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BME

The stable matching problem and its generalizations: an algorithmic and game theoretical approach

PhD Thesis

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Introduction

We say that a market situation is stable in a general sense, if there is no set of agents such that all of them are interested in creating a new cooperation (and breaking their other eventual cooperations). As a special question, Gale and Shapley [19] introduced and studied the problem of stable marriages. Here a matching, that corresponds to a set of marriages, is stable, if there exists no man and woman, who would both prefer to marry each other (after leaving their eventual partners). Gale and Shapley described a natural algorithm that finds a stable matching for the marriage problem, so when the graph, that models the possible partnerships, is bipartite.

The stable matching problem and its generalizations have been extensively studied both in combinatorial optimization and game theory. The main reason is that these models seem to be useful to describe economic and social situations. Moreover, as real applications, centralized matching programs have been established in several areas to solve problems like hospitals-residents matching, student admission or kidney exchange.

The stable matching problem for bipartite graphs is often studied in the context of stable marriages. Actually, whenever we use the marriages as a natural terminology for the above problem, explicitly we should have at least three assumptions: payment (dower) is not allowed, only men and women can marry each other, and everybody can have at most one partner. The most important generalizations of the stable matching models can be obtained in fact, by relaxing these conditions.

From a game theoretical point of view the stable matching problems with or without sidepayments are equivalent to the problem of finding a core element in some corresponding so-called TU or NTU-games, (i.e. cooperative games with or without transferable utility,) respectively. The stable marriage problem can be considered as a basic NTU-game. If we allow transfers between the agents, then the corresponding TU-game is called the assignment game. Shapley and Shubik [33] proved that these games always have a nonempty core.

There are many interesting connections between graph and game theory. The core elements of a TU-game are in fact minimum weight covers if their value is equal to the maximum weight of matchings, thus, when the core of the game is nonempty. So, the result of Shapley and Shubik is actually an easy consequence of the theorem of Egerváry [18] about maximum weight matching in bipartite graphs. If the size of the basic coalitions may be more than 2, then instead of graphs, we can model these problems with hypergraphs. The nonemptyness of the core for some families of NTU-games, like the bipartite matching games, can be proved by using a general game theoretical lemma of Scarf [32] and a theorem of Lovász [25] on normal hypergraphs.

2 Stable roommates problem

We model the *stable matching problem* with a graph G, where the agents are represented by vertices, and two vertices are linked by an edge if the agents are both acceptable to each other. For every vertex v, let $<_v$ be a linear order on the edges incident with v. That is, every agent has strict preferences on his possible partnerships. We say that agent v prefers edge f to e (in other words f *dominates* e at v) if $e <_v f$ holds. A *matching* M is a set of edges with pairwise distinct vertices. A matching M is called *stable* if every nonmatching edge, $e \notin M$ is dominated by some matching edge, $f \in M$. A stable matching can also be defined as a matching without a *blocking edge*: an edge $e = \{u, v\}$ is blocking for a matching M if u is either unmatched or prefers edge e to the matching edge that covers u in M, and at the same time, v is either unmatched or prefers edge e to the matching edge that covers v in M.

Alternatively, stable matchings can be described with inequalities. If M is a set of edges then let $x_M : E(G) \longrightarrow \{0, 1\}$ be its *characteristic function* i.e.

$$x_M(e) = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$$

Subset M of $E(G)$ is a stable matching if the following conditions hold:

(M) Matching:
$\sum_{v \in e} x_M(e) \leq 1$ for every vertex $v \in V(G)$

(S) Stability:
for every edge $e \in E$ there exists a vertex $v \in e$ such that $\sum_{v \in f, f \geq_v e} x_M(f) = 1$

We consider the *stable marriage* problem if the graph is bipartite, and the *stable roommates* problem if the graph is general. Gale and Shapley [19] showed that stable matching may not always exists in the latter case, so the core of the corresponding NTU-game can be empty. Irving [21] constructed the first polynomial algorithm that finds a stable matching if there exists one in a given instance. Later, Tan [35] showed that a so-called stable half-matching always exists for the roommates problems. Stable half-matchings can be defined easily by using the very same inequalities that preserve the matching, (M) and stability, (S) property by extending the range of the weight-function $x_{hM} : E(G) \longrightarrow \{0, \frac{1}{2}, 1\}$.

