathbb{F}$ stand for an algebraically closed field of arbitrary characteristic. Let n be a positive integer. Denote coordinates in $\mathbb{F}^n$ by $x_1, y_1, \ldots, x_r, y_r$ if $n = 2r$ or by $x_1, y_1, \ldots, x_r, y_r, z$ if $n = 2r+1$. The *standard quadratic form* in n variables is

$$q = x_1y_1 + \cdots + x_ry_r \qquad \text{when } n = 2r,$$
$$q = x_1y_1 + \cdots + x_ry_r + z^2 \qquad \text{when } n = 2r+1.$$

The *orthogonal group* $\mathrm{O}_n(\mathbb{F})$ is defined as the group of linear isomorphisms of $\mathbb{F}^n$ that leave the quadratic form q invariant. The group $\mathrm{O}_n(\mathbb{F})$ is a Zariski closed subvariety of the space $M_n(\mathbb{F})$ of linear transformations of $\mathbb{F}^n$. All elements of $\mathrm{O}_n(\mathbb{F})$ have determinant ± 1. Recall that in the Zariski topology, the irreducible components of any algebraic group coincide with the connected components. The *special orthogonal group* $\mathrm{SO}_n(\mathbb{F})$ is defined as the component of $\mathrm{O}_n(\mathbb{F})$ containing the identity.

The group $\mathrm{O}_n(\mathbb{F})$ is generated by reflections. The group $\mathrm{SO}_n(\mathbb{F})$ is precisely the set of linear transformations that are products of an even number of reflections [F0, T].

1.1.3 Invariants

We work over an algebraically closed field $\mathbb{F}$. We write $R = \mathbb{F}[n \times m]$ for the algebra of polynomials over $\mathbb{F}$ in $n \times m$ variables. We arrange the variables to form the $n \times m$ matrix V. The columns of V are n–dimensional vectors $v^{(1)}, \ldots, v^{(m)}$. A typical element of R can be written as

$$f(V) = f\left(v^{(1)}, \ldots, v^{(m)}\right).$$

We shall write R^G for the subalgebra formed by the polynomials invariant under the group $G = \mathrm{O}_n(\mathbb{F})$ or $G = \mathrm{SO}_n(\mathbb{F})$, where each $g \in G$ acts on $n \times m$ matrices by left multiplication (equivalently, acts on the column vectors by linear substitution):

$$g : V = \left(v^{(1)}, \ldots, v^{(m)}\right) \mapsto gV = \left(gv^{(1)}, \ldots, gv^{(m)}\right).$$

Explicitly,

$$R^G = \{f(V) \in R \mid f(gV) = f(V) \quad (g \in G)\}.$$

Next we define some distinguished elements in R. Set

$$(kk) = (kk)_n = q\left(v^{(k)}\right) \qquad (1 \leq k \leq m)$$

and

$$(kl) = (kl)_n = \beta\left(v^{(k)}, v^{(l)}\right) \qquad (1 \leq k \leq m, \quad 1 \leq l \leq m, \quad k \neq l).$$

More explicitly, for $n = 2r$ we have

$$(kl) = x_1^{(k)}y_1^{(l)} + y_1^{(k)}x_1^{(l)} + \cdots + x_r^{(k)}y_r^{(l)} + y_r^{(k)}x_r^{(l)} \qquad (k \neq l),$$

whereas for $n = 2r + 1$ we have

$$(kl) = x_1^{(k)}y_1^{(l)} + y_1^{(k)}x_1^{(l)} + \cdots + x_r^{(k)}y_r^{(l)} + y_r^{(k)}x_r^{(l)} + 2z^{(k)}z^{(l)} \qquad (k \neq l).$$

Let

$$[i_1, \ldots, i_n] = \det\left[v^{(i_1)}, \ldots, v^{(i_n)}\right]$$

be the determinant of the matrix that has $v^{(i_1)}$, ..., $v^{(i_n)}$ as its columns. Then all (kl) are orthogonal invariants, and all $[i_1, \ldots, i_n]$ are special orthogonal invariants.

