Topology Optimization:
Deterministic and Probabilistic Problems

PhD dissertation

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Declaration of Authenticity

I declare that this dissertation is my original work, gathered and utilized especially to fulfil the purposes and objectives of this study. The work and results of other researchers, which are referred squarely in order to separate from the original work, are specifically acknowledged.

Budapest, 01.09.2018

Erika Pintér
Abstract

The subject of the dissertation is to investigate fundamental mechanical problems in the topology optimization field considering real engineering features. This research work presents the main results of my cooperation in the research project group of my supervisor Dr. János Lógó, which contributed to the intended aims of the project by additions and extensions. Performing parametric studies was a part of the task to check the reliability of the numerical calculation program.

The research work includes a new topology optimization area taking into account uncertainties in optimum structural design thus it deals with the consideration of loading uncertainties with probability variables via a fundamental optimization problem setting. Variability of loading in engineering design is realized e.g. in the action of various load combinations.

Optimization has gained widespread application because efficiency and economical design are essential requirements in modern engineering. Continuous advance can be observed in several areas and applications in the industry not only in terms of practical design but also of fundamental research. The determination of the optimal structural topology is a branch of fundamental research where the optimal topology is constructed in a design space based on given boundary conditions serving as a decision support for design.

The large number of publications with cross-references and the great past of optimization may make it difficult to find the appropriate preliminaries. In the first part of the project one of the tasks is to review the reliable publications and identify the new results. Vast majority of publications refer to Michell’s works in 1904 as the foundation of this branch of mechanics, missing the fact that the first contributions were made by Maxwell as early as 1870, of whom even Michell made reference.

The aim of the tasks has been to supplement previous incomplete topics and to provide new contributions. Based on the fundamental problem statements of topology optimization, basic features of optimal topologies have been confirmed by performing comparative parametric analysis, such as the non-uniqueness of optimum solution, which may affect the statical determinacy of the structure under well-defined boundary conditions and practical considerations.
Összefoglalás

Az értekezés tárgya a topológia optimálás témakörében, alapvető mechanikai problémák vizsgálata valós műszaki jellemzők figyelembe vétele mellett. Támogatott kutatási projekt keretében, Dr. Lógó János témavezetésével végeztem kutatómunkát, melynek eredményeivel segítettem kiegészíteni, kiterjeszteni a projekt célkitűzéseit és a numerikus számítási program megbízhatóságát ellenőrizni parametrikus feladatok vizsgálatával.

Új kutatási területet foglal magába a bizonytalanságok figyelembe vétele az optimális szerkezeti kialakítása során, így az építőmérnöki gyakorlatot alapul véve valószínűségi változókkal adott terhelés esetére levezetett feladatokkal optimális topológiák lettek meghatározva a dolgozatban.

Az optimálás igen elterjedt alkalmazás, mivel a mai világban alapvető követelmény a hatékonyság és a gazdaságosság. Számos tudományágban és az ipari alkalmazásokat illetően folyamatosan fejlődés figyelhető meg, nemcsak konkrét tervezés esetén, hanem az alapkutatás vonatkozásában is. A szerkezeti topológia meghatározása az alapkutatás egy olyan ága, melyben adott peremfeltételek mellett egy tervezési területen adott programmal kialakítható a szerkezet optimális topológiája, mely egy döntéstámogató módszer a konkrét tervezéshez.

A kutatási munka első felében a megfelelő előzmények eredményeit bemutató kiadványok megtalálása volt a cél, hogy egyértelmű kép legyen kialakítható a megbízható irodalomról. A kutatás során megállapításra került, hogy már Maxwell is foglalkozott az optimálás témakörével és nem Michell volt az első, ahogy azt a felelhető cikkek jelentős része említi.

A feladatok célja volt a korábbi kutatások hiányosságait kipótolni és új eredményekkel kiterjeszteni. A korai kutatások feltevéseiből kiindulva a topológia optimálás olyan alapvető tulajdonságai lettek alátámasztva, mint az optimális elrendezés-megoldás nem egyértelműsége és a szerkezet határozottsági foka, gondosan felírt peremfeltételek mellett, továbbá a mérnöki gyakorlat igényeit szem előtt tartva.
# Table of Contents

Acknowledgements ........................................................................................................... 1

Declaration of Authenticity ................................................................................................. 2

Abstract .............................................................................................................................. 3

Összfoglalás ......................................................................................................................... 4

Table of Contents ................................................................................................................. 1

1 Introduction ...................................................................................................................... 4

2 History of Topology Optimization .................................................................................... 5

2.1 Overview of Topology Optimization ............................................................................. 5

2.2 The Origins .................................................................................................................... 7

2.2.1 Maxwell’s Theorem on Minimum Volume Design .................................................... 7

2.2.2 Michell’s Formulation on Minimum Volume Design ................................................. 8

2.3 Basic Optimal Layout Theories .................................................................................. 10

2.3.1 Cox’s Optimal Solution .......................................................................................... 10

2.3.2 Shield’s Optimal Solution ..................................................................................... 12

2.3.3 Optimal Solution of Nagtegaal and Prager ............................................................. 14

2.4 Hungarian Aspect ........................................................................................................ 15

2.5 Scope of the Probabilistic Topology Optimization ....................................................... 18

3 Exact Optimal Topologies ............................................................................................... 22

3.1 Introduction ................................................................................................................ 22

3.2 The Optimal Layout Theory ....................................................................................... 22

3.2.1 Preliminaries ......................................................................................................... 22

3.2.2 Basic Features of the Optimal Layout Theory ......................................................... 24

3.2.3 Simple Example Using the Prager-Shield Condition ............................................. 25

3.2.4 Optimal Regions of the Adjoint Strain Field for Michell Structures .................. 26

3.2.5 Simple Example of Applying the Layout Theory to a Michell-Type Problem ....... 27

3.3 Optimal Topologies for Least/Weight Grillages ......................................................... 29

3.4 Difficulties in Deriving the Adjoint Displacement/Strain Field for O-Regions ........ 29

3.5 Extension of the Layout Theory to Elastic Design with Multiple Load Conditions .... 32
3.6 Extension of the Layout Theory to Probabilistic Design ........................................... 34
3.7 Extension of the Layout Theory to Pre-Existing Members ........................................ 36
3.8 Optimal Elastic Design for a Single Load Case ......................................................... 37
3.9 Superposition for Plastic Design ............................................................................... 40
3.10 Concluding Remarks .............................................................................................. 41

4 Basic Property Changes Affecting Optimal Topologies ............................................. 42

4.1 Introduction ............................................................................................................... 42
4.2 Preliminaries ............................................................................................................. 42
4.3 Methodology ............................................................................................................ 43
  4.3.1 Problem Formulation ............................................................................................. 44
  4.3.2 Lagrange Duality Formulation ............................................................................. 45
  4.3.3 Karush-Kuhn-Tucker Conditions ......................................................................... 45
  4.3.4 Iterative Updating Formulas ............................................................................... 46
  4.3.5 Choosing of Ground Element .............................................................................. 48
4.4 Numerical Examples ................................................................................................. 49
  4.4.1 Example 1 ............................................................................................................. 49
  4.4.2 Example 2 ............................................................................................................. 50
4.5 Conclusions .............................................................................................................. 51

5 Compliance Constraint Design with Loading Uncertainty ........................................ 53

5.1 Introduction ............................................................................................................... 53
5.2 Mathematical and Mechanical Background ............................................................. 53
5.3 Determination of the Deterministic Problems ......................................................... 54
5.4 Determination of the Stochastic Bounded Compliance Problems ........................... 56
5.5 Probabilistic Compliance Design with Uncertain Loading Magnitude .................. 57
  5.5.1 Calculation of the Probabilistic Compliance ....................................................... 57
  5.5.2 Derivation of the Optimization Problem ............................................................. 58
5.6 Probabilistic Compliance Design with Uncertain Loading Positions .................... 59
  5.6.1 Distribution of the Loads ...................................................................................... 59
  5.6.2 Adjoint Design ..................................................................................................... 61
5.7 Numerical Examples for Alternative Type Multiple Loads ..................................... 64
5.7.1 Example 1................................................................. 64
5.7.2 Example 2................................................................. 65
5.8 Conclusions.................................................................... 68

6 Statical Determinacy of Structures.................................................. 69
6.1 Introduction........................................................................... 69
6.2 Simple Example for Equivalent Determinate vs Indeterminate Structures........... 70
6.3 Minimum Volume Design of Structures According to the Optimal Layout Theory .... 72
6.4 Conclusions......................................................................... 75

7 Stress Constraint Design with Loading Uncertainty ............................. 76
7.1 Introduction........................................................................... 76
7.2 General Problem Statement.................................................... 76
7.3 Truss Optimization Problem.................................................... 78
  7.3.1 Truss Layout................................................................... 78
  7.3.2 Method of Calculation...................................................... 79
  7.3.3 Numerical Calculations.................................................... 80
  7.3.4 Analytical Calculations.................................................... 81
7.4 Truss-Like Design ............................................................... 84
7.5 Conclusions......................................................................... 86

8 Further Research ...................................................................... 88

9 Thesis Statements .................................................................... 89

Publications of the Author Related to Thesis ..................................... 92

References.................................................................................... 94

Appendix A.................................................................................. 104
Appendix B.................................................................................. 108
Appendix C.................................................................................. 111
Appendix D.................................................................................. 114
1 Introduction

This research was done in cooperation with my supervisor Dr. János Lógó in the framework of National Research Development and Innovation Office (NKFI) research grant. This dissertation presents the main results of my cooperation in this research project.

The aim of outline is to identify new and incomplete topics and the continuation and extension of earlier started researches.

The first chapter presents an overview of topology optimization history in several points of view. Theoretical works in topology optimization deterministic and probabilistic problems are a basic research with mathematical background thus the bibliography research is essential. The selection of the investigated publications cannot cover every searched field because of the huge number of periodicals and conference proceedings available. The scope covers all cases which are reliable and support to this research.

The research made a modification in the literature review about the origin of topology optimization, because widely published Michell was believed to be the first to work on it but some research papers of Maxwell are found in this topic. Furthermore, more significant publications are found from the concealed years.

Modern engineering optimization has since gained widespread application because efficiency and economical design are essential requirements. Continuous advance can be observed in several areas and applications in the industry not only in terms of practical design but also of fundamental research. The determination of the optimal structural topology is a branch of fundamental research, where the optimal topology is constructed in a design space based on given boundary conditions, serving as a decision support for actual design.

We have to mention about the topology optimization that it is a very complex computational procedure because it includes the elements of the layout optimization and the optimal cross-section design simultaneously.

Until the end of the last century almost one could not find any publication on topology optimization considering uncertainties. Exceptions to this most often use multiple load or reliability constraints. Uncertainty is typically limited to the loading, although recent works have considered extensions to support conditions and material properties.
2 History of Topology Optimization


2.1 Overview of Topology Optimization

In a broader sense the history of structural optimization has started with a cantilever design of Galilei for over half a millennium ago [57].

The topology optimization has more than one hundred years of history and is still an expanding field in optimal design and a great popular topic nowadays with numerous publications thus the project assigns to find origin fundamental works of topology optimization.

Usually the technical papers cite the work of Michell [96] from 1904 as the origin but the reality is that the first important work was presented by Maxwell [92]. Maxwell’s result was extended by Michell [96] some years later. They presented closed form solutions for the minimum volume structures. More details are presented in the next section.

According to the literature search of the theorem of the optimal design, the topology design remained unnoticed for some forty years until the papers of Foulkes [36], Cox [21] [22] [23] and Hemp [47] [48] [49], representatives of the scientific workshop at the Cambridge University. These years around 1955 can be called as the “golden ages” of the optimal layout design of trusses. More than half century ago several papers were published in this topic. Independently from the English school among others the papers written by Sved [143] and Barta [10] are one of the most significant ones. It is also important to note that the minimum weight designs of different types of structures were studied by Drucker and Shield [29], Mróz [99], Prager and Shield [106], Shield [137] and their results are significant to understand the optimality in case of complex problems.

The real expansion in layout theory was in the 60s and the 70s. In the 60s the significant publications helped to derive optimality conditions for minimum volume designs. Shield [138] presented optimum design methods for multiple loading. He used variational principles to prove the optimality. In 1960 Schmit [136] applied Full Stress Design (FSD) to statically indeterminate structures and found that FSD provides exact optimum in a single sizing operation for statically determinate structures where the internal forces remain constant.
during resizing but for indeterminate structures the number of resizing iterations can vary from a few to many as a function of the sensitivity of internal forces to changes in member sizes. However, Lansing [75] applied FSD to statically indeterminate structures and obtained good results. The reason why these different results were obtained is known now: it mainly depends on the redundancy of structures or in other words, it depends on whether the internal forces remain almost constant during resizing as it happens in most well designed practical structures. In 1966 Gellatly and Gallagher [43] suggested that FSD should be used to create a “starting point” of nonlinear programming methods. Furthermore, some important remarks of FSD were reported by Gallagher [39] who pointed out that FSD were inadequate for minimum weight design. Berke and Khot [17] concluded that minimum weight design should be at the point including “fully stressed elements, lower bound elements, and neither of them”. During these years Cox [21] [22] [23], Hemp [47] [48] [49], Prager and co-workers [46], [106] elaborated several theories which can be named as the origins of the exact structural topologies. Nagtegaal and Prager [100] investigated the optimal truss layout theory in case of alternative loads. Prager [107] derived an optimality condition for beams and frames subjected to alternating loading by the use of Foulkes mechanism. His results are based on the extension of the optimality conditions presented by Chan [20]. Prager and Rozvany [109] extended the existing optimal layout theory, originally used for low volume fraction, for grid-like structures (trusses, grillages, shell-grids, etc.). The method was validated for the case of not restricting to low volume fraction structures. Rozvany, Olhoff and Bendsoe et al. [117] provided solutions on exact optimal topologies of perforated plates. Achtziger’s papers [1] [2] [3] [4] [5] present significant achievements in the field of truss designs in the last decades. The three-bar truss example was also investigated by Sokol and Lewinski [141].