2.1 "Almost stable" matchings

For a stable roommates problem, where no stable matching exists, it is a natural question to find a matching that admits the fewest number of blocking pairs: it can be regarded as a matching that is "as stable as possible". If an agent may be indifferent between two partnerships, then we consider the stable matching problem *with ties.*

Here, a matching is *weakly stable* if there exist no blocking edge where both agents strictly prefer the other to his actual partner. The following results of Abraham, Biró and Manlove have been published in [1].

Theorem 2.1. [1] *Given an instance of stable roommates problem. The problem of finding a matching for which the number of blocking edges is minimal is not approximable within* $n^{\frac{1}{2}-\varepsilon}$*, for any* $\varepsilon > 0$*, unless* $P = NP$*.*

Theorem 2.2. [1] *Given an instance of stable roommates problem with ties. The problem of finding a matching for which the number of blocking edges is minimal is not approximable within* $n^{1-\varepsilon}$*, for any* $\varepsilon > 0$*, unless* $P = NP$*.*

Below, we collect some related results in a table. To clarify the connections we refer these problems with indices [Ri] in the table. We have already mentioned the results of Gale and Shapley [19] and Irving [21] that have reference indexes [R1] and [R2], respectively. The problem of finding a weakly stable matching in the roommates case with ties is NP-complete [R3]. This was proved first by Ronn [28] for complete graphs and later by Irving and Manlove [23] for incomplete lists.

Beside the stability, the cardinality of the matching can also be an important goal that we may want to maximize. Manlove *et al.* [26] proved that the decision problem related to finding the maximum size of weakly stable matching for a given instance of stable marriage problem with ties is NP-complete [R4]. Finally, Manlove proved that given an instance of stable marriage problem, the problem of finding a maximum cardinality matching for which the number of blocking edges is minimal is also inapproximable [R5] (personal communication).

The problem is to	where	bipartite graph		arbitrary graph	
find a matching M,	M	(strict)	with ties	(strict)	with ties
s.t. M	(arb.)	Yes [R1]	(Yes)	P [R2]	NPc [R3]
is stable	max	(P)	NPc [R4]	(P)	(NPc)
s.t. M has min	(arb.)	(=0)	(=0)	**NPc (2.1)**	**NPc (2.2)**
no. of blocking pairs	max	NPc [R5]	(NPc)	(NPc)	(NPc)

In this table, P denotes that the problem is polynomial time solvable, NPc denotes that the (related) problem is NP-complete, (NPc) denotes that the NP-completeness of the problem is obvious from the mentioned results.

2.2 The dynamics of the stable matchings

For the stable marriage problem, Knuth [24] asked whether it is possible to obtain a stable matching by starting from an arbitrary matching and successively satisfying blocking pairs. Roth and Vande Vate [31] gave a positive answer by a decentralized algorithm, in which pairs or single agents enter the market in a random order, and stability is achieved by a proposal-rejection process. Later, Diamantoudi, Miyagawa and Xue [17] proved the same for solvable stable roommates problems.

However the original goal of Roth and Vande Vate was different, their algorithm can be used to model the dynamics of the two-sided matching market as well. In fact, this mechanism also yields an algorithm to find a stable matching for a market by letting the agents enter the market in a random order and by restoring the stability in each such *active phase*. Independently, Tan and Hsueh [36] constructed an algorithm, that finds a stable half-matching for general graphs by using a similar incremental method. In the bipartite case, the Tan-Hsueh algorithm is equivalent to the Roth-Vande Vate algorithm. In the nonbipartite case infinite repetitions can occur, these are handled by the introduction of half-weighted cycles in the obtained stable half-matchings. We call these two algorithms "incremental algorithms".

