Stability analysis of delaminated composite plates and shells by classical and first-order shear deformation theories
Delaminált kompozit héjak és lemezek stabilitásvizsgálata klasszikus és elsőrendű elmélet alapján

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Abstract

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Today fiber reinforced layered composites are playing a more and more important role on every fields of mechanical engineering. They are applied as material even for such critical load carrying elements as air-plane wings, thanks to their good strength-to-weight ratio, and to the fact that with the proper choice of the reinforcement structure their properties can be set according to the needs. Besides the rapid development of the ingredients and the production technologies, the available theories are evolving fast, which can be used for the design. Because of the anisotropic and heterogeneous material behaviour even the analysis of the non-damaged structures involves challenging problems. The existent interlaminar fractures, also called as delaminations, make the behaviour of these structures quite complex. These delaminations can occur because of manufacturing errors, impacts or fatigue during the service life-time. The aim of this work is to extend the available knowledge regarding the stability of delaminated shell structures. Among the concerned problems the followings can be found: free-vibration and linear buckling problems, dynamic stability analysis, local static and dynamic stability loss of the delaminated zones, and with the extension of the model with geometric non-linearities imperfection analysis have also been carried out. The presented solution methods cover both improved analytical and numerical finite element techniques, which are available in the literature.
Introduction

As environment protection and cost reduction is more and more important in the design of machines, engineers are applying new materials which are optimized regarding the load requirements. This results more slender parts which allows material and weight savings. Fiber-reinforced composites are proper materials for these design principles, because their strength and load-carrying capacity can be set according to the needs, and regarding strength to weight ratio they are outperforming conventional metals. Therefore they are widely used in many fields of the industry: such as cars, airplanes, ships, space vehicles, sport equipments, etc. Compared to metals the material behaviour of composites is strongly anisotropic. This fact implies that the proper and accurate modelling of composites is still an open question in the literature.

One of the main failure modes of layered composites is the interlaminar fracture, also called as delamination. The triggering cause of the delamination can be for example geometric or material discontinuities because of machining errors, foreign body impacts, or interlaminar stresses caused by structural loading in curved laminates. Compared to matrix cracking which is also a common failure mode, the reduction of the stiffness and load-bearing capacity caused by the delamination is more significant. Since the stability loss in filigree designs is more potential, shells have to be designed very carefully. Moreover as the existence of delamination does not cause the immediate collapse of the structure, just affects the stability limit, the need of accurate mechanical models for delaminated composites are substantial for safe designs.

Research objectives

From the previously mentioned reasons it follows that the objective of this research work is to develop improved solutions for the stability analysis of delaminated composite structures. This involves application of the system of exact kinematic conditions (SEKC) for plates [1] and other shell structures, and also the generalization of the method for specimens having multiple delaminations. This approach was used to derive the energy functions of general doubly curved delaminated shells considering geometric nonlinearities in the von Kármán sense. For validation purpose an analytical solution was derived using the Lévy method to estimate the eigenfrequencies and critical buckling loads in case when two opposite edges are simply supported. Numerical solution of the problems was also created which allows the use of more general boundary conditions (BC). However the methods based on the Lévy-type solution are only applicable for certain level of anisotropy, the shell finite element solution is capable to handle any kind of of lay-ups. To further extend the knowledge about the buckling of delaminated structures nonlinear buckling analysis was also done. In case of plates measurements was also carried out to verify the models.
Basic principles of the model

This section gives a short introduction about the main parameters and considerations which were used for describing the layered delaminated shells. The shells can consist of any number of orthotropic plies with different ply thickness $h_i$. For describing a 3D layered shell first it will be divided to separate equivalent single layers (ESLs) along the ply interfaces to reduce the problem into a 2D one. These ESLs are indexed with an increasing number from the inner surface of the shell. Figure 1 shows an example where the shell is divided into two ESLs along the mid-surface. Each ESL has a local orthogonal curvilinear coordinate system (CS) $(O_j, x_j, y_j, z_j)$, where $x_j$ and $y_j$ are describing curvilinear distances and $z_j$ is the surface normal so it denotes the position along the thickness of ESL$_j$ as Fig. 1 shows. The local CSs can be derived from the global reference CS of the shell by linear translation along its $z$ axis.