The most important milestones in deterministic layout should also be reviewed. The origins of the numerical solution technique of the constrained optimality criteria methods (COC) were published by Berke and Khot [17] in 1974. This provided the mathematical background of an effective solution technique in topology optimization. The first numerical procedure for FE (finite element) based topology optimization was elaborated by Rossow and Taylor [112] in 1973, but the real expansion started at the end of 80s with Bensoe, Sigmund [14] and Rozvany [123]. During these years several optimal topologies were numerically calculated but the analytical confirmations – which have come recently – are mostly missing.
Dunning et al. [31] [32] introduced an efficient and accurate approach to robust structural topology optimization. The objective is to minimize expected compliance with uncertainty in loading magnitude and applied direction where uncertainties are assumed to be normally distributed and statistically independent. This approach is analogous to a multiple load case problem where load cases and weights are derived analytically to accurately and efficiently compute the expected compliance and sensitivities.

The layout theory plays primary importance in structural optimization. The main difficulties are concerned with whether the obtained solution is unique or not, the extremal point is a local or global one. This question is more difficult when several loading cases are taken into consideration. During the 50s–70s of the past century, several papers were published to investigate the questions above. Generally, the variational calculus was the tool to prove the optimality and the uniqueness. Here a limited overview is presented by use of the papers of Nagtegaal and Prager [100] and of Shield [139]. The overview of these works is based on Hemp [47] presentation in 1958.

The “modern” topology optimization of continuum type structures was first published independently by Rozvany and Zhou [153], [154], and Bendsoe and Kikuchi [12] at the end of the eighties. One can see that topology optimization as a research field has achieved significant results, especially in the last two decades.

2.2 The Origins

2.2.1 Maxwell’s Theorem on Minimum Volume Design

The topology design historically started with the problem class of layout optimization of trusses and the works were called ‘minimum volume design of frames’ where the term ‘frame’ was historically used for what we now call a truss. The first important achievement in truss optimization was made by Maxwell [92], which deserves presentation here not only due to its significance but also because it is still unknown to many in the field. Maxwell considered the problem of attracting and repulsive forces between points set in the plane and proved a theorem regarding the equilibrium of the force system: “In any system of points in equilibrium in a plane under the action of repulsions and attractions, the sum of the products of each attraction multiplied by the distance of the points between which it acts, is equal to the sum of the products of the repulsions multiplied each by the distance of the points between which
it acts.” This statement can be formulated as \( \sum_t T_t L_t - \sum_c T_c L_c = 0 \) where \( T \) and \( L \) denote the force and the distance between two points, respectively, and indices \( t \) and \( c \) refer to tension (attraction) and compression (repulsion), respectively. For the derivation of this statement the principle of virtual work was applied with a uniform virtual deformation field.

Maxwell commented upon scientific significance of his theorem by the use of the following words: “The importance of the theorem to the engineer arises from the circumstance that the strength of a piece is in general proportional to its section, so that if the strength of each piece is proportional to the stress which it has to bear, its weight will be proportional to the product of stress multiplied by the length of the piece. Hence these sums of products give an estimate of the total quantity of material which must be used in sustaining tension and pressure respectively.” We have to notice that Maxwell uses the word “stress” for what we should term “load”. His result or comment has drawn the practical conclusion about the required weight of the truss.

The Maxwell’s problem can be described as follows: consider a truss which maintains equilibrium with a set of forces \( \vec{F}_i \) acting at the points \( \vec{r}_i, \) \( (i = 1, 2, ..., n) \). Denote by \( T_t \) the load carried in a typical tension member with length \( L_t \) and the section area \( A_t \) while in the case of typical compression members these variables are \( T_c, L_c \) and \( A_c \). The permissible stresses were denoted by \( f_t \) and \( f_c \), respectively. By the use of the principle of virtual work Maxwell derived the optimality condition of the lightest structure, which has the volume given by \( V = V_c \left( 1 + \frac{f_c}{f_t} \right) + \frac{1}{f_t} \sum i \vec{F}_i \vec{r}_i = V_t \left( 1 + \frac{f_t}{f_c} \right) - \frac{1}{f_c} \sum i \vec{F}_i \vec{r}_i \). Here \( V_t \) is the volume of all the tension members and \( V_c \) is the volume of all the compression members.

2.2.2 Michell’s Formulation on Minimum Volume Design

In contradiction to the generally believed knowledge Michell [96] generalized the Maxwell’s theorem and did not invent the theory of topology optimization. He recognized the importance of Maxwell’s result and applied to the calculation of the optimum structural weight. This led him to sufficient conditions for a structure to be an optimum. Michell proved the geometric restriction which determines the classes of orthogonal sets of curves along which the members of an optimum structure must lie.

The Michell problem can be described as follows: in addition to Maxwell’s problem above, let us consider a series of external forces \( \vec{F}_i \) acting at the points \( \vec{r}_i, \) \( (i = 1, 2, ..., n) \). Let \( D \) be a
domain of space containing the points \( r_i \). It should be noted that \( D \) can be the whole of feasible space. Consider then all possible frameworks (trusses) \( S \), contained in \( D \), which equilibrate the forces \( F_i \) and which satisfy the limiting conditions on stresses. Let us assume that there is a framework \( S^* \) which satisfies the following condition of Michell: “There exists a virtual deformation of the domain \( D \) such that the strain along all members of \( S^* \) is equal to \( \pm e \), where \( e \) is a small positive number, and where the sign agrees with the sign of the end load carried by the particular member, and further that no linear element of \( D \) has strain numerically greater than \( e \).” Michell’s theorem states that the volume \( V^* \) of \( S^* \) is less than or equal to the volume \( V \) of any of the frameworks \( S 
abla = \frac{(f_t+f_c)}{2f_{fc}} \left( \sum_t L_t T_t + \sum_c L_c T_c \right) - \frac{(f_t-f_c)}{2f_{fc}} \sum_i F_i r_i.
(2.1)

The actual value of \( V^* \) follows from the principle of virtual work. If the virtual displacements corresponding to Michell’s statement above are \( e \bar{v}_i \) at \( r_i \) this volume \( V^* \) is:

\[
V^* = \frac{(f_t+f_c)}{2f_{fc}} \sum_i F_i \bar{v}_i - \frac{(f_t-f_c)}{2f_{fc}} \sum_i F_i r_i.
(2.2)
\]

The character of the deformation \( e \) imposes certain restrictions upon the layout of members in \( S^* \). At a node of this framework the directions of the strains \( \pm e \), which are along the lines of members of \( S^* \) and are principal directions of strain and must satisfy certain orthogonality conditions. In a three-dimensional truss, at a node with three members, there are no restrictions, if the load in members have the same sign, since in that case the virtual deformation is a pure dilatation and therefore isotropic. If one load is of opposite sign to the others, it must be at right angles to them. For a node with four members, there is again no restriction if all the loads have the same sign. If one member has an opposite load to the other three, then it must be orthogonal to them all and so forces them to lie in a plane. Finally, if the members fall into pairs with opposite-signed loads then one of these pairs must be in line and normal to the other two.

Michell also presented a very important property of the optimal structure, namely, the optimum structure \( S^* \) has greater stiffness than any other structures of \( S \). He also presented the value of the strain energy stored in the optimal structure.

Michell’s key invention is the so-called Michell structure consisting of members aligned with the principal axes of stresses in accordance with the optimality condition mentioned above.
Michell’s results did not induce further advance in this field at that time and it was not until the 1950s that works on minimum volume design restarted at various schools. Within the diverse literature, we only focus on achievements regarding trusses and truss-like structures, which relate to the topic of this study. One of those works is made by Sved [143], which dealt with frictionless joint truss design and suggested a method for determining the minimum weight structures belonging to a specific configuration, but he dealt with the case of a single fixed load only. Sved [143] showed that the minimum weight structure is always statically determinate. A similar theorem but of more general validity was proven by Barta [10]. The two works have a lot of similarity in both contents and conclusions. In Barta’s paper the minimum volume design of plane and space structures (trusses) were discussed. He proved the following theorem: by removing a given number of properly chosen redundant bars from a given network, it is possible to obtain such a statically determinate structure, which yields a structure with the least weight. It has to be noted that this conclusion was stated for single and deterministic loads, as well. Barta also concluded that the proof did not guarantee that only statically determinate structure could be the least weight solution.

It is noted that the original formulas of Michell are not valid for different allowable stresses in tension and compression. In 1960 Chan [19] wrote down correctly the Michell’s Theorem and the validity and the critical examination of the Michell’s theory can be read in Rozvany’s work [123] almost 40 years later.

2.3 Basic Optimal Layout Theories

2.3.1 Cox’s Optimal Solution

Applications of the theorems of Maxwell and Michell to simple design problems have been made by Cox [22]. He has considered first of all the problem of three coplanar forces. In the case where their point of intersection lies within the triangle formed by their points of application, the optimum framework can consist of tension or compression members only. Some of his layouts are given in Figure 1 and we have to note that all these structures have equal weight. One can recognise the non-uniqueness of the optimal layout and we can have an infinity of solutions ranging from mechanisms through simple stiff structures to structures of any degree of redundancy.
Cox extended the theory presented above to build up a structure for the transmission of a bending moment. He showed that if the proportion of length over height of the structure is greater than 4, this structure is considerably lighter than a “simple tie and strut” and that for larger values of length/height multiple constructions, on the lines of Figure 2–Figure 4 can be even lighter. He produced a competitive 14-bar framework and a variation of Figure 2 in which the circles are replaced by spirals which for length/height > 4 is lighter than any other construction considered. These structures for the transmission of bending moments are not Michell’s optimum structures since they fail to satisfy the orthogonality conditions for members with opposite signed loads.

The derivation of the optimal layout of pure bending and the optimal solution of Cox’s beam problem in Figure 2 was presented by Chan [19] (see Figure 3) while the solution of Shield [139] for the same problem can be seen in Figure 4.

Figure 1: Simple tension structures by Cox

Figure 2: Cox’s optimal beam for bending
2.3.2 Shield’s Optimal Solution

As it was indicated in the previous section by Cox, the optimal solutions are not unique in the case of Michell’s structures. By the use of variational calculus, Shield [139] presented appropriate necessary conditions for the structural volume to be stationary but he noted that there is no guarantee to get the global optimum. This section is based on Shield’s original work and below the outline of the original paper is presented.

Shield investigated the so-called Michell’s structures and declared that the Michell type design fails when kinematic constraints are taken into consideration. He presented an alternative approach which does not have the limitation of the Michell method. The procedure is based on the idea to design a frame compatible with a reduced virtual small deformation in which the principal strains are of magnitude $e$/limit-tensional-stress (in case of tensioned members) and $e$/limit-compressive-stress, the directions of frame elements coinciding with the principal $c$ directions of strain as before. Here $e$ is the virtual deformation indicated by Michell. The virtual deformation must satisfy any imposed kinematic constraints.
In the following some examples are presented that Shield gave in [139] to show that minimum-volume frames are not necessarily unique, and he described some new additions to the list of Michell’s structures. The diagram at the top of Figure 5 indicates the layout given by Michell [96] for a single force applied at the midpoint $C$ of the line $AB$ and balanced by equal parallel forces at $A$ and $B$.

The struts $AD$, $EB$ and the curved bar $DE$ carry a uniform compressive force and a quadrantal fan of tie-bars from $C$ to $DE$ maintains the equilibrium of the curved bar. The layout is symmetrical about $AB$ with tie-bars replacing struts and vice-versa. The virtual deformation with principal strains $\pm e$ associated with the layout can be adjusted so that the displacement is zero at points $A$ and $B$. If one assumes the equal tensional and compressive limit stress condition one can use this virtual deformation for the case when one have the same force at $C$ but now $A$ and $B$ are fixed points of support. The optimum structure has the same volume as the structure with specified parallel forces at $A$, $B$ but the optimum design is not unique.

For example, the load at $C$ can be carried by a frame entirely above $AB$ as indicated in the middle diagram of Figure 5. An infinity of optimum designs results from arbitrarily assigning a fraction of the load at $C$ to be carried by a structure above the line $AB$ and the remainder by a structure below the line $AB$. It was noted that if one had specified that the load at $C$ be carried by a beam with centreline $AB$ and built-in at $A$ and $B$, the optimum design would have bending moments at $A$ and $B$. The Michell’s structure has no moments at the fixed points $A$, $B$. The minimum-volume design indicated at the bottom of Figure 5 uses the same virtual deformation with principal strains $\pm e$ but here it is specified that distributed loads at $A$ and $B$ balance the load at $C$.

![Figure 5: Examples of non-uniqueness by Shield](image)

It has to be noted that Figure 5 is presented with the intention to illustrate the non-uniqueness of design. However, this figure also presents the dependence of design on support conditions: the top diagram sliding support direction (vertical reaction); the middle diagram hinge support (reaction following member direction); the bottom diagram distributed sliding support. In
addition to it one can read about the uniqueness theorem related to Michell’s structure design in the paper of Kozlowski and Mróz [74] where the authors presented the use of Michell’s structures to disc design with thickness constraints.

Continuum modelling as a new branch of research in topology optimization became into focus in the 1970s by the advance of computational tools, see e.g. Rossow and Taylor [112]. The modern continuum type optimization is derived from the works of Bendsoe and Kikuchi [12], and later in the 1990’s the previous optimality conditions (OC) algorithm has become well known by the acronym SIMP.

2.3.3 Optimal Solution of Nagtegaal and Prager

In optimization the investigation of the convergence of the applied procedure and the uniqueness of the solution are of primary importance. This question is more difficult whenever there are multiple loadings, and/or the loading uncertainty is considered. In this latter case, the load can be considered as a quantity given in an interval with a certain possibility of location or/and direction, and/or magnitude. The design loads are usually selected on the basis of worst loading cases.

The main achievements of the paper of Nagtegaal and Prager [100] are discussed briefly at first. The starting point is a minimum volume design of a truss. The results of this layout optimization, coming from this uniaxial case, can be generalized and a continuum type topology optimization problem with bi-axial stress state is investigated later. The paper of Nagtegaal and Prager is concerned with the following problem: two alternative loads with the same point of application are to be transmitted to a rigid foundation by a plane truss of minimum weight whose load factors for plastic collapse under one or the other load are not to exceed a given value. A necessary and sufficient condition for global optimality is established and used to determine the optimal layout of the truss. According to their optimality criteria (global optimality condition) in an active truss member (being non-zero cross-section) the sum of the normalized strain rates is unit while in vanishing members the summation of the strain rates results in a smaller value than unit. In addition to the optimality conditions Nagtegaal and Prager gave a short overview how the optimal layout looks like in a special case of alternative loads (see Figure 6). They considered a fixed force (say $P'$) and discussed the optimal types of trusses for all possible other force ($P''$). In Figure 6 “$A$” is the common application point of these two forces. Here vector $AB'$ represents the fix force $P'$. 
The lines $B'C$ and $B'D$ form angles of $45^\circ$ with the horizontal direction and lines $EF$ and $EG$ are obtained by mirroring the lines $B'C$ and $B'D$ with respect to point “$A$”.