Recall, that Gale and Shapley [19] proved that the stable matching obtained by the deferred-acceptance algorithm is man-optimal if men make the proposals, (i.e. no man can have a better partner in any other stable matching, so each man gets his *best possible partner*). Based on these results, Blum, Roth and Rothblum [13] described some properties of a dynamic two-sided matching market. They showed the output of the proposal-rejection process is predictable: if some men enter the market then each man either remains matched with the same partner (if it is possible) or gets a worse (but his best) stable partner for the new market.

Blum and Rothblum [14] pointed out that these results imply that the lastcoming agent gets his best stable partner in the Roth-Vande Vate algorithm. Biró, Cechlárová and Fleiner [11] generalized most of these results for nonbipartite graphs based on a new, so-called Key Lemma.

Lemma 2.3 (Key Lemma)**.** [11] *If hM_v is a stable half-matching for $G - v$, and edge $\{v, u\}$ is not blocking hM_v, then v and u cannot be matched in a stable half-matching for G.*

Theorem 2.4. [11] *Suppose that an agent v enters the market and stability is restored by a proposal-rejection process along the sequence $S = (A|B)$. Then each agent $a \in A (b \in B)$, who became matched by making (accepting) a proposal gets his best (worst) stable partner in the obtained stable half-matching.*

Corollary 2.5. [11] *If an agent enters the market last and becomes matched, then he gets his best stable partner.*

Theorem 2.6. [11] *Each matched agent, that gets a partner in the last active phase by making (accepting) a proposal, receives his best (worst) stable partner in the stable solution output by the incremental algorithm.*

Corollary 2.7. [11] *A stable matching, where no matched agent gets his best stable partner, cannot be output by the incremental algorithm.*

Blum and Rothblum [14] proved that an agent can only benefit from entering the two-sided market later. The corresponding results we proved are the following in the roommates case.

Theorem 2.8. [11] *Let in the incremental algorithm two arrival orders σ and σ' differ only in one agent v in such a way that v arrives later in σ. Let hM and hM' be the outputs of the algorithm realized with the orders σ and σ' respectively. If v is a matched agent, then he gets at least as good partner in hM as in hM'.*

Gale and Sotomayor [20] showed that if some man expands his preference-list then no other man is better off in the new men-optimal stable matching. This implies that the same statement is true if a number of men enter the market. Roth and Sotomayor [30] proved that if a man arrives and becomes matched, then certain women will be better off, and some man will be worse off under any stable matching for the new market than at any stable matching for the original market. We generalize this theorem by using an improved version of a result of Irving and Pittel [27] on the core configuration. (A stable half-matching hM_v is a *core configuration* relative to v if after adding v to the graph, the associated proposal-rejection sequence $S(hM_v)$ is as short as possible.)

Theorem 2.9. [11] *If hM_v is a core configuration relative to v, then the associated proposal-rejection sequence $a_0(=v), b_1, a_1, \ldots, a_{k-1}, b_k(, a_k)$ consists of distinct persons, it is uniquely defined, and for every agent in the sequence, who is matched for G, the following properties are true:*

a) b_i is the worst stable partner of a_i for $G-v$ and b_{i+1} is the best stable partner of a_i for G;

b) a_i is the best stable partner of b_i for $G-v$ and a_{i-1} is the worst stable partner of b_i for G.

Theorem 2.9 implies the following nonbipartite generalization of a results by Roth and Sotomayor [30] proved for the marriage problem.

Theorem 2.10. [11] *Suppose that a new agent is added to the market. There may exist some agents that are better off, and some other agents that are worse off under any stable half-matching for the new market than at any stable half-matching for the original market. We can efficiently find all of these agents.*

3 Stable allocation problems

The stable allocation problem for hypergraphs is defined as follows. A hypergraph H and for each vertex v a strict preference order over the edges incident with v is given. Suppose that nonnegative *bounds* on the vertices $b : V(H) \to \mathbb{R}_+$, and nonnegative *capacities* on the edges $c : E(H) \to \mathbb{R}_+$ are fixed. A a nonnegative function x on the edges in an *allocation*, if $x(e) \leq c(e)$ for every edge e and $\sum_{v \in h} x(h) \leq b(v)$ for every vertex v. An allocation is *stable* if every *unsaturated* edge e (i.e. $x(e) < c(e)$) contains a vertex v such that $\sum_{v \in h, e \leq_v h} x(h) = b(v)$. In this case, we say that e is *dominated at* v.