![Figure 1. Parameters and notations for the model of doubly curved shells.](image)

The side view of the multiply delaminated doubly curved composite shell is shown in Fig. 2. The model contains a delaminated zone $\textcircled{2}$ where the delaminations are under each other and they fill out zone $\textcircled{2}$. It is known in the literature that in case of impact induced delaminations usually more layers are delaminating. The orientation of these delaminations is usually the same as the orientation of the adjacent layers and they are located approximately under each other as Fig. 2 shows. Therefore the presented model can be applicable for real situations where the delamination front can be approximated with straight lines. Along the $x$ direction before and after the zone $\textcircled{2}$ there are non-delaminated zones ($\textcircled{1}$ and $\textcircled{3}$) therefore the delaminations are belonging to the family of through-the-width (TTW) delaminations. Furthermore it is assumed that there is perfect adherence between each plies and the delaminations are not propagating during the examinations.
As modelling principle the SEKC will be used. In this work it was applied to doubly curved shells using classical and first-order theories, and it was extended to multiple delaminations. To apply the SEKC the system has to be divided into ESLs along the surfaces of the delaminations. In case when $M$ delaminations are present the system will consists of $M + 1$ ESLs with thickness $t_i$ as Fig. 2 shows. The displacement field of each ESL can be given in its local CS with the use of the angles $\varphi, \theta$ and the local $z_i$ transversal coordinate:

$$U_i(\varphi, \theta, z_i, \tau) = u_0(\varphi, \theta, \tau) + u_i(\varphi, \theta, \tau) + z_i \phi_{\varphi i}(\varphi, \theta, \tau)$$  \hspace{0.5cm} (1a) \\
$$V_i(\varphi, \theta, z_i, \tau) = v_0(\varphi, \theta, \tau) + v_i(\varphi, \theta, \tau) + z_i \phi_{\theta i}(\varphi, \theta, \tau)$$  \hspace{0.5cm} (1b) \\
$$W_i(\varphi, \theta, z_i, \tau) = w_0(\varphi, \theta, \tau)$$  \hspace{0.5cm} (1c)

where $U_i$ and $V_i$ are the in-surface displacements along $x_i, y_i$ and $W_i$ is the transverse displacement along $z_i$, and $\tau$ denotes the time for dynamical problems. The terms $\phi_{\varphi i}$ and $\phi_{\theta i}$ represent the rotations of transverse normals about the $y_i$ and $x_i$ axes. The $u_0, v_0$ terms on the right hand side are the global, $u_i, v_i$ are the local membrane displacements. The displacement has to be continuous along the non-delaminated zones, whereas there are discontinuities along the delaminated zone. As the ESLs are working together along zones 1 and 3 the in-surface displacements has to be fitted to each other. This can be done by writing the equations of SEKC which states that the in-surface displacements are common along the neighbouring interfaces. The transverse displacements are prescribed to be common in all zones. This implies a constrained model which eliminates the kinematical non-admissible mode shapes, e.g. the intersection of the ESLs along the delaminations. Using this approach there is no need for contact definitions, between the interfaces, which significantly simplifies the calculations. But as the in-surface displacements are individual for all ESLs the model is able to capture the global behaviour of the delaminated structure.

Based on these considerations the governing equations of the multiply delaminated shells were derived using the fundamental lemma of variational calculus. The obtained equations were solved by using analytical and finite element methods. The solutions involve free-vibration and linear, and nonlinear structural stability problems.
Main Results

This chapter summarizes the main results obtained in this work. Every result begins with a short statement written with bold letters. After the statement the obtained results and the theoretical background is summarized briefly. The related publications are listed after the summary of the result.

1\textsuperscript{st} Result

For obtaining the governing equations of the delaminated doubly curved shells, I developed the displacement fields based on the system of exact kinematic conditions using $M + 1$ number of equivalent single layers for modelling $M$ number of delaminations, where $M \in \mathbb{N}$. The displacement fields were derived based on angular coordinates. To develop the displacement field the constrained mode model was applied, therefore the transverse deflections are the same for each equivalent single layers. For the governing equations, I developed an improved version of the Sander’s shell theory. The improvement is based on the series expansion of the thickness dependent terms, which results in more precise curvature terms.