These lines divide the plane of the figure into nine regions. If the load $P''$ is represented by the vector $AB''$, the label of the region that contains $B''$ indicates the type of the optimal truss. Where one of the loads is dominant, that is, exclusively determines the optimal design, the optimal truss consists of two bars that form angles of $45^\circ$ with the wall.

![Figure 6: Truss type as function of the alternative loading](image)

The optimal layout problem of a minimum weight truss design problem with a single vertical force load was presented by Save [135] in the case of stress constraints. Singular situations in minimum-volume elastic design are analysed and illustrated as they occur in optimization of a three-bar truss. Relations between minimum-volume design with bounds on stress intensity, assigned load factor at collapse, and assigned elastic compliance are analysed. The conclusions of his results stated that the optimal layout can be one, two and three bar trusses depending on the design conditions.

These general layout theories can act as reference studies for the case of topology optimization of truss-like structures, in the case multiple and/or stochastic loading. It is to note that in the case of trusses the uniaxial stress state is considered, but the truss-like structures belong to the biaxial stress state problems.

2.4 Hungarian Aspect

On several field of topology optimization Hungarian researchers have worked from the beginning continuously as well with great results.
In this study we have to mention the works of Kazinczy – the inventor of the plastic hinge theory [71] in 1914 – who made significant results in this field but unfortunately these publications have remained hidden because of written in Hungarian. The volume minimization of trusses as an object of the economical design was investigated by him. Kazinczy was also among the first researchers who investigated the problem of the statically indeterminate trusses in case of multiple load conditions [72]. Using the Cremona-type solution procedure, he investigated the case of the pre-stressing technique to reach the uniform collapse of the member forces in the case of statically indeterminate structures. With this technique he used the shakedown theory without naming it, and much earlier than Melan [94] published it in 1936. Kazinczy also discussed the questions of safety and reliability designs much earlier than anybody else in the world.

From the early stage of topology optimization history, in the 1950s Sved and Barta, who are Hungarians, worked on to find the minimum weight of bars in case of single load system and statically determinate or indeterminate structures. Above detailed publication of Sved, who worked abroad was a Hungarian scientist as was John Taylor, who first researched using finite elements in topology optimization, but without any penalization for density.

László Berke was also a Hungarian researcher but abroad published the constrained optimality criteria methods (COC) with Knot in 1974.

A research group (with members e.g. Kaliszky, Gáspár, Rozvany, Vásárhelyi, Lógó, Pomezanski, Ghaemi, Movahedi Rad) was organized at the Budapest University of Technology and Economics around the turn of the millenium with the project to investigate in the field of topology optimization, e.g. development of numerical problems, further development and extension of basic theories with recognized results. This research has been going on since then including the work on the present dissertation.

An essential basis for calculation of exceptional and seismic load cases is the work of Kaliszky on computational methods based on plasticity [55]-[59], [62], [63] and plastic optimality design [60], [61]. Plasticity research of Kaliszky has been developed with elastic, elastic-plastic and plastic optimal design on statically or dynamically loaded elastic-plastic material structures e.g. [64]-[68], [81]-[84], [44]. Doctoral dissertation of Lógó [88] deals with optimal design in case of deterministic input data. This work contains enhanced cases with stochastic
This research is not independent from previous works because of further development.

Rozvany’s work is on the analytical topology optimization from the 1970s for instance with development of SIMP method [109], [117], [118], [153], [154]. Rozvany published several comprehensive papers to summarize developments [123], [132].

One of the research fields on topology optimization in Hungarian aspect is the numerical problems. In the development of structural topology optimization procedure internationally acknowledged researcher G. Rozvany worked with Gáspár, Lógó, Pomezanski on the so-called checkerboard problems and published significant results [128]. The discretization error is a calculation problem which can result in the checkerboard topology. Based on research of Gáspár, Pomezanski, Rozvany and Querin [125], [126] on numerical problems, Gáspár [127] has proven that the exact theoretical strain energy of the checkerboard pattern converges to infinity. Procedure CO-SIMP penalizes the checkerboard pattern to correct the discretization error [105].

Serviceable research in the industry field is made by Farkas and Jármait [33]-[35] for instance via complex non-linear functions optimization in case of steel stiffening elements, plates and shell structures for minimizing the cost or weight including the costs of manufacturing, assembly (e.g. welding) and post-production (e.g. painting). In the mechanical and energy engineering the recent researches of Szabó [144], [145] and [146] about the optimization of hydrodynamic journal bearings using Random Virus Algorithm (RVA) could decrease the lubricant costs significantly therefore have very important environmental protection advantages. Further investigation would be the multidisciplinary optimization (MDO) of structures i.e. gear tooth geometry for higher safety against seizure. Moreover, using topology optimization groups, results would be compared and analysed by proposed evolutionary based system for qualification and evaluation of Szabó [147].

Recently new research direction in the robust optimization with probabilistic loading directions using automatic symbolic computation is investigated by the work of Csébfalvi, e.g. [24], [25], [26], [27], [28].

Further to topology optimization, optimization is an important design aspect in various fields, e.g. in discrete geometry of polyhedra or spatial structures, etc [42], [76], [148], [149].
2.5 Scope of the Probabilistic Topology Optimization

The solution to topology optimization problems poses significant technical challenges in spite of a clear concept. Problems are typically large-scale and discrete, and often exhibit some numerical difficulty associated with underlying mechanics (such as instability of members, checkerboard patterns in continua). For these reasons, the majority of topology optimization research has focused on deterministic design problems, neglecting the uncertainty that arises in most engineering applications.

Until the end of the last century one could hardly find any publications on topology optimization which considered uncertainties. Exceptions to this most often use multiple load or reliability constraints. Uncertainty is typically limited to the loading, although recent works have considered extensions to support conditions and material properties. We address the review of the fundamental types of uncertainty in this paper to allow optimization problems to more closely resemble real-world engineering design problems.

The first type is where the structure is assumed to be built precisely as designed (no uncertainty in geometry) and the load is uncertain. The uncertainties in loads are often the most significant types of uncertainties for structural systems. The loading uncertainties can be handled as the magnitude, line of action, or the point of application of the loads. The second type is where the load is deterministic and the nodal locations that are used to define the geometry of the structure are uncertain. The load uncertainty problem for a finite number of load patterns with discrete probabilities has been analysed previously using a slightly expanded form of the optimization problem. After this form of the load uncertainty problem is briefly reviewed herein, it is shown how it can be extended to include loads characterised by continuous joint probability densities. The nodal uncertainty problem is significantly more complex because it involves randomness in the inverse of the stiffness matrix. To make the results analytically tractable, it is assumed that the uncertainties in the nodal locations are small relative to the length scale of the structural elements. An important application of such an uncertainty model is in representing fabrication errors. In most structural problems, this class of small uncertainties has negligible influence on design. For structures containing thin substructures under axial loads, however, it is shown herein that these uncertainties can have significant effects, particularly when members with collinear axial load under perfect manufacturing conditions are misaligned as a result of manufacturing error. The third type of
group is rather a computational strategy which is proposed for robust structural topology optimization in the presence of uncertainties with known second order statistics. The strategy combines deterministic topology optimization techniques with a perturbation method for the quantification of uncertainties associated with structural stiffness, such as uncertain material properties and, or structure geometry. The use of perturbation transforms the problem of topology optimization under uncertainty to an augmented deterministic topology optimization problem. This in turn leads to significant computational savings when compared with simulation-based (Monte Carlo-based) optimization algorithms which involve multiple formations and inversions of the global stiffness matrix.

The mechanical background of the topology design was presented by Bendsoe and Sigmund [14]. The optimality criteria-based methods were introduced by Rozvany [118], [123]. Last publications by Rozvany [132], [133] give a general view of the past, present and future of deterministic topology design, including the questions of symmetry and non-uniqueness in exact topology optimization. Sokol and Lewinski [141] provide analytical solutions to prove the correctness of the numerically obtained optimal topologies. As a consequence of the achievements of the great number of deterministic topological investigations about a decade ago, something has emerged from the field of probability-based topology optimization. Researchers have used the advantages that reliability-based design optimization (RBDO) and stochastic optimization provided and have achieved significant results in all fields of engineering and mathematics during the last fifty years. During the last twenty years several books have been published where the general reliability theorems and numerical procedures have been discussed and presented – amongst which the most significant works (books, edited volumes, proceedings) were published by Califore and Dabbene [18], Frangopol et al. [37], [38], Jendo and Dolinski [54], Melchers [95], Rackwitz [111] and Thoft-Christensen [150]. The mathematical background of the probabilistic optimization methods has been presented by Kall [69], Marti [91], and Prékopa [110], amongst others. A very useful work by Aoues and Chateauneuf [7] provides a benchmark study of selected numerical methods of RBDO - the single loop approach, the two-level approach, and the decoupled approach.

What was missing was the work that would integrate these achievements into topology optimization. At that time works by Ben-Tal and Nemirovski [15], [16] had provided a strong foundation for probability-based topology optimization, almost a decade earlier than the topic
became “popular”. Before the investigation of the publications in the field of probabilistic topology optimization, the general achievements of the reliability-based optimization are discussed with some limitation.

During the last years before the end of the millennium almost there were no publications in the topic of probability-based topology optimization. The stochastic optimization works of Marti and Stöckl [89], [90] provide early information about this topic. The paper of Duan et al. [30] is among the very first publications in the field of uncertainty-based topology optimization. This work presents an entropy-based topological optimization method for truss structures by the use of iteration technique. Dunning et al. [31], [32] introduce an efficient and accurate approach to robust structural topology optimization. The objective is to minimize expected compliance with uncertainty in loading magnitude and applied direction, where uncertainties are assumed normally distributed and statistically independent. This approach is analogous to a multiple load case problem where load cases and weights are derived analytically to accurately and efficiently compute expected compliance and sensitivities. Illustrative examples using a level-set-based topology optimization method are then used to demonstrate the proposed approach.

Topology optimization with uncertainty in the magnitude and locations of the applied loads and with small uncertainty in the locations of the structural nodes is the object of the paper of Guest and Igusa [45]. Their method is based on the assumption that the loading uncertainties are taken into consideration as “safety factors” of the deterministic load cases in the load combination. The effects of geometric uncertainty were estimated using second order stochastic perturbation and uncertainties in the stiffness of the structure were transformed into a mathematically equivalent system of auxiliary loads. This technique is extended for nonlinear effects of global instability [51] and material property uncertainties [8], to put more control on the variability of the final design via including variance of the compliance [9]. Asadpoure et al. [9] present a computational strategy that combines deterministic topology optimization techniques with a perturbation method for the quantification of uncertainties associated with structural stiffness, such as uncertain material properties and/or structure geometry. The applied technique leads to significant computational savings when compared with Monte Carlo-based optimization algorithms. Jalalpour et al. [51] extend the perturbation-based topology optimization procedure [45] to
approximate the effect of random geometric imperfections on the second order response of trusses. Monte Carlo simulation together with second-order elastic analysis is used to verify that solutions offer improved performance in the presence of geometric uncertainties.

Lógó [83] and Lógó et al. [82], [85] elaborated a rather powerful method for the stochastic topology optimization where the magnitude of the loads or the compliance bounds are given by their mean values, covariances and distribution functions. By the use of direct integration technique for the calculation of the uncertain bounds or applying an appropriate approximation for the loading uncertainties the stochastic expressions are substituted by equivalent deterministic ones to make the optimization problem robust. The loading positions as uncertain data in the topology optimization problem is considered in [86]. In [86] two computational models and the corresponding algorithms are elaborated. Both models use simple transformations to substitute the original load position problem with uncertain loading magnitude ones. This work is a continuation of the research of above cited papers.
3 Exact Optimal Topologies

Related research papers: [EP10], [EP9], [EP7]

3.1 Introduction

The optimal layout design of trusses was developed in the 1950s which can be called the “golden ages” as it was shown in detail in Section 2.1. The real expansion in layout theory started in the 1960s when significant publications helped to derive optimality conditions for minimum volume designs, e.g. works by Hemp, Cox, Shield or Prager. Then in the 1970s the analytical research was proceeding and this chapter essentially falls in line with it.

First an overview of generalizations of truss topology optimization is given via the Prager-Rozvany optimal layout theory [108] to multiple load conditions, probabilistic design and optimization with pre-existing members, also briefly reviewing optimal grillage theory and cognitive processes in deriving exact optimal topologies. Secondly, the optimal regions and difficulties with so-called O-regions are discussed in detail.

3.2 The Optimal Layout Theory

3.2.1 Preliminaries

It has been shown (e.g. Drucker [29]) that for elastic-ideal plastic structures a lower bound on the collapse load is given by any statically admissible stress field, which nowhere violates the yield condition. “Statically admissible” means that a stress field satisfies static boundary conditions and equilibrium. A design based on the above “lower bound theorem” is termed "plastic lower bound design". It need not take kinematic conditions into consideration.

In the problem in Figure 7 we have a beam with equal positive and negative yield moments which are constant over the entire beam length. The moment diagram shown in thick line is statically admissible but kinematically inadmissible for a linearly elastic beam violating kinematic boundary conditions. Yet in "plastic lower bound design" it can be used because the bending moments nowhere exceed the yield moments.

Sved [143] has shown that optimal (least-weight) elastic trusses under a stress constraint and a single load condition are always "statically determinate" (or a convex combination of statically determinate solutions). This term means that (i) there are no "redundant" members
in the structure (a "redundant member" can be removed without making the structure unstable), (ii) its internal forces can be computed on the basis of static equations only, ignoring kinematic (compatibility) requirements, and therefore (iii) the internal forces do not depend on member sizes.

