Calculation of the Spatial Deformations of Rods without Tensile Strength

A Brief Summary of the Dissertation Submitted to the Budapest University of Technology and Economics in partial fulfillment of the requirements for the degree of

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Problem statement

The deformations of linearly elastic rods with tensile strength and the computation methods of such problems is a widely investigated field in mechanics [1]. The investigation of rods without or with limited tensile strength appears rarer in the literature, it is mainly included in the publications about reinforced concrete beams or columns [9, 12]. The suggested methods are typically computer algorithms and they either contain a strict limit on the input data (e.g. the shape of the cross section is limited) [22] or they apply such an algorithm component which is unreliable even for problems being relevant in the engineering practice [2]. A reliable solution of this problem does not exist. The reason for this fact might be that the mathematical foundations of the calculation of spatial deformations of rods were laid in the first decades of the 20^{th} century, but the technical development allowed only about 50 years later to apply computers for the numerical calculations. During this period the elementary model for reinforced concrete had been exceeded, since much more sophisticated methods have appeared [7] to give a more realistic description. To introduce a mathematically consistent, reliable method for computing the spatial deformations first we have to apply the fundamental model of reinforced concrete based on the elementary facts of strength of materials. After we have managed to establish such a method, we can include the later achievements of researchers in this field. In this way my thesis is devoted to developing an algorithm to calculate spatial deformations of rods *without* or *with limited tensile strength.*

Beyond the theoretical interest, the work is inspired by practical reasons, too, namely the application to reinforced concrete beams and columns. The proposed algorithm is *robust*, i.e. the deformations can be calculated in a reliable manner, there is no danger of bad solutions or unexpected halts of the computation. This is an essential requirement for any algorithm being used in the engineering practice. The first chapter of my thesis contains the description of the proposed algorithm and some analytical proofs concerning the convergence of the core of the algorithm. By these proofs we ensure the algorithm is robust. The summary of this chapter is given under the title 'Inspiration of Mechanics'. The second chapter contains examples of the theoretical and practical applications on reinforced and prestressed concrete columns and beams. Here an experimental verification is presented, too. This chapter is summed under the title 'Inspiration of Reinforced Concrete'.

Inspiration of Mechanics

The literature on the spatial deformations of elastic rods with tensile strength is rich. The first rod theory by Bernoulli and Euler in 1732 arose from the investigation of the buckled compressed rod, the elastica. The three-dimensional equilibria of elastic rods was first described by Kirchhoff in 1859. He showed the analogy between the equations of the infinite rod and the dynamics of a spinning top. Love [10] noticed that for rods with circular symmetric cross section the Kirchhoff equations are completely integrable. Timoshenko in 1921 extended this theory by taking the shear deformations into account [19]. The calculation of warping was introduced by Vlasov in 1949 [21]. The basics of a more general approach based on continuum-mechanical considerations for arbitrary constitutive laws, called the Cosserat Rod Theory was worked out in the early 20$^{\text{th}}$ century, however for engineering problems it was applied only in the early sixties [1]

As we stated before, no mathematically consistent, convergent algorithm is available for rods without (or with limited) tensile strength. In these problems *material nonlinearities* are combined with *geometrical nonlinearity*, the two effects cannot be separated. Due to the limited tensile strength, the stiffness varies along the bar axis, and this variation is influenced by the final geometry of the rod. According to [13], having a non-circular cross section typically destroys the complete integrability of the governing equations. For rods without tensile strength, an initially circular cross section may become non-circular due to the appearance of cracks in the tension zone, i.e. we lose integrability of the system. In my thesis I introduce a reliable numerical method to determine the shape of rods in non-integrable cases applying the Kirchhoff rod theory.

Spatial deformations can be computed by integration of the curvature and the rate of twist along the bar axis. As the core of the algorithm, one has to calculate the stress distribution of each cross section, which is a highly nonlinear problem for arbitrary cross sections without tensile strength. In the literature, authors focus on finding the solution for design purposes, i.e. they seek for the ultimate load of the cross section [20] by giving a procedure to determine the failure surface of the cross section [6, 17] or apply an optimization technique to design reinforced concrete [3, 15] or prestressed [16] members.