Using the system of exact kinematic conditions and the improved shell theory the governing equations of the multiply delaminated doubly curved shells was expressed. With the use of angular coordinates the descriptor variables are the same along the reference surfaces. In this case the transformation between the coordinate systems of the reference surfaces and the global reference surface is a linear translation along the normal direction. So the displacement functions of the non-delaminated zones resulted in the following by applying the first-order assumption for the rotational terms:

$U_k = u_0 + \left[ -z_R + \sum_{i=1}^{j-1} t_i \phi_{\varphi j} - \sum_{i=k}^{j-1} (t_i \phi_{\varphi i}) + \sum_{i=j}^{k-1} (t_i \phi_{\varphi i}) + z_{Rk} \phi_{\varphi k} \right] + z_k \phi_{\varphi k}$  \hspace{1cm} (2a)

$V_k = v_0 + \left[ -z_R + \sum_{i=1}^{j-1} t_i \phi_{\theta j} - \sum_{i=k}^{j-1} (t_i \phi_{\theta i}) + \sum_{i=j}^{k-1} (t_i \phi_{\theta i}) + z_{Rk} \phi_{\theta k} \right] + z_k \phi_{\theta k}$  \hspace{1cm} (2b)

$W_k = w_0; \quad k = 1..(M + 1)$  \hspace{1cm} (2c)

The application of the classical shell theory to delaminated composite shells resulted in the following displacement functions:

$U_k = u_0 + P_k \left( \frac{u_0}{R_{\varphi}} - \frac{w_0}{A_{\varphi}} \right) + z \left( \frac{u_0}{R_{\varphi}} - \frac{w_0}{A_{\varphi}} \right)$  \hspace{1cm} (3a)

$V_k = v_0 + P_k \left( \frac{v_0}{R_{\theta}} - \frac{w_0}{A_{\theta}} \right) + z \left( \frac{v_0}{R_{\theta}} - \frac{w_0}{A_{\theta}} \right)$  \hspace{1cm} (3b)
based on the same series expansion technique:

\[ P_k = \frac{-t}{2} + \frac{t_k}{2} + \sum_{i=1}^{k-1} t_i \]  

It has been shown that the displacement field of plates based on the first-order theory is directly applicable for delaminated shells if angular coordinates are used. The displacement field of thin shells was obtained by expressing the rotations based on the thin shell assumption, or in other words the zero shear strain conditions were applied: \( \gamma_{\varphi z} = \gamma_{\theta z} = 0 \). This condition is derived based on the improved version of the Sanders shell theory. The obtained equations are valid for the modelling of delaminated structures wherein the delaminations are placed exactly below each other. In real life applications such delaminations occur due to impacts, which is one of the main cause of the interlaminar fracture.

Based on the the system of exact kinematic conditions, I improved the Sander’s shell theory which includes the von Kármán type nonlinear model. The improvement is based on the series expansion of the thickness dependent terms shown below:

\[
\frac{1}{(1 + \frac{z}{R\varphi})} = \sum_{i=0}^{n} \left( \frac{z}{R\varphi} \right)^i ; \quad \frac{1}{(1 + \frac{z}{R\varphi})^2} = \sum_{i=0}^{n} \left( \frac{(i+1)z}{R\varphi} \right)^i
\]  

where \( R\varphi \) is the principal radius corresponding either to the \( \varphi \) or to the \( \theta \) coordinate. With the aid of series approximation the improved reference-surface curvatures can be given as:

\[
\kappa^0_{\varphi} = \frac{\kappa_{\varphi}}{R_{\varphi}} - \frac{\varepsilon_{\varphi}}{R_{\varphi}}; \quad \kappa^0_{\theta} = \frac{\kappa_{\theta}}{R_{\theta}} - \frac{\varepsilon_{\theta}}{R_{\theta}}
\]

\[
\kappa^0_{\varphi\theta} = \frac{\kappa_{\varphi\theta}}{R_{\varphi}}; \quad \kappa^0_{\theta\varphi} = \frac{\kappa_{\theta\varphi}}{R_{\theta}}
\]  