![Diagram showing a beam with reactions and bending moments](image)

*Figure 7: Example of using the lower bound theorem of plastic design*

Barta [10] extended Sved’s theorem on statical determinacy to trusses with local buckling and McKeown [93] to any combination of stress and displacement constraints under a single load. Pedersen [102] obtained more rigorous proofs of Barta’s [10] theorem and extended the statical determinacy property to trusses with variable support conditions [103], [104]. The equivalent of Sved’s theorem was also proved rigorously by Achtziger [2] in a comprehensive review on optimization of discrete structures. According to Wasiutynski and Brandt [152] the first proof of statical determinacy of single-load trusses, expressed as the ‘theorem on the non-existence of statically indeterminate lattices of uniform strength’ (meaning fully stressed trusses), is due to Lévy [77].

Since the optimal solution for certain classes of redundant structures (e.g. trusses, grillages, rigid frames) with one load condition is statically determinate, we can enlarge our feasible set for optimal elastic design (with static and kinematic admissibility) to include all statically admissible solutions (also those violating kinematic admissibility). This is because by the above theorems the final solution will be statically determinate, and that automatically satisfies elastic compatibility. In other words, the optimal solution for the enlarged feasible set will always be contained in the original, smaller feasible set. This means that optimal elastic design of certain structures with one load condition reduces to the optimal plastic design of these structures. Moreover, a stress-based design of a statically determinate truss can only be optimal, if all members develop the permissible stress.
3.2.2 Basic Features of the Optimal Layout Theory

Rozvany’s first book [114] deals with optimal layout theory for flexural systems (e.g. grillages, shells) and formulated in a more concise form later [109]. It is based on the Prager-Shield [106] optimality condition for plastic design. However, the new element in the layout theory is, that it also gives optimality conditions for “vanishing” members (of zero cross-sectional area) in terms of “adjoint” strains along these members. In other words, optimal layout theory starts off with a ground structure or structural universe of all potential members and selects the optimal members (of non-zero cross-sectional area) out of those.

Using either one of the above theories, we need to find

- a kinematically admissible ‘adjoint’ displacement/strain field (satisfying kinematic continuity and boundary conditions),
- a statically admissible stress field (satisfying equilibrium and statical boundary conditions),
- such that certain (‘static-kinematic’) optimality criteria are also fulfilled.

In optimality criteria the adjoint strains are given by the subgradient of the specific cost function with respect to stresses or stress resultants (member forces F or beam moments M).

The subgradient of a function is the usual gradient but at discontinuities of the gradient any convex combination of the adjacent gradients can be taken.

For sign-independent stress-based design of trusses and grillages of given depth for example the specific cost functions, representing cross sectional areas A, are

\[ A = k|F| \text{ and } A = k|M| \]  \hspace{1cm} (3.1)

where \( k \) is a constant, \( F \) is a member force and \( M \) is a bending moment.

Then, e.g. for trusses, the optimality conditions reduce to those of Michell [96]

\[ \bar{\varepsilon} = k \, \text{sgn} F \text{ (for } F \neq 0) , \, |\bar{\varepsilon}| \leq k \text{ (for } F = 0) \] \hspace{1cm} (3.2)

where \( \bar{\varepsilon} \) is the adjoint strain.
Figure 8: Examples of specific cost functions and the corresponding adjoint strains

For grillages the optimality conditions are

$$\bar{\kappa} = k \text{sgn} M \text{ (for } M \neq 0)$$

$$|\bar{\kappa}| \leq k \text{ (for } M = 0)$$

(3.3)

where $\bar{\kappa} = -\bar{u}''$ is the adjoint beam curvature and $\bar{u}$ is the adjoint beam deflection.

The specific cost function and adjoint strain values for Michell trusses are shown in Figure 8a.

These relations are extended to trusses with a prescribed minimum cross section and respectively, equal and unequal permissible stresses in tension and compression in Figure 8b and c.

3.2.3 Simple Example Using the Prager-Shield Condition

Consider a clamped beam of constant depth and variable width with a central point load (Figure 9a). Then by Equation (3.3) the adjoint curvatures (i.e. second derivatives of the adjoint beam deflections) for positive and negative moments respectively, are $\bar{\kappa} = k$ and $\bar{\kappa} = -k$.

For any system of downward forces on the beam the sign of the moment diagram ($M$) may only change at two places. The simplest of such loads, a single point load is considered in Figure 9. For two negative and one positive segments of the moment diagram one gets the adjoint curvatures in Figure 9b. For a simple load this gives the moment diagram in Figure 9c but the zero moment points would be the same for most downward loads.
Figure 9: Example of applying the Prager-Shield optimality condition (problem of Heyman, 1959)

The general feature of earlier applications of the Prager-Shield condition was that no members or components disappeared from the structure. This condition was often applied to plates or shells with several stress components (see [114]). In layout optimization however, most of the original members vanish from the ground structure but optimality criteria (usually inequalities) must be satisfied along vanishing members also.

3.2.4 Optimal Regions of the Adjoint Strain Field for Michell Structures

It follows from Equation (3.2) that in Michell trusses the members are in the direction of the adjoint principal strains of a constant magnitude \( k \) and the adjoint strains may nowhere exceed this value. The optimal topology usually consists of several “regions”. Denoting the principal adjoint strains by \((\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)\) at any point of a 2D truss for equal permissible stresses in tension and compression one may have a

- **T**-region with a tensile and a compression member at right angles, \( \tilde{\varepsilon}_1 = -\tilde{\varepsilon}_2 = k \),

- **S**-region with members having forces of the same sign in any direction, \( \tilde{\varepsilon}_1 = \tilde{\varepsilon}_2, |\tilde{\varepsilon}_i| = k \) \( (i = 1, 2) \),

- **R**-regions with only one member at any point, \( |\tilde{\varepsilon}_1| = k, |\tilde{\varepsilon}_2| \leq k \) or

- **O**-region with no members \( |\tilde{\varepsilon}_1| \leq k, |\tilde{\varepsilon}_2| \leq k \) with \( k = 1/\sigma_p \), where \( \sigma_p \) is the permissible stress in both tension and compression.

The symbols used for optimal regions are shown in Figure 10a.

For unequal permissible stresses in tension and compression, one has
for $T$-regions $\bar{\varepsilon}_1 = k_T, \bar{\varepsilon}_2 = -k_C$,

for $R$-regions $\bar{\varepsilon}_1 = k_T$ or $\bar{\varepsilon}_1 = -k_C$ and $k_T \geq \bar{\varepsilon}_2 \geq -k_C$ with $1/\sigma_C = k_C, 1/\sigma_T = k_T$.

Depending on the sign of the forces $S$ and $R$ regions may be further subdivided into $S^+, S^-, R^+$ or $R^-$ regions.

Note that most of the Michell literature deals with $T$-regions because of the link with slip-lines in plasticity (Hencky-Prandtl nets) which was developed earlier.

Optimal regions were also derived for 2D perforated and composite continua by Rozvany, Olhoff, Bendsoe, Ong and Szeto [117], for a review see Rozvany, Bendsoe and Kirsch [122] or a book by Rozvany [118]. The optimal regions for 3D perforated continua were presented in an outstanding paper by Olhoff, Ronholt and Scheel [101].

3.2.5 Simple Example of Applying the Layout Theory to a Michell-Type Problem

Figure 10b shows some optimal topologies and Figure 10c the corresponding optimal adjoint displacement fields. There are two line supports at right angles. A normalized formulation is used with $k = 1$.

Elementary calculations show that for the adjoint displacements $\bar{u}$ and $\bar{v}$ in the upper region we have the principal adjoint strains

$$\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = 1$$

at $\pm 45^\circ$ to the horizontal. This corresponds to a $T$-region, denoted by an arrow cross in Figure 10c. In the lower region one has

$$\bar{\varepsilon}_1 = 0, \quad \bar{\varepsilon}_2 = -1$$

which signifies an $R^-$ region denoted by a double-arrow. Both satisfy the optimality conditions in Equation (3.2).
Figure 10: (a) Symbols used for optimal regions. (b) and (c) Example of application of optimal layout theory to Michell trusses

It can be readily seen that along the line supports with $x = 0$ and $y = 0$ one has $\bar{u} = \bar{v} = 0$ satisfying the kinematic boundary conditions. Moreover, along the region boundary with $y = -2x$ both adjoint displacement fields in Figure 10c gives

$$\bar{u} = 0, \quad \bar{v} = -2x$$

satisfying kinematic continuity.

It can be seen from Figure 10b that in the upper region one has two-bar trusses with one tension and one compression bar, in the lower region one has single bar trusses in compression. If the load is acting on the region boundary one may have any convex combination of a two-bar truss and a single bar truss.
3.3 Optimal Topologies for Least/Weight Grillages

The theory of the optimal topology of grillages (beam systems) actually preceded the development of the optimal layout theory [109]. The grillage theory (e.g., [113]) was developed but it has some elements of Morley’s [97] theory for optimal reinforcement in concrete slabs. Reviews of the grillage theory may be found in several texts, e.g., [115] [122].

Exact optimal grillage topologies should actually be used more often as benchmarks because they have the following advantages.

- The optimal grillage theory has advanced much further than the truss theory because exact optimal grillage topologies are available for almost all possible support and load conditions and these analytical solutions can even be generated by computer, e.g., [122]. A complex optimal grillage topology is shown on the cover of the book [114].
- Michell trusses ignore buckling which plays a much more important role for trusses than for grillages.
- Grillages may have three types of boundary conditions (clamped and simply supported boundaries and free edges whilst trusses may only have two (support or no support).
- A large number of extensions of the grillage theory exist [120].
- Whilst most Michell trusses are mechanisms, most optimal grillages are stable structures.

This section is not going to review the grillage theory in detail it is discussed at considerable length at the CISM Advanced Course [120].

The optimal layout theory was also used for least-weight shell-grids including self-weight for a review, see e.g., [118].

3.4 Difficulties in Deriving the Adjoint Displacement/Strain Field for O-Regions

The optimal topology for a completely new problem cannot be derived by a deductive process (using logical reasoning). The rough arrangement of optimal regions has to be first “dreamt up” [95] i.e. assumed by “intelligent” guessing, or invention and then checked if such a solution satisfies all the optimality criteria. During this second stage the exact shape of the
region boundaries is also assessed. The first stage requires considerable insight the second stage a lot of high level mathematical work.

Recently however, it has been possible to compute highly accurate numerical solutions e.g. for Michell trusses using over a billion potential members in the ground structure. These numerical solutions give a very good general idea about the adjoint displacement field for the exact optimal topology except for O-regions which have regions without members.

O-Regions were used by Rozvany in some contributions to papers by Lewinski, Zhou and Rozvany [78], Rozvany, Gollub and Zhou [124], Sokol and Rozvany [142].

It is explained in the last of these papers that O-Regions may contain the usual R-Regions and T-Regions but also the following new types of regions Z-Regions in which both principal strains are zero (rigid region with $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0$), and V-regions (with $|\bar{\varepsilon}_1| \leq k, \bar{\varepsilon}_2 = 0$).

Examples of O-regions consisting of T, R and Z-Regions are given in Figure 11b, c and d (after [142]). An example of an O-Region consisting of T and V-Regions is given in Figure 12. The above diagrams show only one quarter of the domain. Axes of symmetry are indicated in dash-dot lines. They are generalizations of Michell’s solution shown in Figure 11a.

The state of adjoint strains in these T and Z-Regions in Figure 12a, respectively, is represented by the Mohr-circles in Figure 12b and c. It can be seen from Figure 12b that the strain along the boundary of the T- and V-regions is

$$\bar{\varepsilon}_B = k \cos(2\alpha) \quad (3.7)$$

Moreover, one can infer from Figure 12c that for the V-region one has

$$k \cos(2\alpha) = (\bar{\varepsilon}_1/2)(1 + \cos(2\alpha)) \Rightarrow \bar{\varepsilon}_1 = \bar{\varepsilon}_y = k(1 - \tan^2 \alpha) \quad (3.8)$$

It can be seen that for $\alpha = 0$ and $\alpha = 45^\circ$ (Eq. (3.10)) gives the correct $\bar{\varepsilon}_1$ values for the limiting cases in Figure 11a and b. It is important to note that in O-regions the adjoint strain/displacement fields may be non-unique.
A simple example of non-uniqueness of the adjoint strain field is shown in Figure 13. In this problem one has two vertical line supports at a distance of $3L$ from each other and a horizontal point load at a distance $L$ from the right-hand support. To the permissible stresses the same value is assigned in tension and compression. The optimal layout obviously consists of a single horizontal bar between the load and the nearer support. The principal adjoint strain in the vertical direction is everywhere zero: $\bar{\varepsilon}_y = 0$. In Figure 13a one has an R-region on the right-hand side and an O-region on the left-hand side where the inequality in Equation (3.2) admits a horizontal strain of $-k/2$. An alternative discontinuous adjoint strain field is given in Figure 13b in which a Z-region and an R-region are on the left both admissible by Equation (3.2).
Another new type of region within an O-region is a scaled T-region termed T’ region having the property $\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = \lambda k$ with $\lambda < 1$ for equal permissible stresses and $\bar{\varepsilon}_1 = \lambda k_T$, $\bar{\varepsilon}_2 = -\lambda k_C$ for unequal permissible stresses. Figure 14 shows an optimal topology with a T’ region for which one has

$$\lambda = \frac{(4\pi/3 + 2\sqrt{3})}{(8\pi/3 + 2\sqrt{3})}$$  \hspace{1cm} (3.9)

The solution in Figure 14 has been verified numerically to a high degree of accuracy by Sokol [142].

Similarly, one has scaled R-regions termed R’-regions.

**Figure 13:** A trivially example of non-uniqueness of adjoint strain/displacement fields in O-Regions

**Figure 14:** Optimal topology with three T’-regions (in the O-region)

### 3.5 Extension of the Layout Theory to Elastic Design with Multiple Load Conditions

The optimal layout theory [109] has been extended to many additional classes of problems including combined stress and displacement constraints.
General optimality conditions for trusses with several load conditions and several displacement constraints were derived by Rozvany [119]. A compliance constraint puts a limit on the sum of the scalar product of external forces and the corresponding displacements for any one of the load conditions (sometimes called the “worst scenario” constraint). This is a special case of displacement constraint in which a weighted combination of the displacements at the external loads is constrained and the weighting factors are the external forces.