Service loads are not calculated often, for rectangular cross section closed formulas [11, 14] or equations based on experimental data are given [18]. For arbitrary shapes, the problem is either solved by the Newton-Raphson, the Quasi-Newton or the Finite Element (FE) methods [2, 4]. The FE method is, in general, not consistent with the Kirchhoff rod theory, furthermore the robustness is not investigated in the literature. The FE solutions also contain some non-linear solvers such as the Newton-Raphson method [8], in this way the reliability is questionable. Other solutions which are not based on the FEM definitely show divergent behaviour [2], therefore these methods can hardly serve as the core of a robust algorithm to calculate spatial deformations. Some authors suggest to use an iterative technique [22], but they do not investigate the convergence features of their proposed method, or they prove the convergence for symmetric sections under symmetric load. For asymmetrical sections or load, the extension of the iterative procedure is not evident, because in this case chaotic behaviour is expected.

To handle the geometrical nonlinearity, several approaches are known. One can use iterative path following procedures [9]. In this case, the load is increased at each step of the iteration, and the new bar shape is determined by solving the equations of equilibrium according to the second order theory. These solutions are sensitive to the load increment and they cannot find disconnected equilibria. Other solutions in the literature are either limited in the shape of the cross section, or one can show that they are not robust. A good example can be found in [12]: the authors suggest finding the stress distribution and consequently the curvature of the cross section by an *inner iteration*, and determine the shape of the bar by an *outer iteration*. Both of these procedures can be halted by divergence. My goal is to introduce an algorithm which is robust both in solving the cross section and calculating the shape of the rod.

Curvature of the cross section

In my work I determine the neutral axis of an arbitrary cross section of a rod without tensile strength under compression and biaxial bending by a direct recursion derived from the equations of equilibrium. The cross section can contain areas with tensile strength (reinforcement). The stress-strain relation of the part without tensile strength (concrete) and the reinforcement can be given generally by

$$\sigma_c(\varepsilon) = \left\{ \begin{array}{cl} q_1\varepsilon + q_2\varepsilon^2 + \cdots + q_k\varepsilon^k, & \text{if} \quad \varepsilon > 0 \\ 0 & \text{if} \quad \varepsilon \leq 0 \end{array} \right\} \tag{1}$$

$$\sigma_s(\varepsilon) = r_1\varepsilon + r_2\varepsilon^2 + \cdots + r_l\varepsilon^l,$$

where $q_1 \geq 0$, $r_1 \geq 0$, k and l are positive integers.

For calculating the neutral axis of an arbitrary cross section under compression and biaxial bending assuming linear material law in the compressed concrete zone ($q_2 = q_3 = \cdots = q_k = 0$) and linear material law for the reinforcement ($r_2 = r_3 = \cdots = r_l = 0$) the method can be associated with a two dimensional map. I proved analytically the following statements:

1. there is one, and only one fixed point,
2. the fixed point is stable (i.e. the method is locally convergent),
3. the recursion in the case of symmetrical cross section and a compressive load on the axis of symmetry is *globally convergent*.
4. In the general, two dimensional case (asymmetrical cross section or load) according to the literature we expect chaotic behaviour. In contrary, systematic numerical trials show the recursion based on the two dimensional map is *globally convergent*.

