Calculation of the Spatial Deformations of Rods without Tensile Strength

A Dissertation Submitted to the Budapest University of Technology and Economics in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

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Abstract in Hungarian
Magyarnyelvű összefoglaló

1. A probléma felvetése

A húzó- és nyomószilárdsággal rendelkező, lineárisan rugalmas rúd alakváltozásaival számtalan publikáció foglalkozik mind elméleti, mind gyakorlati oldalról [2, 11, 28, 35, 48]. A húzószilárdság nélküli, vagy csak korlátozott húzószilárdsággal rendelkező, például vasbeton rúdak térbeli deformációinak számítása lényegesen ritkábban kerül elő a szakirodalomban [31, 43]. A deformációk számítására eddig javasolt eljárások (jellemzően számítógépes algoritmusok) vagy csak a probléma részleges megoldását teszik lehetővé (például a keresztmetszet alakja korlátozott) [14, 75], vagy pedig olyan számítási módszereket vezetnek, melyek a mérnöki gyakorlat számára releváns példákat sem képesek megbízhatóan megoldani [6, 8]. A bizonyítottan megbízható eljárás hiánya arra vezethető vissza, hogy mire a térbeli rúdalak meghatározásához szükséges számítás-technikai kapacitás megjelen, addigra a vasbeton kutatás már túllépett az elemi szilárdságtanban alkalmazott feltételeken, és a vasbeton viselkedésének minnél valósághűbb leírását kívánta megragadni, igaz, egyre bonyolultabb elméletekkel. Ahhoz, hogy egy matematikailag konzisztens, megbízható eljárást tudjunk bevezetni, vissza kell nyúlni az elemi szilárdságtanban használt vasbeton modellhez. Amennyiben ezen egyszerű megközelítés segítségével sikerül eljárást adunk, természetesen megnyílik az út az elemind bonyolultabb modellke viszgálatára, illetve egyes jelenségek modellbe történő beépítésére is. A megcélzott algoritmus robustus, vagyis a rúd deformációi megbízhatóan számíthatók, nem kell tartani hanis megoldástól, vagy divergens viselkedéstől, azaz az eljárás eredmény nélküli leállásától.

A rudak modellezése a XVIII. század végére nyúlik vissza, az első következetes rúdmellék, igaz, csak sziklabeli alakváltozás feltételezésével, Bernoulli és Euler nevéhez fűződik [20]. Kirchhoff 1859-ben publikálta modelljét [33], mely már tartalmazza az ún. szabad, vagy más néven de Saint-Venant féle csavarás számítását, így térbeli deformációk számítására is alkalmas. A XX. század elején Love ismerte fel, hogy a Kirchhoff-egyenletek kör keresztmetszetű rúd esetén teljesen integrálhatóak [40]. Mielke később megmutatta, hogy más keresztmetszetre az egyenletrendszere tipikusan nem integrálható [44]. Timoshenko Kirchhoff elméletét a nyírási deformációk számításával egészítette ki [66], a gátolt csavarás számítása pedig Vlasov nevéhez fűződik [69]. A jelenleg leginkább alkalmazott rúdelméletek kontinuum-mechanikai megfontolásokat tartalmaznak, ezeket összefoglaló néven Cosserat-féle rúdelméleteknek nevezzik. Ám bár ezen elméletek alapjait a Cosserat fivérek már a XX. század elején megadták, mérnöki
alkalmazásukra csak jóval később, a hatvanas években került sor [2].

Mivel célunk egy robusztus algoritmus leírása, munkánk során Kirchhoff űrdemelé-
téből indulunk ki, de szemben a klasszikus modellel, az anyagtörvény a hűzott beton
beregéde miatt nemlineáris. A űrdemeléti szakirodalomban nem létezik korlátozott
húzósílárdsgá sudak számítására megbízható algoritmus, ennek a már említettekén túl
vélhetően az is oka, hogy a hűzott zónában megjelenő repedések miatt a ród merevse-
ge függ a ród alakjától, a geometria és anyagi nelinearitás együtt határozza meg az
eyensúlyi rúdalakot. Love nyomán a térbeli deformációkat leíró egyenlentrendszer
korlátozott húzósílárdsgás esetén már eredetileg kör keresztmetszetű ród esetén sem inte-
grálható a megjelenő repedések miatt. A feladat megoldására ezért numerikus megoldás
keresünk.

Korlátozott húzósílárdsgággal rendelkező sudak vizsgálata elsősorban a vasbeton
szakirodalomban forró el. Itt először a keresztmetszet optimális méretezésére
álunk eljárást [5, 9, 53, 55, 74], illetve ezzel összefüggésben a vasbeton összetétel
viselkedését leíró modelleket publikáltak [29]. A tetszőleges alakú vasbeton ród gör-
bületén számításával kevésbé foglalkoztak [6, 8, 12]. Ezen eljárások bizonyítottan
nem konvergensek [7].

A mérnöki gyakorlatban a vasbeton rudak alakváltozásainak számítására szolgáló
ejlárások általában csak szimmetrikus keresztmetszetű, egyenes halítással, illetve szim-
metriatengelyen működő, különböző erővel terhelt rudakra használhatóak. Ezen megol-
dások egy része kísérleti eredményekből levezetett összefüggés [5, 65], az általánosan
elterjedt közelítő megoldás az ún. effektív merevség segítségével adja meg a ró-
d alakváltozását [41, 47]. A módszer ferde halításra, illetve aszimmetrikus keresztmetszetekre
nem alkalmazható. Ha mégis ilyen feladatot kell kiszámítani, akkor jelenleg a gyakor-
latban leginkább valamilyé, a végesen módszeren alapuló szoftvert használnak fel.
A végesesemes alkalmazásokat ismeretétő publikációk viszont mellőzik a módszer meg-
bízhatóságának vizsgálatát, ráadásul gyakran használnak fel olyan algoritmusokat (pé-
lául a Newton-Raphson módszert harmad-, vagy magasabb fokú egyenletek megol-
dására) [30], melyek bizonyítottan divergensek lehetnek egyes esetekben.

Dolgozatomban tehát egy, a húzósílárdsgá néhány, vagy korlátozott húzósílárd-
ságú, például vasbeton rudak térbeli deformációinak számítására szolgáló algoritmust
és annak lehetséges alkalmazásait ismertetem. A robusztus működés minden, a mérnöki
gyakorlatban használt algoritmusnál szemben elemi követelmény. Ez persze nem jelenti
a, hogy ne használjánk olyan eljárásokat, melyek nem minden esetben megbízhatóak,
de törekedni kell az alkalmazott eljárások megbízhatóságának igazolására. Ezért dolgo-
zatomban előre vettem a kritikus problémát, vagyis az algoritmus megbízhatóságának
vizsgálatát, ami elsősorban a húzósílárdsgá néhány ród görbületét meghatározó eljárás
meglátogatási igazolását jelenti. Az első rész tartalmazza továbbá a ród deformá-
cióit számító algoritmus ismertetését is. A dolgozat második részében a kifejlesztett
eljárást vasbeton oszlopkok és gerendák pesszkritikus viselkedésének és térbeli alakvál-
tozásainak számítására alkalmazom. Itt kerül sor a numerikus számítások analitikus és
kísérleti eredményekkel történő összevetésére is.
2. HÚZÓSZILÁRDSÁG NÉLKÜLI RUDAK TÉRBELI DEFORMÁCIÓIT SZÁMÍTÓ ALGORITMUS

Rudak térbeli deformációt a hajlításból származó gőbület és az elcsavarodás hosszanti integrálásával lehet számítani (összhangban a Kirchhoff rüdelméettel, a nyírásból és összenyomódásból származó deformációkat figyelmen kívül hagyom). Munkám során felteztem, hogy a keresztmetszetek a deformáció után is súlyos és merőlegesek maradnak a rúd tengelyére (Beronulli-Navier hipotézis). Az általunk javasolt eljárást az elcsavarodás számításakor nem veszi figyelembe a rúd öblösődését, ezért vékonyfalú szelvények és káros I-keresztmetszetek számítása nem alkalmazható. További megkötés a keresztmetszet alakjára vonatkozóan, hogy azt egyetlen zárt gőrbe határolja. Ekkor de Saint-Venant nyomán a rúd egy keresztmetszeténél az elcsavarodás közelítésképpen számítható az azonos területű és poláris inerciájú ellippszis elcsavarodásának meghatározásával [66], az elcsavarodást a berepelt keresztmetszet merésével számítom.

A vizsgált rúd keresztmetszete jellemzően húzószilárdság nélküli anyagból áll, ezt a továbbiakban betonnak, a húzószilárdsággal rendelkező tartományokat veszünk neveznem. A rúd egy keresztmetszeténél a rúd gőbület számításához szükséges a keresztmetszet semleges tengelyének ismerete. A semleges tengely meghatározása egy nemlineáris egyenletrendszert megoldását követeli meg, hiszen a dolgozó betonkeresztmetszet a berepelt zónától az ismeretlen semleges tengely választja el. Az általános gyökerek módszereket (pl. a Newton-Raphson módszert) használó eljárások bizonyíthatóan divergensek bizonyos keresztmetszetek esetén, ezért a megoldásra egy, a hajlítónyomatékok és rúdirányú feszültségek egyensúlyát leíró egyenletekből származó atott rekurziót javasolok. Lineárisnak nevezzem azt az anyagtörvényt, melyben a nyomott beton zónában, illetve a vasakban a feszültségek lineárisan függnek a megnyúlásoktól, egyéb esetekben az anyagtörvény nemlineáris. Természetesen mindkét esetben a beton húzószilárdsága korlátozott, ilyen értelemben ezeket a fogalmakat nem a szokásos értelemben használom.

Lineáris anyagtörvény esetén a javasolt rekurzió egy kétdimenziós leképzésnek feleltethető meg. A leképzés az i. lépésben az aktuális dolgozó keresztmetszethez húzószilárdság feltételezése mellett az egyensúlyi egyenletek megoldásával a semleges tengelyre vonatkozó (i + 1). becslést rendeli hozzá. Az (i + 1). semleges tengely az eredeti keresztmetszetből a húzott beton zóna elhagyásával adja meg a soron következő lépés dolgozó keresztmetszetét. Nemlineáris anyagmodell feltételezése esetén az eljárás egy ötdimenziós, szem-implicit leképzésnek feleltethető meg.

**Lineáris anyagtörvény**

Lineáris anyagtörvény esetén az alkalmazott kétdimenziós leképzésről indirekt bizonyítással igazolom, hogy egy és csak egy semleges tengelyre teljesülnek a nyomató és vetületi egyensúlyi feltételek. A megoldás lokális stabilitását a leképzést lineárizálva, a stabilitási mátrix sajátértékeinek meghatározásával bizonyítható. Ha a keresztmetszet szimmetrikus és a különböző nyomóerő dőfspontja a szimmetria tengelye esik, a semleges tengely helye egyetlen adattal megadható (ez az adat a semleges tengely és a dőfspont távolsága). Ez azt jelenti, hogy a rekurzió egy dimenziós leképzéssel is megváltozható, feltéve, hogy a semleges tengelyre vonatkozó első becsles merőleges volt a szimmetria tengelyre. Ezen speciális eset globális konvergenciáját bebizonyítom. Az egydimenziós leképzésre vonatkozó bizonyítás nem ál-
talánosítható a kétdimenziós esetre, mivel ez utóbbiban két, egymást követő lépésben a semleges tengelyre vonatkozó becsélések a keresztként konvex burkán belül metszhetik egymást. Szásmatematikus numerikus szimulációk alapján azonban állítható, hogy a kétdimenziós leképzés is **globálisan konvergens**.

### Nemlineáris anyagtörvény

A nemlineáris anyagtörvény esetén a berepedt keresztként semleges tengelyét és a görbületet számítható, ötödimenziós szemi-implicit leképzés a lineáris esetben használt kétdimenziós eljárásra épül. A módszer alapja, hogy egy nemlineáris anyagtörvény, különböző nyomott keresztként semleges tengely megoldása egy tipikusan másik dőfespontról, de lineáris anyagtörvény feladatnak. A módszer lényegében a semleges tengelyt és a lineáris feladathoz tartozó dőfespontot határozza meg. Az eljárás folyamán az \(i\)-lépésben először az aktuális dőfesponthoz lineáris anyagtörvény feltételezésével meghatározó az \((i+1)\)-lépés semleges tengelyét. A teljes, nemlineáris anyagtörvény-nél felüli vetítőegyenletből kiszámítható a rövid \(\kappa^{+1}\) görbületet. Nemlineáris anyagtörvény esetén sem teljes a megoldások unicitása, ezért általában a görbületre több valós gyököt és kapok eredménye, ezek közül a legkisebb pozitív értékkel folytatom a számítást. Ez a szabály lényegében nem más, mint a teher történetére vonatkozó feltevés: a teher a szerkezet adott keresztként a tervezési értéket soha nem haladja meg. A görbület és a semleges tengely együtt egyértelműen meghatároz egy feszültségi testet. Ennek segítségével kiszámítható a feszültségekből ébreszkedő, ki nem egyensúlyozott nyomatékok, mely meghatározza az \((i+1)\)-lépés dőfespontját. Az eljárást numerikusan vizsgálhatjuk úgy, hogy a beton anyagtörvénye másodfokú, az acél elsőfokú. Ebben az esetben minden különböző nyomott keresztként is adott helyzetű semleges tengelyhez létezik a \(P_{\text{max}}\) elnövéleti maximális határéért (ami nem feltétlenül azonos a keresztként teherbírásával), ezen erőt meghaladó teher esetén az egységsúlyi egyenleteknek nincs valós megoldása. Az eljárás megvalósítására három lehetőség kínálkozik:

1. Adott \(P\) különböző nyomóerő feltételezése minden lépésben, ha \(P > P_{\text{max}}^{i}\) akkor az eljárás leáll,

2. Minden lépésben \(P_{\text{max}}^{i}\) meghatározása és a semleges tengely számítása ezen a tehenszinten (ekkor teljesül a megoldás unicitása),

3. Amíg \(P < P_{\text{max}}^{i}\) az 1. eljárás, egyébként a 2. eljárás használata.

Numerikus szimulációk alapján mindhárom megközelítés globálisan konvergens.

### A rüd deformációinak számítása

A térbeli deformációkat számító algoritmus maga a rüd egy keresztként a rüd tengely görbülét számíthatja, a fenti alapján lineáris esetben a kétdimenziós, nemlineáris esetben az ötödimenziós leképzés felhasználásával. Mindkét eljárás robusztus, és igen gyorsan konvergál: tipikusan 5-10 lépés elegendő a semleges tengely és az egységsúlyú helyzethez tartozó görbület számítására.

A rüd térbeli deformációinak számítása a \(\kappa\) görbület és a \(\gamma\) elcsavaródás hosszanti integrálásával számítható. Ha a rüd egyik végének geometriai helyzete, illetve az ott működő erők és nyomatékok ismertek, akkor az integrálást végre tudjuk hajtani, ebben
az estben kezdőérték-feladatról beszélünk. Tipikusan a röd mindkét végén vannak ismeretlen geometriai vagy statikai mennyiségek, ekkor egy peremérték-feladatot kell megoldani. Ez lényegesen nagyobb számítási munkát igényel. A peremérték-feladat megoldására az ún. Párhuzamos Simplex Algoritmus (PSA) továbbfejlesztett változata, a Párhuzamos Hibrid Algoritmust (PHA) használtán fel, mely a kezdőérték-feladat megoldására vezeti vissza a problémát. A röd egyik végén a peremfeltételek által nem rögzített $n$ számú alakváltozási vagy statikai jellemzőt változóknak, a röd másik végén a peremfeltételek által előírt $n$ számú mennyiségre vonatkozó egyenletet pedig függvényeknek nevezzük. A megoldásokat a változók és a teherragondoló $d = n+1$ dimenziós terüben keressük, melyet a továbbiakban a probléma Globális Reprezentációs Terének (GRS) nevezzük. A GRS egy-egy pontja tehát egy-egy kezdőérték-feladatnak felel meg, és ezek között keressük a vizsgált mechanikai probléma peremérték-feladatának megoldásait.

A szimplex módszernél első lépésben a GRS-t diszkretizáljuk egy szimplex-hálóval. (Például egy síkbeli konzol 2D-s GRS-ében ez háromszögekre való felosztást jelent.) Ezután a szimplezek összes csúcsjánal kiintézük a kezdőérték-feladatot, míg ha nagyobb a függvények pontképessége.

A szimplezek belsejében lineáris interpolációt alkalmazva keressük a peremértékeket előiró egyenletek közelmű megoldásait. A példa teljes megoldásához a GRS összes szimplexében el kell végezni a számítást, ami rendkívül munkaigényes, a számítási igény a GRS dimenziójával exponenciálisan nő. Ezért érdemes a módszert nagy teljesítményű, párhuzamos, illetve GRID technológiájú rendszerekre implementálni. A szimplex módszer különösen alkalmas erre, mivel a számításokat a GRS tetszőleges tartományainak (akár szimplexenként) külön, egymástól függetlenül lehet végezni, tehát a számítási munka kis, önálló egységekre bontható. A PSA egy letapogató algoritmus, hiszen a GRS egyes részének számítása tartománynál tartománynak történik. A PHA algoritmus a véletlenszerű letapogatást útkövetéssel ötvözi: amennyiben a GRS egy tartományban megoldást talált, a szomszédos tartományokban folytatja a számítást, az egyensúlyi út megtalált szakaszát a tér határáig követi, majd újra véletlenszerű letapogatással folytatja a számítást.

A kifejlesztett algoritmust ipari feladatok számítására is fel szeretnénk használni ezért szükséges, hogy a párhuzamos környezetben futó algoritmust az ipari felhasználók is el tudják érni. Pasztuhov Dániellel közösen kifejlesztettünk egy, az interneten elérhető portál felületet, ahol a számításhoz szükséges adatok megadhatóak, a számítás elindítását lehetővé tesz. Létrehoztuk továbbá egy grafikus alkalmazást a számítási eredmények megjelenítésére. Az elkészült programcsomag a párhuzamos számítástechnika, illetve a GRID technológia egyik első hazai alkalmazása ipari feladat megoldására.

Az ipari felhasználás miatt szükség volt néhány további szubrutin beépítésére. Ezek az EUROCODE 2 szabvány alapján veszik figyelembe az ún. tension-stiffening jelenséget, a beton zsugorodását és kűzsását, továbbá a feszítőerőben fellépő azonnal és időbeni veszteségeket. A kűzsás számítására lehetőség van az ún. Trost modell használatára is. A szubrutinok az algoritmus robustus voltát nem befolyásolják, a deformációk valóságúbb számítását teszik lehetővé.

### 3. AZ ALGORITMUS ALKALMAZÁSA

A dolgozat második részében a kifejlesztett algoritmusban numerikusan vizsgálom vasbeton oszlopopok és gerendák viselkedését. Mindkét fejezetben a posztkritikus tartomány vizsgálatát a lehetséges mérnöki alkalmazások bemutatása követi.
ÖSZLOPOK

A központosan nyomott vasbeton oszlop kritikus ereje megegyezik a lineárisan rugalmas rúd kritikus erejével amit a kifejlesztett algoritmus 1%-nál kisebb hibával számít a pontosnak tekinthető, analitikus megoldásnak viszonyítva. A korlátozott húzószilárdság miatt a posztkritikus ág egy kritikus pont után instabilá válik. Vasalt keresztmetszet esetén egy második kritikus pontot is találunk, innen az ág újra elmekelő, és fokozatosan megközelíti a tiszta hajlítás alatt megrepedt keresztmetszetekkel számított rúd egyensúlyi útját. Korlátozott húzószilárdság esetén a rúd posztkritikus tartományban felvett alakja nem csak a rúd keresztmetszetének meredekségtől, hanem a keresztmetszet alakjától is függ.

Az algoritmus segítségével olyan rudak kritikus terhe is számítható, melyek már az előgazdas kis környezetében is térbeli alakot vesznek fel. Dolgozatomban példát mutatok kis környezetikus geometriájú rúdra, mely a kihajlás alatt szembeni biztonsága alapján optimális szerkezeteknek tekintető.

Külontosan nyomott, húzószilárdság nélküli oszlopokra numerikus számítással megmutatom, hogy a központos nyomásnál meglevő két kritikus pont az ún. $e_{krit}$ kritikus külponthosszág mellett, $e > e_{krit}$ külponthosszáról az instabilitás eltűnik. Az előfeszítés a külontosan nyomott oszlop teherbírássát és kritikus külponthosszagraminárt növeli.

Oszlopok alakváltozásainak számításán túl a kifejlesztett algoritmus felhasználható befogott oszlopokkal épített keretek másodrendű nyomatékaiknak számítására. A módszer azon oszlopok (pl.: aszimmetrikus keresztmetszet...stb.) megjósolására is alkalmas, melyek sem a szabvány előírásaival, sem más szoftverekkel nem számíthatóak.

GERENDÁK

Húzószilárdsággal nem rendelkező, állandó nyomatékkal terhelt, kettámaszú gerendák kritikus nyomatéka jelentősen eltér a húzószilárdsággal rendelkező gerendák kritikus nyomatékától. Húzószilárdság feltételezésével a numerikus számítás a pontosnak tekintető analitikus megoldás középértékében. A posztkritikus viselkedés vizsgálatára a rúd térbeli deformációinak számítását követel meg. Megmutattam, hogy a másodlagos egyensúlyi út, mely az analitikus képletekkel nem lehet követni a húzószilárdság hiányában a bifurkáció környezetében szubkritikus. A gerenda előfeszítése húzószilárdsággal rendelkező anyagmodell esetén a kritikus nyomaték értékét a gerenda felgörbülése miatt kismértékben, húzószilárdság hiányában a meredekség növekedés miatt jelentősen befolyásolja.

Az algoritmus által számított deformációkat kísérleti eredményekkel is összevetem. A kísérlethez három darab aszimmetrikus előfeszített gerendát használtam. Egyérszorát a feszítőerő ráengezése után kialakuló rúdalakot, másérszorát ugyanezen gerendák eredelmezése során fellépő deformációkat hasonlítom össze, különös tekintettel a berepelt tartó alakváltozásaira. A kísérletnek elsősorban azt kívánom benneztetni, hogy az algoritmus a vasbeton előfeszítéses során felmerülő pontatlanságok esetén is megfelelő jósolja meg a gerendák várható alakváltozásait mindaddig, amíg azok II. feszültségi állapotban vannak. A kifejlesztett algoritmus az előfeszítés hatására berepelt gerendák kiegészítését és lehajlását a gyaloglás számára megfelelő pontoszággal (10%-os hibahatáron belül) számítja. Ámában a leggyártott három kísérleti elem statisztikai elemzéshez kevés, az algoritmus a mérési eredményekkel jól egyező adatokat szolgáltatott.
INTRODUCTION

The deformations of linearly elastic rods with tensile strength and the computation methods of such problems is a widely investigated field in mechanics [2, 11, 28, 35, 48]. The investigation of rods without or with limited tensile strength appears rarer in the literature, it is mainly included in the publications about reinforced concrete beams or columns [31, 43]. The suggested methods are typically computer algorithms and they either contain a strict limit on the input data (e.g. the shape of the cross section is limited) [14, 75] or they apply such an algorithm component which is unreliable even for problems being relevant in the engineering practice [6, 8]. A reliable solution of this problem does not exist. The reason for this fact might be that the mathematical foundations of the calculation of spatial deformations of rods were laid in the first decades of the 20th century, but the technical development allowed only about 50 years later to apply computers for the numerical calculations. During this period the elementary model for reinforced concrete had been exceeded, since much more sophisticated methods have appeared [29] to give a more realistic description. To introduce a mathematically consistent, reliable method for computing the spatial deformations first we have to apply the fundamental model of reinforced concrete based on the elementary facts of strength of materials. After we have managed to establish such a method, we can include the later achievements of the researchers of reinforced concrete. In this way this work is devoted to developing an algorithm to calculate spatial deformations of rods without or with limited tensile strength. Beyond the theoretical interest, this work is inspired by practical reasons, too, namely the application to reinforced concrete beams and columns. The proposed algorithm is robust, i.e. the deformations can be calculated in a reliable manner, there is no danger of bad solutions or unexpected halts of the computation. This is an essential requirement for any algorithm being used in the engineering practice, however, sometimes we have to apply solutions which cannot handle every relevant problem. The first chapter of this work contains the description of the proposed algorithm and some analytical proofs concerning the convergence of the core of the algorithm. By this proof we ensure the algorithm is robust. The second chapter contains examples of the theoretical and practical applications. Before that we place our work in a wider context.