For compliance constraint Rozvany’s [119] optimality conditions reduce to following [121].

\[
\varepsilon_{ik} = \frac{F_{ik}}{E_i A_i}, \quad \bar{\varepsilon}_{ik} = \frac{v_k F_{ik}}{E_i A_i}, \quad A_i = \sqrt{\sum_k (v_k F_{ik}^2 / E_i \rho_i)},
\]

\[
(E_i / \rho_i) \sum_k v_k \varepsilon_{ik}^2 = 1 \quad (\text{for } A_i > 0),
\]

\[
(E_i / \rho_i) \sum_k v_k \bar{\varepsilon}_{ik}^2 \leq 1 \quad (\text{for } A_i = 0),
\]

where \(\varepsilon_{ik}\) and \(\bar{\varepsilon}_{ik}\) are kinematically admissible real (elastic) and adjoint strains in the member \(i\) under the load \(k\), \(v_k\) Lagrange multipliers, \(F_{ik} = \bar{F}_{ik}\) the real and adjoint member forces whilst \(A_i = E_i = \rho_i\) denote the cross-sectional area, Young’s modulus and density respectively, for the member \(i\).

For the particular case of a vertical line support and of two alternative loading cases consisting of concentrated forces at \(\pm \beta\), the above optimality criteria lead [121] to an optimal topology of two bars shown in Figure 15 with a very neat closed form result for the bar angles (top of Figure 15).

The optimal volume of the truss is given by

\[
V_{\text{opt}} = \frac{L^2 P^2}{C E \cos^2 \alpha} \left( \frac{\cos^2 \beta}{\cos^2 \alpha} + \frac{\sin^2 \beta}{\sin^2 \alpha} \right)
\]

(3.11)

where \(L\) is the horizontal distance of the load from the vertical support, \(P\) is the magnitude of the loads, \(E\) is Young’s modulus and \(C\) is the given compliance value.

In addition to global proof via layout theory in this rather elaborate paper (Rozvany, Zhou and Birker [121]) optimality of the solutions was also checked numerically by

- using a dense grid of potential truss members,
- optimizing a perforated plate by means of SIMP [14], [153], [121],
- deriving for the given topology (two-bar truss) the optimal angle from the Kuhn-Tucker condition.
Figure 15: Optimal topology for two alternative loads

All solutions derived by different papers have shown a complete agreement thus for this reason “reliability” of the above analytical solutions is fairly high.

3.6 Extension of the Layout Theory to Probabilistic Design

The aim of reviewing the problem in the last section was to extend the same optimal topology to probabilistic loads. This topic was considered by Rozvany and Maute [134].

The general form of the considered problem class is as follows.

\[
\text{minimize } V = \sum A_i L_i \quad (3.12)
\]

subject to

\[
\text{Pr}[C \leq K] \geq R \quad (3.13)
\]

\[
C = \sum \frac{F_i^2 L_i}{A_i E_i} \quad (3.14)
\]

where \(V\) is truss volume, \(A_i\) is cross-sectional area of member \(i\), \(L_i\) is length of member \(i\), \(\text{Pr}\) is probability, \(C\) is total compliance, \(K\) is limiting value of total compliance, \(R\) is limiting value of probability, \(F_i\) is force in member \(i\) and \(E_i\) is Young’s modulus of member \(i\).

The particular example considered is shown in Figure 16a. The topology of a truss is to be optimized within the design domain \(ABDG\) with supports along \(AB\) subject to a non-random vertical load of \(V = 300\) and a random horizontal load \(H\) with a mean of zero and a normal distribution having a standard deviation of \(\sigma = 100\). The truss volume is to be minimized for the conditions in Equation (3.15)-(3.16). It will be shown that the solution for the considered problem is a symmetric two-bar truss (as shown in Figure 16b). For simplicity one assigns a unit value to Young’s modulus.
Figure 16: Elementary benchmark example

Since this problem is symmetric in the only random variable \( H \), condition Eq. (3.15) is fulfilled if one considers the range of values for the random variable:

\[-H_0 \leq H \leq H_0\]  
(3.15)

where the value \( H_0 \) can be calculated from the inverse normal distribution cumulative probability function (also called “quantile” or “probit”) function \( \Phi^{-1} \) and \( R \) in Eq. (3.15):

\[H_0/\sigma = \Phi^{-1}\left(\frac{1+R}{2}\right)\]  
(3.16)

In Eq. (3.18) \((1 + R)/2\) is used since the failure may occur at both ends of the interval of Eq. (3.17). This implies

\[R = \Phi(\mu + H_0/\sigma) - \Phi(\mu - H_0/\sigma) = \Phi(\mu + H_0/\sigma) - (1 - \Phi(\mu + H_0/\sigma)) = 2\Phi(\mu + H_0/\sigma) - 1\]  
(3.17)

from which with \( \mu = 0 \) follows Eq. (3.18).

For example, if one requires a probability of 0.99999 (failure probability of \( 10^{-4} \)) then one has \( \Phi^{-1}(1 + 0.99999)/2) = 3.89 \) and hence in Eq. (3.18)

\[H_0 = 3.89\sigma = 389.\]  
(3.18)

It can be shown [134] that for the considered problem only \( H = \pm H_0 \) can be critical for Eq. (3.15). Therefore, once the value of \( H_0 \) is known and the optimal topology and the optimal volume can be calculated from the relation in Figure 16 and Eq. (3.13).

The above results were confirmed by Maute both analytically and by a first order reliability approach (FORM) combined with a material distribution method (Bendsoe [13], Zhou and Rozvany [153], Rozvany, Zhou and Birker [121]).
3.7 Extension of the Layout Theory to Pre-Existing Members

This problem was discussed in a paper by Rozvany, Querin, Lógo and Pomezanski [129]. If one has some already existing members of cross-sectional area $B$ and wants to add new members to satisfy some stress condition, then the following optimality criteria for equal permissible stresses are (see Figure 8b)

$$
\begin{align*}
\text{(for } |F| < B/k \text{)} & \quad \bar{\varepsilon} = 0, \\
\text{(for } |F| > B/k \text{)} & \quad \bar{\varepsilon} = k \text{sgn}F \\
\text{(for } |F| = B/k \text{)} & \quad 0 \leq |\bar{\varepsilon}| = k, \quad \text{sgn}\bar{\varepsilon} = \text{sgn}F
\end{align*}
$$

(3.19)

For unequal permissible stresses Figure 8c shows similar criteria. Figure 17a shows the optimal topology without pre-existing members and Figure 17b, Figure 17c are for pre-existing members along $QR$ with equal and unequal permissible stresses, respectively.

These analytical solutions were obtained by Rozvany with independent numerical confirmations by Querin, Lógo and Pomezanski.
It is rather remarkable that the first example of Michell [96] shown in Figure 18a has not been extended from one-point load to two-point loads for over a century. This extension discussed in greater detail also for other aspect ratios in Sokol and Rozvany [142].

3.8 Optimal Elastic Design for a Single Load Case

In this stage a simple task has been set out and investigated analytically to clarify fundamental features.

A single point load is given with its magnitude, direction and point of application as well as a line support at a certain distance from the point of application. The optimal truss design is either a 2-bar statically determinate truss with bars enclosing angles $\pi/4$ with the line of support (if the load encloses an angle less than $\pi/4$ with the line of support) or a single bar parallel to the load (if the load encloses an angle greater than $\pi/4$ degrees with the line of support).

Let the load $Q$ act at the origin of the Cartesian coordinate system $(x, y)$ enclosing angle $\beta$ with axis $x$. Let the line support be parallel to axis $y$ given at $x = h$, where $h$ is a positive constant. The two regions containing the line of the load are shown in Figure 19(a) by red and blue patches, respectively.

The total volume of the optimal single-bar or 2-bar truss is computed.
Consider first the single-bar truss. Equilibrium obviously implies that angle $\theta$ enclosed by the bar with axis $x$ is $\theta = \beta$ if $-\pi/2 < \beta < \pi/2$ and $\theta = \beta - \pi$ otherwise, see Figure 19(b). The bar force is equal to the load $S = Q$ the minimum cross-sectional area is calculated with the allowable stress $\sigma_p$ as $A = S/\sigma_p$ and the bar length is $l = h/\cos(\theta)$. Since $\cos(\pi - \beta) = -\cos(\beta)$, the total volume is expressed as

$$V = lA = \frac{Qh}{\sigma_p \cos(\theta)}$$

for all values of $\beta \in (-\pi, \pi)$.

Consider now a truss with two bars connecting the point of application and the line support. Let $\theta_1$ and $\theta_2$ denote the angles the bars enclose with the axis $x$, respectively, see Figure 19(c). Obviously $-\pi/2 < \theta_1, \theta_2 < \pi/2$ and without any limitations on generality we assume that $\theta_1 < \theta_2$.

The equilibrium of the joint at point $O$ is formed by the force $Q$ and the bar forces $S_1$ and $S_2$, which are represented by a vector triangle. Angles enclosed by the pairs $(Q, S_1)$, $(Q, S_2)$ and $(S_1, S_2)$ are $|\theta_1 - \beta|$ or its complementary angle, $|\theta_2 - \beta|$ or its complementary angle and $|\theta_2 - \theta_1|$ or its complementary angle, respectively. According to the law of sines, the bar forces can be computed as

$$S_1 = Q \cdot \frac{c_1 \cdot \sin(\theta_2 - \beta)}{\sin(\theta_2 - \theta_1)}$$
$$S_2 = Q \cdot \frac{c_2 \cdot \sin(\theta_1 - \beta)}{\sin(\theta_2 - \theta_1)}$$

where $c_1$ and $c_2$ are constants with values $\pm 1$ to account for the sign of angles $\theta_2 - \beta$ and $\theta_1 - \beta$, respectively, since $\sin(-\alpha) = -\sin(\alpha)$. If the angle enclosed by two force vectors in the triangle is the complementary angle of those indicated above, then the formulas are not affected, since $\sin(\pi - \alpha) = \sin(\alpha)$. Corresponding to the value of $\beta$ with respect to angles $\theta_1$ and $\theta_2$, five cases are distinguished as follows:

*Figure 19: Truss design for a single point load. (a) Regions of the direction of the load. (b) Line of forces, single-bar truss. (c) Line of forces, 2-bar truss*
1. \( \beta < \theta_1 (< \theta_2) \): \( c_1 = 1, c_2 = 1 \);
2. \( \beta = \theta_1 \): degenerate case with a single bar (1);
3. \( \theta_1 < \beta < \theta_2 \): \( c_1 = 1, c_2 = -1 \);
4. \( \theta_2 = \beta \): degenerate case with a single bar (2);
5. \( (\theta_1 <) \theta_2 < \beta \): \( c_1 = -1, c_2 = -1 \).

The minimum cross-sectional areas required are calculated with the allowable stress \( \sigma_p \) as

\[ A_1 = S_1/\sigma_p \quad \text{and} \quad A_2 = S_2/\sigma_p, \]

the bar lengths are computed as \( l_1 = h/\cos(\theta_1) \) and \( l_2 = h/\cos(\theta_2) \) for bars 1 and 2, respectively and then the total volume is obtained as

\[ V = l_1 A_1 + l_2 A_2 = \frac{Q h}{\sigma_p} \left[ \frac{c_1 \sin(\theta_2 - \beta)}{\sin(\theta_2 - \theta_1)} + \frac{c_2 \sin(\theta_1 - \beta)}{\sin(\theta_2 - \theta_1)} \right]. \quad (3.22) \]

The optimal design is obtained for angles \( \theta_1 \) and \( \theta_2 \) minimizing the volume \( V \). The necessary condition for the minimum is the vanishing of the derivatives with respect to the angles:

\[ \frac{\partial V}{\partial \theta_1} = \frac{Q h}{\sigma_p} \left[ \frac{c_1 \sin(\theta_2 - \beta) \sin(\theta_1)}{\cos^2(\theta_1) \sin(\theta_2 - \theta_1)} + \frac{c_1 \sin(\theta_2 - \beta) \cos(\theta_2 - \theta_1)}{\cos(\theta_1) \sin^2(\theta_2 - \theta_1)} + \frac{c_2 \cos(\theta_1 - \beta)}{\cos(\theta_2) \sin(\theta_2 - \theta_1)} + \right. \]
\[ + \left. \frac{c_2 \sin(\theta_1 - \beta) \cos(\theta_2 - \theta_1)}{\cos(\theta_2) \sin^2(\theta_2 - \theta_1)} \right] = 0 \quad (3.23) \]

\[ \frac{\partial V}{\partial \theta_2} = \frac{Q h}{\sigma_p} \left[ \frac{c_1 \cos(\theta_2 - \beta) \sin(\theta_1)}{\cos(\theta_1) \sin(\theta_2 - \theta_1)} - \frac{c_1 \sin(\theta_2 - \beta) \cos(\theta_2 - \theta_1)}{\cos^2(\theta_1) \sin^2(\theta_2 - \theta_1)} + \frac{c_2 \sin(\theta_1 - \beta) \sin(\theta_2)}{\cos(\theta_2) \sin^2(\theta_2 - \theta_1)} \right. \]
\[ - \left. \frac{c_2 \sin(\theta_1 - \beta) \cos(\theta_2 - \theta_1)}{\cos(\theta_2) \sin^2(\theta_2 - \theta_1)} \right] = 0. \quad (3.24) \]

The solution of the equation system Eq. (3.25), Eq. (3.26) is obtained for the five cases above as follows (solutions were derived using Maple 12):

1. \( \beta < \theta_1 (< \theta_2) \): \( c_1 = 1, c_2 = 1 \);

   The solutions are grouped as follows:
   (a) \( \theta_1 = ..., \theta_2 = \beta \): degenerate case of a single-bar truss;
   (b) \( \theta_1 = \beta, \theta_2 = ... \): degenerate case of a single-bar truss;
   (c) \( \theta_1 = -3\pi/4, -\pi/4, \pi/4, 3\pi/4, \theta_2 = -3\pi/4, -\pi/4, \pi/4, 3\pi/4 \).

   The only feasible solution referring to a 2-bar truss is \( \theta_1 = -\pi/4, \theta_2 = \pi/4 \). This solution is valid for \( \beta < -\pi/4 \) in compliance with the assumptions.