For non-linear material law of the compressed zone (q_2,q_3,...q_k are arbitrary constants) and arbitrary constitutive law of the reinforcement (r_2,r_3,...r_l are arbitrary constants) I derived an algorithm to determine the neutral axis of an arbitrary cross section under compression and biaxial bending. The method can be associated with a 5 dimensional, semi-implicit map.Using this approach I investigated the

$$\sigma_c(\varepsilon) = \left\{ \begin{array}{cc} q_1\varepsilon + q_2\varepsilon^2, & \text{if} \quad \varepsilon > 0 \\ 0 & \text{if} \quad \varepsilon \leq 0 \end{array} \right\} \tag{2}$$

second order stress-strain relation of the concrete with $\sigma_s(\varepsilon) = r_1\varepsilon$ material law of the reinforcement, which is sufficient for practical purposes. In this case there exists a theoretical maximal load P^i_{max} for each estimate of the neutral axis. I defined three possibilities for carrying out the calculation:

1. The load P is constant at each step of the iteration, for $P > P^i_{max}$ the method halts,
2. In each step we determine P^i_{max} and the neutral axis is calculated with this load (in this case the solution is unique, for practical calculations this approach is a safe one, becasue it slightly overestimates the curvature of the cross section),
3. The load P is constant until $P < P^i_{max}$, otherwise the method continues assuming $P = P_{max}$ in each step.

By numerical simulations all the three approaches are *globally convergent*, in the 1. approach this statement concern the $P \leq P^i_{max}$ case in each step.

Calculating the shape of the rod

Based on the analytical results above I developed a new algorithm to calculate the spatial deformations of rods without, or with limited tensile strength. The algorithm is *robust*, i.e. the deformations can be computed in a reliable manner, there is no danger of false solutions or divergent behaviour. In the frame of this work I implemented the algorithm to determine the neutral axis. It calculates the neutral axis and the curvature of the cracked cross section rapidly: typically in 5-10 steps it estimates the curvature within a 1% error. I embedded this algorithm into the core of the Parallel Hybrid Algorithm [5], which is an iteration-free solver of boundary value problems. Due to the global convergence of the algorithms determining the curvature, and the features of the PHA the whole algorithm is robust. The price of the robustness is the high computational cost (for a single reinforced concrete column approximately 10^6 rod shape must be computed to identify the practically relevant set of the equilibrium path), which is partially balanced by the parallelization of the code. About the computation in the parallel environment see the last paragraphs of this document.

I extended the developed algorithm by subroutines to take the tension stiffening, the shrinkage and creep of concrete and the losses of prestress into account. These subroutines are based on the EUROCODE 2 standard, in the case of creep the widely used Trost model is also included. I showed these subroutines do not influence the convergence properties of the algorithm.

Inspiration by Reinforced Concrete

Reinforced concrete became a widespread structural material in the dawn of the 20th century. Today it is the most often applied material in buildings and other engineering structures. In the previous century, outstanding buildings were created by some architects who managed to make the most of the possibilities offered by reinforced concrete. Good examples are the viaducts by *R. Maillart* or the monumental halls by *P. L. Nervi* (Figure 1). Some excellent work by *F. L. Wright* or *Le Corbusier* would have been impossible to construct without this material. Among contemporary architects, the works by *S. Calatrava* show there are still new possibilities in reinforced concrete.

(a) R. Maillart: Arve Bridge, 1936 (b) P. L. Nervi: Palazzetto dello Sport, 1958

Figure 1: *Some masterpieces made of reinforced concrete*

Due to the wide application, research has been very intensive in this field. Recently, concrete containing Fiber Reinforced Polymer (FRP), the rheology and sustainability of reinforced concrete structures have been the most researched areas; these works are mainly experimental studies. Although the foundations of modeling and the calculation of reinforced concrete were laid in the first decades of the 20th century by French (*F. Hennebique, E. Coignot*) and German (*E. Mörsch, F.I.E. von Emperger*) researchers, there are still unsolved problems. Moreover, some of these open questions also appear in design. The calculation of the deflection of a cracked, asymmetrically prestressed concrete beam or, second order moments of the clamped frame-columns under compression and biaxial bending can be mentioned as good examples. My final goal is to apply the algorithm to handle some of these problems.