MECHANICAL BACKGROUND

The literature on the spatial deformations of elastic rods with tensile strength is rich. The first rod theory by Bernoulli and Euler [20] in 1732 arose from the investigation of the buckled compressed rod, the elastica. The three-dimensional equilibria of elastic
rods was first described by Kirchhoff in 1859 [33]. He showed the analogy between the
equations of the infinite rod and the dynamics of a spinning top. Love [40] noticed that
for rods with circular symmetric cross section the Kirchhoff equations are completely
integrable. Timoshenko in 1929 extended this theory by taking the shear deformations
into account [66]. The calculation of warping was introduced by Vlasov in 1949 [69].
The basics of a more general approach based on continuum-mechanical considerations
for arbitrary constitutive laws, called the Cosserat Rod Theory was worked out in the
early 20th century, however for engineering problems it was applied only in the early
sixties [2] (Table 1).

<table>
<thead>
<tr>
<th></th>
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<th>flexure</th>
<th>free torsion</th>
<th>warping</th>
<th>shear</th>
<th>extension</th>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kirchhoff (1859)</td>
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<td>spatial</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Timoshenko (1921)</td>
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<td>spatial</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vlasov (1949)</td>
<td>linear</td>
<td>spatial</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Cosserat (∼ 1962)</td>
<td>arbitrary</td>
<td>spatial</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 1: Rod theories

These theories are inspired as well by the engineering praxis [28, 49] as by the
mechanical modeling of DNA [11, 48, 60] and the explanation of other biological phe-
nomena [24, 25]. In the works about spatial chaos [35, 39] the computation of spatial
deformations plays an important role, although so far only isotropic rods have been
considered.

As we stated before, no mathematically consistent, convergent algorithm is avail-
able for rods without (or with limited) tensile strength. In these problems material non-
linearities are combined with geometrical nonlinearity, the two effects cannot be
separated. Due to the limited tensile strength, the stiffness varies along the bar axis,
and this variation is influenced by the final geometry of the rod. According to [44],
having a non-circular cross section typically destroys the complete integrability of the
governing equations. For rods without tensile strength, an initially circular cross sec-
tion may become non-circular due to the appearance of cracks in the tension zone, i.e. we lose integrability of the system. In this work we introduce a reliable numerical
method to determine the shape of rods in non-integrable cases applying the Kirchhoff
rod theory.

Spatial deformations can be computed by integration of the curvature and the rate
of twist along the bar axis. As the core of the algorithm, one has to calculate the stress
distribution of each cross section, which is a highly nonlinear problem for arbitrary
cross sections without tensile strength. In the literature, authors focus on finding the
solution for design purposes, i.e. they seek for the ultimate load of the cross section
[68] by giving a procedure to determine the failure surface of the cross section [22, 58] or apply an optimization technique to design reinforced concrete [9, 53, 73, 74] or pre-stressed [5, 55] members.

Service loads are not calculated often, for rectangular cross section closed formulas [41, 47] or equations based on experimental data are given [5, 65]. For arbitrary shapes, the problem is either solved by the Newton-Raphson, the Quasi-Newton or the Finite Element (FE) methods [6, 8, 12]. The FE method is, in general, not consistent with the Kirchoff rod theory, furthermore the robustness is not investigated in the literature. The FE solutions also contain some non-linear solvers such as the Newton-Raphson method [30], in this way the reliability is questionable. Other solutions which are not based on the FEM definitely show divergent behaviour [7], therefore these methods can hardly serve as the core of a robust algorithm to calculate spatial deformations. Some authors suggest to use an iterative technique [75], but they do not investigate the convergence features of their proposed method, or they prove the convergence for symmetric sections under symmetric load [14]. For asymmetrical sections or load, the extension of these iterative procedures is not evident.

![Figure 1: Some outstanding structures made of reinforced concrete](image)

To handle the geometrical nonlinearity, several approaches are known. One can use iterative path following procedures [31, 54, 61]. In this case, the load is increased at each step of the iteration, and the new bar shape is determined by solving the equations of equilibrium according to the second order theory. These solutions are sensitive to the load increment and they cannot find disconnected equilibria. Other solutions in...
the literature are either limited in the shape of the cross section, or one can show that they are not robust. A good example can be found in [43]: the authors suggest finding the stress distribution and consequently the curvature of the cross section by an inner iteration, and determine the shape of the bar by an outer iteration. Both of these procedures can be halted by divergence. Our goal is to introduce an algorithm which is robust both in solving the cross section and calculating the shape of the bar.

**INSPIRATION BY REINFORCED CONCRETE**

Reinforced concrete became a widespread structural material in the dawn of the 20th century. Today it is the most often applied material in buildings and other engineering structures. In the previous century, outstanding buildings were created by some architects who managed to make the most of the possibilities offered by reinforced concrete. Good examples are the viaducts by R. Maillart or the monumental halls by P. L. Nervi (Fig. 1). Some excellent work by F. L. Wright or Le Corbusier would have been impossible to construct without this material (Fig. 2 and Fig. 3) [23]. Among contemporary architects, the works by S. Calatrava show there are still new possibilities in reinforced concrete (Fig. 4).

Due to the wide application, research has been very intensive in this field. Recently, concrete containing Fiber Reinforced Polymer (FRP), the rheology and sustainability of reinforced concrete structures have been the most researched areas; these works are mainly experimental studies [3]. Although the foundations of modeling and the calculation of reinforced concrete were laid in the first decades of the 20th century by French (F. Hennebique, E. Coignet) and German (E. Mörsch, F.I.E. von Emperger) researchers, there are still unsolved problems. Moreover, some of these open questions also appear in design. The calculation of the deflection of a cracked, prestressed concrete beam or, second order moments of the clamped frame-columns under compression and biaxial bending can be mentioned as good examples. Our final goal is to apply the algorithm to handle some of these problems.

Among the Hungarian researchers E. Reuss worked on the classical problems of strength of materials. Gy. Miháilich pointed out, that locating the neutral axis of a symmetrical cross section of a beam without tensile strength requires a solution of a
third degree equation. I. Menyhárd gave an approximation to take the second order effects and the initial imperfections in the eccentricity into account. The lateral torsional buckling of reinforced concrete beams with several boundary conditions was intensively studied by P. Csonka [56].

By developing the algorithm we propose to demonstrate the advantages of parallel computation in solving industrial problems. Although the roots of parallel computation can be traced back to the middle sixties [76, 77], even the contemporary applications (high performance calculations in High Energy Physics, Nanoscience, Biomedicine...etc.) are rather more concerned with scientific research than industrial application. The spread of Internet usage established the possibility using shared computer power and data storage capacity over the Internet for complicated problems requiring high computational effort. Grid technology is the service for sharing computer capacity, such as the WWW is the service for sharing information. The Grid goes well beyond simple communication between computers, and aims ultimately to turn the global network of computers into one vast computational resource. Although Grid technology is in a developmental phase, it is expected to revolutionise computer usage. In our work we would like to demonstrate the efficiency of the Grid in structural calculations.

Figure 3: Two masterpieces by F. L. Wright

Figure 4: S. Calatrava: Concert Hall, Tenerife, 2003
Chapter 1

The Algorithm for Calculating Spatial Deformations of Rods Without Tensile Strength
1.1 CROSS SECTION: FUNDAMENTALS

The main goal of this work is to develop an algorithm to calculate spatial deformations of rods without or with limited tensile strength. This first chapter describes the algorithm and the corresponding analytical proofs. The first three sections deal with the calculation of one cross section of the rod: after describing the assumptions and the notations we analytically investigate the numerical method of determining the curvature of a cross section under biaxial bending and axial force with linear material law. The third section introduces a method for the same problem with non-linear constitutive relation. The last section of this chapter focuses on the rod itself: we describe the algorithm to calculate the spatial deformations including the applications needed for the parallel computation and some additional subroutines for industrial purposes.

1.1.1. ASSUMPTIONS

In our work we apply the Kirchhoff rod theory, i.e. we assume the rod to be unshearable and inextensible. Our approach differs from the classical theory in the constitutive law: instead of linear material law we assume a non-linear stress-strain relation, namely the tensile stresses are limited or vanish. If the tensile stress in a point exceeds the assumed tensile strength, then we consider the point to be cracked. We distinguish between two cases: linear material law in this work means linear stress-strain relation in the compressed zone, keeping the assumption for the limited tensile strength in the tension zone. Non-linear material law refers to a non-linear relation between compressive strains and stresses with limited tensile strength in the tension zone.

A typical cross section of the bar is under compression with biaxial bending and torsion. Due to the limitation of tensile stresses, even an originally circular cross section can become non-circular destroying the integrability of the governing equations. Therefore, we aim to develop a numerical method which can handle arbitrary cross sections with some limitations: we neglect warping and assume the torsional center coincides with the centroid of the section. Due to these assumptions we exclude the thin-walled profiles in this research and for the same reason we do not investigate slender I-beams where the warping effect can be significant [19, 57]. The torsional center typically does not coincide with the centroid of an arbitrary cross section; however, according to de Saint-Venant’s investigations, a substitution of a cross section bordered by a singly connected line by an ellipse gives a good approximation of the torsional rigidity and the rate of twist [67] (For details see subsection 1.1.3).

For the sake of practical application the considered cross section of a rod typically consists of two materials: the set of points without or with limited tensile strength is called concrete, the set of points with unlimited tensile strength is called reinforcement (if it is present). Subscript c denotes a quantity belonging to the concrete part, subscript s denotes a quantity of the reinforcement, respectively. We assume continuous distribution of the cracks in the tension part of the bar, i.e. the discrete placement of the cracks is not included in our model.

In a reinforced concrete cross section the reinforcement has constant area and position along the bar axis, the effect of the possibly applied shear reinforcement is neglected in the calculation. We assume perfect bonding between the reinforcement and the concrete. The proportion of the tangent moduli of the two materials is called the modular...
ratio:

\[ n = \frac{E_s}{E_c} \]  

(1.1)

The ratio \( n \) is used to simplify some of the following calculations by producing the so-called equivalent cross section. The area of the equivalent cross section is obtained as

\[ A = A_c + nA_s, \]  

(1.2)

where \( A_c \) and \( A_s \) denote the area of the concrete and the reinforcing bars, respectively. The curvature \( \kappa \) of the equivalent cross section can be calculated by the material constants of concrete. The uncracked part of the concrete section with the total area of the reinforcement is called the working part of the cross section, while the whole concrete section with the reinforcement is the total cross section, the latter will be distinguished by the subscript \( t \) (Fig. 1.1 (a), (b)).

![Figure 1.1: The cross section of the bar](image)

We assume that plane cross sections remain plane after deformation and they are perpendicular to the rod axis (Hypothesis of Bernoulli-Navier). In the case of zero tensile strength, the cracked and uncracked concrete zone is separated by the so-called neutral axis; for limited tensile strength they are separated by the border line. Due to the hypothesis of plane sections the border line and the neutral axis are parallel, straight lines (Fig. 1.1 (c)).

### 1.1.2. Calculation of the Curvature

To compute the curvature \( \kappa \), we have to determine the neutral axis of the cross section under biaxial bending and compression. To locate the neutral axis and the stress distribution, a non-linear system of equations must be solved, since the unknown neutral axis determines the border of the working concrete part, it influences the stiffness of the cross section. The methods suggested in the literature [6, 8, 12] can exhibit divergent...
<table>
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<th>total concrete section</th>
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<td>( A_{e,t} = \int_{A_{e,t}} A , dA )</td>
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<tr>
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<tr>
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<td>( I_{x,c,t} = \int_{A_{e,t}} y^2 , dA )</td>
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<td>( I_{y,c} = \int_{A_e} x^2 , dA )</td>
<td>( I_{y,c,t} = \int_{A_{e,t}} x^2 , dA )</td>
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<tr>
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<td>( D_{xy,c,t} = \int_{A_{e,t}} xy , dA )</td>
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<th>reinforcement otherwise</th>
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<td>( A_s = \int_{A_s} dA )</td>
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</tr>
<tr>
<td>( I_{x,s,e} = n \int_{A_s} y^2 , dA )</td>
<td>( I_{x,s} = \int_{A_s} y^2 , dA )</td>
</tr>
<tr>
<td>( I_{y,s,e} = n \int_{A_s} x^2 , dA )</td>
<td>( I_{y,s} = \int_{A_s} x^2 , dA )</td>
</tr>
<tr>
<td>( D_{xy,s,e} = n \int_{A_s} xy , dA )</td>
<td>( D_{xy,s} = \int_{A_s} xy , dA )</td>
</tr>
<tr>
<td>( J_{x,s,e} = n \int_{A_s} y^3 , dA )</td>
<td>( J_{x,s} = \int_{A_s} y^3 , dA )</td>
</tr>
<tr>
<td>( J_{xy,s,e} = n \int_{A_s} y^2 x , dA )</td>
<td>( J_{xy,s} = \int_{A_s} y^2 x , dA )</td>
</tr>
<tr>
<td>( J_{yx,s,e} = n \int_{A_s} x^2 y , dA )</td>
<td>( J_{yx,s} = \int_{A_s} x^2 y , dA )</td>
</tr>
<tr>
<td>( J_{y,s,e} = n \int_{A_s} x^3 , dA )</td>
<td>( J_{y,s} = \int_{A_s} x^3 , dA )</td>
</tr>
</tbody>
</table>

Table 1.1: The moments of area of the parts of the cross section distinguished in the text

behaviour for some cross sections and loading. We propose to solve the problem by a direct recursion based on the equations of equilibrium. The input of the \( i \)-th step of the recursion is the current estimate on the neutral axis or the border line, the output is the \((i + 1)\)-th estimate derived from the conditions of equilibrium. The recursion is derived as follows.

The cross section is loaded by a compressive force \( P \) at the point \( D \) (Fig. 1.2). Since we seek a physically objective quantity, namely the neutral axis, the statements on the iterative procedure are independent of the applied coordinate system. The origin of the \([xy]\) coordinate system in the plane of the cross section coincides with point \( D \), the direction of the axes are arbitrary. The surface integrals of the cross section are calculated according to the Tables 1.1 and 1.2 in this coordinate system. We can define
the strains $\varepsilon(x, y)$ in the $[xy]$ coordinate system as

$$\varepsilon = -\kappa(x \cos \omega + y \sin \omega - t), \tag{1.3}$$

where $\omega$ is the (clockwise) angle between the neutral axis and the axis $y$, $t$ denotes the distance between the origin of the coordinate system and the neutral axis, thus $t \geq 0$, $\kappa$ is the curvature (Fig. 1.2).

In the $i$-th step of the recursion, the neutral axis and the border line are characterized by their intersections with the coordinate axes ($x^i_n, y^i_n$) and ($x^i_b, y^i_b$). The quantities $\omega$ and $t$ in eq. 1.3 in the $i$-th step of the recursion can be calculated as

$$\omega^i = \arctan \frac{x^i_n}{y^i_n}, \tag{1.4}$$

$$t^i = \frac{x^i_n y^i_n}{\sqrt{x^i_n^2 + y^i_n^2}}. \tag{1.5}$$

In eqs. 1.4 and 1.5 the denominator formally can be 0, but one can show, that the recursion cannot produce solutions, where the neutral axis contains point $D$, i.e. division by zero cannot happen (see Fig. 1.7 in subsection 1.2.3).

In our calculation we assume elasticity, i.e. unloaded structure is free of deformations even if it had been loaded earlier. We denote compressive stresses and strains by the positive sign. Let $\sigma_e(\varepsilon)$ denote the stress-strain relation of the concrete and $\sigma_s(\varepsilon)$ the relation of the reinforcing steel, respectively. In the literature we can find other closed formulas ([63]) to describe the material law, but these solutions would make the following analytical investigations too difficult. To ensure the robustness of the method, instead of the complicated formulas or numerical procedures we choose to describe the material law of the concrete and the steel with polynomials. Let $f_{ctm}$ and $\varepsilon_{ctm}$ denote

<table>
<thead>
<tr>
<th>area</th>
<th>equivalent working cross section</th>
<th>equivalent total cross section</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = A_c + A_{s,e}$</td>
<td>$A_t = A_{c,t} + A_{s,e}$</td>
<td></td>
</tr>
<tr>
<td>the first moments of area</td>
<td>$S_x = S_{x,c} + S_{y,s,e}$</td>
<td>$S_{x,t} = S_{x,c,t} + S_{y,s,e}$</td>
</tr>
<tr>
<td>$S_y = S_{y,c} + S_{y,s,e}$</td>
<td>$S_{y,t} = S_{y,c,t} + S_{y,s,e}$</td>
<td></td>
</tr>
<tr>
<td>the second moments of area</td>
<td>$I_x = I_{x,c} + I_{x,s,e}$</td>
<td>$I_{x,t} = I_{x,c,t} + I_{x,s,e}$</td>
</tr>
<tr>
<td>$I_y = I_{y,c} + I_{y,s,e}$</td>
<td>$I_{y,t} = I_{y,c,t} + I_{y,s,e}$</td>
<td></td>
</tr>
<tr>
<td>$D_{xy} = D_{xy,c} + D_{xy,s,e}$</td>
<td>$D_{xy,t} = D_{xy,c,t} + D_{xy,s,e}$</td>
<td></td>
</tr>
<tr>
<td>the third moments of area</td>
<td>$J_x = J_{x,c} + J_{x,s,e}$</td>
<td>$J_{x,t} = J_{x,c,t} + J_{x,s,e}$</td>
</tr>
<tr>
<td>$J_{xy} = J_{xy,c} + J_{xy,s,e}$</td>
<td>$J_{xy,t} = J_{xy,c,t} + J_{xy,s,e}$</td>
<td></td>
</tr>
<tr>
<td>$J_{yx} = J_{yx,c} + J_{yx,s,e}$</td>
<td>$J_{yx,t} = J_{yx,c,t} + J_{yx,s,e}$</td>
<td></td>
</tr>
<tr>
<td>$J_y = J_{y,c} + J_{y,s,e}$</td>
<td>$J_{y,t} = J_{y,c,t} + J_{y,s,e}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.2: The moments of area of the equivalent and the total cross sections
the tensile strength of concrete and the corresponding strain, respectively. In problems without tensile strength we take \( f_{ctm} = \varepsilon_{ctm} = 0 \). We assume that both \( \sigma_c \) and \( \sigma_s \) have non-negative roots, they can be given as

$$
\sigma_c(\varepsilon) = \begin{cases} 
q_1 \varepsilon + q_2 \varepsilon^2 + \cdots + q_k \varepsilon^k, & \text{if } \varepsilon > \varepsilon_{ctm} \\
0 & \text{if } \varepsilon \leq \varepsilon_{ctm}
\end{cases}
, \quad (1.6)
$$

$$
\sigma_s(\varepsilon) = r_1 \varepsilon + r_2 \varepsilon^2 + \cdots + r_l \varepsilon^l, \quad (1.7)
$$

where \( q_1 > 0, \ r_1 > 0, \ q_2 \ldots q_k \) and \( r_2 \ldots r_l \) are arbitrary constants. We demand non-negative roots since in each step of the recursion we assume temporarily tensile strength for the concrete (Fig. 1.3 (b)) and the negative roots may cause uncertainties in the
calulcation with non-linear material law. For the imaginary situation the stress-strain relation of the concrete is

\[ \sigma'_c(\varepsilon) = q_1 \varepsilon + q_2 \varepsilon^2 + \cdots + q_k \varepsilon^k. \]  

\( (1.8) \)

![Graph of stress-strain relation](image)

**Figure 1.3:** The constitutive law of the concrete

We neglect the deformations originating from shear, consequently we do not have equations describing the equilibrium of shear forces. The equations of equilibrium are the following:

\[
\begin{align*}
\int_{A_c} \sigma_c(\varepsilon) \, dA + \int_{A_s} \sigma_s(\varepsilon) \, dA - P &= 0, \\
\int_{A_c} x \sigma_c(\varepsilon) \, dA + \int_{A_s} x \sigma_s(\varepsilon) \, dA &= 0, \\
\int_{A_c} y \sigma_c(\varepsilon) \, dA + \int_{A_s} y \sigma_s(\varepsilon) \, dA &= 0, \\
\int_{A_c + A_s} \Phi dA - M_z &= 0,
\end{align*}
\]

where \( \Phi \) is a stress function, \( M_z \) is the torsional moment. The stress function \( \Phi \) can be determined for some cross sections, but it cannot be given in a closed form generally [67]. Eq. 1.12 can be solved independently, we describe it in the next subsection (1.1.3).

The system of equations 1.9-1.11 is highly non-linear, since \( A_c \) is determined by the unknown neutral axis or the border line, i.e. in the \( i \)-th step of the recursion the value of \( A^i_c \) is determined by \( (x^i_n, y^i_n) \) for zero and \( (x^i_b, y^i_b) \) for limited tensile strength. Typically, the neutral axis (or the border line) cuts the total concrete section into two parts, the working part \( A_c \) contains point \( D \). For emphasizing the dependence of the area \( A^i_c \) on the current estimate on the neutral axis \( (\omega^i, t^i) \) we write it in the form \( A_c(\omega^i, t^i) \). In each step of the recursion we assume temporarily tensile strength for the concrete, so we apply \( \sigma'_c \) as stress-strain relation for the concrete. Since we determine the working part of the cross section \( (A^i_c) \) at each step, this substitution provides the solution of 1.9-1.11 as far as the recursion is convergent. After substituting 1.3 into eqs. 1.7 and 1.8 we aim to solve the following system of equations.
\[
\int_{A_s(\omega, t')} \sigma'_c(-\kappa(i+1)(x \cos \omega(i+1) + y \sin \omega(i+1) - t(i+1))) \, dA + \\
+ \int_{A_s} \sigma_s(-\kappa(i+1)(x \cos \omega(i+1) + y \sin \omega(i+1) - t(i+1))) \, dA - P = 0,
\]
\[(1.13)\]

\[
\int_{A_s(\omega, t')} x \sigma'_c(-\kappa(i+1)(x \cos \omega(i+1) + y \sin \omega(i+1) - t(i+1))) \, dA + \\
+ \int_{A_s} x \sigma_s(-\kappa(i+1)(x \cos \omega(i+1) + y \sin \omega(i+1) - t(i+1))) \, dA = 0,
\]
\[(1.14)\]

\[
\int_{A_s(\omega, t')} y \sigma'_c(-\kappa(i+1)(x \cos \omega(i+1) + y \sin \omega(i+1) - t(i+1))) \, dA + \\
+ \int_{A_s} y \sigma_s(-\kappa(i+1)(x \cos \omega(i+1) + y \sin \omega(i+1) - t(i+1))) \, dA = 0,
\]
\[(1.15)\]

for the unknowns \(\omega(i+1), t(i+1)\) and \(\kappa(i+1)\). The angle \(\omega(i+1)\) and the distance \(t(i+1)\) define the neutral axis of the \((i+1)\)-th step of the recursion, we remark, that the magnitude of load \(P\) is kept constant during the iteration. So the system of equations 1.13-1.15 is equivalent to a map

\[
\begin{bmatrix}
\omega(i+1) \\
t(i+1) \\
\kappa(i+1)
\end{bmatrix} = \hat{F} \begin{bmatrix}
\omega^i \\
t^i \\
\kappa^i
\end{bmatrix},
\]
\[(1.16)\]

For the easier implementation the neutral axis is given by it’s intersections \((x_n^i, y_n^i)\) with the coordinate axes, i.e. we include eqs. 1.4-1.5 and their inverse into the map

\[
\begin{bmatrix}
x_n^{(i+1)} \\
y_n^{(i+1)} \\
\kappa^{(i+1)}
\end{bmatrix} = F \begin{bmatrix}
x_n^i \\
y_n^i \\
\kappa^i
\end{bmatrix},
\]
\[(1.17)\]

and we iterate on this map until

\[
|x_n^{(i+1)} - x_n^i| < \delta, \quad |y_n^{(i+1)} - y_n^i| < \delta,
\]
\[(1.18)\]

where \(\delta\) is a fixed error. In case of limited tensile strength \((f_{ctm} \neq 0)\), the distance \(o\) between the neutral axis \((x_n^{(i+1)}, y_n^{(i+1)})\) and the border line \((x_b^{(i+1)}, y_b^{(i+1)})\) can be obtained as:

\[
o = \frac{f_{ctm}}{E \kappa}.
\]
\[(1.19)\]

In case of linear stress-strain relation \((q_1 > 0, q_2 = q_3 = ... q_k = 0\) in eq. 1.6 and \(r_1 > 0, r_2 = r_3 = ... r_l = 0\) in eq. 1.7) the method can be reduced to a two dimensional map since eqs. 1.14-1.15 can be solved in \(\omega\) and \(t\) explicitly and \(\kappa\) is cancelled. In this case the map is denoted by \(F_\ell\), the explicit form for \(f_{ctm} = 0\) is given in section 1.2.