The classical "first fundamental theorem" for the (special) orthogonal group asserts that when $\mathbb{F}$ is of characteristic zero, the algebra $\mathbb{F}[n\times m]^{\mathrm{O}_n(\mathbb{F})}$ is generated by the inner products (kl) of the indeterminate vectors under consideration, and the algebra $\mathbb{F}[n \times m]^{\mathrm{SO}_n(\mathbb{F})}$ is generated by the inner products and the determinants. This has been discussed along with the analogous results for the other classical groups by Hermann Weyl in [W2]. De Concini and Procesi [CP] gave a characteristic free treatment to the subject, in particular, they proved that the first fundamental theorem for the (special) orthogonal group remains unchanged in odd characteristic. Concerning characteristic 2, Richman [Ri] proved later that the algebra $\mathbb{F}[n\times m]^G$ for the group G preserving the bilinear form $x_1^{(1)}x_1^{(2)}+\cdots+x_n^{(1)}x_n^{(2)}$ is generated in degree 1 and 2. However, though this group preserves the quadratic form $x_1^2+\cdots+x_n^2$, it is not the orthogonal group in characteristic 2, not even up to change of basis: the quadratic form $x_1^2 + \cdots + x_n^2$ is the square of a linear form, hence is degenerate. So in characteristic 2 the question about vector invariants of the orthogonal group remains open. We shall see that the field of rational O_n–invariants is generated by the obvious quadratic invariants even in characteristic 2. However, the behaviour of polynomial invariants turns out to be very much different, see Section 1.2.

1.2 Constructing indecomposable invariants

In this section, we explicitly construct indecomposable m–linear invariant polynomials for all possible values of m. The constructions rely on analyzing the Pfaffian of the skew-symmetric matrix whose entries above the diagonal are the inner products of the vector variables. We observe basic properties of our invariants that serve as the basis for the proofs of indecomposability, which are postponed to Section 1.4.

1.3 Separation of orbits

This section contains results that are similar in characteristic 2 as in all other characteristics. The theorems in this section are valid over any algebraically closed field $\mathbb{F}$, but most of them are well known if char $\mathbb{F} \neq 2$. The interesting part is that they are valid also in characteristic 2. The proofs use Witt's theorem [T, Theorem 7.4], standard facts concerning reductive groups, and basic algebraic geometry.

1.3.1 The null-cone

Recall that the null-cone corresponding to a graded algebra of polynomials is defined to be the locus of common zeros of its homogeneous elements of positive degree.

Theorem 1.3.1 *Over any algebraically closed field, the null-cones corresponding to the algebras of (special) orthogonal vector invariants are defined already by the quadratic invariants.*

Corollary 1.3.2 *Over any algebraically closed field, the algebras of (special) orthogonal vector invariants are finitely generated as modules over the subalgebra generated by the quadratic invariants.*

1.3.2 Algebro-geometric lemmas

There is no new result in this subsection. We only recall some well-known facts from algebraic geometry that are used in the next subsection.

1.3.3 Rational invariants

Let $G = \mathrm{SO}_n(\mathbb{F})$ or $G = \mathrm{O}_n(\mathbb{F})$. We define $K = \mathbb{F}(n \times m)$ to be the fraction field of the polynomial algebra $R = \mathbb{F}[n \times m]$. The elements of K are called rational functions in $n \times m$ variables. Invariant rational functions are defined in the same manner as invariant polynomials.

In this subsection, we look at the field K^G of invariant rational functions. Note that K^G is the fraction field of R^G.

Theorem 1.3.3 *Over any algebraically closed base field $\mathbb{F}$, the fields of (special) orthogonal rational vector invariants are purely transcendental over $\mathbb{F}$.*

Explicit transcendence bases, consisting of polynomials with a degree bound independent of the number m of vector variables, are given in the dissertation.

1.3.4 The case $m \leq n$

In this subsection, we describe — in terms of generators and relations — the invariant algebras $\mathbb{F}[n \times m]^{\mathrm{SO}}$ and $\mathbb{F}[n \times m]^{\mathrm{O}}$ for $m \leq n$. We find that their behaviour in characteristic 2 is essentially the same as in all other characteristics, i.e., there are no 'exotic' invariant polynomials for $m \leq n$. This is the content of Theorems 1.3.16, 1.3.17 and 1.3.18 in the dissertation.

1.4 Proofs of indecomposability

In this section, the main results of Chapter 1 are completed: the high degree multi-linear invariants constructed in Section 1.2 are shown to be indecomposable.

1.5 The orthogonal group scheme

We define the orthogonal group scheme and prove that its invariants are the same as the naively defined invariants discussed in the previous sections.

2 Character formulae for classical groups

2.1 Introduction

This chapter is essentially identical to the paper [F1]. We give formulae relating the value $\chi_\lambda(g)$ of an irreducible character of a classical group G to entries of powers of the matrix $g \in G$. This yields a far-reaching generalization of a result of J. L. Cisneros-Molina concerning the GL_2 case [C].

The Weyl character formula [FH, GW, W1, W2] tells us how to compute the character χ_λ of an irreducible finite dimensional representation V_λ with highest weight λ of a (complex, semisimple, connected) Lie group G. In the case of the classical matrix groups, the Weyl character formula expresses the value of the character χ_λ at a group element $g \in G$ in terms of the eigenvalues of g.