Both the first-order and the classical variants of the curvatures can be given using the system of exact kinematic conditions. The improved curvatures are valid both for the delaminated and for the non-delaminated zones. The radii dependent stiffness terms can be also given based on the same series expansion technique:

\[
\bar{A}_{ij} = \left( A_{ij} + \frac{B_{ij}}{R_{\varphi}} \right) \quad \overline{\bar{A}}_{ij} = \left( \bar{A}_{ij} + \frac{\bar{B}_{ij}}{R_{\theta}} \right) \quad i, j = 1, 2, 4, 5, 6
\]

\[
\bar{B}_{ij} = \left( B_{ij} + \frac{D_{ij}}{R_{\varphi}} \right) \quad \overline{\bar{B}}_{ij} = \left( \bar{B}_{ij} + \frac{\bar{D}_{ij}}{R_{\theta}} \right) \quad i, j = 1, 2, 6
\]

\[
\bar{D}_{ij} = \left( D_{ij} + \frac{F_{ij}}{R_{\varphi}} \right) \quad \overline{\bar{D}}_{ij} = \left( \bar{D}_{ij} + \frac{\bar{F}_{ij}}{R_{\theta}} \right) \quad i, j = 1, 2, 6
\]

\[
[A_{ij}, B_{ij}, D_{ij}, F_{ij}]^T_k = L_k \sum_{i=1}^{n_k} \mathcal{C}_{ij} \begin{bmatrix} z_k, z_k^2, z_k^3 \end{bmatrix}^T dz_k
\]  

The series expansion makes it possible to avoid the logarithmic terms in the stiffness coefficients, which can make the solution of the governing equations of multiply delaminated composite shells too complicated. With the use of the stiffness terms and the strain-field the linear \( (N_{\varphi}, N_{\theta}, N_{\varphi\theta}, M_{\varphi}, M_{\theta}, M_{\varphi\theta}, M_{\theta\varphi}) \) and nonlinear stress resultants \( (N_{\varphi}, N_{\theta}, N_{\varphi\theta}, N_{\theta\varphi}) \) can be derived based on the constitutive equation of the laminae. Similarly the inertia terms are:

\[
\mathcal{T}_{0k} = \left( I_0 + \frac{I_1}{R_{\varphi}} + \frac{I_2}{R_{\theta}} + \frac{I_3}{R_{\varphi}R_{\theta}} \right)_k
\]
\[ T_{1k} = \left( I_1 + \frac{I_2}{R_{\phi}} + \frac{I_2}{R_{\theta}} + \frac{I_3}{R_{\phi}R_{\theta}} \right) \]

\[ T_{2k} = \left( I_2 + \frac{I_3}{R_{\phi}} + \frac{I_3}{R_{\theta}} + \frac{I_4}{R_{\phi}R_{\theta}} \right) \]

\[ [I_0, I_1, I_2, I_3, I_4]_k^T = \sum_{l=1}^{L_k} \int_{-\frac{h_k}{2}}^{\frac{h_k}{2}} \rho_l \begin{bmatrix} 1, z_k, \frac{z_k^2}{4}, \frac{z_k^4}{24} \end{bmatrix}_k^T \, dz_k \]

With these terms the Lagrange function of the doubly curved delaminated shells can be derived:

\[ \mathcal{L} = \sum_{k=1}^{M+1} (\delta U_k + \delta V_k - \delta T_k) \]

By applying the fundamental lemma of variational calculus on the Lagrange function the governing equations can be obtained. These are valid for doubly curved shells even in cases when the principal radii of curvatures and the Lamé parameters are not constant, but dependent from the angular coordinates. I showed that if the thickness terms are not neglected in the thin shell theory, then: \( N_{\phi\theta} \neq N_{\theta\phi} \) and \( M_{\phi\theta} \neq M_{\theta\phi} \). Consequently, the thin version is valid even for cases when the two principal radii of curvature are not the same, or not negligible as in case of shallow shells. Regarding the applicability of the classical shell theory it has to be mentioned that the conditions for the \( t \) ratio given in literature is only true in case of the anisotropy is small. For anisotropic shells there is no exact limit for the applicability of the classical shell theory, but it can be said that in these cases the first-order or the higher-order theories are providing more accurate results therefore it is advised to use these ones in for anisotropic shells. The obtained governing equations have been successfully applied both in case of analytical and finite element solutions of multiple shell types.