For non-linear stress-strain relation in the compressed zone we assume \(q_1 > 0\) and
$r_1 > 0$ (if reinforcement is present) and the other constants ($q_2, q_3, \ldots, q_k, r_2, r_3, \ldots, r_l$) are arbitrary. In this case the solution of eqs. 1.13-1.15 demands to calculate surface integrals of trigonometric functions. Moreover, if there were more roots in $\omega^{(i+1)}$, $t^{(i+1)}$ and $\kappa^{(i+1)}$ we should define, which solution to accept as the curvature of the cross section.

We developed an alternate method, which carries out the calculation without a numeric procedure to determine the surface integrals of trigonometric functions, moreover, it contains a consideration which root to be accepted as a solution. The basic idea of the method is to locate the neutral axis in each step of the recursion by assuming linear stress distribution, but replace the location of the load according to the unbalanced bending moments arising from the non-linear part of the constitutive law. Here we have altogether five unknowns: $x_n$ and $y_n$ describe the neutral axis, $\kappa$ is the curvature, $x_D$ and $y_D$ are the distances between the original and the current location of the load (Fig. 1.2 (c)). This method can be associated with a five dimensional, semi-implicit map denoted by $F_{nl}$. This method (assuming $f_{cm} = 0$) will be investigated in details in section 1.3.

With some strict limitations, (see Table 1.3) the problem can be reduced to a one-dimensional map denoted by $f_1$ in the linear and $f_{nl}$ in the non-linear case. In subsections 1.2.3 and 1.3.1 we will investigate these reduced maps instead of the previously defined, more sophisticated two and five dimensional ones.

<table>
<thead>
<tr>
<th>Linear Material Law</th>
<th>Non-linear Material Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_l$</td>
<td>$F_{nl}$</td>
</tr>
<tr>
<td>2D map (section 1.2)</td>
<td>5D semi implicit map (subsection 1.3.2)</td>
</tr>
<tr>
<td>special case</td>
<td></td>
</tr>
<tr>
<td>$f_1$</td>
<td>$f_{nl}$</td>
</tr>
<tr>
<td>1D map* (subsection 1.2.3)</td>
<td>1D map** (subsection 1.3.1)</td>
</tr>
</tbody>
</table>

**Table 1.3:** The applied maps for several problems without tensile strength. For the special case the maps denoted by (*) and (**) can be applied for symmetrical cross sections under a load located at the axis of symmetry. In addition, the application of the map marked by (**) requires zero reinforcement ($A_s = 0$).

### 1.1.3. Calculation of the Rate of Twist

We only take the so-called Saint-Venant torsion into account, the warping of the cross section is neglected. The torsion (eq. 1.12) of prismatic bars is calculated by the following approximation. The working cross section part, including the reinforcement, is substituted by an ellipse with the same cross sectional area ($A$) and polar moment of inertia ($I_p = I_x + I_y$), in this way the effect of the cracks on the torsional rigidity is taken into account. The rate of twist $\gamma$ calculated by the stress function $\Phi$ derived for an elliptical cross section gives a good approximation [67]. We obtain the rate of twist of the equivalent cross section by
\[ \gamma = \frac{M_x}{C}, \]  
(1.20)

where \( M_x \) denotes the torsion and \( C \) is the torsional rigidity. \( C \) can be calculated as

\[ C = \frac{G A^4}{4\pi^2 I_p}, \]  
(1.21)

where \( G \) is the modulus of elasticity in shear. We assume an identical Poisson’s ratio for both the concrete and the reinforcement. Having the \( \mu \) Poisson’s ratio of the material, \( G \) can be obtained as

\[ G = \frac{E_c}{2(1 + \mu)}. \]  
(1.22)

### 1.1.4. Prestressed Concrete Cross Section

The cross section may contain prestressed wires or other tendons. The curvature \( \kappa \) of such cross sections can be determined by considering the prestressing force among the external loads [37], i.e. for each section the point \( D \) and the magnitude of the load \( P \) is calculated by adding the effect of the prestressing force to the effect of the resultant loads on the structure. This calculation is accurate when the stress-strain relations for both the concrete and the reinforcement are linear. The calculation of the losses in prestress is given in subsection 1.4.5.

### 1.2 Cross Section: Linear Material Law

For an arbitrary cross section assuming \( f_{ctm} = 0 \), the neutral axis and the curvature \( \kappa \) can be determined by a two dimensional map if the stress-strain relation in the compressed zone is linear (i.e. in eqs. 1.7 and 1.8 \( q_1 > 0, r_1 > 0 \) and \( q_2 = q_3 = \cdots = q_k = r_2 = r_3 = \cdots = r_l = 0 \)). In this case eqs. 1.13-1.15 can be simplified by applying the equivalent cross section with \( n = r_1/q_1 \). First we substitute eq. 1.3 into 1.7 and into 1.8 and do the simplification gaining the following system of equations:

\[ -q_1 \kappa^{i+1} \int_{A(\omega',\mu)} (x \cos \omega^{i+1} + y \sin \omega^{i+1} - t^{i+1}) \, dA - P = 0, \]  
(1.23)

\[ -q_1 \kappa^{i+1} \int_{A(\omega',x')} x (x \cos \omega^{i+1} + y \sin \omega^{i+1} - t^{i+1}) \, dA = 0, \]  
(1.24)

\[ -q_1 \kappa^{i+1} \int_{A(\omega',x')} y (x \cos \omega^{i+1} + y \sin \omega^{i+1} - t^{i+1}) \, dA = 0. \]  
(1.25)

Equations 1.24-1.25 can be divided by \( \kappa^{i+1} \), meaning that the inclination \( \omega^{i+1} \) and the distance \( t^{i+1} \) of the neutral axis can be calculated from these two equations. Eq. 1.23 is needed to determine the curvature \( \kappa^{i+1} \). Applying the surface integrals
of Tables 1.1 and 1.2 and the inverse of eqs. 1.4 and 1.5 the explicit form of the two dimensional map can be given as follows:

\[
\begin{bmatrix}
  x_{n+1}^i \\
  y_{n+1}^i
\end{bmatrix} = F_i \left( x_n^i, y_n^i \right) = \begin{bmatrix}
  G(x_n^i, y_n^i) \\
  H(x_n^i, y_n^i)
\end{bmatrix} = \begin{bmatrix}
  D_{xy}^i - I_x^i I_y^i \\
  D_{xy}^i S_y^i - I_y^i S_x^i \\
  D_{xy}^i S_x^i - I_x^i S_y^i \\
  D_{xy}^i S_y^i - I_y^i S_x^i
\end{bmatrix}. \tag{1.26}
\]

We remark, that the counter of functions $G$ and $H$ cannot equal 0, since by algebraic transformations one can show that in an arbitrary coordinate system

\[ D_{xy}^2 - I_x I_y = -I_1 I_2, \tag{1.27} \]

where $I_1$ and $I_2$ are the stiffnesses around the principal axes of the cross section. The denominators of functions $G$ and $H$ can be 0, but not simultaneously: one can show, that if $D_{xy} S_y - I_y S_x = 0$, then $D_{xy} S_x - I_x S_y \neq 0$. The algorithm must handle this singularity. The curvature can be obtained from eq. 1.23 as:

\[ \kappa^{(i+1)} = -\frac{P}{q_1 \int_{A(w, t')} (x \cos \theta + y \sin \theta) dA}, \tag{1.28} \]

however, it is not needed to run the recursion, so we calculate it after the neutral axis has been found.

In numerical simulations we found the two-dimensional map given in eq. 1.26 to be \textit{globally convergent}. Since a two-dimensional map is typically chaotic \cite{45}, we investigated the convergence features of the map analytically. We prove the uniqueness of the solution in subsection 1.2.1, the local stability in 1.2.2 and finally the global convergence for symmetric problems in 1.2.3.

### 1.2.1. Uniqueness of the Solution

We prove uniqueness of the solution by the proof of contradiction: we assume two different stress distributions fulfilling the conditions of equilibrium; the corresponding neutral axes are denoted by $n_1$ and $n_2$. The $[xy]$ coordinate system is located so that the axis $x$ contains the intersection point $N$ of lines $n_1$ and $n_2$. (If the lines were parallel lines, the axis $x$ is parallel to them.) We denote the regions of the cross section according to Fig. 1.4 by $A_1, \ldots, A_4$. (If the point $N$ is outside the convex boundary of the cross section, then $A_3$ or $A_4$ is an empty set.)

Stresses can be scaled so that stress $\sigma_1(D) = 1$. The stress distributions belonging to the neutral axes $n_1$ and $n_2$ are the following:

\[
\sigma_1 = \begin{cases}
  \left( \frac{1}{n} x + b_1 y \right) & \text{if} \quad (x, y) \in A_1 \cup A_2 \cup A_3 \\
  \left( \frac{1}{n} x + b_1 y \right) & \text{if} \quad (x, y) \in A_4 \\
  0 & \text{otherwise}
\end{cases}, \tag{1.29}
\]

\[
\sigma_2 = \begin{cases}
  \left( \frac{1}{n} x + b_2 y \right) & \text{if} \quad (x, y) \in A_1 \cup A_2 \cup A_4 \\
  \left( \frac{1}{n} x + b_2 y \right) & \text{if} \quad (x, y) \in A_1 \\
  0 & \text{otherwise}
\end{cases}, \tag{1.30}
\]
where \( d \) is the distance of points \( N \) and \( D \). Due to eq. 1.27 we can exclude the \( d = 0 \) case. Without restricting generality, we can assume that \( b_1 > b_2 \), resulting in \((\sigma_1 - \sigma_2) \geq 0\) for \( y > 0 \). We introduce a new stress distribution:

\[
\sigma^* = \begin{cases} 
(1 + \frac{1}{2}x + b_1y) & \text{if } (x, y) \in A_1 \\
(1 + \frac{1}{2}x + b_2y) & \text{if } (x, y) \in A_2 \\
\sigma & \text{if } (x, y) \in A_3 \\
0 & \text{otherwise}
\end{cases}
\]  \hspace{1cm} (1.31)

The bending moments due to the stresses in \( \sigma_1 \), \( \sigma_2 \) and \( \sigma^* \) around axis \( x \) are denoted by \( M_1 \), \( M_2 \) and \( M_0 \), respectively. \( \sigma_1 \) and \( \sigma_2 \) are equilibrium solutions, consequently \( M_1 \) and \( M_2 \) must equal zero:

\[
M_1 = \int_A \sigma_1 y \, dA = M_0 + \int_A (\sigma_1 - \sigma^*) y \, dA = M_0 + \int_{A_2 \cup A_3} (\sigma_1 - \sigma^*) y \, dA = 0, \hspace{1cm} (1.32)
\]

\[
M_2 = \int_A \sigma_2 y \, dA = M_0 + \int_A (\sigma_2 - \sigma^*) y \, dA = M_0 + \int_{A_1 \cup A_4} (\sigma_2 - \sigma^*) y \, dA + \int_{A_3} (\sigma_2 - \sigma^*) y \, dA = 0. \hspace{1cm} (1.33)
\]

In the region \( A_2 \cup A_3 \) \((\sigma_1 - \sigma^*) \geq 0\) and \( y \geq 0 \) so \( M_0 \) must be negative. In the region \( A_1 \cup A_4 \) \((\sigma_2 - \sigma^*) \geq 0\) and \( y \leq 0 \); for the reinforcement \((\sigma_2 - \sigma^*) \geq 0\) for \( y \leq 0 \) and \((\sigma_2 - \sigma^*) \leq 0\) for \( y \geq 0 \) so \( M_0 \) must be a positive value. Consequently eqs. 1.32 and 1.33 cannot be fulfilled for the same cross sectional. We came to a contradiction, the original assumption was false: there is only one neutral axis fulfilling the conditions of
equilibrium. According to this proof the 2D map in eq. 1.26 has only one fixed point\textsuperscript{1}. In the next section we focus on the stability of this single fixed point.

1.2.2. Local Stability

The stability of the fixed point \((x_0, y_0)\) of the map given in 1.26 can be investigated by the \(\lambda_1(x_0, y_0)\) and \(\lambda_2(x_0, y_0)\) eigenvalues of the stability matrix \(L(x_0, y_0)\) (Jacobian) \[45\], where

\[
L(x_0, y_0) = \begin{bmatrix}
\frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \\
\frac{\partial H}{\partial x} & \frac{\partial H}{\partial y}
\end{bmatrix}_{x=x_0 \\ y=y_0}.
\]

If

\[
|\lambda_2(x_0, y_0)| \leq |\lambda_1(x_0, y_0)| < 1,
\]

then the fixed point is an attractor. We will show a coordinate system, where

\[
\left| \frac{\partial G}{\partial x} \right|_{x=x_0 \\ y=y_0} = \left| \frac{\partial H}{\partial x} \right|_{x=x_0 \\ y=y_0} = \left| \frac{\partial G}{\partial y} \right|_{x=x_0 \\ y=y_0} = \left| \frac{\partial H}{\partial y} \right|_{x=x_0 \\ y=y_0} = 0,
\]

consequently \(\lambda_1 = \lambda_2 = 0\) as long as the map is differentiable. The chance for cutting a reinforcing bar by the neutral axis is small. Since we will compare the first and second moments of area of arbitrary small regions beside the neutral axis, we neglect the reinforcement. We use the \([xy]\) global coordinate system, where \(D_{xy} = 0\). The intersections of the neutral axis belonging to the fixed point with the convex boundary of the cross section are denoted by \(A\) and \(B\) (Fig. 1.5). Typically \(A \neq B\) and for an asymmetrical cross section and loading the line \(AB\) is generally not parallel with the axes of the \([xy]\) coordinate system. If they were parallel, the equations used in the proof would become more sophisticated, but the statement for the \(\lambda_1(x_0, y_0)\) and \(\lambda_2(x_0, y_0)\) eigenvalues remains true. We will give the proof for the typical case.

For the fixed point in the chosen global coordinate system the following is valid:

\[
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix} = \begin{bmatrix}
I_{y,0} \\
S_{y0}
\end{bmatrix}.
\]

The neutral axis can be given by

\[
y = y_0 - \frac{y_0}{x_0} x.
\]

A line, denoted by \(e\) is arbitrarily close to the neutral axis:

\[
y = y_0 + dy_0 - \frac{y_0 + dy_0}{x_0 + dx_0} x.
\]

The region of the cross section between the neutral axis and line \(e\) is denoted by \(\Delta A\). We introduce the following notation:

\textsuperscript{1}We remark that this proof can be extended to cases with \(\sigma_c(\varepsilon) = \varepsilon^k\) and \(\sigma_s(\varepsilon) = \varepsilon^k\), where \(k\) is a positive integer. In this work we do not apply such a constitutive law.
\[
\int_{\Delta A} (.) \, dA = \int_{x=\alpha x_0}^{x=\beta x_0} \int_{y=\gamma_0}^{y=\gamma_0+\frac{\gamma_1-y_0}{x_0-x_0_0} x} (.) \, dx \, dy. 
\]

(The area calculated according to eq. 1.40 differs from the area shaded in Fig. 1.5 only in the second degree terms.) Applying eq. 1.40, the first and second moments of area belonging to line \(e\) can be given as:

\[
S_x = \frac{I_{x,0}}{y_0} + \int_{\Delta A} y \, dA,
\]

\[
S_y = \frac{I_{y,0}}{x_0} + \int_{\Delta A} x \, dA,
\]

\[
I_x = I_{x,0} + \int_{\Delta A} y^2 \, dA,
\]

\[
I_y = I_{y,0} + \int_{\Delta A} x^2 \, dA,
\]

\[
D_{xy} = \int_{\Delta A} xy \, dA.
\]

We substitute these values into eq. 1.26 and differentiate it using the computer program MAPLE. For example, the partial derivative of the function \(G(x, y)\) with respect
to $x$ can be obtained by the following command:

\[ >\text{\texttt{simplify}}(\text{\texttt{subs}}(dx=0, dy=0, \text{\texttt{diff}}(G, dx))) \].

Calculating both partial derivatives of the functions $G(x, y)$ and $H(x, y)$ we gain

\[
\frac{\partial G}{\partial x} \bigg|_{x=x_0, y=y_0} = \frac{\partial H}{\partial x} \bigg|_{x=x_0, y=y_0} = \frac{\partial G}{\partial y} \bigg|_{x=x_0, y=y_0} = \frac{\partial H}{\partial y} \bigg|_{x=x_0, y=y_0} = 0. \tag{1.46}
\]

The proof can be extended for concave cross sections, where the line $AB$ contains points which do not belong to the cross section. In this case line $AB$ contains more, detached areas, bordered with $\alpha_1 x_0$ and $\beta_1 x_0$, $\alpha_2 x_0$ and $\beta_2 x_0$,...etc. The equation 1.40 must be modified according to these conditions. After the substitutions and the partial derivations we gain the results of eq. 1.46. We found all of the elements and the eigenvalues of the stability matrix $L$ are zero, the fixed point $(x_0, y_0)$ is (super)stable and the map is locally convergent.

### 1.2.3. Partial Results on Global Stability

In numerical simulations we found the map given in eq. 1.26 to be globally convergent. We give a proof for global convergence for symmetrical cross sections under compression and uniaxial bending. This case can be associated with a 1D map, provided the first estimate on the neutral axis is perpendicular to the axis of symmetry. Although we were unable to extend this proof for the general two dimensional case, we are convinced that a similar mechanism governs the iteration.

A load is applied on the axis of symmetry of the cross section at point $D$. We assume that point $D$ is outside of the kernel of the cross section, but it is inside the convex boundary of it. (If it was inside the kernel, we would have a neutral axis outside the cross section, for this case we have to solve the equilibrium equations only once.) The axis $x$ of the global coordinate system is at the axis of symmetry; its direction is given in such a way that the coordinate $x$ of the centroid $C$ is positive. Due to the symmetry, the estimates on the neutral axis and the final solution are parallel with each other. The one dimensional map associated with the recursion [14] is:

\[ x_{n+1} = f_i(x_n) = \frac{I_y}{S_y} x_n. \tag{1.47} \]

We will investigate the convergence properties of the map given in eq. 1.47. In each step of the iteration, the cross section is cut into a maximum of two parts by the current estimate of the neutral axis. We solve eq. 1.47 for the working cross section part, i.e. the part containing the point $D$. Due to our assumptions there must be an estimate, where the first momentum $S_y$ of the working cross section part equals zero. This estimate is denoted by $e_2$. For an estimate between the axis $y$ and $e_2$, $S_y$ is negative, otherwise it is positive. $I_y$ is a positive number for any estimate. This implies, that $f_i(x)$ has a singularity at $x = e_2$. Due to the rule on the working cross section part, there is another singularity at $x = 0$. Except the two singularities, function $f_i(x)$ is continuous. Summing up, we have the following statements for the sign of $f_i(x)$:
\[ f_l(x) > 0 \quad \text{if} \quad x < 0 \quad \text{or} \quad x > e_2, \]
\[ f_l(x) < 0 \quad \text{if} \quad 0 \leq x < e_2, \]
\[ \lim_{x \to e_2^-} f_l(x) = -\infty, \]
\[ \lim_{x \to e_2^+} f_l(x) = +\infty. \] (1.48)

To determine the shape of \( f_l(x) \), we investigate the first derivative by assuming there is a limit point, where

\[ f_l(x + dx) = f_l(x). \] (1.49)

This latest can be written as

\[ \frac{I_y + dh(x + \frac{dx}{2})^2}{S_y + dh(x + \frac{dx}{2})} = \frac{I_y}{S_y}, \] (1.50)

where \( h \) is the height of the arbitrarily narrow slice according to Fig. 1.6. Neglecting the higher degree terms in \( dx \) and solving 1.50, we obtain that the first derivative can be zero iff

\[ x = \frac{I_y}{S_y}. \] (1.51)

By the proof of the uniqueness of the solution there must be one point, where \( x = f_l(x) \), this point is denoted by \( x_0 \). Eq. 1.48 ensure, that \( x_0 > e_2 \), by 1.51 in this point \( df_l(x)/dx = 0 \). Making an inequality from eqs. 1.50 and 1.51 we can state that:

1. for \( S_y > 0 \)
   - if \( f_l(x) < x \) then \( f_l(x) \) increases monotonously,
   - if \( f_l(x) > x \) then \( f_l(x) \) decreases monotonously;
2. for \( S_y < 0 \)
   - if \( f_l(x) < x \) then \( f_l(x) \) decreases monotonously,
   - if \( f_l(x) > x \) then \( f_l(x) \) increases monotonously.

---

**Figure 1.6:** A symmetric cross section under uniaxial bending and compression

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Due to these results there is only one limit point, namely $x_0$. To give the shape of $f_i(x)$ we take the $e_1$ and $e_3$ borders of the cross section into account. The function $f_i(x)$ can be given as follows:

$$
x[-\infty, e_1] \quad f_i(x) = q_0 \\
x(e_1, 0] \quad 0 < f_i(x + dx) < f_i(x) \\
x(0, e_2] \quad f_i(x + dx) < f_i(x) < 0 \\
x(e_2, x_0] \quad 0 < f_i(x + dx) < f_i(x) \\
x(x_0, e_3] \quad 0 < f_i(x) < f_i(x + dx) \\
x(e_3, +\infty] \quad f(x) = q_0
$$

(1.52)

where $q_0$ is a positive constant. The shape of $f_i(x)$ is given in Fig.1.7. The only fixed point is at $x_0$. Since $f(x)$ monotonously rises in $(x_0, e_3]$ and it is constant for $x > e_3$ an iteration started at $x > x_0$ definitely converges to the fixed point and cannot produce $x < x_0$. One can show that the iteration started elsewhere gets into this convergent region in a finite number of iterative steps, although in this paper we neglect the detailed proof [59].

**Figure 1.7: The $f_i(x)$ function of the map**

The behaviour of the general, two dimensional recursion is very similar to the one dimensional case, although we were unable to extend the analytical proof. The basic difference is that in the two dimensional case the $(x_n^i, y_n^i)$ and $(x_n^{i+1}, y_n^{i+1})$ estimates
of the neutral axes can intersect inside the convex boundary of the cross section (Fig. 1.8). This intersection happens quite often during the iteration process. Due to this occurrence, we were unable to show any quantity which is monotonous during the iteration. One can show counterexamples, where the working area, the $t$ distance between point $D$ and the neutral axis increases, or the stress at point $D$ decreases, although these quantities typically change in just the other way. We even found that the recursion is very sensitive to small perturbations, i.e. there are examples where due to a small "artificial" change in the $\omega$ inclination of the neutral axis a cycle occurs. Without perturbation there is a convergent behaviour to the fixed point. To find a divergent or cyclic solution we carried out systematic numerical trials, but we were unable to find any counterexample. The map given in eq. 1.26 was globally convergent even for very extreme cross sections including cross sections consisting of more detached regions, or very unusual, concave shapes.