2. \( \beta = \theta_1 \): degenerate case with a single bar (1).

3. \( \theta_1 < \beta < \theta_2 \): \( c_1 = 1, c_2 = -1 \).

   No solution is obtained.
4. $\theta_2 = \beta$: degenerate case with a single bar (2).

5. $(\theta_1 <)\theta_2 < \beta$: $c_1 = -1, c_2 = -1$.

The solutions are grouped as follows:

(a) $\theta_1 = \ldots, \theta_2 = \beta$: degenerate case of a single-bar truss;
(b) $\theta_1 = \beta, \theta_2 = \ldots$: degenerate case of a single-bar truss;
(c) $\theta_1 = -3\pi/4, -\pi/4, \pi/4, 3\pi/4, \theta_2 = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$.

The only feasible solution referring to a 2-bar truss is $(\theta_1 = -\pi/4, \theta_2 = \pi/4)$. This solution is valid for $\beta > \pi/4$ in compliance with the assumptions.

The solutions of the five cases above imply that there is no optimal 2-bar truss for $-\pi/4 < \beta < \pi/4$ and the optimal 2-bar truss has bars enclosing angles $\pm \pi/4$ with the line of support for $\beta < -\pi/4$ or $\beta > \pi/4$. In the latter case the total volume is computed as follows.

The bar forces are now

$$S_1 = \begin{cases} +Q \sin(\pi/4 - \beta), & \beta < -\pi/4, \\ -Q \sin(\pi/4 - \beta), & \beta > \pi/4 \end{cases}$$

$$S_2 = \begin{cases} +Q \sin(-\pi/4 - \beta), & \beta < -\pi/4, \\ -Q \sin(-\pi/4 - \beta), & \beta > \pi/4 \end{cases}$$

The total volume is expressed as

$$V = \begin{cases} -2 \frac{Qh}{\sigma_p} \sin(\beta), & \beta < -\pi/4, \\ \frac{2 Qh}{\sigma_p} \sin(\beta), & \beta > \pi/4 \end{cases}$$

Comparing the single-bar truss Eq. (3.22) and the 2-bar truss Eq. (3.29) in the intervals $\beta < -\pi/4$ and $\beta > \pi/4$ reveals that the 2-bar truss has less volume than the single-bar truss hence it is the optimal design. In the interval between the single-bar truss is the only optimal design.

### 3.9 Superposition for Plastic Design

Two alternative point loads with magnitude and direction at the same point of application and a line support at a given distance from the point are given. In this case, if either the lines of action of the component loads both enclose angles less than $\pi/4$ with the line of support or both enclose angles less than $\pi/4$ with the line perpendicular to the line of support, then the optimal plastic truss design is statically determinate and therefore it is also the optimal elastic truss design.
Let $\mathbf{P}_1$ and $\mathbf{P}_2$ denote the two-point loads acting at the origin of the Cartesian coordinate system $(x, y)$. Let the line support be parallel to axis $y$ given at $x = h$, where $h$ is a positive constant. The component loads $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are calculated by the superposition principle as

$$
\mathbf{Q}_1 = \frac{1}{\sqrt{2}} (\mathbf{P}_1 + \mathbf{P}_2), \quad \mathbf{Q}_2 = \frac{1}{\sqrt{2}} (\mathbf{P}_1 - \mathbf{P}_2)
$$

(3.28)

The regions are the same as those shown in Figure 19(a). The conditions are satisfied if the $\mathbf{Q}_i$ component load vectors both lie either in the blue regions or both lie in the red regions. According to the single load case above, if both component loads enclose angles less than $\pi/4$ with axis $x$, then the individual optimal solutions comprise of a single bar each and hence the superposition is a 2-bar truss. On the other hand, if they both enclose angles less than $\pi/4$ with axis $y$, then the individual optimal solutions are 2-bar trusses with angles $\pm \pi/4$ and hence the superposition is also a 2-bar truss with angles $\pm \pi/4$. In both cases the optimal design is a 2-bar statically determinate truss. The obtained result is in accordance with the article of Nagtegaal and Prager.

3.10 Concluding Remarks

Several exact solutions have been found by guessing the geometry of the nets of bars, thereafter the static and kinematic admissibility with the given loading, and the correctness has been checked on this geometry by satisfaction of optimality conditions. It is even more surprising that Michell’s [96] truss theory has not been extended until now to several load conditions (stress constraints, elastic design).

In this chapter the optimal layout theory by Prager and Rozvany [109] is reviewed and some of its extensions to various design constraints. Difficulties in obtaining solutions were also explained and new types of optimal regions were introduced for overcoming them.

**Thesis 1**

I derived, independently, analytical solution for plastic stress-based optimization of truss-like structures with two, alternative loads of different magnitude and direction satisfying the least-weight optimality conditions. It is found that for certain load cases the optimal plastic design became statically determinate, therefore it is also the optimal elastic design and the obtained result is in accordance with the Nagtegaal-Prager solutions, the conclusion is in agreement with Sved.
4 Basic Property Changes Affecting Optimal Topologies

Related research paper: [EP1]

4.1 Introduction

This chapter has a parametric study in tasks with great number of degrees of freedom to examine some effects on the optimal topologies. The aim of the chapter is to investigate the influence of parameter changes to optimal design property. The task includes influence of material parameter $\nu$ (Poisson’s ratio) as well as the size of the ground element which is commonly applied during the discretization. Increasing the size of the ground elements while the total number of the finite elements is constant, the computational time is significantly reduced. Therefore, study on changing accuracy versus ground element resolution may be an important factor in choosing ground element size. In addition to it the effective properties of arrangements of the strong and weak materials (black and white elements) in a checkerboard fashion are also investigated. The Michell-type problem is investigated by the minimization of the weight of the structure subjected to a compliance condition. It is shown that when four-node quadrilateral elements are involved, and the size of the ground elements are varied, these constraints result in a numerically induced, artificially high stiffness and different optimal solution patterns. This can account for the formation of checkerboard patterns in continuous layout optimization problems of compliance minimization.

4.2 Preliminaries

The engineer designers have to take into consideration external (loading, design domain) and internal (effect of the numerical approximations) uncertain data and effect during this procedure. Sometimes the initial loading information has to be recalculated (optimize) before the design [50] or due to the multiple solutions the designers have to select the most appropriate one. In engineering one can find an effective tool for these questions in topology optimization [131].

The “modern” structural optimization period has been counted since the seminal paper of Bendsoe and Kikuchi in 1988 [12].
The programming method and the different solution techniques can be followed in several publications e.g. [11], [14], [83], [131], [153]. It has reached a rather high level of reputation in almost all fields of life including many industrial fields and it has widespread academic use for structural optimization problems and for material, mechanism, electromagnetics and other coupled field of design as well. Despite the level of research in topology optimization, several problems still exist concerning convergence, checkerboards and mesh-dependence which are subject to debate in the topology optimization community, e.g. [128], [130], [131].

During its hundred years of history it has become clear that the non-unique solution property of the method is affected by the material parameters (Poisson’s ratio) and the ways of the discretization.

The applied finite element technique and the selected type of finite elements (generally four-node quadrilateral elements are used) can overcome numerical difficulties, e.g. [83], [87], [153]. From the very first start of the numerical solution technique of topology optimization, a serious problem with it was the erroneous appearance of corner contacts between solid elements in the solution (checkerboards, diagonal element chains, isolated hinges).

### 4.3 Methodology

Great number of design variables is a great challenge in optimization. Most of the methods are not able to solve that very demanding task. Thus, there is a need for developing an algorithm to solve such a large topological optimization. This very important and popular topic was of interest to many scientists. Our approach is based on Karush-Kuhn-Tucker conditions described in detail in [81]. This SIMP (Solid Isotropic Material with Penalization) type algorithm is rather popular and has a long history [14], [153]. It was found that the quality of the discretization has an influence on the optimal solution (on the optimal topology).

It was found in earlier studies (Rozvany [130]) that the quality control can be performed by mesh refinement. Contrary to the theoretical trends the “efficiency” decreased with refinement of the element in numerical experiments. The reason was that the number of finite elements and the number of ground elements were increased simultaneously, keeping the number of finite elements per ground element constant. The coarser net of larger size elements increased the discretization error, erroneously increasing the stiffness and decreasing the compliance (even without corner contacts in the solution). The above problem
was overcome in the improved experiments (Rozvany [130]) in which the total number of the finite elements was kept constant in all calculations, but the number of ground elements was progressively increased (i.e. the number of FEs per ground element decreased).

In the following the ground element size and the variation of the Poisson’s ratio is investigated in connection with the optimal topology. It is noted that the original Michell structures composed of members having uniaxial stress and generally zero Poisson’s ratio is used in the numerical problems for the perforated plates.

4.3.1 Problem Formulation

This section is based on our SIMP type algorithm [81]. Let us consider a plane stress or plane strain structure with rectangular element discretization using uniform rectangular mesh with elements \( g = 1, \ldots, G_g \). Due to the “checkerboard effect” described later (see Figure 21) each element is usually subdivided to several sub-elements (Figure 22). The structure is subjected to static load and boundary condition. The structure has also been imposed displacement constraints. The objective function is the weight of the structure which can be expressed as:

\[
W = \sum_{g=1}^{G} y_g A_g t_g,  \tag{4.1}
\]

where \( y_g \) is the weight of the ground element, \( A_g \) is area of the ground element, and \( t_g \) is the thickness factor. The last one is also a design variable in topology optimization tasks. The factor takes values from the range \([0,1]\), however, we strive to have values of either \( t_{\min} = 10^{-6} \) or \( t_{\max} = 1 \). It is caused by numerical reasons. An optimized structure should have only two states: either material is present or there is void. Thus, optimization process tends to eliminate intermediate values of thickness. To achieve this effect thickness in Equation (4.1) is penalized as follows:

\[
W = \sum_{g=1}^{G} y_g A_g \frac{t_g^p}{t_g^{p-1}},  \tag{4.2}
\]

Penalization minimizes value of the weight for limit values (0 or 1).

The topology optimization problem can be formulated as follows:

- \( \min W \),
- simple bounds: \( t_{\min} \leq t_g \leq t_{\max} \),
- inequality constraint: \( u_D \leq \Delta_D \),
where $u_D$ is a chosen displacement in the structure and $\Delta_D$ is the prescribed permitted value of this displacement.

The inequality constraint can be also written in the form:

$$\tilde{u}_d^T Ku - \Delta_d \leq 0 \quad (d = 1, ..., D)$$

(4.3)

where $\tilde{u}_d$ is the virtual displacement vector of virtual loads, $u$ is the displacement caused by static load vector $P$, $\Delta_d$ is the prescribed displacement threshold. $K$ is the structural linear stiffness matrix. As it was proven in [11] the inequality condition Eq. (4.3) can be rewritten as:

$$u^T Ku - C \leq 0$$

(4.4)

where $C$ is the compliance of the structure. In the case of static load compliance is a monotonic function of load intensity. The topology optimization problem is to minimize penalized weight Eq. (4.2) subjected to inequality constraint:

$$\min W = \sum_{g=1}^{G_e} \gamma_g A_g t_g^{\frac{1}{p}}$$

subject to

$$\begin{cases}
    u^T Ku - C \leq 0, \\
    t_{\min} - t_g \leq 0, \\
    t_g - t_{\max} \leq 0.
\end{cases}$$

(4.5)

4.3.2 Lagrange Duality Formulation

Using formulation Eq. (4.5), Lagrange function can be written in the form as follows:

$$\mathcal{L}(t_g, \nu, \alpha_g, \beta_g, h_1, h_{2g}, h_{3g}) =$$

$$\sum_{g=1}^{G_e} \gamma_g A_g t_g^{\frac{1}{p}} + \nu(u^T Ku - C + h_1^2) + \sum_{g=1}^{G_e} \alpha_g (t_{\min} - t_g + h_{2g}^2) + \sum_{g=1}^{G_e} \beta_g (t_g - t_{\max} + h_{3g}^2)$$

(4.6)

where $\nu$, $\alpha_g$, $\beta_g$ are the Lagrange multipliers and $h_1$, $h_{2g}$, $h_{3g}$ are slack variables. Using the standard numerical procedure, the Karush-Kuhn-Tucker conditions can be written as follows.

4.3.3 Karush-Kuhn-Tucker Conditions

Stationary conditions are based on derivatives with respect to Lagrange multipliers. Differentiating Eq. (4.6) we obtain:

$$\frac{\partial \mathcal{L}}{\partial t_g} = \frac{1}{p} \gamma_g A_g t_g^{\frac{1-p}{p}} - \nu \left( \frac{\partial u^T Ku}{\partial t_g} + u^T \frac{\partial K}{\partial t_g} u + u^T K \frac{\partial u}{\partial t_g} \right) - \alpha_g + \beta_g = 0 \quad (g = 1, ..., G_e)$$

(4.7)
Due to symmetry of the stiffness matrix and linear relation between $K$ and $t_g$ the above equation can be simplified in the form below:

$$\frac{\partial L}{\partial t_g} = \frac{1}{p} y_g A_g t_g^{\frac{1-p}{p}} - \nu \frac{R_g}{t_g} - \alpha_g + \beta_g = 0,$$

where

$$R_g = t_g^2 \sum_{e=1}^{E_g} u_g^T K_{ge} u_g,$$

and $K_{ge}$ is the finite element stiffness matrix computed for $t_g = 1$. For brevity the rest of the derivatives is presented in simplified form:

$$\frac{\partial L}{\partial \nu} = u^T Ku - C + h_1^2 = 0 \quad \text{and} \quad \frac{\partial L}{\partial h_1} = 2 \nu h_1,$$

$$\frac{\partial L}{\partial \alpha_g} = -t_g + t_{\text{min}} + h_{2g}^2 = 0 \quad \text{and} \quad \frac{\partial L}{\partial h_{2g}} = 2 \alpha_g h_{2g},$$

$$\frac{\partial L}{\partial \beta_g} = t_g - t_{\text{max}} + h_{3g}^2 = 0 \quad \text{and} \quad \frac{\partial L}{\partial h_{3g}} = 2 \beta_g h_{3g},$$

### 4.3.4 Iterative Updating Formulas

Lagrange multipliers and slack variables can be computed iteratively. Thickness, which is also a design variable, is calculated by iterative formulas. There are three possible scenarios of updating a design variable:

First, when $t_{\text{min}} < t_g < t_{\text{max}}$, ground elements are called "active: $A \neq 0$", $\alpha_g = \beta_g = 0$, updating formula can be derived from Eq. (4.8):

$$t_g = \left(\frac{\nu R_g}{A_g y_g}\right)^{\frac{p}{p+1}}.$$

In the second case ("passive: $P \neq 0"), where $t_g = t_{\text{min}}$, Lagrange multipliers are $\alpha_g \geq 0$, $h_{2g} = 0$, and from Eq. (4.8) one can obtain:

$$t_g \geq \left(\frac{\nu R_g}{A_g y_g}\right)^{\frac{p}{p+1}}.$$

Thus, if smaller value of $t_g$ than $t_{\text{min}}$ is calculated, Equation (4.8) is satisfied by $t_g = t_{\text{min}}$.