Biró and Fleiner proved in [7] that Scarf's lemma [32] implies the existence of a stable allocation for every allocation problem in hypergraps.

Theorem 3.1. [7] *Every stable allocation problem for hypergraphs is solvable.*

The stable allocation problem was introduced by Baïou and Balinski [5] for bipartite graphs. Their so-called *inductive algorithm* solves the problem for two-sided markets by $O(n+m)$ augmenting steps (where n and m denote the number of vertices and edges of the input graph, respectively), thus by a strongly polynomial algorithm.

The integral version, (i.e. if the allocation x is required to be integer on every edge for integer bounds and capacities) was called *stable schedule problem* by Alkan and Gale [3], although they considered a more general model, the case of so-called *substitutable* preferences. They showed that a stable solution can always be found by a natural generalization of the Gale-Shapley algorithm. Here, instead of stable schedule problem, we call the integral version of the stable allocation problem simply as *integral stable allocation problem*.

We note, that if every bound and capacity is equal to 1, then we get the stable matching problem. Another important family of problems can be derived from the integral stable allocation problem by setting every edge-capacity to be 1. This is called *stable b-matching problem* (or many-to-many stable matching problem). Furthermore, if the vertex-bounds are also equal to 1 in one side of the market, then we get the many-to-one stable matching (or the college admission or the hospital-resident) problem.

The above described integral stable allocation problems are also solvable by the Gale-Shapley algorithm, in fact, the original goal of the Gale-Shapley paper was to study the college admission problem. Their algorithm is used currently in many established centralized matching program for several kind of two-sided markets. Moreover, the very same algorithm had already been implemented in 1952 in the National Intern Matching Program [29] .

3.1 Integral stable allocation problem on graphs

Biró and Fleiner showed in [7] that Scarf's lemma [32] implies the existence of a half-integer stable allocation for every integral stable allocation problem in nonbipartite graphs. The proof is similar to the one used by Aharoni and Fleiner [2] that verifies the existence of the stable half-matching in the roommates case.

Theorem 3.2. [7] *For every integral stable allocation problem for graphs there exists a half-integer stable allocation.*

In [6] we presented two alternative proofs for the above result. In the first argument we showed that every integral stable allocation problem for graphs can be reduced with graph constructions to a stable roommates problem. Except for a trivial step, these constructions had already been introduced by Cechlárová and Fleiner in [15].

The second proof is constructive: we generalized the inductive algorithm of Baïou and Balinski [5] for nonbipartite graphs. The idea of this algorithm is the following. At the beginning of the inductive algorithm, we set the bounds to be $b^0(u) = 0$ for every $v \in V(G)$. Here $x^0(e) = 0$ for every $e \in E(G)$ is a trivial stable allocation. Then we successively increment the bounds of the vertices, by simultaneously modifying the stable allocation along augmenting paths, until reaching the vertex-bounds b for every vertices.

As the inductive algorithm of Baïou and Balinski is a kind of generalization of the incremental algorithm of Roth and Vande Vate, the general inductive algorithm also generalizes the incremental Tan-Hsueh algorithm. In fact, if we use the inductive algorithm for the roommates problem, then it can be verified that the augmentation is always conducted along the shortest path that corresponds to the proposal-rejection sequence of the incremental algorithm in case of a core configuration.

We proved the following result in [6] on the running time of the generated inductive algorithm.