![Figure 1.8: Two intersecting approximations of the neutral axis in the following steps of the recursion](image)

1.2.4. **The Principal Result of the Recursion with Linear Material Law**

**P.R.1.1.** For calculating the neutral axis of an arbitrary cross section under compression and biaxial bending I derived a direct recursion from the equations of equilibrium assuming linear material law in the compressed concrete zone ($q_2 = q_3 = \cdots = q_k = 0$) and linear material law for the reinforcement ($r_2 = r_3 = \cdots = r_l = 0$). The method can be associated with a two dimensional map. I proved analytically the following statements:

1. there is one, and only one fixed point,

2. the fixed point is stable (i.e. the method is locally convergent),

3. the recursion in the case of symmetrical cross section and a compressive load on the axis of symmetry is *globally convergent*.
P.R.1.2. In the general, two dimensional case (asymmetrical cross section or load) according to the literature we expect chaotic behaviour. In contrary, systematic numerical trials show the recursion based on the two dimensional map is globally convergent.

1.3 Cross Section: 
Non-linear Material Law

During the proof in the previous section we limited our investigation to a linear constitutive relation in the compressed zone. The real behaviour of concrete is rather non-linear, in this chapter we introduce a procedure for a non-linear constitutive law. In the literature there are some rather sophisticated methods to calculate the neutral axis and the curvature in the case of non-linear stress-strain relation [34], they can hardly serve as the core of an algorithm to calculate spatial deformations.

We aim to define a convergent recursion like we did in the linear case to find the neutral axis. We assume that the stress-strain relation can be given in a polynomial form, because we would like to avoid computing surface-integrals of transcedent functions. In the first subsection we show a possible, but limited, extension of the linear case. The second subsection of the present section introduces a much broader approach.

1.3.1. A Limited Solution

In this subsection we investigate a symmetrical cross section without reinforcement under compression and uniaxial bending. We assume that the stress-strain relation of the concrete can be given as

\[
\sigma_c(\varepsilon) = \begin{cases} 
q_1 \varepsilon + q_2 \varepsilon^2, & \text{if } \varepsilon > 0 \\
0 & \text{if } \varepsilon \leq 0 
\end{cases},
\]

(1.53)

where \(q_1\) and \(q_2\) are material constants. The polynomial in eq. 1.53 has two real roots (the polynomials with fewer roots are without interest for our proposal). It means that for one stress body there are two lines where the stresses vanish. To simplify the problem we describe the stress body by its maximum line, which is the line of the points with the maximum stress. After the maximum line has been found, the location of the neutral axis can easily be calculated. We use the first line of eq. 1.53 in the following form:

\[
\sigma_c(\varepsilon) = q_2 \varepsilon (\varepsilon - 2 \varepsilon_1) = q_2 \varepsilon (\varepsilon - 2 \sqrt{-\frac{f_{cm}}{q_2}}),
\]

(1.54)

where \(\varepsilon_1\) is the strain at the maximal compressive stress \(f_{cm}\) and \(q_2 < 0\) (Fig. 1.9).

In the recursion based on eq. 1.54 we have to solve two quadratic equations, while the method derived by eq. 1.53 would require solving a fourth degree equation. We note that with a well-chosen material constant \(q_2\) the non-linear stress-strain relation suggested by the EUROCODE 2 standard [21] can be well approximated.

\(^2\text{Here we do not take the ultimate strain of the concrete into consideration, but after the neutral axis and the curvature of the cross section has been found, the requirements of any standard can be checked.}\)
For symmetrical cross sections the maximum line and consequently the neutral axis can be calculated by a recursion associated with a one dimensional map. For defining the recursion we have to fulfill the equilibrium of forces and bending moments\(^3\). The origin of the \([xy]\) coordinate system is point \(D\). The strains are given as

\[ \varepsilon = -\kappa (x - t_m) + \varepsilon_1, \]

where \(t_m\) is the distance between the point \(D\) and the maximum line (\(t_m\) might be a negative number). By substituting eq. 1.55 into eq. 1.54 we obtain

\[ \sigma_c = q_2 \kappa^2 (x - t_m)^2 - q_2 \varepsilon_1^2 = q_2 \kappa^2 (x - t_m)^2 + f_{cm}. \]

The two equations of equilibrium\(^4\):

\(^3\)In the linear case the equation of the bending moments was enough to define the map.
\(^4\)the equation for the equilibrium of bending moments around axis \(x\) is fulfilled due to the symmetrical arrangement in each step of the recursion.
\[ \int_{A_e} \sigma dA - P = \int_{A_e} (q_2 \kappa^2 (x - t_m)^2 + f_{cm}) dA - P = \]
\[ = q_2 \kappa^2 (I_{y,c} - 2t_m S_{y,c} + t_m^2 A_e) + f_{cm} A_e - P = 0, \quad (1.57) \]

\[ \int_{A_e} \sigma x dA = \int_{A_e} (q_2 \kappa^2 (x - t_m)^2 x + f_{cm} x) dA = \]
\[ = q_2 \kappa^2 (J_{y,c} - 2t_m I_{y,c} + t_m^2 S_{y,c}) + f_{cm} S_{y,c} = 0. \quad (1.58) \]

We express \( \kappa^2 \) from eq. 1.57 as
\[ \kappa^2 = -\frac{f_{cm} A_e - P}{q_2 (I_{y,c} - 2t_m S_{y,c} + t_m^2 A_e)}, \quad (1.59) \]
and substitute it into eq. 1.58:
\[ \frac{(f_{cm} A_e - P)(J_{y,c} - 2t_m I_{y,c} + t_m^2 S_{y,c})}{I_{y,c} - 2t_m S_{y,c} + t_m^2 A_e} + f_{cm} S_{y,c} = 0. \quad (1.60) \]

Finally we express the force \( P \) as a function of \( t_m \):
\[ P(t_m) = \frac{f_{cm}(J_{y,c} A_e - 2t_m I_{y,c} A_e - I_{y,c} S_{y,c} + 2t_m^2 S_{y,c})}{J_{y,c} - 2t_m I_{y,c} + t_m^2 S_{y,c}}. \quad (1.61) \]

On the right side the counter is linear, the denominator is second degree in \( t_m \). The root of the counter is denoted by \( t_{c1} \), the roots of the denominator are denoted by \( t_{d1} < t_{d2} \), if they exist. The limit values of \( P(t_m) \) in the infinities:
\[ \lim_{t_m \to -\infty} P(t_m) = \lim_{t_m \to +\infty} P(t_m) = 0 \quad (1.62) \]

The counter of the first derivative of \( P(t_m) \) is second degree in \( t_m \), consequently there can be a maximum of two extrema of the \( P(t_m) \) function. There are three cases to be distinguished:

1. The denominator of eq. 1.61 is positive for any \( t_m \), which can be expressed as
\[ I_{y,c}^2 - J_{y,c} S_{y,c} < 0. \quad (1.63) \]

In this case the function \( P(t_m) \) is continuous. \( P(t_m) = 0 \) at \( t_m = t_c \). There must be a maximum and a minimum point to fulfill eq. 1.62, and the derivative ensures there are no other extrema. The location of the maximum is denoted by \( t_0 \), the value of the function here is denoted by \( P_{max} \), and called **theoretical ultimate load**\(^5\). The shape of the function is given in Fig.1.11(a).

---

\(^5\)The theoretical ultimate load typically is not the ultimate load of the cross section, because it is possible, that at \( P = P_{max} \) the maximum compressive strain at several points of the section exceeds the limit value approved by the standard, i.e. concrete grains.
2. There is one root of the denominator, i.e. it is non-negative for any \( t_m \). In this case \( J_{y,c}^2 - J_{y,c}S_{y,c} = 0 \). We prove, that \( t_c < t_{d1} \):

\[
t_c = \frac{I_{y,c}S_{y,c} - A_cJ_{y,c}}{2(S_{y,c}^2 - I_{y,c}A_c)} < \frac{I_{y,c} - \sqrt{J_{y,c}^2 - J_{y,c}S_{y,c}}}{S_{y,c}} = t_{d1}. \tag{1.64}
\]

By \( J_{y,c}^2 - J_{y,c}S_{y,c} = 0 \) the 1.64 inequality can be simplified:

\[
\frac{I_{y,c}}{2(S_{y,c})} < \frac{I_{y,c}}{S_{y,c}}, \tag{1.65}
\]

which is fulfilled, since \( I_{y,c} \) and \( S_{y,c} \) are positive values. The shape is given in Fig.1.11(b).

3. The denominator has two real roots ( \( J_{y,c}^2 - J_{y,c}S_{y,c} > 0 \)). In this case the function has two discontinuities at \( t_m = t_{d1} \) and \( t_m = t_{d2} \). We prove \( t_c < t_{d1} \), consequently eq. 1.64 is valid for any cross section. First we point out, that

\[
S_{y,c}^2 - I_{y,c}A_c = -A_cI_{y,c,0} < 0, \tag{1.66}
\]

where \( I_{y,c,0} \) is the second moments of area around the axis \( y' \) of the centroid \( C \) (Fig. 1.10). For \( J_{y,c} = 0 \) the statement in eq. 1.64 is true, since in special case \( t_{d1} = 0 \) and \( t_c < 0 \) due to eq. 1.66 and the facts \( I_{y,c} > 0 \), \( S_{y,c} > 0 \). Let us assume there exist a value of \( J_{y,c} \) (it can be either positive or negative) for which \( t_{d1} = t_c \). In this imaginary case inequality 1.64 becomes an equation:

\[
t_c = \frac{I_{y,c}S_{y,c} - A_cJ_{y,c}}{2(S_{y,c}^2 - I_{y,c}A_c)} = \frac{I_{y,c} - \sqrt{J_{y,c}^2 - J_{y,c}S_{y,c}}}{S_{y,c}} = t_{d1}. \tag{1.67}
\]

By algebraic transformations eq. 1.67 is a second degree expression in \( J_{y,c} \), its two roots are:

\[
J_{y,c1,2} = \frac{3A_cI_{y,c}S_{y,c}^2 - 2S_{y,c}^4 \pm 2S_{y,c}(S_{y,c}^2 - A_cI_{y,c})\sqrt{S_{y,c}^2 - A_cI_{y,c}}}{A_c^2S_{y,c}^2}. \tag{1.68}
\]

According to eq. 1.66 the expression under the square root is negative, thus 1.68 leads to an imaginary expression. We got to contradiction with our original assumption, there is no possible value of \( J_{y,c} \) to fulfill eq. 1.67. Since \( t_c \) and \( t_{d1} \) depends continuously on \( J_{y,c} \), inequality 1.64 is fulfilled for any \( J_{y,c} \). Due to this fact\(^6\), the \( P(t_m) \) function must have a (local) maximum and a (local) minimum point. The \( P_{\text{max}} \) local maximum is at \( t_m = t_0 \). We note, that at the local minimum \( P > P_{\text{max}} \), but one can prove, that the \( \kappa \) curvature is imaginary between \( t_{d1} \) and \( t_{d2} \), the values of \( P_{i,m} \) are physically irrelevant here (Fig.1.11(c)).
Figure 1.11: The possible shapes of $P(t_m)$

From this investigation we conclude, that for any symmetric cross section and loading the theoretical maximum force $P_{\text{max}}$ exists. Eq. 1.60 is a second degree expression in $t_m$, its solutions are

$$t_{m,1,2} = \frac{2PI_{y,c} - 2f_{cm}A_cI_{y,c} + 2f_{cm}S_{y,c}^2}{2PS_{y,c}} \pm \sqrt{(P - f_{cm}A_c)^2I_{y,c}^2 - f_{cm}^2I_{y,c}^2S_{y,c}^2A_c + f_{cm}^4S_{y,c}^4 + S_{y,c}(f_{cm}A_c - P)(PJ_{y,c} - S_{y,c}I_{y,c}f_{cm})}$$

$$\pm \frac{\sqrt{(P - f_{cm}A_c)^2I_{y,c}^2 - f_{cm}^2I_{y,c}^2S_{y,c}^2A_c + f_{cm}^4S_{y,c}^4 + S_{y,c}(f_{cm}A_c - P)(PJ_{y,c} - S_{y,c}I_{y,c}f_{cm})}}{PS_{y,c}}.$$  

We remark without a detailed proof, that the expression under the square root is positive, if $P < P_{\text{max}}$ and negative if $P > P_{\text{max}}$. For both real values of $t_m$ there exists a stress body fulfilling the conditions of equilibrium. We have to define which solution

\(^a\)Without this statement one could show examples on a $P(t_m)$ function without local extrema.
should be accepted. At \( P = P_{\max} \) there is only one solution in \( t_m \). There are three approaches to define the recursion:

1. We give a rule stating which neutral axis to choose if there is more than one solution and halt if there is no solution. This rule can be to always choose the stress distribution for the smaller curvature \( \kappa \), which can be seen as an assumption based on the load history: we assume the considered \( P \) has not been exceeded in the past.

2. At each step of the recursion we calculate the value of \( P_{\max} \) by the first and second derivatives of \( 1.6l \), and calculate the neutral axis for this force. The advantage of this approach is that at \( P = P_{\max} \) the uniqueness of the solution is fulfilled. This method varies the magnitude of the load, but it can be seen as a safe solution since it predicts a higher curvature and bigger cracked zone for the cross section.

3. We keep \( P \) at the prescribed value until \( P < P_{\max} \), and take \( P = P_{\max} \) otherwise.

![Diagram](image)

**Figure 1.12:** An example: a rectangle cross section under uniaxial bending and compression

We demonstrate the three approaches on a rectangular cross section. We use the signs of Fig. 1.12 and assume \( q_2 = -1 \) GPa, \( g < 0 \) and \( h > -g \). If \( x_n^i > h \) then we take \( x_n^i = h \). For \( x_n^i < h \) eqs. 1.57-1.58 can be written as:

\[
-k^2 b \frac{1}{3} (x_n^i - g)^3 - (x_n^i - g^2)t_m + (x_n^i - g)(x_n^i)^2 + f_{cm} b (x_n^i - g) - P = 0, \quad (1.70)
\]

\[
-k^2 b \frac{1}{4} (x_n^i - g)^4 - \frac{2}{3} (x_n^i - g^3)t_m + \frac{1}{2} (x_n^i - g^2)(x_n^i)^2 + \frac{1}{2} f_{cm} b (x_n^i - g^2) = 0. \quad (1.71)
\]

**Recursion with a Constant Load**

By substituting \( k^2 \) from eq. 1.70 into eq. 1.71 we have two roots for \( t_m \). Two imaginary roots imply the \( P > P_{\max} \) case, coinciding roots means \( P = P_{\max} \). Having two different
real roots we have to determine both the corresponding values of $\kappa^2$, the smaller value and the belonging root in $t_m$ should be accepted as a solution. Analytically both the positive and the negative values of the square root of $\kappa^2$ fulfill the conditions of equilibrium, but considering a load history without exceeding the present load, only the positive value of the curvature $\kappa$ can be accepted as a solution.

After substituting $\kappa^2$ and $t_m$ into eq. 1.56 material law we obtain the distance of the neutral axis by solving the second degree equation. Since we seek the neutral axis belonging to the positive value of $\kappa$, the solution is the bigger root of the second degree equation:

$$x_n^{(i+1)} = t = \frac{1}{6(x_n^i + g)} \left[ f_{cm} (g - x_n^i)^3 - Z + 4g^2P + 4gx_n^iP + 4P_{x_n^i}^2 + \sqrt{2} \left( f_{cm} (g - x_n^i)^3 + (g - x_n^i)^2P - Z \right) \right], \quad (1.72)$$

where

$$Z^2 = \left[ f_{cm}^2 g^4 + 2g^3Pf_{cm} - 4g^2f_{cm}^2x_n^i + 6g^2f_{cm}^2x_n^i^2 - 6g^2Pf_{cm}x_n^i - 2g^2P^2 - 8gP^2x_n^i + 6gPf_{cm}x_n^i^2 - 4gf_{cm}^2x_n^i^3 - 2Pf_{cm}x_n^i^3 - 2P^2x_n^i^2 + f_{cm}^2x_n^i^4 \right] (g - x_n^i)^2. \quad (1.73)$$

Equation 1.72 is the $f_{ul}(x)$ function of the corresponding, one dimensional map.

**Recursion with the Maximal Load**

For a section we can calculate $P_{\text{max}}$ by eq. 1.61:

$$P(t_m) = \frac{g^4 - 2g^3t_m - 2g^2x_n^i + 6g^2x_n^i^2t_m + 2gx_n^i^3 - 6g^2x_n^i^2t_m + 2x_n^i^3t_m - x_n^i}{3g^3 - 8t_mg^2 + 3g^2x_n^i + 6t_m^2g - 8gx_n^i t_m + 3gx_n^i^2 + 6t_m^2x_n^i - 8t_m x_n^i^2 + 3x_n^i^3 f_{cm} b}, \quad (1.74)$$

and we obtain the local extrema of $P(t_m)$ from the following equation:

$$\frac{dP(t_m)}{dt_m} = 0. \quad (1.75)$$

There are two solutions:

$$t_{0,1} = \frac{1}{2}(g + x_n^i) - \frac{1}{6} \sqrt{3}(x_n^i - g), \quad (1.76)$$

$$t_{0,2} = \frac{1}{2}(g + x_n^i) + \frac{1}{6} \sqrt{3}(x_n^i - g). \quad (1.77)$$

By the second derivatives of $P(t_m)$ one can show, that $t_{0,1}$ is a maximum and $t_{0,2}$ is a minimum, for any suitable $g$ and $x_n^i$ values.

By substituting $t_{0,1}$ into $P(t_m)$ we get the theoretical maximum load $P_{\text{max}}$: 

**Spatial Deformations of Rods without Tensile Strength**
\[ P_{\text{max}} = -\frac{(x^n_i - g)^2}{\sqrt{3}g + g - \sqrt{3}x^n_i - x^n_i^2} f_{cm} b. \] (1.78)

The values of \(P_{\text{max}}\) and \(t_{0,1}\) define the stress distribution, \(\kappa\) can be obtained from eq. 1.70. The strains can be calculated by substituting \(\kappa\) and \(t_{0,1}\) into eqs. 1.56 and 1.55. The neutral axis can be obtained as

\[
x^{(i+1)}_n = t = \frac{3 (g + x^n_i)^2 - \sqrt{3} (x^n_i^2 - g^2) + \sqrt{2} \sqrt{3(g + x^n_i)^2 + \sqrt{3} (x^n_i^2 - g^2) (-x^n_i + g)^2}}{6 (x^n_i + g)}, \tag{1.79}
\]

which is the \(f_{nl}(x)\) function defining the map. It resembles the linear case since it depends only on the geometrical values \(g\) and \(x^n_i\), and not on \(f_{cm}\).

**Figure 1.13:** The functions \(f_{nl}(x)\) and \(f_l(x)\) for several cases mentioned in the text

As an example we take a rectangle with \(g = -20\) cm, \(h = 60\) cm, \(b = 10\) cm and \(f_{cm} = 48\) MPa. In Fig. 1.13 we compare the function \(f_{nl}(x)\) assuming the maximum load at each step of the recursion to the curves belonging to different loads kept constant during the iteration. The figure contains the function for the linear case \(f_l(x)\) as well. Without detailed explanation the following statements can be done:

1. As the load \(P\) is kept constant during the recursion, the function \(f_{nl}(x)\) has a local minimum at the fixed point, i.e. the recursion is convergent if the first estimate on the neutral axis was on the proper side (i.e. \(x^n_1 > x^n_\infty\)) of the fixed point.
2. If the recursion is carried out by the current maximum load $P_{\text{max}}$, then the local minimum and the fixed point do not coincide any more. Perhaps for some cross sections this approach can produce cycles or chaotic behaviour, although we could not produce any example.

The basic problem of the approach described in this section is that it cannot be generalized. An asymmetrical load or cross section, or even the presence of reinforcing bars cannot be included in this model since the system of equations can hardly be solved and attempting to define a rule to choose the proper root is hopeless. Using the results gained by this approach we show a method to solve the general problem in the next subsection.

## 1.3.2. General Solution

In this subsection we aim to define a recursive procedure to determine the neutral axis of an arbitrary cross section under compression and biaxial bending if the material law of the compressed concrete zone is non-linear, i.e. $q_1 > 0$ and $r_1 > 0$ (if reinforcement is present) and the other constants ($q_2, q_3, \ldots, q_k, r_2, r_3, \ldots, r_l$) are arbitrary. The location of the first maximum of eq. 1.6 is denoted by $\varepsilon_1$, the $0 < \varepsilon < \varepsilon_1$ is called the *monotonic* part of the stress-strain relation (Fig 1.14).

![Graph](image)

**Figure 1.14:** A non-linear stress-strain relation of the compressed zone

The solution of the equations of equilibrium 1.13-1.15 in $\omega$, $t$ and $\kappa$ cannot be given without calculating surface integrals of trigonometric functions. In addition, there are many roots in $t$ and $\omega$ and it seemed impossible to give a strategy for deciding which one to choose.\(^7\)

We seek an *equivalent linear problem*, which is the same cross section with linear stress-strain relation under biaxial bending and compression with the same neutral axis as the non-linear problem has, but with a different location of the load. We seek the neutral axis $(x_n, y_n)$, the curvature $\kappa$ and the location $(x_D, y_D)$ of the load of the equivalent linear problem. The semi-implicit procedure can be given in the following form:

\(^7\)This problem also arises if we seek the neutral axis of a cross section with tensile strength. Applying a similar recursion to the one given here without leaving the cracked zones one can find the neutral axis in the same manner.
\[
\begin{bmatrix}
 x_{n+1}^i \\
y_{n+1}^i \\
\kappa_{n+1}^i \\
x_{D_{n+1}}^i \\
y_{D_{n+1}}^i
\end{bmatrix} = \begin{bmatrix}
 \hat{G} (x_n^i, y_n^i, \kappa_n^i, x_D^i, y_D^i) \\
 \hat{H} (x_n^i, y_n^i, \kappa_n^i, x_D^i, y_D^i) \\
 I (x_n^i, y_n^i+1, \kappa_n^i, x_D^i, y_D^i) \\
 J (x_n^i+1, y_n^i+1, \kappa_n^i+1, x_D^i, y_D^i) \\
 K (x_n^i+1, y_n^i+1, \kappa_n^i+1, x_D^i, y_D^i)
\end{bmatrix}.
\] (1.80)

Although this approach seems to be more complicated than the two dimension recursion applied in the linear case, it proved to be successful.

We locate the actual neutral axis \((x_{n+1}^i, y_{n+1}^i)\) by assuming linear constitutive law and apply the functions \(\hat{G}\) and \(\hat{H}\), which are formally identical with the \(G\) and \(H\) functions of eq. 1.26, but they depend on the current location \(D_i\) of the load, not only on the current estimate of the neutral axis \((x_n^i, y_n^i)\). Having the neutral axis, we take the non-linear part of the material law into account. By substituting \((x_{n+1}^i, y_{n+1}^i)\) into eqs. 1.4 and 1.5 we obtain \(\omega(i+1)\) and \(t(i+1)\). Applying eq. 1.13 we gain a \(k\)-th degree equation in \(\kappa\). This equation contains higher order moments of area for the working part of the cross section. Having a \(k\)-th degree material law the \((k+1)\)-th order moments of area must be calculated. We assume, that during the load history the monotonous part of the stress-strain relation has not been exceeded, i.e. we seek the smallest root in \(\kappa\), which is the \((i+1)\)-th estimate on the curvature.