In the dissertation, we consider the connected classical groups and we prove variants of the Weyl character formula that express the value $\chi_\lambda(g)$ in many different, explicit rational ways in terms of the entries of powers of the generic matrix $g \in G$. It seems likely that these formulae provide the fastest and most straightforward way of calculating $\chi_\lambda(g)$ for generic g.

We work over the field $\mathbb{C}$ of complex numbers. The classical groups we deal with are the following.

- The general linear group $\mathrm{GL}_n(\mathbb{C})$, consisting of all $n \times n$ invertible matrices. It is discussed in Section 2.2 of the dissertation.
- The special linear group $\mathrm{SL}_n(\mathbb{C})$, consisting of all $n \times n$ matrices with determinant 1. It is discussed in Section 2.3 of the dissertation.
- The special orthogonal group $\mathrm{SO}_n(\mathbb{C})$, consisting of all $n \times n$ matrices with determinant 1 that leave the standard non-degenerate quadratic form invariant. It is discussed in Sections 2.4 and 2.6 of the dissertation.
- The symplectic group $\mathrm{Sp}_{2r}(\mathbb{C})$, consisting of all $2r \times 2r$ matrices that leave the standard symplectic form invariant. It is discussed in Section 2.5 of the dissertation.

We write $M_n(\mathbb{C})$ for the space of $n \times n$ complex matrices and $\mathbf{1}$ for the identity matrix.

2.2 General linear group

Let $G = \mathrm{GL}_{r+1}(\mathbb{C})$ with $H = (\mathbb{C}^*)^{r+1}$ the maximal torus consisting of all invertible diagonal matrices, and $\mathrm{Hom}(H, \mathbb{C}^*) = \mathbb{Z}^{r+1}$ the weight lattice. For $\lambda = (\lambda_0, \ldots, \lambda_r) \in \mathbb{Z}^{r+1}$, write $z^\lambda : H \to \mathbb{C}^*$ for the corresponding multiplicative character of the torus H, and, when λ is dominant, i.e. $\lambda_0 \geq \cdots \geq \lambda_r$, write $\chi_\lambda : G \to \mathbb{C}$ for the character of the irreducible representation with highest weight λ.

Theorem 2.2.1 *Let* $\lambda = (\lambda_0 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^{r+1}$ *and* $g \in \mathrm{GL}_{r+1}(\mathbb{C})$. *Then*

$$\bigwedge_{i=0}^{r} g^{\lambda_i + r - i} = \chi_\lambda(g) \cdot \bigwedge_{i=0}^{r} g^{r-i}.$$

When $\lambda_r \geq 0$, *we may allow arbitrary (possibly non-invertible) matrices* $g \in M_{r+1}(\mathbb{C})$.

Corollary 2.2.2 *Let* Ω *be an alternating* $(r+1)$*–linear form on the space* $M_{r+1}(\mathbb{C})$. *Then, for* $\lambda = (\lambda_0 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^{r+1}$, *we have*

$$\Omega\left(g^{\lambda_0 + r}, g^{\lambda_1 + r - 1}, \ldots, g^{\lambda_r}\right) = \chi_\lambda(g) \cdot \Omega\left(g^r, g^{r-1}, \ldots, \mathbf{1}\right).$$

To express $\chi_\lambda(g)$ as a rational function in entries of powers of g, we must choose Ω such that the right hand side is not identically zero. For example, $\Omega(g_0, \ldots, g_r)$ could be the determinant of the matrix formed by the diagonals, or by the first rows, etc. of the argument matrices.

Corollary 2.2.3 *On the space* $M_{r+1}(\mathbb{C})$, *consider an alternating* r*–linear form* ω *such that* ω *vanishes if an argument is* $\mathbf{1}$. *Then, for* λ *as above and with* $\lambda_r = 0$, *we have*

$$\omega\left(g^{\lambda_0 + r}, g^{\lambda_1 + r - 1}, \ldots, g^{\lambda_{r-1} + 1}\right) = \chi_\lambda(g) \cdot \omega\left(g^r, g^{r-1}, \ldots, g\right).$$

To express $\chi_\lambda(g)$ as a rational function in entries of powers of g, we must choose ω such that the right hand side is not identically zero. For example, $\omega(g_0, \ldots, g_{r-1})$ could be the determinant of the $r \times r$ matrix formed by the truncated (i.e., leftmost entry omitted) first rows of the argument matrices.

Corollary 2.2.3, for $r = 1$, is the result of J. L. Cisneros-Molina's paper [C] mentioned in the Introduction.