Columns

I investigated numerically the behaviour of compressed *RC columns* by the developed algorithm assuming linear stress-strain relation ($k = 1$, $l = 1$ in eq.(1)) in the compressed zone. I demonstrated that for centrally compressed columns the postcritical branch has typically two limit points. The first one after the bifurcation appears due to the cracking, after this limit point the branch is unstable. For reinforced columns the second limit point makes the branch stable again. The branch asymthotically reaches the postcritical branch of the compressed column with cracked cross sections under pure bending. I showed that in the case of eccentric compression with eccentricity e the two limit points unify at a critical value $e = e_{cr}$ in a catastrophe point (i.e. a cut-off point), thus for $e > e_{cr}$ the instability disappears. By prestressing the reinforcing bars of the column the load belonging to the first critical point (i.e. the critical load of the column under eccentric compression) increases and the value of the critical eccentrity e_{cr} is higher, too.

Among these investigations I introduced a slightly asymmetrical structural example which, contrary to the engineer's intuition, can be considered to be optimal, i.e. the risk of buckling is minimal. The example is a symmetrical compressed column between planar hinges, the symmetry breaking variable is the offset of two bars of reinforcement. I showed the optimal value of the offset of the reinforcement as a function of the geometrical ratio of the concrete cross section.

I applied the algorithm to investigate reinforced concrete frames. Recently, these frames have often been built without additional bracing, in this case the columns are clamped at the bottom and pinned at the top. The edge columns are under compression and biaxial bending. Calculations often neglect the effect of the limited tensile strength of the concrete, in this way the second order moments are underestimated. On the other hand the method of EUROCODE 2 significantly overestimates the second order moment. Although some software tools are capable of calculating second order moment in a more accurate way by taking the effective stiffness of the columns into account, to the best of our knowledge they cannot handle cross sections with arbitrary shape.

Beams

In the second chapter of my thesis I also investigated numerically the behaviour of compressed *RC beams* by the developed algorithm assuming linear stress-strain relation ($k = 1$, $l = 1$ in eq.(1)) in the compressed zone. Although the critical bending moment for lateral torsional buckling of a beam without tensile strength can be calculated by the analytical solutions of the literature, the investigation of the postcritical behaviour requires the calculation of the spatial deformations of the beam. I showed that for beams without tensile strength the bifurcation is subcritical. The prestressing influences slightly the critical bending moment of a beam with tensile strength due to the upward deflection caused by the prestress. For beams without tensile strength the effect of prestressing is more significant due to the increase in the stiffness of the beam.

To verify my computations I carried out experiments on symmetrically and asymmetrically prestressed beams to compare the predicted and measured deflections. I measured the deflections in three points of the three specimens. My method predicts the vertical and lateral deflections of the beam just after transferring prestress in a reliable manner. I loaded the beams with concentrated forces and measured the vertical and lateral deflections for three levels of the load (Figure 2). I determined the enveloping curves for each specimen which takes the uncertainities of the manufacturing (the modulus of ealsticity of the concrete, the real value of the prestressing force and the location of the prestressing tendons) into account. The measured values with their 1 mm error are kept in the zone determined by the enveloping curves as long as the stresses are kept in the elastic range.