Finally we have to determine our next approximation on the location of the load of the equivalent linear problem. By assuming the \textit{equivalent linear force} \(P_i^l\) is located in point \(D^l\) instead of \(D^0\) (the location of the load \(P\)) in the calculation of the equilibrium of bending moments we have to consider the bending moment arising from the difference of these two points. Taking this fact into account and substituting the values of \(\omega(i+1), t(i+1)\) and \(\kappa(i+1)\) into eqs. 1.14-1.15 the so called \textit{unbalanced bending moment} \((\Delta M_y, \Delta M_x)\) can be obtained as:

\[
\Delta M_y^{(i+1)} = - \int_{A_{c(x',y')}} x_c' (-\kappa^{(i+1)}(x \cos \omega^{(i+1)} + y \sin \omega^{(i+1)} - t^{(i+1)})) \, dA - \\
- \int_{A_s} x \sigma_s (-\kappa^{(i+1)}(x \cos \omega^{(i+1)} + y \sin \omega^{(i+1)} - t^{(i+1)})) \, dA + P x^i_D. \tag{1.81}
\]

\[
\Delta M_x^{(i+1)} = - \int_{A_{c(x',y')}} y_c' (-\kappa^{(i+1)}(x \cos \omega^{(i+1)} + y \sin \omega^{(i+1)} - t^{(i+1)})) \, dA - \\
- \int_{A_s} y \sigma_s (-\kappa^{(i+1)}(x \cos \omega^{(i+1)} + y \sin \omega^{(i+1)} - t^{(i+1)})) \, dA + P y^i_D. \tag{1.82}
\]

The unbalanced bending moment contains the effect of the non-linear part of the material law, since the current estimate of the neutral axis provided equilibrium between the equivalent linear force \(P_i^l\) and the stresses arisen due to the linear term of the material law. \(P_i^{l(i+1)}\) can be determined as follows:

\[
P_i^{l(i+1)} = - q_k \kappa^{(i+1)} \int_{A_{c(\omega^{i+1},\mu^{i+1})}} (x \cos \omega^{(i+1)} + y \sin \omega^{(i+1)} - t^{(i+1)}) \, dA - \\
- r_k \kappa^{(i+1)} \int_{A_s} (x \cos \omega^{(i+1)} + y \sin \omega^{(i+1)} - t^{(i+1)}) \, dA. \tag{1.83}
\]
The ratio of the unbalanced bending moments and the equivalent linear force gives the new approximation on the location $D_{i+1}$ of the load for the equivalent linear problem:

$$
\begin{bmatrix}
x_{D_{i+1}}^{i+1} \\
y_{D_{i+1}}^{i+1}
\end{bmatrix} = \begin{bmatrix}
J(x_{n}^{i+1}, y_{n}^{i+1}, \kappa^{i+1}, x_D, y_D) \\
K(x_{n}^{i+1}, y_{n}^{i+1}, \kappa^{i+1}, x_D, y_D)
\end{bmatrix} = \begin{bmatrix}
\frac{\Delta M_y^{(i+1)}}{P^{(i+1)}_t} + x_D^{i} \\
\frac{\Delta M_x^{(i+1)}}{P^{(i+1)}_t} + y_D^{i}
\end{bmatrix}.
(1.84)
$$

**AN EXAMPLE**

As an example we take a second degree $\sigma_c$ (eq. 1.53) and a linear $\sigma_s$ stress-strain relation. In this case the reinforcement can be included applying the equivalent cross section with $n = r_1/q_1$ and the following equations are applied. After the $(i+1)$-th neutral axis is determined the curvature $\kappa^{(i+1)}$ can be calculated as follows:

$$\kappa^{(i+1)} = \min \left| \frac{-b_0 \pm \sqrt{b_0^2 - 4a_0 c_0}}{2a_0} \right|, \quad (1.85)$$

where

$$
a_0 = q_2 \left( I_{y,c}^{i} \cos^2 \omega^{(i+1)} + I_{x,c}^{i} \sin^2 \omega^{(i+1)} + t^{(i+1)} l^{(i+1)} A_{y,c}^{i} + 2D_{xy,c}^{(i)} \sin \omega^{(i+1)} \right), \quad (1.86)
$$

$$
b_0 = -q_1 (S_{y,c}^{i} \cos \omega^{(i+1)} + S_{x,c}^{i} \sin \omega^{(i+1)} - t^{(i+1)} A_{y,c}^{i}), \quad (1.87)
$$

$$
c_0 = -P. \quad (1.88)$$

The components of the unbalanced bending moment and the equivalent linear force:

$$
\begin{align*}
\Delta M_y^{(i+1)} &= -q_2 \kappa^{(i+1)^2} \left( J_{y,c}^{i} \cos^2 \omega + J_{xy,c}^{i} \sin^2 \omega + J_{x,c}^{i} \sin^2 \omega + J_{y,c}^{i} \cos^2 \omega + L^{(i+1)} S_{y,c} + 2J_{x,c}^{i} \cos \omega - 2t I_{x,c} \cos \omega - 2t D_{xy,c} \sin \omega \right) + P x_D^{i},
\end{align*}
(1.89)
$$

$$
\begin{align*}
\Delta M_x^{(i+1)} &= -q_2 \kappa^{(i+1)^2} \left( J_{x,c}^{i} \sin^2 \omega + J_{xy,c}^{i} \sin^2 \omega + J_{y,c}^{i} \cos^2 \omega + J_{x,c}^{i} \cos^2 \omega - J_{y,c}^{i} \sin^2 \omega - 2J_{x,c}^{i} \sin \omega \right) + P y_D^{i},
\end{align*}
(1.90)
$$

$$
P^{(i+1)}_t = -q_1 \kappa^{(i+1)} (S_y \cos \omega + S_x \sin \omega - A_t).
(1.91)$$

Point $D^{(i+1)}$ is obtained by applying eq. 1.84.

Finally, we give an example. The "T" shaped cross section contains 6 bars of 020 reinforcement, $n = 6.36$, $q_1 = 20$ GPa, $q_2 = -1$ GPa. The location of the centroid of the cross section in the $[xy]$ coordinate system is $x_c = 13$ cm and $y_c = 8$ cm. Fig. 1.15 contains the solution for the linear stress-strain relation (a). Two solutions for the second degree material law with constant load $P = 17500$N and $P = 35000$N are given in subfigures (b) and (c). The (d) subfigure is the solution assuming $P = P_{max}$. For the non-linear case the offset of point $D$ is also displayed.

---

8For higher degree stress-strain relation of the reinforcement the equivalent cross section can be applied only for the linear term of the material law, the higher degree terms must be integrated seperately for the concrete and the reinforced part of the cross section.
Figure 1.15: The neutral axes determined by the methods mentioned in the text
1.3.3. The Principal Result of the Recursion with Non-linear Material Law

P.R.2.1. For non-linear material law of the compressed zone \((q_2, q_3, \ldots, q_k)\) are arbitrary constants) and arbitrary constitutive law of the reinforcement \((r_2, r_3, \ldots, r_l)\) are arbitrary constants) I derived an algorithm to determine the neutral axis of an arbitrary cross section under compression and biaxial bending. The method can be associated with a 5 dimensional, semi-implicit map.

P.R.2.2. Using this approach I investigated the second order stress-strain relation of the concrete with \(\sigma_2(\epsilon) = r_1 \epsilon\) material law of the reinforcement, which is sufficient for practical purposes. In this case there exists a theoretical maximal load \(P_{\text{max}}^i\) for each estimate of the neutral axis. I defined three possibilities for carrying out the calculation:

1. The load \(P\) is constant at each step of the iteration, for \(P > P_{\text{max}}^i\) the method halts,

2. In each step we determine \(P_{\text{max}}^i\) and the neutral axis is calculated with this load (in this case the solution is unique),

3. The load \(P\) is constant until \(P < P_{\text{max}}^i\), otherwise the method continues assuming \(P = P_{\text{max}}\) in each step.

By numerical simulations all the three approaches are globally convergent.

1.4 Rod: Calculation of the Shape

We carry out calculations only on rods supported at the ends, there are no middle supports. The cross section of the rod along the bar axis is constant, the rod is free from initial imperfections. Deformations are calculated by integrating the curvature and the rate of twist along the bar axis, provided that the spatial position, the forces and the bending moments are known at one end of the bar. In this case we solve an initial value problem (IVP). In practice there are several boundary conditions at both ends of the bar leading to a boundary value problem (BVP). Solving a BVP requires a much higher computational effort than calculating an IVP. Therefore we use an algorithm which applies the advantages of the parallel computation.

We would like to apply the algorithm to several types of problems so the required functionality can be different. We use a notation of four letters to identify the applied algorithm component. The first letter distinguishes between the two basic types of the material law of the concrete

- L - the material law is linear in the compressed zone,
- N - the material law is nonlinear in the compressed zone.

The second letter gives information about the tensile strength of the material as follows:
• T - the material has tensile strength,
• 0 - the material has zero tensile strength \( f_{ctm} = 0 \), it cracks under tensile stresses,
• L - the material has limited tensile strength \( f_{ctm} < 0 \).

The third letter indicates the type of the applied reinforcement:
• 0 - there is no reinforcement in the rod,
• S - there is reinforcement without prestress,
• P - there is at least one prestressed tendon in the cross section.

The fourth letter refers to the special material features:
• C - The effects of tension stiffening, shrinkage and creep are taken into account,
• 0 - The special material features are omitted.

These notations are used in the second part of this work to identify the applied algorithm for each application. For example the L0SC notation refers to a problem calculated by linear constitutive law assuming zero tensile strength, where the cross section contains un prestressed reinforcing bars and the effect of the special material features are taken into account. Theoretically there are 36 combinations of these letters which indicate 36 different algorithms. However, the presence of the reinforcement and the special material properties influence only the call of some subroutines. These subroutines are described in subsection 1.4.5.

The differences caused by the material law and the tensile strength effect only the calculation of the curvature of one cross section. The known features of the algorithms to determine the curvature are summarized in Table 1.4.

<table>
<thead>
<tr>
<th>Material Law</th>
<th>L</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>uniqueness of the solution no iteration needed</td>
<td>several solutions local stability convergent</td>
</tr>
<tr>
<td>Tensile</td>
<td>O</td>
<td>local stability globally convergent</td>
</tr>
<tr>
<td>Strength</td>
<td>L</td>
<td>uniqueness of the solution local stability globally convergent</td>
</tr>
<tr>
<td></td>
<td></td>
<td>several solutions ?</td>
</tr>
</tbody>
</table>

**Table 1.4:** The algorithms to determine the curvature. Features in italics are analytical, the others are numerical results.

In the case of the LT algorithm the neutral axis is determined by solving eq. 1.26 only once. The NT and the NL algorithms are beyond the scope of this work, although
the table contains their known features. We describe the other three algorithms in detail in the first subsection of this section. The solution of the IVP and the BVP are the same procedure in all algorithms, they are topics of the next two subsections. In the fourth subsection we introduce additional algorithm components to enable industrial users to run the computation in parallel architectures and grid systems. Finally we describe the subroutines to calculate the effect of the special material features of concrete and to determine the magnitude of the prestress.

1.4.1. Determining the Curvature

At the core of the algorithm one has to calculate the curvature and the rate of twist of one cross section. The load is typically compression with biaxial bending and torsion; however the algorithm must be able to handle pure bending. It can be well approximated by placing a load of small magnitude far away from the cross section and handling the problem as eccentric compression. The torsion is calculated by applying the equations of subsection 1.1.3. In all of our calculations we assumed $\mu = 0.2$, consequently $G = 0.42E_c$, which is a generally accepted approximation for concrete. In the following algorithms we accept an estimate as solution if the two neutral axes in the following steps of the recursion are closer to each other than a previously chosen, arbitrarily small, fixed error $\delta$.

Linear Material Law [L0]

The analytical proofs in Section 1.2 concern the L0 algorithm, the corresponding map $F_l$ (eq. 1.26) is implemented as 3 embedded procedures:

$$F_l = F_{l,3} \circ F_{l,2} \circ F_{l,1}. \quad (1.92)$$

The input of the $i$-th step of the recursion is the $i$-th approximate location $(x_i^i, y_i^i)$ of the neutral axis in the $[x,y]$ coordinates system (Fig. 1.2 (a)), the matrix $K_0$ with the measure $2 \cdot N_0$ contains the $N_0$ contour points $(x_j^i, y_j^i, j = 1, 2, ..., N_0)$ of the total concrete section and the matrix $V_0$ contains the location, the area and the type of the reinforcement bars. The type can be normal or prestressed. The output is the $(i + 1)$-th location of the neutral axis $(x_n^{i+1}, y_n^{i+1})$. The concrete section is cut into a maximum of two parts by the actual neutral axis. The two parts are determined by the Weiler-Atherton Polygon Clipper [72]. The part containing the external load is the working concrete part, this part is stored in the matrix $K^i$ with the measure $2 \cdot N^i$, where $N^i$ is the number of the contour points of the working part in the $i$-th step.

$$K^i = F_{l,1} \left( \left[ \begin{array}{c} x_n^i \\ y_n^i \end{array} \right], K_0, V_0 \right). \quad (1.93)$$

Since the matrices $K_0$ and $V_0$ are constants at each step, we will not indicate them as independent variables. The matrix $K^i$ is the input of the subroutine determining the cross sectional areas ($A^i$, $S_x^i$, $S_y^i$, $I_x^i$, $I_y^i$, $D_{xy}^i$). This calculation is carried out according to [52], where only the working cross section part (including the reinforcement) is needed. These values are stored in the $c^i$ vector, which has 6 elements for the linear stress-strain relation.
\[ c^i = F_{l,2} (K^i) = F_{l,2} \left( F_{l,1} \left( \begin{bmatrix} x_n^i \\ y_n^i \end{bmatrix} \right) \right). \]  

(1.94)

In the next step of the algorithm we assume tensile strength for the actual concrete cross section. We determine the neutral axis of the next step by solving the equations of equilibrium with the constitutive law of this imaginary situation:

\[ \begin{bmatrix} x_n^{(i+1)} \\ y_n^{(i+1)} \end{bmatrix} = F_{l,3} \left( c^i \right) = F_{l,3} \left( F_{l,2} \left( F_{l,1} \left( \begin{bmatrix} x_n^i \\ y_n^i \end{bmatrix} \right) \right) \right). \]  

(1.95)

For arbitrary sections the explicit form of \( F_{l,3} \) is given in eq. 1.26, for symmetrical sections under compression and uniaxial bending \( F_{l,3} \) is given by eq. 1.47. In the recursion \( x_n^i \) or \( y_n^i \) can attain infinite value, this happens if the function \( F_{l,3} \) has a singularity for the input data. In the implementation, these cases require additional subroutines to handle the division by zero, for details we refer to [39]. Once we have the neutral axis the curvature can be determined in both cases from the equilibrium of forces (eq. 1.23).

**Limited Tensile Strength [LL]**

For the LL algorithm the \( F_l \) map contains 4 subroutines:

\[ F_l = F_{l,4} \circ F_{l,3} \circ F_{l,2} \circ F_{l,1}, \]

(1.96)

since we have to determine the neutral axis and the border line. Beyond using the border line instead of the neutral axis to determine the working cross section, the \( F_{l,1}, F_{l,2}, F_{l,3} \) procedures are the same as in the case of the L0 algorithm. The \( F_{l,4} \) procedure determines the \((i+1)\)-th border line from the neutral axis applying eq. 1.19 and finding the intersection of the border line with the coordinate axes:

\[ \begin{bmatrix} x_n^{(i+1)} \\ y_n^{(i+1)} \end{bmatrix} = F_{l,4} \left( \begin{bmatrix} x_n^{(i+1)} \\ y_n^{(i+1)} \end{bmatrix} \right) = F_{l,4} \left( F_{l,3} \left( F_{l,2} \left( F_{l,1} \left( \begin{bmatrix} x_n^i \\ y_n^i \end{bmatrix} \right) \right) \right) \right). \]  

(1.97)

**Non-Linear Material Law [N0]**

For the N0 algorithm the \( F_{nl} \) map contains 6 subroutines:

\[ F_{nl} = F_{nl,5} \circ F_{nl,4} \circ F_{nl,3} \circ F_{nl,2} \circ F_{nl,1} \circ F_{nl,0}. \]  

(1.98)

To determine the neutral axis we use the same steps as we did in the L0 algorithm, consequently

---

\(^9\)A good example is the symmetrical cross section under compression and uniaxial bending: if we solve this problem with eq. 1.26 instead of eq. 1.47, then \( y_n^i \) is infinite in each step if the first estimate of the neutral axis was perpendicular on the axis of symmetry, consequently \( y_n^i = y_n^{i+1} = \pm \infty \) for any \( i \). In the final implementation instead of the additional subroutines we applied the \( \text{atan2}(y, x) \) of the C++ language, which gives the arcus tangent of \( y/x \) even if \( x = 0 \).
Figure 1.16: The flow charts of the L0 and N0 algorithms. Observe the shaded boxes: they represent the same subroutines in both algorithms.
\begin{align*}
F_{nl,1} &= F_{l,1}, \\
F_{nl,2} &= F_{l,2}, \\
F_{nl,3} &= F_{l,3}.
\end{align*}

The only difference is that in $F_{nl,2}$ we need higher terms of the cross sectional areas as well, so $c^i$ has more elements\textsuperscript{10}. If $q_1, q_2, \ldots, q_k \neq 0$ and all the other constants in the material law equal zero then the number of the elements is $0.5 \cdot (k + 2)(k + 1)$.

The $F_{nl,0}$ procedure is a coordinate transformation, since point $D^i$, consequently the $[x y]$ coordinate system is replaced at each step of the recursion. The elements of the matrices $\mathbf{K}_0$ and $\mathbf{V}_0$ are changed in this step.

The function $F_{nl,4}$ calculates the $\kappa^i$ curvature from the equilibrium of forces by assuming the actual neutral axis. Here we seek the smallest root of a higher order equation (subsection 1.3.2). For this propose the Laguerre’s method [1] can be applied as a fast and reliable method.

\begin{equation}
\kappa^i = F_{nl,4}\left(\begin{bmatrix} x_{n+1}^i \\ y_{n+1}^i \end{bmatrix}\right).
\end{equation}

The last function, $F_{nl,5}$ is applied to determine the new location of the point $D^{i+1}$. Here we calculate the unbalanced bending moment, the equivalent linear force and apply eq. 1.84:

\begin{equation}
\begin{bmatrix} x_{D+1}^i \\ y_{D+1}^i \end{bmatrix} = F_{nl,5}\left(c^i, \kappa^i\right).
\end{equation}

The flow charts of the L0 and N0 algorithm can be compared on Fig. 1.16.

### 1.4.2. Solution of the IVP

We solve the Initial Value Problem by the generally accepted method of determining large deformations (\textit{geometrically nonlinear calculation}). In the procedure the bar is sliced into arbitrarily narrow, ds wide slices, the cut plane is normal to the bar axis, the actual slice is denoted by $j$. Each cross section is described by a position vector ($\mathbf{r}^j$) and a rotating matrix ($\mathbf{T}^j$). We use an arbitrary global $[XYZ]$ and a local $[xyz]$ coordinate system fixed to the cross section. The origin of $[xyz]$ is the centroid of the uncracked section, the axis $z$ is perpendicular to the plane of the cross section (Fig. 1.17).

For one cross section the curvature $\kappa$, the location $(x_n, y_n)$ and the inclination $\omega$ of the neutral axis are calculated according to the previous section, the rate of twist $\gamma$ is calculated by eq. 1.20. The matrix $\mathbf{T}$ is rotated around the axis $z$ by the rotating matrix $\mathbf{S}_z$ such that axis $y$ becomes parallel to the neutral axis. Next, $\mathbf{T}$ is rotated around the axis $y$ (i.e. the neutral axis) by the angle $\kappa ds$ applying matrix $\mathbf{S}_y$, and finally it is rotated back around axis $z$ by the angle $(\omega - \gamma ds)$. The connection between the rotating matrices of the $j$-th and $(j + 1)$-th cross section:

\textsuperscript{10}In [52] the cross sectional areas are given up to the second order in our work we extended that formulas to the third order case and included the reinforcement. This extension can be continued to higher terms if it is required.
\[ T^{(j+1)} = T^j S_z(\omega) S_y(\kappa \, ds) S_z(-\omega + \gamma \, ds), \] (1.104)

where

\[ S_y(\alpha) = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix}, \] (1.105)

\[ S_z(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \] (1.106)

The \((j + 1)\)-th position vector can be obtained as

\[ r^{(j+1)} = r^j + T^{(j+1)} \begin{bmatrix} 0 \\ 0 \\ ds \end{bmatrix}. \] (1.107)

### 1.4.3. Solution of the BVP

In practical problems the routine in the previous subsection is not sufficient: it can be applied only for rods where the spatial position of the cross section, the forces and the moments at the end point are known. Typically, there are prescribed geometrical and statical boundary conditions at both ends of the bar. The solution of the nonlinear Boundary Value Problem requires a higher computational effort. Due to the cracks, the system of differential equations describing the shape of the bar has variable coefficients, so we do not seek an analytical solution. In the literature, BVPs are commonly solved by incremental-iterative techniques, starting from the trivial configuration and following the equilibrium path in small steps [13]. Since we consider cases where the equilibrium path contains several, detached sets of points, we aimed a scanning algorithm in this work. A suitable algorithm is the Parallel Simplex Algorithm (PSA), which applies the
routine for the IVP to solve the BVP. A more advanced approach is the *Parallel Hybrid Algorithm* (PHA) which starts as a scanning algorithm, but after finding a point of the equilibrium path, it follows it [15, 16, 26]. The PSA and PHA reduce the BVP to a nonlinear algebraic system of equations by applying a forward integrator, i.e., this approach does not contain corrective iterative steps.

![Figure 1.18: A compressed cantilever beam](image)

We describe the Simplex Algorithm in a simple example (Fig.1.18): we have to determine the equilibrium path of a linear elastic, planar cantilever beam with span \( L \) and bending stiffness \( EI \). A concentrated load \( P \) is located at point \( B \). The shape of the bar is given by the

\[
EI \frac{d^2 \alpha(s)}{ds^2} + P \sin \alpha = 0
\]

(1.108)

differential equation, where \( \alpha(s) \) is the slope of the bar axis. The boundary conditions are

\[
\alpha(0) = 0, \\
M(l) = \frac{d\alpha(l)}{ds} = 0,
\]

(1.109) (1.110)

where \( M(s) \) denotes the bending moment. We assume that the load parameter \( P \) is known, the only unknown of the problem is the

\[
v_1 = \kappa(0) = \frac{M(0)}{EI} = \frac{d\alpha(0)}{ds}
\]

(1.111)

curvature at the clamped end. If we give a value for \( v \) and take 1.109 into account, then we have to integrate the IVP belonging to the differential equation 1.108. The result is the shape of the bar and the \( M(l) \) bending moment at the far end. This latest can be seen as a \( w(P,v) \) function, and the solutions of the original BVP according to eq. 1.110 can be obtained by the solution of the \( w = 0 \) equation. Generally beyond the \( P \) load parameter there are \( n \) variables \( (v_i, i=1,2,...,n) \) and \( n \) equations to be fulfilled:

\[
w_i(P,v_1,v_2,...,v_n) = 0 \quad i = 1, 2, ..., n.
\]

(1.112)

For example, for a structure of one bar the variables are the free rotations which are not fixed by the boundary conditions at the close end, the functions are the quantities prescribed by the boundary conditions at the far end. We seek the solution in the \( d = n + 1 \) dimensional space of the variables and the load parameter. This space is called the *Global Representation Space* (GRS) of the problem. A point of the GRS can be associated with an IVP. The first step of the simplex method is the discretization...
of the GRS by a simplicial grid (In the example above this means a discretization of the two dimensional GRS by triangles). The next step is the integration of the IVPs belonging to eq. 1.108 at all points of the simplexes. The results are the values of the \( w_i \) functions. Inside the simplexes we seek the solution of eq. 1.112 by linear interpolation (here we solve a linear system of equations). For the entire solution we have to carry out the calculation at each simplex of the GRS, which requires a high computational effort. With increasing dimension of the GRS the computational need grows exponentially. That is why the implementation of parallel architecture or GRID technology is advantageous. The simplex method is very suitable for this propose, since the calculation of the simplexes or the sets of simplexes are independent.