2.3 Special linear group

Let $G = \mathrm{SL}_{r+1}(\mathbb{C})$ with $H \simeq (\mathbb{C}^*)^r$ the maximal torus consisting of all unimodular diagonal matrices, and $\mathrm{Hom}(H, \mathbb{C}^*) = \mathbb{Z}^{r+1}/\mathbb{Z}$ the weight lattice. Write

$$\rho = (r, r-1, \ldots, 0) + \mathbb{Z} \cdot (1, \ldots, 1) \in \mathbb{Z}^{r+1}/\mathbb{Z}$$

for the half-sum of the positive roots. When $\lambda = (\lambda_0, \ldots, \lambda_r) \in \mathbb{Z}^{r+1}/\mathbb{Z}$, write $z^\lambda : H \to \mathbb{C}^*$ for the corresponding multiplicative character of the torus H, and, when λ is dominant, write $\chi_\lambda : G \to \mathbb{C}$ for the character of the irreducible representation with highest weight λ.

For $\ell \in \mathbb{Z}^{r+1}/\mathbb{Z}$ and $g \in G$, the anti-symmetric tensor

$$\bigwedge_{i=0}^{r} g^{\ell_i} \in M_{r+1}(\mathbb{C})^{\wedge(r+1)}$$

is well defined.

Theorem 2.3.1 *Let* $\lambda = (\lambda_0 \geq \cdots \geq \lambda_r) + \mathbb{Z} \cdot (1, \ldots, 1) \in \mathbb{Z}^{r+1}/\mathbb{Z}$ *and* $g \in \mathrm{SL}_{r+1}(\mathbb{C})$. *Then*

$$\bigwedge_{i=0}^{r} g^{\ell_i} = \chi_\lambda(g) \cdot \bigwedge_{i=0}^{r} g^{r-i},$$

where $\ell_i = \lambda_i + r - i$.

2.4 Odd special orthogonal group

Let $G = \mathrm{SO}_{2r+1}(\mathbb{C})$ be the connected group preserving the quadratic form

$$x_1y_1 + \cdots + x_ry_r + z^2.$$

We take the maximal torus $H = (\mathbb{C}^*)^r$ consisting of all special orthogonal diagonal matrices

$$\mathrm{diag}\left(z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}, 1\right).$$

In the weight lattice $\mathrm{Hom}(H, \mathbb{C}^*) = \mathbb{Z}^r$, we take $\lambda = (\lambda_1, \ldots, \lambda_r)$ to correspond to the monomial

$$z^\lambda = \prod_{j=1}^r z_j^{\lambda_j}.$$

Write

$$\rho = (r - 1/2, r - 3/2, \ldots, 3/2, 1/2)$$

for the half-sum of the positive roots.

Theorem 2.4.1 *Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r_{\geq 0}$ and $g \in \mathrm{SO}_{2r+1}(\mathbb{C})$. Then*

$$\bigwedge_{i=1}^r \left(g^{\ell_i} - g^{-\ell_i}\right) = \chi_\lambda(g) \cdot \bigwedge_{i=1}^r \left(g^{r+1/2-i} - g^{-(r+1/2-i)}\right),$$

where $\ell = \lambda + \rho$, i.e. $\ell_i = \lambda_i + r + 1/2 - i$, and the powers are defined using any, but always the same value of $\sqrt{g} \in \mathrm{SO}_{2r+1}(\mathbb{C})$.

2.5 Symplectic group

Let $G = \mathrm{Sp}_{2r}(\mathbb{C})$ be the group preserving the skew bilinear form

$$\sum_{i=1}^r (x_i'y_i'' - y_i'x_i'')$$

on $\mathbb{C}^{2r}$. We take the maximal torus $H = (\mathbb{C}^*)^r$ consisting of all symplectic diagonal matrices

$$\mathrm{diag}\left(z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}\right).$$

In the weight lattice $\mathrm{Hom}(H, \mathbb{C}^*) = \mathbb{Z}^r$, we take $\lambda = (\lambda_1, \ldots, \lambda_r)$ to correspond to the monomial

$$z^\lambda = \prod_{j=1}^r z_j^{\lambda_j}.$$

Write $\rho = (r, r-1, \ldots, 1)$ for the half-sum of the positive roots.

Theorem 2.5.1 *Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r_{\geq 0}$ and $g \in \mathrm{Sp}_{2r}(\mathbb{C})$. Then*

$$\bigwedge_{i=1}^r \left(g^{\ell_i} - g^{-\ell_i}\right) = \chi_\lambda(g) \cdot \bigwedge_{i=1}^r \left(g^{r+1-i} - g^{-(r+1-i)}\right),$$

where $\ell = \lambda + \rho$, i.e., $\ell_i = \lambda_i + r + 1 - i$.