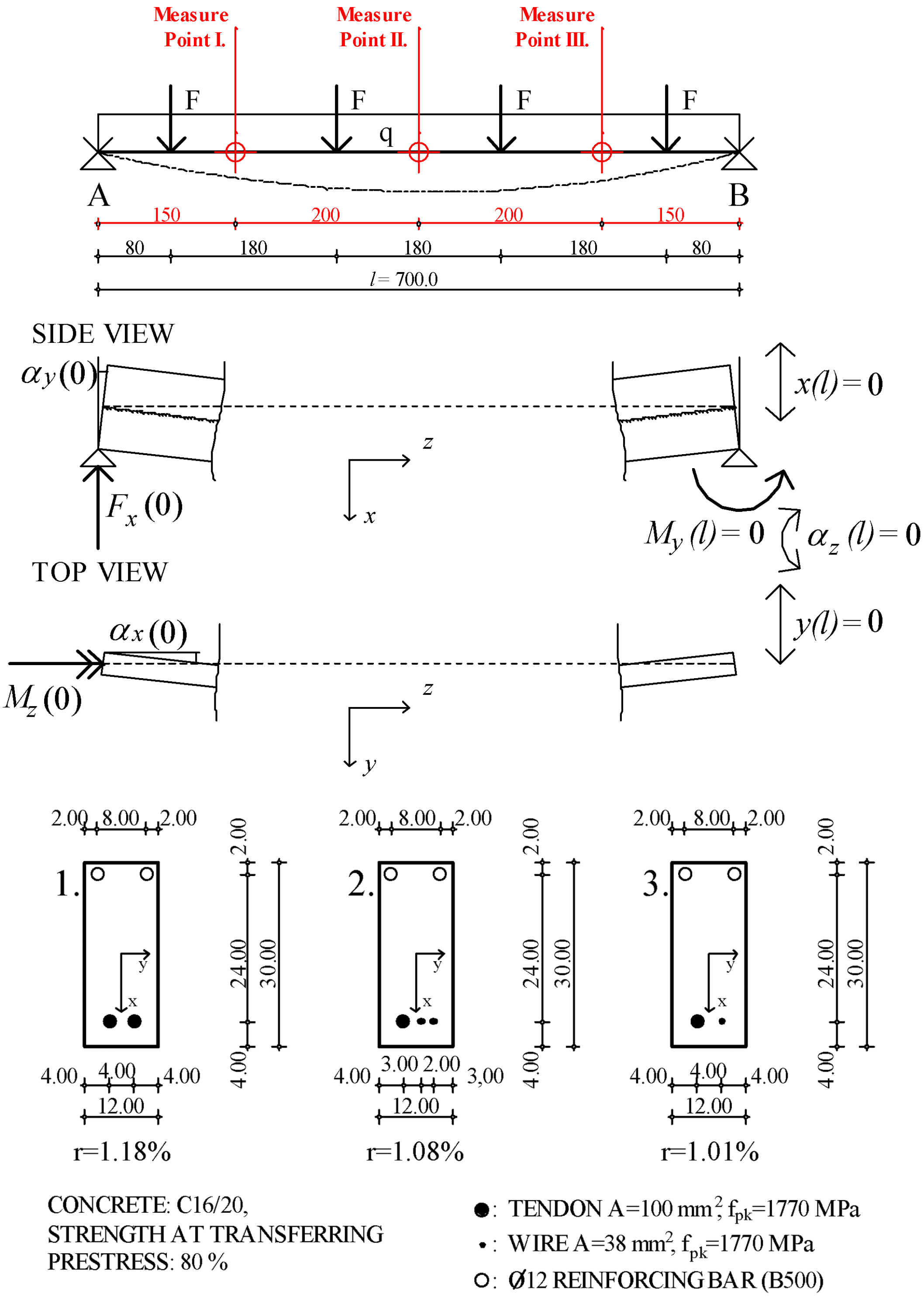

Figure 2: *The simply supported beam of the experiments, measures are in cm*

Computations under the Grid

By developing the algorithm I also proposed to demonstrate the advantages of parallel computation in solving industrial problems. Although the roots of parallel computation can be traced back to the middle sixties, even the contemporary applications (high performance calculations in High Energy Physics, Nanoscience, Biomedicine...etc.) are rather more concerned with scientific research than industrial application. The spread of Internet usage established the possibility using shared computer power and data storage capacity over the Internet for complicated problems requiring high computational effort. Grid technology is the service for sharing computer capacity, such as the WWW is the service for sharing information. The Grid goes well beyond simple communication between computers, and aims ultimately to turn the global network of computers into one vast computational resource. Although Grid technology is in a developmental phase, it is expected to revolutionise computer usage.

In my work I demonstrated the efficiency of the Grid in structural calculations. Since the parallel computational environment has not offered the accustomed facilities of a desktop PC, I took part in developing a user interface for the algorithm to enable industrial users to carry out calculations easily via a web browser. I defined the requirements of the user and the needed functionality. I developed and implemented a graphical tool to display the results of the calculation. The developed software package is one of the first applications of parallel computation and GRID technology for industrial proposes in Hungary.

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