In this work we investigate spatial deformations of one bar, this problem generally can be solved as a 6D problem, although sometimes the problem can be simplified. A planar problem is typically 3 dimensional. The algorithm can be extended to the calculation of frames, then the problem can exceed six dimensions. Some applications to mechanical problems can be found in [42, 27]

### 1.4.4. Calculation in the Parallel Environment

![Diagram of the algorithm](image)

**Figure 1.19:** The additional components of the algorithm to be used in the parallel environment
The PHA algorithm is implemented for parallel environments (PVM, GRID) [62], since the solution of the BVP requires a high computational effort. Typically the computation of a spatial rod is a 6D problem, however in some special cases the dimension of the GRS can be reduced. In the 6D problem a sufficient division of the GRS results in a computation of about 100 million bar shapes.

We implemented the recursions described in subsections 1.4.1 and 1.4.2 into the core of the existing PHA code. The parallel environment does not yet offer those facilities which are usual for a desktop computer (i.e. graphical applications, easy file manipulation, etc). Our final goal is to enable the algorithm to be used by industrial users for engineering purposes. We added two components to the algorithm to reach our goal (Fig. 1.19). We developed a user interface, which can be accessed via the internet by a web browser to input all needed data and to start the calculation. The other component is a graphical application to display the equilibrium path and a corresponding bar shape for a chosen point of the path. In this section we describe both of these components.

**The User Interface**

The interface is based on the Conflit system and has been developed by D. Pasztuhov [51]. The development included a definition of a new language to describe the behaviour of the interface. Using this language several requirements of the user can be easily fulfilled. In our work we prescribed the following requirements of the industrial user:

- input about 100-1000 parameters describing the problem, the number of the input parameters basically depends on the geometrical complexity of the problem,
- save the given data into a file,
- load earlier given data from a file,
- reconfigure the GRS of the problem,
- start the calculation,
- download the file containing the computational result.

The groups of parameters can be given in the 7 tabs of the user interface (Fig. 1.20):

1. **tab: Geometry.** The total concrete cross section can be given by the coordinates of its contour points. A maximum of 200 points with their $x$ and $y$ coordinates can be given. This number of points enables users to define sophisticated cross sections (i.e. a hollow core slab, etc.). The reinforcement is described by the location and the cross sectional area of the reinforcing bar. There is a check box for each bar to take it into account as a prestressed element. The interface shows the given cross section graphically for checking.

2. **tab: Boundary Conditions.** The user can choose the boundary conditions at both ends of the bar. The boundary condition can be: free, clamped, planar hinge around axis $x$ and $y$, or spatial hinge. The free-free, free-planar hinge, free-spatial hinge and the spatial hinge–spatial hinge combinations are not allowed.

---

11Conflit=Configurable Portlet
Figure 1.20: The 1. tab of the user interface to define the geometry of the rod

3. tab: Environmental Variables and Materials. The environmental variables are several quantities needed for the calculation according to the EUROCODE 2 standard, for example the relative humidity or the time between manufacturing and loading. The material grade of the concrete and the reinforcement can be chosen from lists containing the approved grades of the standard, for example the concrete can be C16/20. The interface assigns the material properties automatically ($f_{cm}$, $E_{cm}$, etc.)

4. tab: Prestressing. This tab is active only if there was at least one prestressed bar on the 1. tab. The material characteristics of the prestressed tendon and some circumstances (heat curing, etc.) of the prestressing and casting procedure can be given here.

5. tab: Loads. There can be 10-10 concentrated loads in both directions $x$ and $y$, respectively. The location of these loads along the bar axis is arbitrary. Furthermore, there can be a distributed load in both directions as well.
6. **tab:** Configuration. The GRS for the PHA can be modified here. For the PHA algorithm the weight between the scanning and the path following mode can be given, too. The user can submit the job from this tab.

7. **tab:** File Management. The user can save the given data or load an earlier file. The output file of the calculation can be downloaded.

**Displaying the Results**

![Graphical tool to display computational results](image)

**Figure 1.21:** The graphical tool to display the computational results. On the left the equilibrium path in the defined subspace of the GRS is plotted, choosing one of its points the corresponding bar shape is plotted in the right box.

The result of the calculation is a set of points where eq. 1.112 is fulfilled, this set is called the **bifurcation diagram** of the problem. If \( d > 2 \) then we can display the two dimensional projection of the diagram. For each point of the bifurcation diagram there is a corresponding bar shape and by choosing one of the points, the shape can be displayed by solving the IVP. This calculation does not require the parallel environment since it can be easily calculated on a desktop PC. However, the graphical tool has been implemented to be run from the user interface. In this case, the user needs only a web browser for the whole procedure. The graphical tool contains a routine to read the output file of the calculation and display the bifurcation diagram. After the user has chosen a point, the corresponding IVP is solved by an identical routine to the one in the parallel code. The calculated shape and the possibly cracked zone are plotted and the maximal deflections are given numerically. The functions of the curvature and the rate of twist along the bar axis can be plotted, too (Fig. 1.21). The tool
was implemented by the non-commercial edition of QT 2.2. [78], which is a complete C++ application development framework including a class library and tools for cross-platform development. The bar shapes and the functions of curvature in the second part are made by this application.

1.4.5. ADDITIONAL SUBRoutines

The algorithm presented in the previous subsections can be extended by some subroutines to calculate reinforced concrete bars according to the EUROCODE 2 standard [21]. The special material properties, such as tension stiffening, shrinkage and creep can be taken into account without losing the robustness of the method. The non-linear stress-strain relation can be handled according to Section 1.3. Even by the second order stress-strain relation (eq. 1.53) a good approximation of the non-linear material law suggested by the standard can be given. The losses of the prestress are calculated according to the rules of the standard,

TENSION STIFFENING

Tension stiffening appears for loads between approximately $0.7 \cdot M_{cr}$ and $10.0 \cdot M_{cr}$, where $M_{cr}$ denotes the bending moment causing first cracking in the cross section. Below $0.7 \cdot M_{cr}$ the deflection can be calculated assuming uncracked, over $10.0 \cdot M_{cr}$ assuming a fully cracked cross section. According to experiments between the two values, the measured deflection of the structure is intermediate of the values calculated by assuming uncracked and fully cracked sections along the rod [46]. EUROCODE 2 suggests to use an intermediate value between the curvatures of the fully cracked and the uncracked section as follows. The method could be applied directly to the deflection or any other deformations parameter; in our case the curvature seems to be the most adequate quantity. The equation given in the Point 7.4.3 of the standard is

$$\kappa = \zeta \kappa_{II} + (1 - \zeta) \kappa_{I},$$

(1.113)

where $\kappa_{II}$ and $\kappa_{I}$ are the curvatures of the cracked and the uncracked cross sections, respectively. $\zeta$ is a distribution coefficient, it can be obtained as

$$\zeta = 1 - \beta \left(\frac{\sigma_{cr}}{\sigma_s}\right)^2,$$

(1.114)

where $\beta$ depends on the duration and the repetition of the load. $\beta = 1.0$ for a single short-term load and $\beta = 0.5$ for cycles of repeated load. $\sigma_s$ and $\sigma_{cr}$ are the stresses in the tension reinforcement of the cracked section, and the section under the load causing first cracking. If we take tension stiffening into account, then instead of eqs. 1.28, 1.59 and 1.85 the curvature is calculated by applying eq. 1.113. The algorithms in Sections 1.2 and 1.3 calculate $\kappa_{II}$ directly; $\kappa_{I}$ can be determined by calculating the curvature in the first step of the recursion if we have started the iteration by estimating a neutral axis which does not cut the total cross section. Furthermore, the stress in the tension reinforcement must be determined for both cases. Taking the tension stiffening into account does not influence the robustness of the method since the recursion is not changed, just the value of the curvature is modified as described above.
1.4. ROE: CALCULATION OF THE SHAPE

Shrinkage

The shrinkage of the concrete can be handled as an additional curvature. According to the standard, the shrinkage curvature can be obtained as

\[ \kappa_{cs} = \frac{\varepsilon_{cs} S}{I}, \]  \hspace{1cm} (1.115)

where \( \varepsilon_{cs} \) is the free shrinkage strain, \( S \) is the first moment of area of the reinforcement about a line parallel to the neutral axis and containing the centroid of the cross section, \( I \) is the second moment of area of the cross section around the same line, \( n_e \) is the effective modular ratio: \( n_e = E_e/E_{c,eff} \). The calculation of \( \varepsilon_{cs} \) is given in the Point 3.1.4. and the Annex B2 of EUROCODE 2. \( E_{c,eff} \) is the effective modulus of elasticity, see eq. 1.116. The standard suggest applying eq. 1.113 in calculating the shrinkage curvature. It means, that \( \kappa_{cs} \) is calculated for both the cracked and the uncracked section and added to \( \kappa_{II} \) and \( \kappa_I \) in eq. 1.113, respectively. The calculation of the shrinkage does not influence the robustness of the method, since it does not affect the recursion calculating the neutral axis. It increases the value of the bending curvature leading to a higher deflection of the rod.

Creep

The creep of concrete structures is a time-dependent phenomenon. In our work we assume the validity of linear creep theory and linear stress-strain relation [4, 50]. For concrete an accurate model has to take the memory and the aging of the material into account. Memory means the deformations after a given time \( t_1 \) depend on the permanent stresses before \( t_1 \). If the deformations depend on the time of first loading, then we talk about aging [38]. The standard allows to use the simplest model, the so-called Fritz method. In this approach the curvature is calculated by the \( E_{c,eff} \) effective modulus of elasticity:

\[ E_{c,eff} = \frac{E_c}{1 + \varphi}, \]  \hspace{1cm} (1.116)

where \( \varphi \) is the creep coefficient, it can be determined by the equations of Annex B2 of the standard. According to the Fritz method the curvature is calculated by \( E_c = E_{c,eff} \). The memory and the aging are omitted by this approach, and consequently the calculated final deformations are underestimated. This approach definitely does not affect the robustness of the algorithm, since it change the material constant \( q_1 \) of the material law. Next we describe two more accurate approaches.

The Trost Method

The Trost Method can be seen as a development of the Fritz method, it suggests the following modification:

\[ E_{c,eff} = \frac{E_{c(t_0)}}{1 + \chi \varphi(t, t_0)}, \]  \hspace{1cm} (1.117)

where \( \chi \) is the aging factor. \( \chi \) depends on the cross section and the variation of the stress in time. Generally we take \( \chi = 0.8 \). The standard suggests taking the duration of loading by modifying the creep coefficient:
\[ \varphi(t, t_0) = \varphi_0 \beta_c(t, t_0), \]

where \( \varphi_0 \) is the basic value of the creep coefficient, \( \beta_c(t, t_0) \) is the coefficient to describe the development of creep with time after loading. The Tresca Method takes the aging of concrete into account, but cannot handle the variation of the stress in time. It can be easily incorporated into the algorithm, instead of \( E_{cm} \) we take the modulus of elasticity by eq. 1.117 into account.

A more accurate application of this method can also be given. In this case there can be different stress at the beginning \( (t_0) \) and at the end \( (t) \) of the investigated period. In our work we only consider the stress variation from the additional deformation of the bar. For this calculation we have to run the algorithm twice: first we calculate the initial deformations without creep, and in the second run we substitute the load parameter with the time and run the calculation again with the following linear material law:

\[ \sigma_c = \varepsilon_0 E_c(t_0) + [\varepsilon - \varepsilon_0 (\varphi(t, t_0)) - \frac{E_c(t_0)}{1 + \chi \varphi(t, t_0)}], \]

where \( \varepsilon_0 \) is the strain at \( t = t_0 \). The steps of calculation:

1. We determine the spatial deformations of the rod by the initial load, the variables \( v_i \) and the functions \( w_i \) of the problem are defined by the boundary conditions.

2. We choose a level of the load parameter and store all variables for this value fulfilling the boundary conditions (i.e. we store one chosen point of the equilibrium path).

3. We define a new GRS, where the load parameter is substituted by the time \( t_i \), the variables and the functions are identical with the problem solved previously.

4. We calculate the final deformations at the time \( t \) with the creep coefficient \( \varphi(t, t_0) \) using the material law in eq. 1.119

The robustness of the algorithm is kept, since we determine the curvature in both calculations (step 1. and 4.) applying linear constitutive relations.

**A more accurate approach** A more realistic model of creep requires the integration of the deformations in time originating from stresses in the past. This fact means that the accurate approach cannot be included in the proposed algorithm, since the shape of the bar at a given time is influenced by the whole stress history in the past \(^{12}\). However, one can calculate the initial shape by the PHA algorithm, and follow the changes originating from creep by a path-following algorithm. The steps of this algorithm could be:

1. We determine the spatial deformations of the rod by the initial load; the variables \( v_i \) and the functions \( w_i \) of the problem are defined by the boundary conditions.

2. We slice the \( (t_0, t_1) \) period into \( T \) slices

\(^{12}\) i.e. the points of the equilibrium path cannot be calculated separately.
3. For each slice:
   3.1. We calculate the increment of the creep coefficient
   3.2. We determine the $\varepsilon_c$ additional deformations from creep
   3.3. By the new shape of the bar we determine the additional stresses and strains

**Losses of Prestress**

In our calculations we take only prestressed beams with pre-tensioned bonded tendons into account. The force applied to a wire or other tendon is denoted by $P_{\text{max}}$ and it can be calculated from the material properties according to the Point 5.10.2 of EUROCODE 2, or it can be given by the manufacturer. The $P_{\text{m,0}}$ initial prestressing force contains the immediate losses of prestress due to short term relaxation and heat curing. The mean value of the prestress force $P_{\text{m,t}}$ contains the time dependent losses in prestress at the time $t$ according to the point 5.10.6 of the standard. Here the effects of creep, shrinkage and final relaxation are taken into account.

1.4.6. **The Principal Result of the Algorithm to Calculate Spatial Deformations**

I developed a new algorithm to calculate the spatial deformations of rods without, or with limited tensile strength. The algorithm is robust, i.e. the deformations can be computed in a reliable manner, there is no danger of false solutions or divergent behaviour. In the frame of this work

**P.R.3.1.** I implemented the algorithm to determine the neutral axis according to the 1st and 2nd principal results. The algorithm calculates the neutral axis and the curvature of the cracked cross section rapidly: typically in 5-10 steps it estimates the curvature within a 1% error. I embedded this algorithm into the core of the Parallel Hybrid Algorithm, which is an iteration-free solver of boundary value problems. Due to the global convergence of the algorithm determining the curvature, and the features of the PHA the whole algorithm is robust.

**P.R.3.2.** I extended the algorithm by subroutines to take the tension stiffening, the shrinkage and creep of concrete and the losses of prestress into account. These subroutines are based on the EUROCODE 2 standard, in the case of creep the widely used Trost model is also included. I showed these subroutines do not influence the convergence properties of the algorithm.

**Industrial Application of the Results**

Since the parallel computational environment has not offered the accustomed facilities of a desktop PC, I took part in developing a user interface for the algorithm to enable industrial users to carry out calculations easily via a web browser. I defined the requirements of the user and the needed functionality. I developed and implemented a graphical tool to display the results of the calculation. The algorithm was implemented in C++ language and the developed software package is one of the first applications of parallel computation and GRID technology for industrial proposes in Hungary. (Common work with Dániel Paszthuhov)
Chapter 2

Applications
2.1 COLUMNS

Theoretical and practical applications of the algorithm presented in the first chapter will be introduced in two sections: the first one deals with reinforced concrete columns, the second one with beams. We consider a column as a member that sustains an external axial load. Lateral loads may be present, i.e. we do not deal with beam-columns separately. We define the beam as a member without external axial load; however, we allow prestressing of the beam. We investigate the postcritical behaviour of perfect elements and the affect of asymmetrical geometry or loading. We demonstrate some possible industrial applications for both columns and beams.

For each numerical example we give the variables and the functions of the PHA calculation, for details see subsection 1.4.3. As a computational result we plot the bifurcation diagrams in two or three dimensional subspaces of the GRS. In most of the examples we plot the deformed shapes for several points of the equilibrium path. The functions of bending curvatures and rate of twist along the bar axis are also given for these shapes.

2.1.1. POSTCRITICAL BEHAVIOUR OF CENTRALLY COMPRESSED COLUMNS

COLUMNS WITH DOUBLE SYMMETRICAL CROSS SECTION

As a first example we take a compressed cantilever with double symmetrical cross section. The shape of the bar is determined by the following system of differential equations [36]:

\[ u^{IV} + \left( \frac{P}{EI_y} u' \right)' = 0, \]
\[ v^{IV} + \left( \frac{P}{EI_x} v' \right)' = 0, \]
\[ -C\phi^{IV} + \left( \frac{P T_p}{A} \phi' \right)' = 0, \]

where \( u \) and \( v \) denotes the displacement in the directions \( x \) and \( y \), \( \phi \) is the angle of torsion. For a compressive force located at the free end of the cantilever, for constant cross section along the bar axis and for a material with tensile strength eqs. 2.1-2.3 are a system of uncoupled ordinary differential equations with constant coefficients. In this case the critical loads for both Euler buckling around the principal axes of the section and the critical torque can be obtained analytically. The shape of the bar in the postcritical range is a planar curve for the Euler modes. It is a good opportunity to verify the algorithm by comparing the numerically and analytically calculated critical loads.

For a material without tensile strength the eqs. 2.1-2.3 have variable coefficients.

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1To plot these diagrams we used GNUPLLOT 4.0.
2The bar shapes and the functions were plotted by the graphical tool described in subsection 1.4.4.
the stiffness \((EI_x, EI_y, C)\) depends on the deformed shape of the bar i.e. the geometrical and material nonlinearities combine resulting in a highly non-linear, non-integrable problem. [17] contains the investigation of a compressed column with rectangular cross section and pointed out, that occurrence of the cracks may lead to the instability of the structure. Here we investigate this problem in more details.

In the PHA calculation the cantilever can be described as a 3 dimensional problem, since the equilibrium conditions of the horizontal forces and the torsional moment can be fulfilled easily. In our example the horizontal loads and the torsional moment, consequently the horizontal reactions and the torsional reaction equal zero. The two variables of the problem are the bending moments at the clamped end, the load parameter is the compressive force \(P\):

\[
v_1 = M_x(0), \quad v_2 = M_y(0).
\]  

The functions prescribe the zero bending moments at the free end of the cantilever:

\[
w_1 = M_x(l), \quad w_2 = M_y(l).
\]  

We carried out the calculation on columns with rectangle cross section consisting of different sizes of reinforcement bars, the proportion of the reinforcement area is denoted by \(r\). For Column-1. on Fig. 2.1 we assumed tensile strength, the other five examples
are without tensile strength. The applied algorithms are LT00 for Column-1, L000 for Column-2 and L080 for the remaining cases. We took $E_c = 20.0$ MPa and $n = 10$. The width and height of the rectangle sections are determined in such a way that the stiffnesses in both principal directions are identical for each total cross section. The equilibrium path of the bar with tensile strength is given in Fig. 2.2 (a) in the space of $(P, M_x, M_y)$. We plotted the postcritical branches for the weakest modes in both directions. The critical loads around axes $x$ and $y$ analytically: 

$$P_{krit,y} = \frac{\pi^2 EI_y}{(2l)^2} = \frac{\pi^2 \cdot 2000 \cdot 3456}{(2 \cdot 200)^2} = 425.93 \text{ kN},$$  

$$P_{krit,x} = \frac{\pi^2 EI_x}{(2l)^2} = \frac{\pi^2 \cdot 2000 \cdot 13824}{(2 \cdot 200)^2} = 1703.73 \text{ kN}. \quad (2.6)$$

By the numerical calculation we got $P_{krit,y} = 422.45 \text{ kN}$ and $P_{krit,x} = 1689.16 \text{ kN}$. The error is $0.82\%$ for $P_{krit,y}$ and $1.09\%$ for $P_{krit,x}$. The values of the critical loads do not change assuming zero tensile strength for the concrete: the equilibrium path of Column-2 differs only in the postcritical range from the path of Column-1: due to the appearance of cracks the postcritical branch reaches a limit point and becomes unstable (Fig 2.2 (b)).

We compare the equilibrium paths for the Columns 3-6 in Fig. 2.3. For the the 3rd and the 4th columns we determined the cracked cross sections under pure bending denoted by $3m$ and $4m$. We determined the equilibrium paths belonging to the columns with these cross sections assuming tensile strength. The result of these computations are also plotted in Fig. 2.3 (a). We draw the following conclusions for columns with double symmetrical cross section without tensile strength:

1. As long as the compressive load $P$ is inside the kernel of each cross section, then there is no importance in the tensile strength, since all points are under compression. The Euler critical load of the rods with and without tensile strength coincide and the local neighborhood of the bifurcation is identical (Fig. 2.2 b.).

2. For the bar without tensile strength there is a limit point on the postcritical branch due to the appearance of cracks, beyond this point the branch falls. We denote the value of the axial load at the limit point to $P_{l,1}$ and call it the load at limit point-1. The branch reaches the $P = 0$ level at $M = 0$ (Fig. 2.2 b.). The postcritical shape remains planar after the cracks occurred since the working cross section is still symmetrical, the principal axes of the uncracked and the cracked cross sections are parallel or identical.

3. Having an arbitrary small amount of reinforcement there is another limit point on the postcritical branch, which increases beyond this point (Fig. 2.3). The load at this limit point is the load at limit point-2, $P_{l,2}$. The branch after this second limit point asymptotically reaches the postcritical branch belonging to a column with a cross section completely cracked under pure bending.  

4. Between the two limit points the postcritical branch is unstable.

---

3This statement can be explained in that for large deflections the majority of the cross sections are dominantly under bending, i.e. the neutral axis gets very close to the neutral axis of the cross section under pure bending.
Figure 2.2: The effect of the limited tensile strength on the postcritical response of the column. In the case of Column-1 the bifurcations belonging to the lowest critical loads in both principal directions are plotted. For Column-2 only the first bifurcation and its postcritical path is plotted, without reinforcement the path reaches the $P = 0$ level at $M_y = 0$. 
2.1. COLUMNS

Figure 2.3: The equilibrium path of Columns-3,-4,-5 and -6.
Due to the reinforcement, a second limit point occurs on the postcritical branch. It asymptotically reaches the branch belonging to a column with a cross section completely cracked under pure bending (Columns-3m and -4m)

Cross Sections with a Maximum of one Axis of Symmetry

Next, we turn to cross sections with fewer axes of symmetry. The postcritical shape of the cantilever with tensile strength and constant cross section along the bar axis under
central compression is always planar: the bar buckles around the weaker principal axis of the cross section, the geometry of the cross section does not influence the post-buckling behaviour. In the next example we investigate rods with a maximum of one axis of symmetry without tensile strength (Columns-7,8 and -9 of Fig. 2.4). Although in the following examples the torsional center does not coincide with the centroid of the cross section, the rate of twist is small for the applied loads, the method described in subsection 1.1.3 gives a good approximation. We applied the L000 algorithm to this example. The variables and the functions of the PHA calculation are identical to the ones in the previous example, see eqs. 2.4 and 2.5.

![Diagram](image)

Figure 2.4: A compressed cantilever with triangular cross section without tensile strength

The equilibrium paths and typical bar shapes for each case are given in Fig. 2.5. We can draw the following conclusions for columns with a cross section with a maximum of one axis of symmetry without tensile strength:

1. For cross sections with one axis of symmetry the postcritical shape after the bifurcation of the primary critical load is planar if the cracked section remains symmetrical. This condition is met then and only then if the axis of symmetry coincides with the stronger principal axis of the cross section (e.g. Column-8)

2. The shape of a rod with asymmetrical cross section becomes spatial after the cracks have occurred because the principal axes of the uncracked and cracked sections are typically different.