In the third case ("passive: $P \neq 0"), where $t_g = t_{\text{max}}$, the corresponding Lagrange multipliers are $\beta_g \geq 0$, $h_{3g} = 0$ and Eq. (4.8) implies:

$$t_g \leq \left(\frac{\nu R_g}{A_g y_g}\right)^{\frac{p}{p+1}}.$$
To avoid some numerical problems, it is strongly recommended to set a minimum value to a sort of small value, for example $t_{\text{min}} = 10^{-6}$. Next step the final iterative formula is derived on which the topology optimization algorithm is based on.

While compliance condition Eq. (4.5) is active this condition can be rewritten as:

$$C - \sum_{g=1}^{G_e} \frac{R_g}{t_g} = 0. \quad (4.16)$$

Substituting into Eq. (4.16) relation Eq. (4.13) yields:

$$C - \sum_{g \in \mathcal{P}} \frac{R_g}{t_g} = \sum_{g \in \mathcal{A}} \frac{R_g}{t_g} = \sum_{g \in \mathcal{A}} \frac{R_g}{(v_p R_g)^{p+1}}, \quad (4.17)$$

re-arranging with respect to $v$:

$$v_p^{p+1} = \frac{\sum_{g \in \mathcal{A}} (A_g r_g)^{p+1}}{C - \sum_{g \in \mathcal{P}} \frac{R_g}{t_g}} \quad (\text{for } \mathcal{A} \neq 0). \quad (4.18)$$

To update $t_g$ Lagrange multiplier has to be computed from Eq. (4.18) and new thickness from formula Eq. (4.13).

Iterative algorithm of topology optimization can be enumerated as follows:

1. Create FE space model together with ground elements and boundary conditions.
2. Assume simple bound values of design values $t_g$, $t_{\text{min}} = 10^{-6}$, $t_{\text{max}} = 1$.
3. Specify $C_{\text{max}} = 150\% \times C^*$ – corresponding to $t_g = t_{\text{max}}$ for all elements.
4. Initialize penalty parameter $p = 1$. The parameter will evolve during optimization progress.
5. Solve FEM task
6. Compute element compliance $C_e = u_e^T \mathbf{K}_e u_e$, where $u_e$ is current element displacement vector, and $R_g$ from Equation (4.9).
7. Compute Lagrange multipliers.
8. Calculate new element thicknesses from Eq. (4.13).
9. Update sets of active and passive elements according to:

<table>
<thead>
<tr>
<th>Value</th>
<th>If</th>
<th>Element set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{g,\text{new}} = t_{\text{min}}$</td>
<td>$t_{g,\text{new}} \leq t_{\text{min}} = 10^{-6}$</td>
<td>$eeP$</td>
</tr>
<tr>
<td>$t_{g,\text{new}} = t_{\text{max}}$</td>
<td>$t_{g,\text{new}} \geq t_{\text{max}} = 1$</td>
<td>$eeP$</td>
</tr>
<tr>
<td>$t_{g,\text{new}} = t_{g,\text{new}}$</td>
<td>$t_{g,\text{new}} \geq t_{\text{min}} = 10^{-6}$</td>
<td>$eeA$</td>
</tr>
</tbody>
</table>

10. If there are some changes in set of active and passive element go to 5 (FEM computations).
11. If there are no changes in passive and active sets change penalty parameter $p$ according to scheme:

<table>
<thead>
<tr>
<th>$p$ value range</th>
<th>Increment</th>
<th>Go to</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - 1.45$</td>
<td>0.1</td>
<td>5</td>
</tr>
<tr>
<td>$1.5 - 3$</td>
<td>0.15</td>
<td>5</td>
</tr>
<tr>
<td>$3 - 15$</td>
<td>0.25</td>
<td>5</td>
</tr>
<tr>
<td>$&gt; 15$</td>
<td>—</td>
<td>12</td>
</tr>
</tbody>
</table>

12. Stop.

4.3.5 Choosing of Ground Element

In plane stress state the thickness of finite element can be treated as the design variable. Unfortunately, if each finite element thickness is treated as an independent design variable, it leads to unexpected effect called “checkerboard effect”. This phenomenon is illustrated in Figure 20. Enlarged part of topology solution is presented in Figure 21. Black and white elements are adjacent to each other alternately. This solution cannot be regarded as a correct one. To avoid this undesirable phenomenon several adjacent finite elements are assigned the same value of design variables (thickness). This group (Figure 22) is called ground element.

*Figure 20: Solution with ground element equal to finite element*

*Figure 21: Checkerboard effect with ground element size = 1*
Usually choosing ground element of size two is enough to obtain proper results (Figure 22). Greater size of the ground element causes faster topology optimization convergence but decreases resolution. In this chapter we will check the influence of the ground element size on the results.

4.4 Numerical Examples

4.4.1 Example 1

Several numerical examples have been investigated to illustrate the topological parametric study. The first example is shown in Figure 24: Beam scheme. It is a plane stress beam with one force acting at the bottom of the beam in the middle distance between supports. We benefit on the symmetry of the beam therefore a square mesh representing half of the beam will be taken into consideration. To achieve accurate results high resolution rectangular mesh will be used with dimension of $1320 \times 1320$ elements. It makes around 3.5 million degrees of freedom.

Topology optimization will be performed for several values of Poisson’s ratio $\nu = 0, 0.1, 0.2, 0.3, 0.4$. The influence of ground element size on optimal shape is observed. Therefore, topology optimization will be performed for a set of ground element sizes: $g_e = (1, 2, 3, 4, 5, 6, 8, 10, 12, 24)$. 
It is worth mentioning that the finite element mesh dimension is always the same (described above) so there is no impact of ground element size on single finite analysis time. Detailed results are presented in the Appendix A Table A 1. Plane stress material was isotropic with Young-modulus $E = 1$.

Computations were performed on HPC to benefit on sparse cluster architecture. In this very demanding task parallel computing plays a crucial role. HPC solution allows for even up to 10 topology optimization analysis performed simultaneously. Another level of parallelism used in computations was the multiprocessor architecture of particular cluster node. Linear equation solver Pardiso subroutine implemented in multithreaded version was able to use this type of hardware significantly decreasing solution time.

Schemas of all solutions for each parameter combinations are presented in Table A 1.

4.4.2 Example 2

Three plane stress problems presented in Figure 25 were computed. These schemas differ only in support mode. The optimal structures as well as the optimal volume parametric study are presented in Appendix A from Table A 2 to Table A 4 and Figure 26.
4.5 Conclusions

Computing time strongly depends on the size of ground elements (see Figure 27) for the small sizes. Although we do not change the dimension of the finite element model, the calculation time varies considerably. Ground element size $E_g = 2$ reduces computational time almost by half. Increasing it to $E_g = 3$ we observe another significant time reduction, approximately half. But beyond that we find the trend change. From size $E_g = 4$ to $E_g = 24$ the speeding up of the calculations is linear and only a few percent. Thus, ground element size of $E_g = 3$ is the recommended choice from efficiency point of view.
Thesis 2

Based on a large number of computer simulations I investigated the effect of two discretization parameters (ground element size and Poisson’s ratio) on the optimal topologies of four plane stress disk benchmark models for the minimization of the weight of the structure subjected to a compliance condition using an iterative algorithm.

(a) I found that the optimal choice for a ground element size is three while the finite element mesh dimension is the same, when four-node quadrilateral elements are involved, to achieve the best efficiency on the computing time in terms of ground element size.

(b) I found that while the finite element mesh dimension is the same, greater size of the ground element causes faster topology optimization convergence but decreases resolution and the boundary constraints result in a numerically induced, artificially high stiffness. Furthermore, different optimal solution patterns are obtained until the result becomes too coarse to be appropriate for design.
5 Compliance Constraint Design with Loading Uncertainty


5.1 Introduction

The consideration of stochastics in topology optimization has become an emerging approach which may relate to many aspects of loading. One possible option to model variability of loading on the structure is to apply resultants of different combinations of loads, for instance variable loads (i.e. wind) can be potentially active simultaneously with the constant dead load.

A simplified model of this concept consisting of two alternative loads is investigated in the examples. The term ‘alternative’ means that the two fixed forces are not acting on the structure simultaneously but alternately, as two independent load cases.

The aim of this chapter is to analyse and compare the validity of the optimal truss and truss-like structures under stochastic loading through the example of a fundamental and popular optimization problem which was stated e.g. by Nagtegaal and Prager [100]. Furthermore, the aim is to demonstrate that non-uniqueness of solutions can also be found in the case of uncertain loading.

In this chapter the uncertainties of the load positions are considered. By the use of a simple simulation technique and the stochastic upperbound theorem of Kataoka [70] a generalized compliance design problem is elaborated. The uncertain quantities are substituted by their statistical measures. To solve this constrained mathematical programming problem an iterative solution technique is derived by the use of the optimality criteria method. Several numerical examples are presented and compared.

5.2 Mathematical and Mechanical Background

The elaborated technique can be described as follows: it is assumed that the load positions are given by their distribution function, mean value and covariance matrix or by the simple values of the probability of the occurrence of a force at a certain location.

The case mentioned first, where the statistical information (distribution functions, mean values and variations) are given, is always done by a simple calculation which results in the
probability values of the occurrence of a force at a certain location – practically it is the second case. Hence each load is considered in an extended loading domain. Since the loading positions are not known precisely an equivalent loading system should be also created around the expected location of each force to perform a “simulation”.

According to the original distribution assumption, mean value and variation of the point of application, an extended force system is set up for each possible loading domain with the original magnitude of the force and given (or calculated) probability values – for sake of simplicity here seven points as “base” points are used with adjustment to the original distribution. Each load is independent and acts as independent load cases in the original loading domain. Applying these forces at these “base” points as loads the stochastic design problem becomes a deterministic one. By the use of the element of this force system one by one, the displacement vectors can be calculated from the usual linear equilibrium equations with several load cases.

Since the material is linearly elastic the additive properties of the displacements and the reciprocity theorem can be applied. Using these vectors and the assigned probability values the expected displacement and its variation can be calculated. By the use of these data the original compliance value which is probabilistic due to the position uncertainties, can be substituted with a deterministic one applying the Kataoka theorem [70].

Using this compliance formulation, a min-max objective function can be formed which is composed by the expected compliance and a certain type of variance of the compliance. In case of Gaussian distribution of the displacement field this objective function is simplified to a function minimization due to also this theorem.

5.3 Determination of the Deterministic Problems

Let us consider first the deterministic compliance design procedure of a linearly elastic 2D structure (disk) in plane stress which is known from literature (e.g. Rozvany [123], Lógó [83]).

For illustration purposes, with the use of the finite element method, the following simple case can be considered:

- the linearly elastic, 2D structure (disk) is subdivided into \((g = 1, \ldots, G)\) ground elements with constant thicknesses \((t_g)\) which are either \(t_g = t_{\text{min}} = 0\) or \(t_g = t_{\text{max}} = 1\), such that each ground element \((g)\) contains several sub-elements \((e = 1, \ldots, E_g)\) whose stiffness
coefficients are linear homogeneous functions of the ground element thickness $t_g$.

Practically, it means that the meshing consists of two parts, a primary and a secondary one.

- single static loading,
- given boundary conditions and
- compliance constraints.

The weight ($W$) of the structure is given by

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g,$$  \hspace{1cm} (5.1)

where $\gamma_g$, $A_g$, $t_g$ denote the specific gravity, area, and thickness of element $g$.

The compliance constraint can be expressed as

$$u^T Ku - C \leq 0;$$ \hspace{1cm} (5.2)

Here $K$ is the system stiffness matrix, $u$ is the nodal displacement vector associated with the load $F$. In the equation above the nodal displacement vector $u$ is calculated from $Ku = F$ linear system. The given compliance value is denoted by $C$.

The side constraints can be stated as

$$-t_g + t_{\min} \leq 0; \text{ (for } g = 1, \ldots, G),$$
$$t_g - t_{\max} \leq 0; \text{ (for } g = 1, \ldots, G).$$ \hspace{1cm} (5.3)

In order to suppress the intermediate thicknesses, the weight calculation formulation is replaced by $W = \sum_{g=1}^{G} \gamma_g A_g t_g^p$, where $p$ is the penalty parameter and $p \geq 1$. This gives the exact weight value for $t_g = 0$ and $t_g = 1$ in the case of any $p$ value. The use of the penalty parameter has a similar effect in the later formulations as this is its role in the classical optimality criteria method.

The deterministic optimization problem is to minimize the penalized weight of the structure, which is subjected to a given compliance and side constraints.

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g^p \leq \min!$$ \hspace{1cm} (5.4a)

subject to \begin{align*}
-u^T Ku - C & \leq 0; \\
-t_g + t_{\min} & \leq 0; \text{ (for } g = 1, \ldots, G), \\
t_g - t_{\max} & \leq 0; \text{ (for } g = 1, \ldots, G).
\end{align*} \hspace{1cm} (5.4b)
The penalization of the ground element thicknesses $t_g$ results in a more distinct material distribution indicating material or no material. Arising from this penalization, the optimization problem is non-unique in some sense but the method is widely applied in engineering optimization.

5.4 Determination of the Stochastic Bounded Compliance Problems

In the case of probabilistic design, the lower and upper compliance bounds are random variables and these follow Gaussian distribution (or any type of distribution). The bounds are given by their distribution functions $\Phi(C_1), \Phi(C_2)$ which mean values and variances $(C_{\min}, \sigma_{\min}, C_{\max}, \sigma_{\max})$, respectively.