Theorem 3.3. [6] *The inductive algorithm produces a stable half-allocation for a given integral stable allocation problem in* $O(n+m)\sum_{v\in V(G)} b(v)$ *augmenting steps.*

So unfortunately, the generalized inductive algorithm does not remain strongly polynomial. Moreover, in [6] it is shown by a special construction that the order of growing in the running time of the general inductive algorithm can be an exponentional function of n. However, we note that a scaling property ensures that this algorithm can be modified with standard techniques to become polynomial for the integral stable allocation problem. But it is still an open question whether there exists a strongly polynomial algorithm for the integral stable allocation or the more general stable allocation problem for nonbipartite graphs.

3.2 Higher education admission in Hungary

Since 1985, the admission procedure of higher education institutions is based on a centralized matching program in Hungary. Hungarian universities have faculties, where the education is organized in different fields of studies, quite independently. So, students apply for fields of studies of particular faculties, referred simply as fields hereafter.

At the beginning of the procedure, students give their ranking lists over the fields they apply for. Students receive scores at each field they applied for according to their final notes at the high school, and entrance exams. Note, that the score of a student can differ at two fields. Universities can admit a limited number of students to each of their fields, these *quotas* are determined by the Ministry of Education. After collecting the applicants' rankings and their scores, a centralized program computes the score-limits of the fields. An applicant is admitted by the first place on his list, where he is above the score-limit.

Formally, let $A = \{a_1, a_2, \ldots, a_n\}$ be the set of applicants and F be the set of field of studies, where q_u denotes the quota of field f_u. Let the ranking of the applicant a_i be given by a preference list P^i, where $f_v >_i f_u$ denotes that f_v preceeds f_u in the list, i.e. applicant a_i prefers the field f_v to f_u. Let s^i_u be a_i's final score at field f_u.

The score-limit l is a nonnegative integer mapping $l : F \to \mathbb{N}$. An applicant a_i is admitted by a university to a field f_u, if he achieves the limit at field f_u, and f_u is the first such place in his list, i.e. $s^i_u \geq l(f_u)$, and $s^i_v < l(f_v)$ for every field $f_v >_i f_u$. A score-limit l is *feasible* if the number of admitted applicants is not more than the given quota for each field. A score-limit is *stable* if no university can decrease the limit of any of its fields without violating its quota (assuming that the others do not change their limits). We note that this definition coincides with the original stability condition of Gale and Shapley [19] if there are no ties in the lists (i.e. if the scores of the applicants are distinct at each field).

The currently used *college-proposing* score-limit algorithm and the applicant-proposing version are described in [8]. Both algorithms are very similar to the original Gale-Shapley algorithms. The only difference is that here, universities cannot select exactly as many best applicants to their fields as their quotas are, since the applicants may have equal scores. Here, instead universities set their score-limits at each field always to be the smallest one, for that their quotas are not exceeded.

If the scores of the applicants are distinct at each field then these algorithms are equivalent to the original ones by Gale and Shapley. That is why it is not suprising that similar statements can be proved for this more general setting:

Theorem 3.4. [8] *Both the score-limit l_F, obtained by the college-proposing algorithm and the score-limit l_A, obtained by the applicant-proposing algorithm are stable.*

We say that a score-limit l is *better* than l_* for the applicants if $l \leq l_*$, (i.e. $l(f_u) \leq l_*(f_u)$ for every field f_u). In this case every applicant is admitted by the same or by a preferred place at score-limit l than at l_*.

Theorem 3.5. [8] *l_F is the worst possible and l_A is the best possible stable score-limit for the applicants, i.e. for any stable score-limit l, $l_A \leq l \leq l_F$ holds.*

4 Exchange of indivisible goods

Assume that a simple digraph $D = (V, A)$ is given, where V is the set of agents. Suppose that each agent has exactly one piece of indivisible goods, and (i, j) is an arc of A if the good of agent i is suitable for agent j. An *exchange* is a permutation π of V such that, for each $i \in V$, $i \neq \pi(i)$ implies $(i, \pi(i)) \in A$. This can be considered equivalently as a directed cycle packing in the digraph. We denote by $C^\pi(i)$ the cycle of π containing i. If $C^\pi(i)$ has length at least 2, then the agent is said to be *covered*.