For rods without tensile strength the postcritical bar shape is influenced by the geometry of the cross section, but just after the cracks occurred, i.e. in the local neighborhood of the bifurcation, the bar shapes are planar and identical with the solution of the bar with tensile strength. Although we demonstrated these effects on a cantilever, these statements are independent of the boundary conditions and we gained the same experiences assuming other end conditions for the column. Next we turn to cases where the spatial shapes occur arbitrarily close to the bifurcation point.
Figure 2.5: The equilibrium paths of Columns 7–9, and the bar shape, the components of the curvature and the rate of twist for the marked points of each figure.
<table>
<thead>
<tr>
<th></th>
<th>End A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>clamped</td>
</tr>
<tr>
<td>End</td>
<td>free end</td>
</tr>
<tr>
<td>B</td>
<td>clamped</td>
</tr>
<tr>
<td></td>
<td>planar hinge</td>
</tr>
<tr>
<td></td>
<td>spatial hinge</td>
</tr>
</tbody>
</table>

Table 2.1: The considered boundary conditions in the algorithm. √: stable structure, the algorithm is implemented -: unstable structure

2.1.2. **Can an Asymmetrical Column be Optimal?**

In this subsection we investigate two examples, where due to the boundary conditions the shape of the bar is spatial in an arbitrary close region of the bifurcation, i.e. the algorithm described in the first chapter is required to determine the primary critical load. Here we focus on the value of the primary critical load itself, not the postcritical branch.

The post critical behaviour of elastic planar rods with several boundary conditions is widely studied in the literature, we only refer to these results [10, 4]. In our work we implemented those boundary conditions which appear regularly in practice, the investigated pairs are summarized in Table 2.1. In the next study we will take only rods between planar hinges, i.e. the rod is pinned around one axis and clamped around the perpendicular axis. We seek a slightly asymmetrical configuration of the column, which can be considered to be optimal. The theory of slightly asymmetrical optima has been laid in [70, 71]. For the theoretical results the following calculation is a good practical example.

We consider the simplest kind of optimization problem: a scalar "optimization" in \( \eta \). Potential \( U(\eta, p) \) will be associated with the structure and we seek local minima of \( U \) as optimum structural configurations. The variable \( \eta \) will refer to the deviation from the symmetric configuration, i.e. \( \eta = 0 \) will be always associated with the symmetric problem. The parameter \( p \) will describe a family of structures, each of which possesses the same symmetry. We seek the optimal value of \( \eta \) for fixed values of \( p \), but we do not optimize for \( p \) itself, i.e. we do not solve a shape-optimization problem. As our final goal, we seek bifurcation diagrams in the \([\eta, p]\) plane, describing how optima evolve as the parameter is varied. In the following examples we investigate the spatial buckling of a centrally compressed column. The compressive load is denoted by \( P \), the lowest critical load of the column is denoted by \( P_{\text{crit}}(\eta, p) \), the optimization potential is

\[
U(\eta, p) = \frac{P}{P_{\text{crit}}(\eta, p)}.
\]  

(2.8)

This is an overall condition, i.e. it represents a global optimization criterion. We expect that at some critical value \( p_c \) of the parameter \( p \) the symmetric, trivial configuration switches from optimal to pessimal. However, we can neither predict whether \( p_c \) will be in the physical range, nor can we predict whether asymmetric optima will emerge at \( p = p_c \). To find out these essential details, we performed numerical compu-
tations in the following two examples.

In the first example we will rotate the hinge at one end of the column, in the second one we will have an asymmetrical cross section between parallel hinges. We use the LT00 algorithm in the first and the LTS0 code in the second example, since the value of the primary critical load of a compressed column does not depend on the tensile strength (See subsection 2.1.1.). The problem is 6 dimensional for both cases, beyond the load parameter $P$ the 5 variables are the $v_1 = F_x(0)$ and $v_2 = F_y(0)$ horizontal forces, the $v_3 = M_x(0)$ bending moment, the $v_4 = M_z(0)$ torsion and the $v_5 = \alpha_y(0)$ rotation of the cross section at point A. At point $B$ the 5 functions are

$$w_1 = x(l), \quad w_2 = y(l), \quad w_3 = M_y(l), \quad w_4 = \alpha_x(l), \quad w_5 = \alpha_z(l), \quad (2.9)$$

where $x(l)$ and $y(l)$ denote the offset of the centroid, $M_y(l)$ is the bending moment around axis $y$, $\alpha_y(l)$ and $\alpha_z(l)$ denote the rotation of the cross section around axes $y$ and $z$, respectively.

### A Column Between Planar Hinges

In our first example we investigate the effect of the position of one hinge on the primary critical load of a prismatic bar with a rectangle cross section (Fig. 2.6). If the axes of the planar hinges are parallel to each other and one of the principal axes of the cross section, then we have the elastic beam investigated by Euler. The axis of the hinge at point $B$ is parallel to the axis $y$, the angle between the axis of the other hinge and the axis $y$ is denoted by $\eta$. First we take three cases: $\eta = 0^\circ$, $\eta = 45^\circ$ and $\eta = 90^\circ$ and calculate them with Cross Section I. The $[x'y'z']$ coordinate system is attached to the planar hinge at point A so that the axis of the hinge is $y'$.

The equilibrium paths of the three cases are given in Fig. 2.7, the curvatures of one point chosen from the 10.a, 11.a and 12.a branches are plotted in Fig. 2.8. For Column-10 the rod buckles by the rotation of the hinges around axis $y'$. The post buckling shape of the bar assuming small deflections can be given by

$$x = Z \sin \frac{\pi \cdot z}{l}, \quad (2.10)$$

where $Z$ is the amplitude of the deflection, it is indeterminate for small displacement assumption. In the case of Column-11 the post buckling shape of the bar is a spatial curve, i.e. after the bifurcation bending moment and rotation of the hinge at point A occur simultaneously. For Column-12 the post buckling shape is planar again.

Although we demonstrated only three cases, a detailed investigation of the previous example shows the parallel placement of the hinges gives the optimal solution. If we take a square cross section then just the other statement is true, the perpendicular placement of the hinges proved to be optimal. A question arises: is there any rectangle cross section, where the intermediate placement ($\eta \neq 0^\circ$ and $\eta \neq 90^\circ$) of the hinge at point A is optimal?

To answer the question we characterize the cross section by $p = a/b$ (Cross Section II. in Fig. 2.6), in the case of Cross Section I. $p = 0.5$. We calculate the $P_{\text{crit}}(\eta, p)$ critical load for several values of $p$ and $\eta$, the computational results are shown in Fig. 2.9. The optimal direction of the hinge is denoted by $\eta_{\text{opt}}$. For $0.71 < p < 1$ the perpendicular placement of the hinges is optimal, $\eta_{\text{opt}} = 90^\circ$, for $p < 0.71$ $\eta_{\text{opt}} = 0^\circ$. At
Figure 2.6: A pinned-pinned column: the hinge at point A can be rotated by the angle $\eta$.

Figure 2.7: The equilibrium paths of the Columns 10-11 and -12 in Fig. 2.6. While the branches 10.a., 11.a. and 12.a. denote the postcritical branches of the primary critical load around the weaker principal axis, then 10.b., 11.b. and 12.b. belong to the critical load around the stronger principal axis. 12.e. belongs to the secondary critical load around the weaker axis.
\[ \eta = 0 \]

\[ \eta \approx 18 \]

\[ p = 0.71 \] the critical load is identical for any \( \eta \). The optimum-bifurcation diagram (Fig. 2.10) of the problem summarises this result. In this example we found a critical value \( p_c \) where the symmetrical arrangement \( (\eta = 0^\circ) \) switches from optimal to pessimal, but asymmetrical optima does not emerge.

**A Column with Asymmetrical Cross Section**

In the previous example we investigated the dependence of the primary critical load on the rotation of one hinge of a pinned-pinned bar. In the next example we influence the symmetry of the cross section by the offset of two reinforcing bars (Fig. 2.11).

The rod is between parallel planar hinges, the axes of the hinges are parallel to the axis \( y \). The cross section is rectangular and contains 8 bars of \( \phi 20 \) reinforcement with the total area \( A_s = 25.12 \text{ cm}^2 \). We take the modular ratio \( n = 10 \) and the width of the cross section \( b = 10 \text{ cm} \). The symmetry-breaking variable \( \eta \) will be interpreted as a simultaneous offset of two reinforcement bars parallel to the \( x \) axis. As before, the parameter \( p \) is the ratio of the sides of the rectangle. Due to the two asymmetrical placed reinforcements, the shape of the column belonging to the primary buckling mode is a spatial one (except the \( \eta = 0 \) case, when it is planar).

The computational results are plotted in Fig. 2.12: for a square cross section the highest critical load belongs to \( \eta \approx 18.5 \text{ cm} \), i.e. the two reinforcing bar are out of the concrete cross section. As \( p \) is decreased gradually, the value of \( \eta \) belonging to the highest critical load decreases and there exists a critical value of the parameter \( p = p_c \) below which the highest value of the primary critical load belongs to the symmetrical \( (\eta = 0) \) configuration. The value of \( p_c \) can be calculated analytically since at \( p = p_c \) the two Euler critical loads in the directions \( x \) and \( y \) coincide:
Figure 2.9: The primary critical load for the rod with rotated hinge at point A. Colours correspond to constant value of the parameter $p$.

Figure 2.10: The optimum bifurcation diagram for the pinned-pinned column. Solid lines represent optima, dashed lines pessima, the dotted line represents that at $p = 0.71$ the critical load is identical for any $\eta$. 

$P_{\text{crit}}(\eta, p) \ [\text{kN}]$
2.1. COLUMNS

**Figure 2.11:** A compressed reinforced concrete column with an asymmetrical cross section.

**Figure 2.12:** The primary critical load for cross sections with increasing asymmetry. Colours correspond to constant values of the parameter $p$. Observe that below $p = p_c = 0.5869$ the symmetrical configuration becomes optimal. The border of the concrete cross section is marked for each curve.
\[ \frac{EI_x(p)}{(0.5l)^2} = \frac{EI_y(p)}{l^2}, \]

where \( I_x(p) \) and \( I_y(p) \) denote the second moments of area of the cross section as a function of \( p \). The solution of eq. 2.11: \( p_c = 0.5869 \). Based on the computational results summarized in Fig. 2.12 we can now plot the optimum-bifurcation diagram; this is illustrated in Fig. 2.13. We learned that for cross sections which are relatively close to the square \((p < p_c)\) the optimal arrangement of the reinforcement is asymmetric. Close to \( p = p_c \) we can observe slightly asymmetric structural optima.

Intuitively, this phenomenon can be summarized as follows. In the symmetric configuration of the square cross section the buckling loads \( P_{\text{crit},x} \) and \( P_{\text{crit},y} \) differ only due to the different boundary conditions, we have \( P_{\text{crit},x} > P_{\text{crit},y} \). If we decrease the parameter \( p \), at some value \((p = p_c = 0.5869)\) the two buckling loads will coincide: \( P_{\text{crit},x} = P_{\text{crit},y} \). Below this critical value \( P_{\text{crit},x} < P_{\text{crit},y} \).

Since the offset \( \eta \) causes increase of the bending stiffness \( I_y \) (while \( I_x \) remains constant) the offset will increase the larger (irrelevant) critical load if \( p < p_c \). In contrast, for \( p > p_c \) the offset will increase the smaller (relevant) critical load, so asymmetry is desirable in this range. (Of course, the principal axes are also rotated by the offset and the computations considered that effect, here we just wanted to give an approximate intuitive explanation.)

Needless to say, this phenomenon is not restricted to reinforced concrete. We picked this example because the arrangement of the reinforcement bars can be easily interpreted as a continuous, symmetry breaking variable. However, similar configurations can be constructed using any other structural material or different cross section.

## 2.1.3. Eccentric Compression

After we have demonstrated the efficiency of the algorithm in calculating the critical load we turn to a more practical problem: we propose to calculate bars under eccentric
compression. Due to the eccentricity some or all the bifurcation on the equilibrium path disappears. First we investigate a rod under uniaxial bending (i.e. the load is located on one of the principal axes), next we take a biaxial case. Finally we investigate the effect of prestressing the reinforcing bars.

**Uniaxial Bending**

We examine a rod under uniaxial bending with Columns-4 and -5 in Fig. 2.1 with the reinforcement ratios \( r = 0.5\% \) and \( r = 1.0\% \). The load has an eccentricity along the axis \( x \) denoted by \( e_x \). We apply the L0S0 algorithm, the variables of the PHA calculation coincides with eq. 2.4. The functions prescribe the bending moment arising from the eccentricity of the axial load \( P \):

\[
w_1 = M_x(l), \quad w_2 = M_y(l) - e_x P.
\]  

(2.12)

![Equilibrium paths for several eccentricities of eccentric compression of Column-4 in Fig. 2.1. Over the optimal eccentricity the two limit points coincide, the instability disappears.](image)

The equilibrium paths of the Column-4 for several eccentricities (\( e_x = 1.0, 2.0, 3.0, 5.0 \) cm) are given in Fig. 2.14 in the \( (P, M_y) \) plane. Over a threshold in the \( e_x \) eccentricity the two critical points of the branch coincide (there is a cut-off point), the instability disappears. We call this threshold critical eccentricity, in the numerical example \( e_{cr} \approx 4 \) cm. By comparing the results of Columns-4 and -5 under the same load, we find that for higher reinforcement ratio the optimal eccentricity is smaller (Fig. 2.15). For example at the eccentricity \( e_x = 3.0 \) cm of the load gives instability for Column-4, but it is over the optimal eccentricity of Column-5.


\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.15.png}
\caption{The effect of the applied reinforcement ratio in the value of the optimal eccentricity $e_{x, opt}$. For higher reinforcement ratio the value of the optimal eccentricity is smaller.}
\end{figure}

**Biaxial Bending**

We compare the effect of uniaxial and biaxial bending on Column-3 of Fig. 2.1. For the uniaxial case $e_x = 1.0$ and $e_y = 0.0$ cm, for the biaxial one $e_x = e_y = 1.0$ cm. The calculation of the uniaxial case is identical with the previous example, the biaxial case differs only in its $w_1$ function:

\[ w_1 = M_x(l) - e_y P, \quad w_2 = M_y(l) - e_x P. \]  \hspace{1cm} (2.13)

Along the branches marked '1' and '2' of Fig. 2.16 the shape of the bar is planar, because the uncracked and cracked sections have the same axis of symmetry, the principal coordinate system does not rotate along the bar. For the branch denoted by '3' we have spatial shapes which appear arbitrary close to the bifurcation point. Here the spatial configuration originates from the eccentricity.

For biaxial bending the shape of the bar is typically spatial along all of the branches (branch '4' in Fig. 2.16). We plotted the bars’ shapes of the 4 marked points in Fig. 2.17.\footnote{This example shows the advantage of applying the PHA algorithm: we found all the disconnected branches of the equilibrium path.}
(a) Equilibrium path for uniaxial bending: The bifurcation at the primary critical load sets apart, however, the shape of the bar in Points 1 and 2 is planar. The bifurcation at the smallest critical load around the stronger principal axis is preserved, but the shape of the bar is spatial along its postcritical branch (Point 3.)

(b) Equilibrium path for biaxial bending. Due to biaxial bending there is no bifurcation any more, the shape of the bar is typically spatial, as in Point 4.

Figure 2.16: Eccentric compression of Column-3 in Fig. 2.1. The shape of the bar in the numbered points (Point 1 - Point 4) is plotted in Fig. 2.17
**Figure 2.17:** The shape of the bar in the four marked points of Fig. 2.16

**Prestressing**

As we demonstrated previously, the limited tensile strength significantly influences the deformation of an eccentrically compressed column. The reduction in the stiffness of the column can be limited by prestressing the reinforcing bars. As we showed, the two limit points of the equilibrium path unify over the $e_{opt}$ critical eccentricity. Can the value of the critical eccentricity be modified by prestressing? To answer the question we carried out the calculation on Column-4 in Fig. 2.1 by the LOP0 algorithm. The variables are given by eq. 2.4, the functions by eq. 2.12. By including the initial losses in the prestress we calculated by a $\sigma_{m,0} = 1290$ MPa prestress in each reinforcing tendon.

According to the numerical results by the prestressing the load at limit point-1 $P_{1,1}$ is higher, this load can be explained as the critical load of the column under eccentric compression. The critical eccentricity $e_{cr}$ also increases by prestressing. In this way, the application of prestressed columns is two-folded: on one hand it gives a higher load bearing capacity before reaching $P_{1,1}$, but the dynamical response in the postcritical range is more likely.

**2.1.4. Second Order Moment of a Clamped Column in a Frame**

Recently, reinforced concrete frames have often been built without additional bracing, in this case the columns are clamped at the bottom and pinned at the top. The edge columns are under compression and biaxial bending. Calculations often neglect the effect of the limited tensile strength of the concrete, in this way the second order moments are underestimated. Although some software tools are capable of calculating...

---

Spatial deformations of rods without tensile strength
Figure 2.18: Effect of the prestressing force for several eccentricities. Colour red denotes the equilibrium path without, green the equilibrium path with prestressing of all reinforcing bars with $\sigma_{m,0} = 1290$ MPa. Both the load at limit point-1 $P_{l,1}$ and the optimal eccentricity $e_{cr}$ is higher for a prestressed column.
second order moment by taking the effective stiffness of the columns into account, to the best of our knowledge they cannot handle cross sections with arbitrary shape. On the other hand the method of EUROCODE 2 significantly overestimates the second order moment. In this subsection we apply the LTSC algorithm to calculate the second order moments of an RC column being an element of a simple frame structure.

![Graph](image)

**Figure 2.19:** Determining the final horizontal displacement of a frame with two columns

Assuming the deformations of the horizontal beam to be arbitrarily small, the planar case can be solved as a 2 dimensional problem; the columns of the frame can be computed separately. (PHA is capable of solving the general problem, i.e. the deformations of the beam are also taken into account, this can be solved as a 6D problem [27].) Let $j$ denote the serial number of a given column. The two variables are the wind force $\lambda$ and the $e_{x,j}$ horizontal displacement of the top point of the $j^{th}$ column. The $(N_j, M_j)$ reactions at the clamped end can be computed from the value of $\lambda$, $N_j$ and $e_{x,j}$. Solving the IVP from the reactions, the horizontal displacement $e_{x,\text{calc},j}$ of the top point can be calculated. The investigated function is

$$w_1 = e_{x,j} - e_{x,\text{calc},j}. \quad (2.14)$$

This solution seems to be more sophisticated, than taking the $(\lambda, M_j)$ space as GRS, but it is preferred for design purposes. In this solution no additional calculation is needed: the bifurcation diagram of the problem contains the movement of the top point of the column. The horizontal displacement of all columns must be equal, the bifurcation diagrams can be added. Having the design value for $\lambda$, one can get the final horizontal displacement $e_{x,\text{final}}$ of the structure from the diagram containing the results of all columns (Fig. 2.19). By $e_{x,j}$ the $M_{2,j}$ second order moment can be obtained as
\[ M_{2j} = N_{j} e_{x,j}. \] (2.15)

In this way, frames with an arbitrary number of columns can be solved, even if the cross sections or the reinforcement of the columns are different\(^5\) (i.e., frames having both prestressed and non-prestressed columns). Having the second order moments the cross sections at the clamped ends are checked in the Ultimate Limits States (ULS) for compression and bending.

![Cross Section of columns 1 and 2:](image)

**Figure 2.20:** An example: a reinforced concrete frame braced with clamped columns.

Here we compute an example, which can be compared to the solution of another type of software (ABACUS-STUR) and to the method applied in the EUROCODE 2 standard. The dimensions of the frame, the cross section of the columns and the material properties are given in Fig. 2.20. In the example we consider a symmetrical arrangement of the vertical loads, i.e., \( N_1 = N_2 = 302.8 \) kN, in this case we have to calculate only one column since the \( \lambda \) vertical load is distributed between the two columns equally. The design value of the wind load is \( \lambda = 34.0 \) kN, so each column has to carry \( \lambda_1 = \lambda_2 = 17.0 \) kN. For this latter horizontal load we find \( e_x = 33.01 \) cm on the bifurcation diagram (Fig 2.21). We calculated the same problem with the ABACUS-STUR software, which is a widely used tool for designing prefabricated reinforced concrete frames. The results of the two types of software are close to each other.

---

\(^5\)In the case of several columns with different stiffness the assumption of rigid horizontal beam can result in a false model of the structure due to the second order effects, this problem can be handled by applying a more accurate approach by taking the deformations of the beam into account.
(Table 2.2), but both of them give a smaller magnitude in the second order moment than the result of the method in the standard 6.

<table>
<thead>
<tr>
<th>$\varepsilon_{x, \text{final}}$ [cm]</th>
<th>Proposed Method</th>
<th>ABACUS-STUR software</th>
<th>Design rules of EUROCODE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>33.01</td>
<td>31.92</td>
<td>75.60</td>
</tr>
<tr>
<td>$M_{2,1} = M_{2,2}$ [kNm]</td>
<td>99.95</td>
<td>96.65</td>
<td>228.91</td>
</tr>
</tbody>
</table>

**Table 2.2: Computational results with the proposed method, the ABACUS-STUR software and the EUROCODE 2 standard**

Finally we investigate the effect of prestressing: we assume that each bar is prestressed to the maximal value approved by the standard, however, this can be disadvantageous in the ultimate load of the cross section. The design of the cross section is not included in our algorithm, that is why we do not seek the optimal value of the prestressing force. The results (Fig. 2.21) show that prestressing is advantageous because the horizontal displacement and consequently the second order moments are decreased significantly. Our method is capable of handling those cases (i.e. asymmetrical or ring-shaped cross sections, biaxial bending with compression) where second order moments cannot be calculated by the equations given in the EUROCODE 2 standard.

![Graph showing horizontal displacement](image)

**Figure 2.21: Horizontal displacement of the top point of the columns of Fig.2.20**

2.1.5. **The Principal Result of RC Columns**

I investigated numerically the behaviour of compressed RC columns by the developed algorithm assuming linear stress-strain relation ($k = 1$, $l = 1$) in the compressed zone. Based on my computations I draw the following conclusions.

---

6We followed the design rules of Point 5.8.8 of EUROCODE-2.
P.R.4.1. I demonstrated that for centrally compressed columns the postcritical branch has typically two limit points. The first one after the bifurcation appears due to the cracking, after this limit point the branch is unstable. For reinforced columns the second limit point makes the branch stable again. The branch asymptotically reaches the postcritical branch of the compressed column with cracked cross sections under pure bending. I showed that in the case of eccentric compression with eccentricity $e$ the two limit points unify at a critical value $e = e_{cr}$ in a catastrophe point (i.e. a cut-off point), thus for $e > e_{cr}$ the instability disappears. By prestressing the reinforcing bars of the column the load belonging to the first critical point (i.e. the critical load of the column under eccentric compression) increases and the value of the critical eccentricity $e_{cr}$ is higher, too.

P.R.4.2. I introduced a slightly asymmetrical structural example which can be considered to be optimal, i.e. the risk of buckling is minimal. The example is a symmetrical compressed column between planar hinges, the symmetry breaking variable is the offset of two bars of reinforcement. I showed the optimal value of the offset of the reinforcement as a function of the geometrical ratio of the concrete cross section.

2.2 BEAMS

For the investigation of beams we follow the same path as we did for columns: first we investigate the postcritical behaviour. If the applied loads reach a certain limit, out of plane deflection and twisting of the cross section occur. For geometrically perfect elastic beams, the limit of the applied loads at which lateral instability occurs is called lateral torsional buckling load. We will compare our numerical prediction for this critical load to analytical solutions in the literature and investigate the effect of the limited tensile strength and the prestress on the postcritical behaviour. In the second subsection of this section we report about experiments on asymmetrical prestressed concrete beams.