2.6 Even special orthogonal group

Let $G = \mathrm{SO}_{2r}(\mathbb{C})$ be the connected group preserving the quadratic form

$$q = x_1 y_1 + \cdots + x_r y_r .$$

We take the maximal torus $H = (\mathbb{C}^*)^r$ consisting of all special orthogonal diagonal matrices

$$\mathrm{diag}\left(z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}\right).$$

In the weight lattice $\mathrm{Hom}(H, \mathbb{C}^*) = \mathbb{Z}^r$, we take $\lambda = (\lambda_1, \ldots, \lambda_r)$ to correspond to the monomial

$$z^\lambda = \prod_{j=1}^{r} z_j^{\lambda_j}.$$

Write $\rho = (r-1, r-2, \ldots, 1, 0)$ for the half-sum of the positive roots. Write

$$\epsilon = (1, 1, \ldots, 1, 1)$$

and

$$e = \epsilon + \rho = (r, r-1, \ldots, 2, 1).$$

Theorem 2.6.1 *Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$ with $\lambda_1 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r|$. Set*

$$\bar{\lambda} = (\lambda_1, \ldots, \lambda_{r-1}, -\lambda_r).$$

Let $g \in \mathrm{SO}_{2r}(\mathbb{C})$. Then

$$2 \bigwedge_{i=1}^{r} \left(g^{\ell_i} + g^{-\ell_i}\right) = (\chi_\lambda + \chi_{\bar{\lambda}})(g) \cdot \bigwedge_{i=1}^{r} \left(g^{r-i} + g^{-(r-i)}\right).$$

Also,

$$\sqrt{-1}^r \,\mathrm{pf}_q\left(g - g^{-1}\right) \cdot \bigwedge_{i=1}^{r} \left(g^{\ell_i} - g^{-\ell_i}\right) =$$

$$= (\chi_\lambda - \chi_{\bar{\lambda}})(g) \cdot \bigwedge_{i=1}^{r} \left(g^{r+1-i} - g^{-(r+1-i)}\right).$$

Throughout, $\ell = \lambda + \rho$, i.e., $\ell_i = \lambda_i + r - i$.

Here the Pfaffian with respect to q of the linear transformation

$$g - g^{-1} \in \mathfrak{so}_{2r}(\mathbb{C})$$

is a square root of the determinant. It is defined by computing the linear transformation's matrix with respect to a positively oriented ordered q–orthonormal basis of the standard vector representation $\mathbb{C}^{2r}$, and taking the Pfaffian of that anti-symmetric matrix. We declare the ordered q–orthonormal bases of the standard vector representation $\mathbb{C}^{2r}$ with determinant $\left(2\sqrt{-1}\right)^r$ to be of positive orientation, as opposed to those with determinant $-\left(2\sqrt{-1}\right)^r$.

3 Inequalities for positive semi-definite matrices

3.1 Introduction

Chapter 3 of the dissertation is based on the paper [F2]. It was motivated by the following question of Benítez, Sarantopoulos and Tonge [BST] (1998). Consider linear functionals f_i with norm 1 on a Euclidean space (i.e., on a vector space endowed with a positive definite inner product). For the norm of the pointwise product, does the inequality

$$||f_1 \cdots f_n|| \geq n^{-n/2} \tag{1}$$

necessarily hold? This has come to be known as the 'real linear polarization constant' problem.

For the complex case, the affirmative answer was proved by Arias-de-Reyna [A] in 1998, based on the complex analog [A, B2, G] of one of Wick's formulas known from quantum field theory, and on Lieb's famous inequality for permanents of positive semi-definite matrices.

In the real case, the affirmative answer to the [BST] question above for $n \leq 5$ was proved by Pappas and Révész [PR] in 2004. For general n, the best result known before [F2] was that of Révész and Sarantopoulos [RS] (2004), based on results of [MST], proving that (1) holds if we replace the right hand side by $2 \cdot (2n)^{-n/2}$. See [Mat1, Mat2, MM, R] for accounts on this and related questions.

In Theorem 3.3.4 of the dissertation, a Hafnian analogue of the Lieb inequality is proved. Namely, if A is a real symmetric positive semi-definite matrix, then

$$\operatorname{haf} \begin{pmatrix} A & A \\ A & A \end{pmatrix} \geq \operatorname{per} A \tag{2}$$

(in fact, Theorem 3.3.4 is more general). Using the original, real Wick formula (labeled 3.11 in the dissertation), this Hafnian inquality is applied to products of jointly normal random variables (Theorem 3.4.1), and then to products of real linear functionals (Theorem 3.4.3):

$$||f_1 \cdots f_n|| \geq \left(\frac{3\sqrt{3}}{e} n \right)^{-n/2}. \tag{3}$$

This improves on the [RS] estimate by an exponential factor.

In Conjecture 3.3.5, a more difficult Hafnian inequality is stated. From this, (1) would follow.