Shapley and Scarf [34] described the exchange problem of indivisible goods as a partitioning NTU-game, referred also as *houseswapping game*. Here, the set of common activities for a coalition S corresponds to the set of permutations of S. Preferences of the agents over the possible permutations are derived from the preferences over the goods they receive. As in an exchange π each agent i receives the good of his *predecessor*, $\pi^{-1}(i)$, agent i prefers an exchange π to another exchange σ, if he prefers $\pi^{-1}(i)$ to $\sigma^{-1}(i)$. Thus, an exchange π is in the core of the game, or it is *stable*, if there is no blocking coalition B and permutation σ of B, such that each agent $i \in B$ prefers σ to π. Shapley and Scarf proved that each such market has a nonempty core. Moreover, they showed that a core solution can always be found by the Top Trading Cycle (TTC) algorithm proposed originally by Gale.

The *permutation game* is the houseswapping game with payments. The nonemptyness of the core of permutation games was proved first by Tijs *et al.* [37], they showed that these TU-games are always balanced. We note that here, every core element corresponds to a maximum weight exchange in the given weighted digraph.

An interesting recent application of this model is the kidney exchange problem. Patients needing transplants may have donors who cannot donate them because of immunological incompatibility. So these incompatible patient-donor pairs may accept an exchange with other pairs. Formally, the set V of nodes represents incompatible patient-donor pairs and an arc (u, v) means that the kidney of u's donor is suitable for $v's$ patient.

Basically, there are three main concepts used already in established programs or studied in the literature. As a first priority, most of the current models want to maximize the number of patients that receive a suitable kidney in the exchange that yields to the problem of finding a maximum size directed cycle packing in the digraph. A more sophisticated model makes a distinction between suitable kidneys and tries to find a solution where the sum of benefits is maximal, thus find a maximum weight directed cycle packing. A third concept requires stability of the solution under various criterias. So in the basic case, we get the problem of finding a core-solution in the corresponding houseswapping game.

In these models, the difficulty of the corresponding problem is due to the fact that the length of the cycles in the exchanges is bounded. The reason is that all surgeries along a cycle have to be done simultaneously. Most programs allow only pairwise exchanges, but sometimes 3-way exchanges are also possible. This motivates the problem of exchange with restricted lengths. Thus here, a possible solution of the problem is an *l-way exchange* that contains no cycle with length more than l, that is equivalent to a *vertex-disjoint packing of directed cycles with length at most l*.

If $l = 2$, so only pairwise exchanges are allowed, then the problem becomes a matching problem in an undirected graph G with the same vertex set. (Here, an edge links two vertices if a pairwise exchange is possible between the corresponding pairs.) Thus, the houseswapping game is equivalent to the stable roommates problem, and the permutation game is equivalent to the stable roommates problem with transferable utilities. These games may have an empty core, but the problem of finding a core-solution, if one exists is solvable in polynomial time in both cases. For $l \geq 3$ the problem of exchange of indivisible goods, becomes theoretically hard for the NTU and TU-games as well.

4.1 Maximum weight exchange with restrictions

Here, we collect some results of Biró and Rizzi [12] about the complexity of maximum size and maximum weight l-way exchange problems.

Theorem 4.1. [12] *The problem of finding a maximum size l-way exchange is APX-hard for any integer $l \geq 3$.*

This result obviously implies that the maximum weight version is also APX-hard, since that is a more general problem. This fact motivates the study of finding approximation algorithms for these problems.

The maximum weight l-way exchange problem can also be reduced to the maximum weight l-set packing problem (that can be considered as a maximum weigth matching problem in a hypergraph). The following approximation result was proved by Arkin and Hassin in [4]. An alternative proof of Biró and Rizzi [12] is based on a less general local search algorithm.

Theorem 4.2 (Ankin-Hassin 1998, [12])**.** *The problem of finding a maximum weight l-set packing is approximable with factor $l-1+\varepsilon$ for any $\varepsilon > 0$.*

Since the application of kidney exchanges requires an optimal solution, creating an exact algorithm is important. The following result is based on a particular algorithm for the 3-way exchange problem.