2.2.1. LATERAL TORSIONAL BUCKLING OF BEAMS

During the lateral torsional buckling the shape of the originally planar beam becomes spatial [57]. We consider only cross sections with negligible warping. Assuming small deflections, the shape of the beam is governed by the following system of differential equations [10]:

\[ EI_x \frac{d^2 v}{dz^2} + M_x = 0, \]  
\[ EI_y \frac{d^2 u}{dz^2} + \gamma M_x = 0, \]  
\[ C \frac{d^2 \gamma}{dz^2} - \frac{du}{dz} M_x = 0 \]  

where $C$ is the torsional rigidity, $u$ and $v$ denote the horizontal and vertical deflection, $\gamma$ is the rotation of the cross section. We deal only with an $M_x = M_0$ uniform bending moment as an external load, resulting eq. 2.16 is independent of the other two equations. In fact, this equation describes the in-plane deflection of the beam. The buckling behaviour of the beam is described by the last two equations. By differentiating eq.
2.18 once with respect to \( z \) and substituting the result into eq. 2.17 the two equations can be combined to give

\[
\frac{EIyC}{M_0} \frac{d^2 \gamma}{dz^2} + \gamma M_0 = 0. \tag{2.19}
\]

This differential equation has the same general solution as the Euler problem, the critical moment is

\[
M_{\text{crit}} = \frac{\pi}{l} \sqrt{EIyC}. \tag{2.20}
\]

For other loading (e.g. distributed or concentrated lateral loads etc.), boundary conditions or sections with significant warping the critical bending moment is given by numerical results [10] which are based on the beam with constant bending moment. For cases where in-plane deflection is not negligible compared to the out-of-plane deflection [32] gives an approximate solution:

\[
M_{\text{crit},a} = \frac{\pi}{l} \sqrt{\frac{EIyC}{1 - \frac{I_y}{I_x}}}. \tag{2.21}
\]

We investigate a simply supported beam with several cross sections (Fig. 2.22). We assume tensile strength for Beams-1, -2 and -3; Beam-2 is reinforced while Beam-3 contains prestressing tendons. Beams-4 and -5 are without tensile strength (for Beam-1 there exists no equivalent without tensile strength). We carry out the calculation with linear stress-strain relation in the compressed zone. The applied algorithms for the computation are summarized in Table 2.3.

<table>
<thead>
<tr>
<th>Beam-1</th>
<th>Beam-2</th>
<th>Beam-3</th>
<th>Beam-4</th>
<th>Beam-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>LT00</td>
<td>LTS0</td>
<td>LTP0</td>
<td>L0S0</td>
<td>L0P0</td>
</tr>
</tbody>
</table>

Table 2.3: The applied algorithms to investigate lateral torsional buckling of beams

The GRS of the problem is 3 dimensional. The load parameter is the bending moment \( M_0 \), the two variables are the rotation of the cross section at the end \( A \) around axes \( x \) and \( y \):

\[
v_1 = \alpha_x, \quad v_2 = \alpha_y. \tag{2.22}
\]

The two functions prescribe the zero displacement at point \( B \):

\[
w_1 = x(l), \quad w_2 = y(l). \tag{2.23}
\]

For Beams-1 and -2 the analytical results on the critical bending moment according to eq. 2.20 are

\[
M_{\text{crit},1} = \frac{\pi}{l} \sqrt{EIyC} = \frac{\pi}{1000} \sqrt{2320 \cdot 5760 \cdot 18793088} = 49760 \text{kNcm}, \tag{2.24}
\]

\[
M_{\text{crit},2} = \frac{\pi}{l} \sqrt{EIyC} = \frac{\pi}{1000} \sqrt{2320 \cdot 7291 \cdot 26531434} = 66519 \text{kNcm}. \tag{2.25}
\]
\[ M_{0} \]

\[ A \quad \text{SIDE VIEW} \quad \alpha_{y}(0) \quad \text{TOP VIEW} \quad \alpha_{z}(0) \]

\[ y(l) = 0 \]

\[ x(l) = 0 \]

---

**Figure 2.22**: A simply supported beam under uniform moment

The approximate solutions take the in-plane deflection into account by eq. 2.21:

\[ M_{\text{crit},1} = \frac{\pi}{l} \sqrt{\frac{EI_{y}C}{1 - \frac{I_{z}}{I_{x}}}} = \frac{\pi}{1000} \sqrt{\frac{2320 \cdot 5760 \cdot 18793088}{1 - \frac{5760}{64000}}} = 52163 \text{ kNcm}, \quad (2.26) \]

\[ M_{\text{crit},2} = \frac{\pi}{l} \sqrt{\frac{EI_{y}C}{1 - \frac{I_{z}}{I_{x}}}} = \frac{\pi}{1000} \sqrt{\frac{2320 \cdot 7291 \cdot 26531434}{1 - \frac{7291}{95012}}} = 69238 \text{ kNcm}. \quad (2.27) \]

By the numerical calculation \( M_{\text{crit},1} = 54146 \text{ kNcm} \) and \( M_{\text{crit},2} = 71930 \text{ kNcm} \). We used the material properties of concrete, the in-plane deflections are relatively high.
at the critical moment, so we compare our results to the solutions gained by eq. 2.21. The error is 3.80% for $M_{\text{crit},1}$ and 3.88% for $M_{\text{crit},2}$.

Figure 2.23: The effect of the zero tensile strength on the critical bending moment and the postcritical behaviour: the equilibrium paths of Beams-2 and -4

The Role of the Constitutive Law  The literature gives approximate solutions ([18]) to determine the critical bending moment of the simply supported beam without tensile strength, but without a proper numerical method the postcritical response could not been investigated. The comparison of the results of Beams-2 and -4 shows that the critical bending moment significantly decreases due to the limited tensile strength (Fig. 2.23), however, with different cross sectional geometry it may lead to an increase of
the critical bending moment. This latter can be explained by eq. 2.21: if the second moments of area of the cracked cross section are close to each other (i.e. \( I_x \approx I_y \)), then the value of \( M_{\text{cr}} \) definitely increases. The postcritical behaviour is affected by the limitation of the tensile strength: the bifurcation is supercritical for the beam with tensile strength, but it is subcritical for the beam without tensile strength.

The comparison of the previous statement and the results about columns shows, we cannot conclude a general rule about the effect of the cracking process on the postcritical behaviour: in the case of compressed column the bifurcation was supercritical for both columns with, or without tensile strength, the effect of the cracking process appeared only along the postcritical branch. In the case of lateral torsional buckling the cracking affected the nature of the bifurcation, too. The investigation of Beam-4 also reveals that the postcritical branch of the equilibrium path of the beam without tensile strength contains a limit point, where the branch becomes stable again.

**The Role of Prestressing** Although a beam cannot buckle under pure prestress, the prestressing can affect the critical bending moment of a loaded beam. As we already mentioned, the deflections of the beam influence the critical bending moment, this statement is true for deflections caused by prestress. First we compare the critical moments of Beams -2 and -3: the deflection from prestress slightly affects the critical bending moment of the beam (Fig. 2.24) with tensile strength.

![Figure 2.24: The effect of prestressing on the critical moment in the case of beam with tensile strength: the equilibrium paths of Beams-2 and -3](image)

The comparison of Beams-4 and -5 shows that for beams without tensile strength the prestressing significantly influences the critical moment: the prestress keeps the beam uncracked for a while thus increasing the critical moment (Fig. 2.25).

This brief study of beams helped to understand the postcritical behaviour better, but it is hardly capable of being used directly for practical purposes: strength failure of common RC beams occurs at lower bending moment than buckling failure. Beams of some composite materials might be good candidates to verify these results experimen-
tally. In the next subsection we turn to calculating such problems that can be verified in experiments, namely determining vertical and lateral deflections of prestressed concrete beams.

\[\alpha_y [\text{rad}]\]

\[M = 19050 \text{ kNm}\]

\[M_0\]

**Figure 2.25:** The effect of prestressing on the critical moment in the case of beam without tensile strength: the equilibrium paths of Beams-4 and -5

### 2.2.2. Spatial Deformations of Prestressed Concrete Beams

In the previous subsection we compared our numerical results to analytical solutions concerning the critical moment for lateral buckling of the beam. Although we stated that for reinforced concrete the strength failure happens at lower bending moment, a both theoretical and practical question arises: how does geometrical asymmetry influence the deflections of a prestressed beam? The question is theoretical, since to the best of our knowledge the imperfection-sensitivity of asymmetrically prestressed concrete beams have been not studied in the literature. The question is practical, too, since according to the experiences of manufacturers, precast bridge beams with a span over 30 m often suffer such high lateral deflection and torsion that makes erection impossible. The technology of prestressing and the inhomogeneity of the fresh concrete are together responsible for asymmetry occurring in structures designed to be symmetrical [64]. With this study our final goal is to calibrate our method to enable it to predict such unfavorable deformations. In this subsection we report about the experiments to compare our numerical results to experimental data.
Figure 2.26: The simply supported beam of the experiments
CHAPTER 2. APPLICATIONS

DESCRIPTION OF THE EXPERIMENTS

As specimens we applied simply supported, prestressed concrete beams (Fig. 2.26). Beam-1 is designed to be symmetrical with two prestressing tendons. Beams-2 and -3 are asymmetrically prestressed members by applying 2 or 1 wires instead of one of the tendons. We applied only the minimal (i.e., 7 pieces of) stirrups along the beam, which was needed to the placing of the longitudinal bars. Basically we verify our method by comparing the calculated and measured initial deformations just after the prestressing force has been transferred to the beam. It was 14 days after the vibration of the concrete. This led to the differences in the achieved concrete quality due to the different weather conditions between the manufacturing and the measuring of the beams. We carried out the calculation with the measured material properties of the concrete. We did not apply heat curing to speed up the hardening of concrete.

The initial deformations occur due to the prestressing force and the weight of the beam. The deformations are measured in three points along the beam: two points were 1.5 m from the ends A and B, the third point is the middle point of the beam. Since the differences in the deflection of the different edges of the beam originating from torsion were under the 1 mm accuracy of measurement, we will compare vertical and lateral displacement of the three investigated points. After the initial deformations had been measured, Beams-2 and -3 were loaded by concentrated forces denoted to $F$ in Fig. 2.26. We measured the deformations of the middle point (Measure point II.) at several load levels for both beams. We mention that Beam-2 was loaded 14 days after transferring the prestress, i.e., the concrete was 28 days old at that time, that’s why we distinguish this measure as Beam-2*.

We carried out the experiments in a reinforced concrete precast factory (BVM Épelem Kft.) because we would have liked to demonstrate that the developed method is suitable for industrial purposes, namely to predict the deformations of members before manufacturing. For this reason we only applied such tools for manufacturing and measuring the beams, which are available and used in a prefabrication factory. That is why we consider three uncertain quantities in the calculation: the strength of the concrete (consequently the $E_c$ modulus of elasticity), the magnitude of the prestressing force and the placement of the prestressing tendons can slightly differ from the designed arrangement.

For measuring the strength of the concrete we made 9-9 tests with a Schmidt hammer at the three measure-points and determined the modulus of elasticity ($E_c$) according to the Hungarian Code MSZ-4715/5-72. At the time of transferring the prestress three cube specimens made of the same concrete were tested in a laboratory. The average and the variation of the experimental results and the corresponding $E_c$ values for the three beams are given in Table 2.4. In each case the results of the Schmidt hammer gave a smaller, and the cubic specimens a higher, concrete strength. Table 2.5 contains information about the density of the concrete which is needed to determine the $q$ weight of each beam.

Although we prescribed 950 MPa prestressing stress, according to the experience of the precast factory there can be about maximum 10% fluctuation in the transferred prestressing force, in the calculation we allow the prestressing force to be between 850 and 1050 MPa. We measured the placement of the prestressing tendons at the end cross sections of each beam and we found that there can be 2 mm offset compared to the designed situation, in the calculation this uncertainty will also be included.
Figure 2.27: Loading Beam-2

<table>
<thead>
<tr>
<th>Schmidt hammer</th>
<th>Cubic specimens</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{ck} ) [MPa]</td>
<td>variation</td>
</tr>
<tr>
<td>Beam-1</td>
<td>3.20</td>
</tr>
<tr>
<td>Beam-2</td>
<td>9.20</td>
</tr>
<tr>
<td>Beam-2*</td>
<td>15.01</td>
</tr>
<tr>
<td>Beam-3</td>
<td>4.85</td>
</tr>
</tbody>
</table>

Table 2.4: The measured concrete strength and the calculated \( E_c \) modulus of elasticity in the experiments. Beam-2 was measured 2 times, since between transferring the prestress and loading the beam 14 days elapsed.

<table>
<thead>
<tr>
<th>Density ( [kg/m^3] )</th>
<th>Weight ( [kN/m] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam-1</td>
<td>2265</td>
</tr>
<tr>
<td>Beam-2</td>
<td>2356</td>
</tr>
<tr>
<td>Beam-2*</td>
<td>2350</td>
</tr>
<tr>
<td>Beam-3</td>
<td>2314</td>
</tr>
</tbody>
</table>

Table 2.5: The density of concrete and the weight of the specimens in the experiments
CHAPTER 2. APPLICATIONS

NUMERICAL CALCULATION

In the numerical calculation we used the L0PC algorithm. The problem is more sophisticated than in the investigation of lateral torsional buckling due to the presence of lateral loads, here a 5D GRS is needed. We take the concentrated force $F$ as load parameter, the four variables are the rotation of the cross section around axes $x$ and $y$, the vertical reaction force and reaction torsion at point A:

$$v_1 = \alpha_x(0), \quad v_2 = \alpha_y(0), \quad v_3 = F_x(0), \quad v_4 = M_z(0). \quad (2.28)$$

The four functions prescribe zero displacement, zero rotation around axis $z$ and zero bending moment around axis $y$ at point B:

$$w_1 = x(l), \quad w_2 = y(l), \quad w_3 = \alpha_z(l), \quad w_4 = M_y(l). \quad (2.29)$$

COMPARISON OF THE RESULTS

In Table 2.6 we compare the initial deformations of all the three beams. $u$ denotes the lateral displacement, $v$ the vertical displacement of a point, respectively.

<table>
<thead>
<tr>
<th>Measure point I.</th>
<th>Measure point II.</th>
<th>Measure point III.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$ [mm]</td>
<td>$v$ [mm]</td>
<td>$u$ [mm]</td>
</tr>
<tr>
<td>Beam-1</td>
<td>measured</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>computed</td>
<td>0</td>
</tr>
<tr>
<td>Beam-2</td>
<td>measured</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>computed</td>
<td>4.5</td>
</tr>
<tr>
<td>Beam-3</td>
<td>measured</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>computed</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 2.6: The initial deformations after transferring prestress

In comparing the deformations under the concentrated forces $F$ we plot the results calculated by the upper and lower values of the modulus of elasticity, prestressing force and offset of the prestressing tendons. First we compare the effect of each uncertain quantity on the lateral deformations (Fig. 2.28), finally we give the enveloping curves of the possible deformations (Fig. 2.29-2.30) compared to the measured values. Although these three experiments are very few to draw statistical conclusions, we are convinced that the result of the experiments demonstrates the method predicts the deformations of RC and PC beams in a reliable manner.
Figure 2.28: The effect of the uncertain quantities on the lateral deformations
Figure 2.29: The envelope figures for Beam-2
Figure 2.30: The envelope figures for Beam-3
2.2.3. The Principal Result of RC Beams

I investigated numerically the behaviour of compressed RC beams by the developed algorithm assuming linear stress-strain relation \((k = 1, \ l = 1)\) in the compressed zone. Based on my computations I draw the following conclusions.

**P.R.5.1.** Although the critical bending moment for lateral torsional buckling of a beam without tensile strength can be calculated by the analytical solutions of the literature, the investigation of the postcritical behaviour requires the calculation of the spatial deformations of the beam. I showed that for beams without tensile strength the bifurcation is subcritical. The prestressing of influences slightly the critical bending moment of a beam with tensile strength due to the upward deflection caused by the prestress. For beams without tensile strength the effect of prestressing is more significant by increasing the stiffness of the beam.

**P.R.5.2.** I carried out experiments on symmetrically and asymmetrically prestressed beams to compare the predicted and measured deflections. I measured the deflections in three points of the three specimens. Our method predicts the vertical and lateral deflections of the beam just after transferring prestress in a reliable manner (the error is smaller, than 10\%). I loaded the beams with concentrated forces and measured the vertical and lateral deflections for three levels of the load. I determined the enveloping curves for each specimen which takes the uncertainties of the manufacturing (the modulus of elasticity of the concrete, the real value of the prestressing force and the location of the prestressing tendons) into account. The measured values with their 1 mm error are kept in the zone determined by the enveloping curves as long as the stresses are kept in the elastic range.
SUMMARY AND PRINCIPAL RESULTS

I determine the neutral axis of an arbitrary cross section of a bar without tensile strength under compression and biaxial bending by a direct recursion derived from the equations of equilibrium. The cross section can contain areas with tensile strength (also reinforcement). The stress-strain relation of the part without tensile strength (concrete) and the reinforcement can generally be given by

\[ \sigma_c(\varepsilon) = \begin{cases} q_1 \varepsilon + q_2 \varepsilon^2 + \cdots + q_k \varepsilon^k, & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon \leq 0 \end{cases} \]

\[ \sigma_s(\varepsilon) = r_1 \varepsilon + r_2 \varepsilon^2 + \cdots + r_l \varepsilon^l, \]

where \( q_1 \geq 0, r_1 \geq 0, k \) and \( l \) are positive integers. We assume that all roots of \( \sigma_c(\varepsilon) \) in \( \varepsilon \) are positive. Although the literature suggests to use iterative procedures to determine the neutral axis and the curvature, the convergence-features of these solutions are not known. The reliability of the methods applied in the engineering praxis is also questionable. The first principal result contains statements about the special, linear method, while the second principal result describes the solution for the general case.

1. PRINCIPAL RESULT

P.R.1.1. For calculating the neutral axis of an arbitrary cross section under compression and biaxial bending I derived a direct recursion from the equations of equilibrium assuming linear material law in the compressed concrete zone \((q_2 = q_3 = \cdots = q_k = 0)\) and linear material law for the reinforcement \((r_2 = r_3 = \cdots = r_l = 0)\). The method can be associated with a two dimensional map. I proved analytically the following statements:

(a) there is one, and only one fixed point,
(b) the fixed point is stable (i.e. the method is locally convergent),
(c) the recursion in the case of symmetrical cross section and a compressive load on the axis of symmetry is globally convergent.

P.R.1.2. In the general, two dimensional case (asymmetrical cross section or load) according to the literature we expect chaotic behaviour. In contrary, systematic numerical trials show the recursion based on the two dimensional map is globally convergent.
2. **Principal Result**

**P.R.2.1.** For non-linear material law of the compressed zone ($q_2,q_3,...,q_k$ are arbitrary constants) and arbitrary constitutive law of the reinforcement ($r_2,r_3,...,r_l$ are arbitrary constants) I derived an algorithm to determine the neutral axis of an arbitrary cross section under compression and biaxial bending. The method can be associated with a 5 dimensional, semi-implicit map.

**P.R.2.2.** Using this approach I investigated the

$$\sigma_c(\varepsilon) = \begin{cases} q_1 \varepsilon + q_2 \varepsilon^2, & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon \leq 0 \end{cases}$$

second order stress-strain relation of the concrete with $\sigma_s(\varepsilon) = r_1 \varepsilon$ material law of the reinforcement, which is sufficient for practical purposes. In this case there exists a theoretical maximal load $P_{max}$ for each estimate of the neutral axis. I defined three possibilities for carrying out the calculation:

(a) The load $P$ is constant at each step of the iteration, for $P > P_{max}$ the method halts,

(b) In each step we determine $P_{max}$ and the neutral axis is calculated with this load (in this case the solution is unique),

(c) The load $P$ is constant until $P < P_{max}$, otherwise the method continues assuming $P = P_{max}$ in each step.

By numerical simulations all the three approaches are **globally convergent**.

3. **Principal Result**

I developed a new algorithm to calculate the spatial deformations of rods without, or with limited tensile strength. The algorithm is **robust**, i.e. the deformations can be computed in a reliable manner, there is no danger of false solutions or divergent behaviour. In the frame of this work

**P.R.3.1.** I implemented the algorithm to determine the neutral axis according to the $1^{st}$ and $2^{nd}$ principal results. The algorithm calculates the neutral axis and the curvature of the cracked cross section rapidly: typically in 5-10 steps it estimates the curvature within a 1% error. I embedded this algorithm into the core of the Parallel Hybrid Algorithm, which is an iteration-free solver of boundary value problems. Due to the global convergence of the algorithms determining the curvature, and the features of the PHA the whole algorithm is robust.

**P.R.3.2.** I extended the algorithm by subroutines to take the tension stiffening, the shrinkage and creep of concrete and the losses of prestress into account. These subroutines are based on the EUROCODE 2 standard, in the case of creep the widely used Trest model is also included. I showed these subroutines do not influence the convergence properties of the algorithm.
4. **Principal Result**

I investigated numerically the behaviour of compressed RC columns by the developed algorithm assuming linear stress-strain relation \( k = 1, l = 1 \) in the compressed zone. Based on my computations I draw the following conclusions.

**P.R. 4.1.** I demonstrated that for centrally compressed columns the postcritical branch has typically two limit points. The first one after the bifurcation appears due to the cracking, after this limit point the branch is unstable. For reinforced columns the second limit point makes the branch stable again. The branch asymptotically reaches the postcritical branch of the compressed column with cracked cross sections under pure bending. I showed that in the case of eccentric compression with eccentricity \( e \) the two limit points unify at a critical value \( e = e_{cr} \) in a catastrophe point (i.e. a cut-off point), thus for \( e > e_{cr} \) the instability disappears. By prestressing the reinforcing bars of the column the load belonging to the first critical point (i.e. the critical load of the column under eccentric compression) increases and the value of the critical eccentricity \( e_{cr} \) is higher, too.

**P.R. 4.2.** I introduced a slightly asymmetrical structural example which can be considered to be optimal, i.e. the risk of buckling is minimal. The example is a symmetrical compressed column between planar hinges, the symmetry breaking variable is the offset of two bars of reinforcement. I showed the optimal value of the offset of the reinforcement as a function of the geometrical ratio of the concrete cross section.

5. **Principal Result**

I investigated numerically the behaviour of compressed RC beams by the developed algorithm assuming linear stress-strain relation \( k = 1, l = 1 \) in the compressed zone. Based on my computations I draw the following conclusions.

**P.R. 5.1.** Although the critical bending moment for lateral torsional buckling of a beam without tensile strength can be calculated by the analytical solutions of the literature, the investigation of the postcritical behaviour requires the calculation of the spatial deformations of the beam. I showed that for beams without tensile strength the bifurcation is subcritical. The prestressing influences slightly the critical bending moment of a beam with tensile strength due to the upward deflection caused by the prestress. For beams without tensile strength the effect of prestressing is more significant due to the increase in the stiffness of the beam.

**P.R. 5.2.** I carried out experiments on symmetrically and asymmetrically prestressed beams to compare the predicted and measured deflections. I measured the deflections in three points of the three specimens. Our method predicts the vertical and lateral deflections of the beam just after transferring prestress in a reliable manner (the error is smaller, than 10%). I loaded the beams with concentrated forces and measured the vertical and lateral deflections for three levels of the load. I determined the enveloping curves for each specimen which takes the uncertainties of the manufacturing (the modulus of elasticity of the concrete, the real value of the prestressing force and the location of the prestressing tendons) into account. The measured values with their 1 mm error are kept in the zone determined by the enveloping curves as long as the stresses are kept in the elastic range.
INDUSTRIAL APPLICATION OF THE RESULTS
Since the parallel computational environment has not offered the accustomed facilities of a desktop PC, I took part in developing a user interface for the algorithm to enable industrial users to carry out calculations easily via a web browser. I defined the requirements of the user and the needed functionality. I developed and implemented a graphical tool to display the results of the calculation. The algorithm was implemented in C++ language and the developed software package is one of the first applications of parallel computation and GRID technology for industrial proposes in Hungary. (Common work with Dániel Pasztuhov)
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In the implementation of the algorithm I received a lot of help from Imre Szeberényi and I thank Dániel Paszthov for his help in developing the user interface. I am very grateful to Antal Tápai and the employees of BVM Építem Ltd. for the contribution in the experiments including designing, manufacturing and measuring the specimens. László Polgár and Eszter Gondár from ASA Ltd. helped me in calculating the frame using the ABACUS-STUR software.

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