Related publication: [1], [2], [3]

2nd Result

I solved the linear governing equations of the multiply delaminated shells by an analytical method for rectangular plates, cylindrical and spherical shells. The solution technique is based on the Lévy-method and the state-space approach. The latter one has been extended for coordinate-variant cases, as the governing PDE system may contain coefficients dependent on the angular coordinates. The calculations were done based on the time variant solution of control systems, in which the state-transition matrix was created based on the decomposed system matrix.

Based on the decomposition the state-transition matrix can be given as follows:

\[ \chi(\varphi, \varphi_0) = \prod_{i=1}^{n} e^{\mathbf{T}_i f_i(\hat{\varphi}) d\hat{\varphi}} \]

where \( \mathbf{T}_i \) can be given as:

\[ \mathbf{T}(\varphi) = \sum_{i=1}^{n} \mathbf{T}_i f_i(\varphi) \]

Where \( \mathbf{T}_i \) is a constant matrix such that \( \mathbf{T}_i \mathbf{T}_j = \mathbf{T}_j \mathbf{T}_i \) and \( f_i \) is a scalar-valued function of \( \varphi \).
The Lévy solution method is applicable in case if two parallel edges are simply supported and moreover the following two conditions must hold for delaminated composite shell structures:

- **I.** \( R_\varphi, R_\theta \) principal radii of curvature and \( A_\varphi, A_\theta \) Lamé parameters are dependent only on the \( \varphi \) angular coordinate. This results in the fact that the strain fields can be expanded into Fourier series.

- **II.** The stiffness-coefficient \( (A, B, D, F)_{16,26} \) and \( A_{45} \) are all zero, which ensure that the stress-resultants can be expanded into Fourier series.

The developed mechanical model was utilized to solve free-vibration and buckling problems of delaminated composite plates, cylindrical and spherical shells. In case of plates and cylindrical shells the decomposition results in only one term, the \( \mathbf{T}_1 \) matrix and the corresponding \( f_1(\varphi) = 1 \) function. In case of the first-order delaminated spherical shell as \( A_9 \) is a function of \( \varphi \) the decomposition results in six coefficient matrices with the use of the following coefficient functions:

\[
\mathbf{f} = \begin{bmatrix} \cot(\varphi), \cot^2(\varphi), \frac{1}{\sin^2(\varphi)}, \cot(\varphi), \frac{1}{\sin(\varphi)}, 1 \end{bmatrix}^T \tag{12}
\]

And for the solution of a spherical shell based on the classical shell theory the following decomposition functions have to be used:

\[
\mathbf{f} = \begin{bmatrix} \cot^2(\varphi), \cot^2(\varphi), \cot(\varphi), \cot(\varphi), \frac{1}{\sin^4(\varphi)}, \frac{1}{\sin^4(\varphi)}, \cot^2(\varphi), \frac{1}{\sin^4(\varphi)}, \cot(\varphi), \frac{1}{\sin(\varphi)} \end{bmatrix}^T \tag{13}
\]

The decomposed matrices were given in the dissertation for the first-order shell theory and applied to the free vibration problem of delaminated spherical shells. The solution method was validated to finite element models based on the Lévy-method and to models created in ABAQUS with S4R shell elements. The non-delaminated models were also compared to results available in the literature. It was shown that the developed solution is applicable whenever the decomposition can be done and the given conditions hold. For the state-space solution the related boundary and continuity conditions of the delaminated shells were also given. For delaminated classical shells the continuity of the equivalent bending moments should be imposed, in other words, the effect of the in-surface forces has to be taken into account because the delamination is closed at both sides. This results in the following continuity condition:

\[
\left\{ \tilde{u}_{kn}, \tilde{w}_{kn} \right\}_{k2} = \left\{ \tilde{u}_0, \tilde{w}_0 \right\} + P_k \left( \frac{\tilde{u}_0}{\alpha_p} - \frac{\tilde{w}_0}{\lambda_p} \right) \quad ; k = 1..M + 1 \tag{14a}
\]

\[
\left\{ \tilde{w}_0 \right\}_2 = \left\{ \tilde{w}_0 \right\}_2 + \left\{ \tilde{w}_0 \right\}_2 \quad \alpha \tag{14b}
\]

\[
\sum_{k}^{M+1} \left\{ \tilde{N}_{\varphi kn} + \tilde{N}_{\theta kn} \right\} = \sum_{k}^{M+1} \left\{ \tilde{N}_{\varphi kn} + \tilde{N}_{\theta kn} \right\} \quad \alpha \tag{14c}
\]

\[
\sum_{k}^{M+1} \left\{ \tilde{M}_{\varphi kn} + P_k \tilde{M}_{\varphi kn} + 2\beta \left( \tilde{M}_{\varphi kn} + \tilde{M}_{\theta kn} + P_k \left[ \tilde{N}_{\varphi kn} + \tilde{N}_{\theta kn} \right] \right) \right\}_{\alpha} = \sum_{k}^{M+1} \left\{ \tilde{M}_{\varphi kn} + P_k \tilde{M}_{\varphi kn} + 2\beta \left( \tilde{M}_{\varphi kn} + \tilde{M}_{\theta kn} + P_k \left[ \tilde{N}_{\varphi kn} + \tilde{N}_{\theta kn} \right] \right) \right\}_{\alpha} \tag{14d}
\]
where all the \( \square \) quantities are the proper Lévy-type displacement fields and stress resultants. Therefore they are only dependent on \( \varphi \) which can take either \((\varphi_0 + L_1)\) or \((\varphi_0 + L_1 + d)\) respectively. The \( P_k \) constant contains only geometric information and \( \alpha \) is either 1 or 3 referring to a non-delaminated zone whereas the 2 in the subscript is referring to the delaminated zone of the shell.

Related publication: [1], [3]

3\textsuperscript{rd} Result

a) A novel set of linear Lévy-type finite elements for delaminated plates and shells subjected to Lévy type boundary conditions was developed. The set contains three elements: the element of the delaminated zone, the element of the non-delaminated zone and a special delamination tip element based on the system of exact kinematic conditions establishing the continuity between the delaminated and non-delaminated zones. Structures having simply-supported edges along the length can be modelled very efficiently with this element set.

The novel finite element set was utilized to solve the dynamic stability problem of single delaminated composite shells subjected to non-conservative loading (edge forces). The latter involves the so-called flutter type instability phenomenon, which requires the solution of the model by varying the load and frequency parameters in wide ranges in order to obtain the stability diagrams.

b) The local stability analysis of the delaminated zones subjected to the in-surface stress-resultants can be carried out by using the novel element set. To obtain solution the distributions along both in-surface coordinates have to be taken into account. Along the length this was done by averaging the normalized distribution for each finite element. The widthwise change of the in-surface stress resultants was also considered and the related finite element equation was solved by the method of harmonic balance.

The widthwise distribution is sinusoidal because of the Lévy type boundary conditions. Thus for the static stability analysis, the so-called Bolotin-matrix becomes:

\[
\begin{bmatrix}
K & \frac{1}{2}N_LK_y & 0 & 0 & \cdots \\
\frac{1}{2}N_LK_y & K & \frac{1}{2}N_LK_y & 0 & \cdots \\
0 & \frac{1}{2}N_LK_y & K & \frac{1}{2}N_LK_y & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
d_0 \\
d_1 \\
d_2 \\
d_3 \\
\vdots
\end{bmatrix}
\Psi = 0
\] (15)

The results of numerical calculations show that the extreme value of the trigonometric distribution is \( \sim 40\% \) higher than the magnitude of the uniform distribution of the in-surface forces. In the dynamic case the delamination is excited with the periodic stress-resultant. To reduce the complexity of the latter problem the trigonometric (half sine wave) widthwise distribution of the in-surface forces was replaced by equivalent uniformly distributed in-surface forces. Thus, the dynamic stability analysis of delaminated shells resulting in a system of Mathieu - Hill equations which can be solved - similarly to the static case with widthwise trigonometric force distribution - based on the method of harmonic balance. The corresponding Bolotin
The obtained solutions provide good approximations for the stability loss of the system under dynamic conditions.