3.2 Old inequalities on determinants and permanents

Recall that the determinant and the permanent of an $m \times m$ matrix $C = (c_{i,j})$ are defined by

$$\det C = \sum_{\pi \in \mathfrak{S}_m} (-1)^\pi \prod_{i=1}^m c_{i,\pi(i)}, \qquad \operatorname{per} C = \sum_{\pi \in \mathfrak{S}_m} \prod_{i=1}^m c_{i,\pi(i)},$$

where $\mathfrak{S}_m$ is the symmetric group on m elements. In Section 3.2 of the dissertation, we review classical inequalities on determinants and permanents of positive semi-definite matrices. In particular, for

$$C = \begin{pmatrix} B' & A \\ A^* & B'' \end{pmatrix} \geq 0,$$

Lieb [L] proved that

$$\text{per } C \geq \text{per } B' \cdot \text{per } B''.$$

The following simple observation seems to have gone unnoticed for 40 years.

Remark 3.2.1 In Lieb's inequality, the condition that the matrix C is positive semi-definite can be replaced by the weaker condition that the diagonal blocks B' and B'' are positive semi-definite. The proof goes through virtually unchanged.

3.3 New inequalities for Pfaffians and Hafnians

3.3.1 Pfaffians

Recall that the Pfaffian of a $2n \times 2n$ matrix $C = (c_{i,j})$ is defined by

$$\text{pf } C = \frac{1}{n!2^n} \sum_{\pi \in \mathfrak{S}_{2n}} (-1)^{\pi} c_{\pi(1),\pi(2)} \cdots c_{\pi(2n-1),\pi(2n)}.$$

When C is anti-symmetric, we have $(\text{pf } C)^2 = \det C$.

For A and B both of size $n \times n$, we consider the polynomial

$$(-1)^{\lfloor n/2 \rfloor} \text{pf} \begin{pmatrix} -\lambda A & B \\ -B & A \end{pmatrix} = \sum_{t=0}^{\lfloor n/2 \rfloor} p_t \lambda^t.$$

Theorem 3.3.1 *Let A and B be real $n \times n$ matrices with A anti-symmetric and B symmetric. If B is positive semi-definite, then $p_t \geq 0$ for all t. If B is positive definite, then $p_t > 0$ for $t \leq (\text{rk } A)/2$ and $p_t = 0$ for $t > (\text{rk } A)/2$.*

Theorem 3.3.2 *Let A and B be real $n \times n$ matrices with A anti-symmetric and B symmetric. Let $\lambda \geq 0$. If B is positive semi-definite, then*

$$(-1)^{\lfloor n/2 \rfloor} \text{pf} \begin{pmatrix} -\lambda A & B \\ -B & A \end{pmatrix} \geq \det B + \lambda^{n/2} \det A.$$

If B is positive definite, then equality occurs if and only if $\lambda A = 0$ or $n = 2$.

3.3.2 Hafnians

Recall that the Hafnian of a $2n \times 2n$ matrix $C = (c_{i,j})$ is defined by

$$\text{haf } C = \frac{1}{n!2^n} \sum_{\pi \in \mathfrak{S}_{2n}} c_{\pi(1),\pi(2)} \cdots c_{\pi(2n-1),\pi(2n)}.$$

For A and B, both of size $n \times n$, we consider the polynomial

$$\operatorname{haf} \begin{pmatrix} \lambda A & B \\ B & A \end{pmatrix} = \sum_{t=0}^{\lfloor n/2 \rfloor} h_t \lambda^t.$$

Theorem 3.3.3 *Let A and B be symmetric real $n \times n$ matrices. If B is positive semi-definite, then $h_t \geq 0$ for all t. If B is positive definite, then $h_t = 0$ if and only if all $2t \times 2t$ diagonal sub-Hafnians of A vanish.*

Theorem 3.3.4 *Let A and B be symmetric real $n \times n$ matrices. Let $\lambda \geq 0$. If B is positive semi-definite, then*

$$\operatorname{haf} \begin{pmatrix} \lambda A & B \\ B & A \end{pmatrix} \geq \operatorname{per} B + \lambda^{n/2} (\operatorname{haf} A)^2.$$

If B is positive definite, then equality occurs if and only if λA is a diagonal matrix or $n = 2$.

(For odd n, we define haf $A = 0$.)

Conjecture 3.3.5 *If $A = (a_{i,j})$ is a positive semi-definite symmetric real $n \times n$ matrix, then the Hafnian of the $2pn \times 2pn$ matrix consisting of $2p \times 2p$ blocks A is at least*

$$(2p-1)!!^n \prod_{i=1}^{n} a_{i,i}^p,$$

with equality if and only if A has a zero row or is a diagonal matrix.