Theorem 4.3. [12] *The problem of finding a maximum weight 3-way exchange is solvable in $O(2^{\frac{m}{2}})$ time (where m denotes the number of arcs in the digraph).*

4.2 Stable exchange problem

A family of stable exchange problems has been introduced and studied in [9]. We have already mentioned, that the basic stable exchange problem was solved by Shapley and Scarf [34] as the problem of finding a core element in the houseswapping game. We note, that according to their definition, in case of ties a core element corresponds to a weakly stable matching.

Cechlárová *et al.* introduced *$\mathcal{L}$-preferences* in [16]. Here, an agent i prefers a permutation π to another permutation σ if either he prefers $\pi^{-1}(i)$ to $\sigma^{-1}(i)$ or he is indifferent between them, but the length of $C^{\pi}(i)$ is smaller than the length of $C^{\sigma}(i)$. They called the corresponding NTU-game as *kidney exchange game*. The solution produced by the TTC algorithm remain stable for this setting too. While Biró and Cechlárová [10] proved that the problem of maximizing the number of agents covered by a stable exchange under $\mathcal{L}$-preferences (denoted by MAXCOVER-$\mathcal{L}$SE) is hard.

Theorem 4.4. [10] MAXCOVER-$\mathcal{L}$SE *is not approximable within $n^{1-\varepsilon}$ for any $\varepsilon > 0$ unless $P = NP$.*

Considering an *l-way exchange problem*, the size of the blocking coalitions can also be restricted. We say that an exchange is *b-way stable* if there exist no blocking coalition of size at most b. Biró [9], [6] proved the following results about 3-way stable 3-way exchange problems.

Theorem 4.5. [9], [6] *The 3-way stable 3-way exchange problem is NP-complete.*

Theorem 4.6. [9], [6] *The decision problem related to finding a 3-way stable 3-way exchange that covers the maximum number of agents is NP-complete, even for three-sided cyclic digraphs (i.e. $V(D) = M \cup W \cup C$ where every arc $(i, j) \in A(D)$ is from either $W \times M$ or $C \times W$ or $M \times C$).*

Hereby, we collect again some related results and we refer these problems with indices [Ri] in the following table. We have already mentioned some results about pairwise stable pairwise exchanges (i.e. stable roommates problems) with indices [R2], [R3] and [R4]. The theorem of Shapley and Scarf [34] on the existence of a nonempty core for the houseswapping game gets index [R5]. Irving [22] proved recently that the problem of finding a (cycle) stable pairwise exchange is NP-complete [R6]. The same result holds for *3-way* stable pairwise exchanges [R7]. Finally, ??? means that we think that these unsolved problems are relevant, the reasons are explained below.

	$l =$	2-way exchange		3-way exchange		exchange	
$b =$		(strict)	ties	(strict)	ties	(strict)	ties
2-way stable	existence	P [R2]	NPc [R3]	???		(Yes)	(Yes)
	MAXCOVER	P	NPc [R4]			???	
3-way stable	existence	NPc [R7]	(NPc)	**NPc (4.5)**	(NPc)	(Yes)	(Yes)
	MAXCOVER	(NPc)	(NPc)	**NPc (4.6)**	(NPc)		
(cycle) stable	existence	NPc [R6]	(NPc)	(NPc)	(NPc)	(Yes)	Yes [R5]
	MAXCOVER	(NPc)	(NPc)	(NPc)	(NPc)	???	

In the present applications of kidney exchange, some programs allow three-way exchanges, and pairwise stability may become a natural expectation. This is why the problem of finding a pairwise stable 3-way exchange in an instance of SE is important. Considering the exchanges without restrictions on the cycle-lengths, the TTC algorithm always provides a stable exchange, that is also a pairwise stable exchange. But the problem of maximizing the number of covered vertices by a stable or by a pairwise stable exchange is still open.

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