Using the presented technique the mixed-mode static stability of simply-supported delaminated shells can be solved, where the global and local mode shapes can be superimposed using a simple arch length criterion:

\[
\int \sqrt{1 + \left( \frac{\partial w_g}{\partial \varphi} + sw_l(\varphi) \right)^2} \, d\varphi - (u_{LR} + u_{CR}) := d
\]

(17)

So the behaviour of the delaminated shells can be estimated also in this case.

Related publication: [1], [4], [5], [6], [7]

4th Result

I developed a linear four-noded shell element set for multiply delaminated composite shells under arbitrary boundary conditions. The set contains the same element types as the set based on the Lévy-type formulation. The elements were defined based on the presented classical and first-order shell theories.

In contrast with the Lévy-type formulation the four-noded shell elements can be applied to shells subjected to any kind of boundary conditions involving free and clamped edges as well as edges loaded by moments and forces. Moreover it can be applied to any kind of lamination types. The results of finite element models built up by the four-noded shell elements were compared to results obtained by experimental modal-analysis and buckling measurements of single delaminated unidirectional plates. The comparison indicates that the developed finite elements and the created models capture very well the effect of the delamination on the frequency and buckling properties of the delaminated shells.

Related publication: [2], [8]

5th Result

The linear shell finite element set was extended to geometric nonlinear problems based on the von Kármán type model. For obtaining accurate results the different compressive and flexural strengths of composite specimens have to be taken into account. The different elastic moduli should be considered by evaluating the extensional- \((A_{ij})\) and bending-stiffnesses \((D_{ij})\) besides the different values.

With this approach the diverse behaviour of composites under compression and bending can
be taken into account through the constitutive equation of the lamina:

$$\begin{bmatrix} N + N' \\ M \end{bmatrix}_k = \begin{bmatrix} N_\varphi + N'_\varphi \\ N_\theta + N'_\theta \\ N_{\varphi\theta} + N'_{\varphi\theta} \\ N_{\theta\varphi} + N'_{\theta\varphi} \\ M_{\varphi} \\ M_{\theta} \\ M_{\varphi\theta} \\ M_{\theta\varphi} \end{bmatrix} = \begin{bmatrix} A^* \\ 0 \\ D \end{bmatrix} \left( \begin{bmatrix} \varepsilon^0 \\ \kappa^0 \end{bmatrix} + \begin{bmatrix} \varepsilon^0 \\ 0 \end{bmatrix} \right)$$

(18a)

$$\begin{bmatrix} Q_\theta \\ Q_\varphi \end{bmatrix} = K_s A^*_s \gamma^0_z \tag{18b}$$

where * denotes the element which were determined using the compression modulus. For making the differentiation of the elastic moduli possible the non-delaminated elements have to be derived from the total potential energies of one equivalent single layer which models the whole non-delaminated zone, as the displacement field resulted by the system of exact kinematic condition contains additional rotational terms. The differentiation was applied for models where the coupling terms ($B_{ij}$) are zero.

Carrying out the uni-axial compression tests using single delaminated unidirectional plate specimens the axial displacement - compression load and the transversal displacement - compression load curves were obtained. According to the experiences the compression stiffness of the used glass fibre reinforced specimens is much lower than the flexural modulus. Applying the differentiation among the stiffness matrices presented in this work the numerical results can be modified and the real behaviour of the systems can be captured very well. Figure 3 shows the comparison of the results to the measurement obtained with both methods.

![Figure 3. Comparison of the results of the delaminated specimen ($d = 100$ mm) to the measurement with and without the differentiation between the extensional- and bending-stiffnesses: a) Compression force - Axial displacement; b) Compression force - Transversal displacement.](image)

Related publication: [2], [9]
Bibliography