3.4 Products of real linear functionals

In this section, we apply Theorem 3.3.4 to products of jointly normal random variables and then to products of real linear functionals, which was the main motivation for the work in this chapter. The ideas in this section are analogous to those that Arias-de-Reyna [A] used in the complex case.

Let $\xi_1, \ldots, \xi_d$ denote independent random variables with standard Gaussian distribution. We write $Ef(\xi)$ for the expectation of a function $f = f(\xi) = f(\xi_1, \ldots, \xi_d)$.

On $\mathbb{R}^d$, we write $(\cdot, \cdot)$ for the standard Euclidean inner product. We recall the well-known [G, S, Z]

Wick formula *Let $x_1, \ldots, x_n$ be vectors in $\mathbb{R}^d$ with Gram matrix $A = ((x_i, x_j))$. Then*

$$E \prod_{i=1}^{n} (x_i, \xi) = \operatorname{haf} A. \tag{4}$$

(For odd n, we define haf $A = 0$.)

The following theorems are easy corollaries of Theorem 3.3.4 together with the Wick formula (4).

Theorem 3.4.1 *If $X_1, \ldots, X_n$ are jointly normal random variables with zero expectation, then*

$$E\left(X_1^2 \cdots X_n^2\right) \geq EX_1^2 \cdots EX_n^2.$$

Equality holds if and only if the X_i are independent or at least one of them is almost surely zero.

The generalization of Theorem 3.4.1 to an arbitrary even exponent $2p$ is equivalent to Conjecture 3.3.5.

Theorem 3.4.2 *For any $x_1, \ldots, x_n \in \mathbb{R}^d$, $|x_i| = 1$, the average of*

$$\prod_{i=1}^{n} (x_i, \xi)^2$$

on the unit sphere $\{\xi \in \mathbb{R}^d \ : \ |\xi| = 1\}$ is at least

$$\frac{\Gamma(d/2)}{2^n \Gamma(d/2+n)} = \frac{(d-2)!!}{(d+2n-2)!!} = \frac{1}{d(d+2)(d+4)\ldots(d+2n-2)},$$

with equality if and only if the vectors x_i are pairwise orthogonal.

Theorem 3.4.3 *For real linear functionals f_i on a real Euclidean space,*

$$||f_1 \cdots f_n|| \geq \frac{||f_1|| \cdots ||f_n||}{\sqrt{n(n+2)(n+4)\cdots(3n-2)}}.$$

Here $|| \cdot ||$ means supremum of the absolute value on the unit sphere. In the infinite-dimensional case, functionals with infinite norm may be allowed. Then the convention $0 \cdot \infty = 0$ should be used on the right hand side.

Note that

$$n(n+2)(n+4)\cdots(3n-2)) < \left(\frac{3\sqrt{3}}{e} n\right)^n,$$

and $3\sqrt{3}/e < 3 \cdot 1.8/2.7 = 2$, so Theorem 3.4.3 is an improvement on [RS].

References

[A] J. Arias-de-Reyna, Gaussian variables, polynomials and permanents, Lin. Alg. Appl. 285 (1998), 107–114.

[Ball] K. M. Ball, The complex plank problem, Bull. London Math. Soc. 33 (2001), 433–442.

[B1] A. Barvinok, Estimating L^∞ norms by L^{2k} norms for functions on orbits, Found. Comput. Math. 2 (2002), 393–412.

[B2] A. Barvinok, Integration and optimization of multivariate polynomials by restriction onto a random subspace, arXiv preprint: math.OC/0502298

[BST] C. Benítez, Y. Sarantopoulos, A. Tonge, Lower bounds for norms of products of polynomials, Math. Proc. Camb. Phil. Soc. 124 (1998), 395–408.

[Bo] A. Borel, Linear Algebraic Groups, W. A. Benjamin Inc., New York, 1969.

[C] J. L. Cisneros-Molina, An invariant of 2×2 matrices, Electronic Journal of Linear Algebra 13 (2005), 146–152.

[CP] C. De Concini and C. Procesi, A characteristic free approach to invariant theory, Adv. Math. 21 (1976), 330–354.

[DF1] M. Domokos, P. E. Frenkel, On orthogonal invariants in characteristic 2, J. Algebra 274 (2004), 662–688. Published online:
`http://authors.elsevier.com/sd/article/S0021869303005131`,
arXiv preprint: `math.RA/0303106`

[DF2] M. Domokos, P. E. Frenkel, Mod 2 indecomposable orthogonal invariants, Adv. Math. 192/1 (2005), 209–217.

[DKZ] M. Domokos, S. G. Kuzmin, A. N. Zubkov, Rings of matrix invariants in positive characteristic, J. Pure Appl. Alg. 176 (2002), 61–80.

[D] D. Ž. Đoković, Simple proof of a theorem on permanents, Glasgow Math. J. 10 (1969), 52–54.

[F0] P. E. Frenkel, Vector invariants of the orthogonal group in characteristic 2 (in Hungarian), MSc thesis,
`http://www.cs.elte.hu/math/diploma/math/index.html`

[F1] P. E. Frenkel, Character formulae for classical groups, Central European Journal of Mathematics 4 (2006), no. 2, 242–249.

[F2] P. E. Frenkel, Pfaffians, hafnians and products of real linear functionals, Math. Res. Lett., to appear.

[FH] W. Fulton and J. Harris, Representation theory, GTM, Springer, New York, 1991.

[Ga] M. Gaudin, Une démonstration simplifiée du théorème de Wick en Mécanique statistique, Nuclear Phys. 15 (1960), 89–91.

[GW] R. Goodman and N. R. Wallach, Representations and invariants of the classical groups, Cambridge University Press, Cambridge, 1998.

[G] L. Gurvits, Classical complexity and quantum entanglement, J. Comput. System Sci. 69 (2004), no. 3, 448–484.

[J] C. R. Johnson, Inequalities for a complex matrix whose real part is positive definite, Trans. Amer. Math. Soc. 212 (1975), 149–154.

[L] E. H. Lieb, Proofs of some conjectures on permanents, J. Math. Mech. 16 (1966), 127–134.

[L1] E. H. Lieb, A theorem on Pfaffians, J. Combinatiorial Theory 5 (1968), 313–319.

[Mar1] M. Marcus, The permanent analogue of the Hadamard determinant theorem, Bull. Amer. Math. Soc. 69 (1963), 494–496.

[Mar2] M. Marcus, The Hadamard theorem for permanents, Proc. Amer. Math. Soc. 15 (1964), 967–973.

[MN] M. Marcus, M. Newman, The permanent function as an inner product, Bull. Amer. Math. Soc. 67 (1961), 223–224.

[Mat1] M. Matolcsi, A geometric estimate on the norm of product of functionals, Lin. Alg. Appl. 405 (2005), 304–310.

[Mat2] M. Matolcsi, The linear polarization constant of $\mathbb{R}^n$, Acta Math. Hungar. 108 (2005), no. 1-2, 129–136.

[MM] M. Matolcsi, G. A. Muñoz, On the real linear polarization constant problem, Math. Inequal. Appl. 9 (2006), no. 3, 485–494.

[Mats] H. Matsumura, Commutative Ring Theory, Cambridge Univ. Press, 1986.

[Mi] H. Minc, Permanents, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1978

[MST] G. A. Muñoz, Y. Sarantopoulos, A. Tonge, Complexifications of real Banach spaces, polynomials and multilinear maps, Studia Math. 134 (1999), no. 1, 1–33.

[N] P. Newstead, Introduction to moduli problems and orbit spaces, Tata Inst. Lecture Notes, Springer-Verlag, 1978.

[PR] A. Pappas, Sz. Révész, Linear polarization constants..., J. Math. Anal. Appl. 300 (2004), 129–146.

[R] Sz. Gy. Révész, Inequalities for multivariate polynomials, Annals of the Marie Curie Fellowships 4 (2006), `http://www.mariecurie.org/annals/`,
arXiv preprint: `math.CA/0703387`

[RS] Sz. Gy. Révész, Y. Sarantopoulos, Plank problems, polarization and Chebyshev constants, J. Korean Math. Soc. 41 (2004) 157–174.

[Ri] D. R. Richman, The fundamental theorems of vector invariants, Adv. Math. 73 (1989), 43–78.

[Se] A. Seidenberg, The hyperplane sections of normal varieties, Trans. Amer. Math. Soc. 69 (1950), 357–386.

[S] B. Simon, The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton Series in Physics, Princeton University Press, 1974

[T] D. E. Taylor, The Geometry of the Classical Groups, Heldermann Verlag, Berlin, 1992.

[W1] H. Weyl, Theorie der Darstellung kontinuerlicher halbeinfacher Gruppen durch lineare Transformationen, I, II, III, und Nachtrag, Math. Zeitschrift 23 (1925) 271–309 and 24 (1925) 328–376, 377–395, 789-791; reprinted in Selecta Hermann Weyl, 262–366, Birkhäuser, Basel, 1956.

[W2] H. Weyl, The classical groups, Their invariants and representations, Princeton University Press, Princeton, 1946.

[Wi] G. C. Wick, The evaluation of the collision matrix, Phys. Rev. 80 (1950), 268–272.

[Z] A. Zvonkin, Matrix integrals and map enumeration: an accesible introduction, Combinatorics and physics (Marseille, 1995), Math. Comput. Modelling 26 (1997), 281–304.