The compliance constraint has a modified form
\[
P(C_1 \leq u^T Ku \leq C_2) \geq q,
\]
where $q \geq 0$ is the given probability value. Substituting Equation (6.5) in Equation (6.4) one can obtain the following optimal design formulation:
\[
W = \sum_{g=1}^{G} \gamma_g A_g t_g^\frac{1}{\gamma} = \text{min!}
\]
subject to \[
\begin{align*}
q - P(C_1 \leq u^T Ku \leq C_2) &\leq 0; \\
-t_g + t_{\min} &\leq 0; \ (\text{for } g = 1, \ldots, G), \\
t_g - t_{\max} &\leq 0; \ (\text{for } g = 1, \ldots, G).
\end{align*}
\]

To suppose $C_1$ and $C_2$ are independent random variables thus Equation (5.6b) is written as
\[
P(C_1 \leq u^T Ku)P(u^T Ku \leq C_2) \geq q.
\]

Then the minimum penalized weight problem subjected to probabilistic constraint is defined following:
\[
W = \sum_{g=1}^{G} \gamma_g A_g t_g^\frac{1}{\gamma} = \text{min!}
\]
subject to \[
\begin{align*}
q - P(C_1 \leq u^T Ku \leq C_2) &\leq 0; \\
-t_g + t_{\min} &\leq 0; \ (\text{for } g = 1, \ldots, G), \\
t_g - t_{\max} &\leq 0; \ (\text{for } g = 1, \ldots, G).
\end{align*}
\]

To simplify the probabilistic constraint (Equation 5.8b) the following standardized forms is introduced for random variables:
\[
q - P\left(\frac{C_1 - C_{\min}}{\sigma_{\min}} \leq x\right)P\left(y \leq \frac{C_{\max} - C_2}{\sigma_{\max}}\right) \geq 0,
\]
where \( x = \frac{u^T Ku - c_{\text{min}}}{\sigma_{\text{min}}} \) and \( y = \frac{c_{\text{max}} - u^T Ku}{\sigma_{\text{max}}} \).

The density functions and the distribution functions are the following:

\[
\text{dens}_x = e^{-\frac{x^2}{2}}, \quad \text{distr}_x = \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz, \quad \text{distr}_y = \int_{-\infty}^{y} e^{-\frac{z^2}{2}} \, dz.
\]

By the use of the standardized forms of random variables constraint Eq. (5.8b) is written as follow:

\[
q - \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz \cdot \int_{-\infty}^{y} e^{-\frac{z^2}{2}} \, dz \leq 0,
\]

The final form for the minimum penalized weight problem subjected to probabilistic constraint as bounds on compliance is defined as follows:

\[
W = \sum_{g=1}^{G} \gamma_g A_g \bar{t}_g \bar{p} = \min
\]

subject to

\[
\begin{align*}
q - \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz \cdot \int_{-\infty}^{y} e^{-\frac{z^2}{2}} \, dz & \leq 0; \\
-t_g + t_{\text{min}} & \leq 0; \quad (\text{for } g = 1, \ldots, G), \\
t_g - t_{\text{max}} & \leq 0; \quad (\text{for } g = 1, \ldots, G).
\end{align*}
\]

The constrained mathematical programming problem can be solved by the use of a SIMP-type algorithm. Details of the solution technique are presented in [82].

5.5 Probabilistic Compliance Design with Uncertain Loading Magnitude

5.5.1 Calculation of the Probabilistic Compliance

Suppose that the loads are given by \( n \) point loads and the loading \( F^T = [f_1, f_2, \ldots, f_n] \) has uncertainties such that the magnitude or the points of applications of the elements of \( F^T \) are random and these follow joint normal distribution. The mean values of the elements of \( F^T \) and the elements of the covariance matrix \( K_{ov} \) are denoted by \( \bar{f}_i = E(f_i) \) (\( i = 1, \ldots, n \)) and \( \kappa_{i,j} \) (\( i = 1, \ldots, n; j = 1, \ldots, n \)), respectively. The nodal displacement vector \( \bar{u}^T = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n] \) associated with the loading \( \bar{F}^T = [\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n] \) is calculated from \( K\bar{u} = \bar{F} \).

According to the principle of virtual forces the displacement \( u_i \) can be determined by the use of a unit virtual force acting at the location and direction of \( f_i \). The applied formulation is \( u_i = u^T K\bar{u}_i \), where the displacement \( \bar{u}_i \) is associated with this unit virtual force.
As it is known the compliance value is calculated as follows:

$$\mathbf{u}^T \mathbf{F} = u_1 f_1 + u_2 f_2 + \cdots + u_n f_n,$$

(5.13a)

where the displacement \((u_i, i = 1, \ldots, n)\) are obtained from \(\mathbf{K} \mathbf{u} = \mathbf{F}\) linear system and the force are denoted by \(f_i (i = 1, \ldots, n)\) in the direction of the load. By the use of appropriate approximation and simplifications (Lógo [83]) a possible “first order” stochastic compliance value is obtained as:

$$\mathbf{u}^T \mathbf{F} \sim 2\bar{\mathbf{u}}^T \mathbf{F} - \bar{\mathbf{u}}^T \bar{\mathbf{F}},$$

(5.13b)

where the first part contains the stochastic expression, the second one is calculated by the use of the mean value. This linearized form is used as a stochastically calculated compliance value, which has an effect on the volume fraction of optimal topology (Rozvany [123]) thus linearization is therefore appropriate to investigate the loading uncertainties.

If Equation (5.4b) is probabilistically constrained and the stochastic compliance calculation is used, the new constraint, similar to Equation (5.6b), is expressed as follow:

$$P\left(2(\bar{u}_1 f_1 + \bar{u}_2 f_2 + \cdots + \bar{u}_n f_n) - (\bar{u}_1 \bar{f}_1 + \bar{u}_2 \bar{f}_2 + \cdots + \bar{u}_n \bar{f}_n) - C \leq 0\right) \geq q,$$

(5.13c)

where \(0 < q < 1\) is given minimum probability value.

5.5.2 Derivation of the Optimization Problem

Using Equation (5.13c) one can obtain the following probability based optimal design formulation:

$$W = \sum_{g=1}^{G} y_g A_g \bar{t}_g^p = \text{min}!$$

(5.14a)

subject to

\[
\begin{cases}
P\left(2(\bar{u}_1 f_1 + \bar{u}_2 f_2 + \cdots + \bar{u}_n f_n) - (\bar{u}_1 \bar{f}_1 + \bar{u}_2 \bar{f}_2 + \cdots + \bar{u}_n \bar{f}_n) - C \leq 0\right) \geq q; \\
-t_g + t_{\text{min}} \leq 0; \quad \text{(for } g = 1, \ldots, G), \\
t_g - t_{\text{max}} \leq 0; \quad \text{(for } g = 1, \ldots, G).
\end{cases}
\]

(5.14b-d)

Introduction a slack random variable \(f_{n+1}\) with mean value \(E(f_{n+1}) = 1\) and the values \(\kappa_{n+1,i} = 0, \quad \kappa_{i,n+1} = 0; \quad (i = 1, \ldots, n + 1)\) of “extended” covariance matrix, the constraint Equation (5.14b) is written in a compact form:

$$P\left(2 \sum_{i=1}^{n+1} x_i f_i - \sum_{i=1}^{n} \bar{u}_i \bar{f}_i \leq 0\right) \geq q,$$

(5.15a)
where \( x_i = \bar{u}_i; \) \((i = 1, \ldots, n)\) and \( x_{n+1} = -C/2. \) By the use of theorem of Prékopa [110] the Equation (5.15a) is equivalent with the following convex formulation:

\[
2 \sum_{i=1}^{n+1} x_i f_i - x_{n+1} \bar{u}_i f_i + 2 \Phi^{-1}(q) \sqrt{x^T \mathbf{K}_{ov} x} \leq 0. \tag{5.15b}
\]

The \( \Phi^{-1}(q) \) is inverse cumulative distribution function (so-called probit function) of the normal distribution, \( \mathbf{K}_{ov} \) is “extended” covariance matrix and the underlined part in Equation (5.15b) is equivalent with the following equation:

\[
2 \sum_{i=1}^{n+1} x_i f_i - \sum_{i=1}^{n} \bar{u}_i f_i = \bar{u}^T \mathbf{K} \bar{u} - C. \tag{5.16}
\]

The displacement vector \( \bar{u} \) is associated with the mean values \( (\bar{f}_i = E(f_i)) \) of loading. The final form of stochastic constraint Equation (5.15b) is:

\[
\bar{u}^T \mathbf{K} \bar{u} - C + 2 \Phi^{-1}(q) \sqrt{x^T \mathbf{K}_{ov} x} \leq 0. \tag{5.17}
\]

Thus, the penalized minimum weight problem subjected to probabilistic compliance constraint has the form:

\[
W = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{2}} = \min \tag{5.18a}
\]

subject to \[
\begin{cases}
\bar{u}^T \mathbf{K} \bar{u} - C + 2 \Phi^{-1}(q) \sqrt{x^T \mathbf{K}_{ov} x} \leq 0; \\
-t_g + t_{\text{min}} \leq 0; \text{ (for } g = 1, \ldots, G), \\
t_g - t_{\text{max}} \leq 0; \text{ (for } g = 1, \ldots, G).
\end{cases} \tag{5.18b-d}
\]

Similar to problem Eq. (5.4a-d) the ground element thicknesses \( t_g \) are the design variables in Eq. (5.18) with lower bound \( t_{\text{min}} \) and upper bound \( t_{\text{max}}, \) respectively. As noted each ground element \( (g = 1, \ldots, G) \) contains several sub-elements \( (e = 1, \ldots, E_g) \) whose stiffness coefficients are linear homogeneous functions of the ground element thickness \( t_g. \) Since the mathematical nature of problem Eq. (5.18) is similar to problem Eq. (5.4) all mathematical statements concerning convexity and differentiability are also valid (Rozvany [123], Lógó [83]).

The traditional optimality criteria method (SIMP) of problem Eq. (5.18).

5.6 Probabilistic Compliance Design with Uncertain Loading

Positions

5.6.1 Distribution of the Loads

Here a simplified mechanical model is created on basis of the original loading domain.
Let us consider the design problem given in Figure 28. Since the loading positions are not known precisely an equivalent loading system should be also created around the expected location $\bar{x}_i$ of each force $f_i$ to perform the simulation. According to the original distribution assumption the mean value and the standard deviation of the point application are determined by the force system $f_{ij}$ ($j = 1, ..., k$) with the original magnitude $f_i$ – for sake of simplicity and to describe the loading domains with seven points – as “base” points are used with symmetrical adjustment $(f_{i1}, f_{i2}, f_{i3}, f_{i4})$. Note that, the minimum number of the points is three. Each load is independent and a well-defined probability value $w_{ij}$ ($j = 1, ..., k = 7$) is assigned to them, in practice it can take as design information. The determination of this probability value $w_{ij}$ ($j = 1, ..., k$) is based on the original distribution and it can be calculated with a simple computation. In this way the loading is given by these doubled parameters $w_{ij}$ ($j = 1, ..., k = 7$), $(f_{i1}, f_{i2}, f_{i3}, f_{i4})$ and applied as independent load cases. The modified topology design problem is given in Figure 28, if the original load and the supports are located on the same line.

If the application points and the supports cannot be connected with a single line the surrogate model of the loading is based on a force and uncertain moment system at the expected location of the original load. Applying these forces at these “base” points as loads the stochastic design problem becomes a deterministic one after this transformation. By the use of the element $f_{ij}$ ($j = 1, ..., k$) of this force system one by one, the displacement vectors $u_{ij}$ ($j = 1, ..., k$) can be calculated from the $Ku_{ij} = f_{ij}$ linear equations. Since the material is linearly elastic the additive properties of the displacements and the reciprocity theorem can
be applied. Using these vectors and the assigned probability values \( w_{ij} \) \((j = 1, \ldots, k)\) the expected displacement \( \bar{u}_i \) and its variation \( D_i^2(\bar{u}_i) \) can be calculated in the following form:

\[
\bar{u}_i = \sum_{j=1}^{k} u_{ij} w_{ij},
\]

(5.19a)

\[
D_i^2(\bar{u}_i) = \sum_{j=1}^{k} (u_{ij})^2 w_{ij} - \bar{u}_i^2.
\]

(5.19b)

These computed values are used to compose the element of the mathematical programming problem Eq. (5.18). Due to the nature of this type of loading the covariance matrix is diagonal.

\[
\mathbf{K}_{ov} = \langle D_1^2(\bar{u}_1), D_2^2(\bar{u}_2), \ldots, D_n^2(\bar{u}_n) \rangle
\]

(5.20)

Interchanging the expected compliance calculation by the generalized expected strain energy formulation the penalized minimum weight problem subjected to probabilistic compliance constraint has the form:

\[
W = \sum_{g=1}^{G} y_g A_g \frac{1}{t_g} = \min!
\]

subject to

\[
\sum_{i=1}^{n} (\bar{u}_i^{(1)} \mathbf{K} \mathbf{u}_i^{(1)} - C + \Phi^{-1}(q) \sqrt{\mathbf{K}_{ov}}} \mathbf{x}_i^{(1)} \leq 0;
\]

\[
\vdots
\]

\[
\sum_{i=1}^{n} (\bar{u}_i^{(k)} \mathbf{K} \mathbf{u}_i^{(k)} - C + \Phi^{-1}(q) \sqrt{\mathbf{K}_{ov}}} \mathbf{x}_i^{(k)} \leq 0;
\]

\[
-t_g + t_{\text{min}} \leq 0; \text{ (for } g = 1, \ldots, G),
\]

\[
t_g - t_{\text{max}} \leq 0; \text{ (for } g = 1, \ldots, G).
\]

(5.21b-d)

5.6.2 Adjoint Design

This section introduces a simple technique. The elaborated technique can be applied when a structure is supported by few (two, three) supports. Actually, most of engineering structures have similar boundary conditions.

Suppose that the structure (the design domain) is supported by two hinges \((A\) and \(B)\) and \(n\) different vertical forces with uncertain locations act as external loads \((f_1, \ldots, f_i, \ldots, f_n)\) on it. The distance of the point of application of the load \(f_i\), indicated by \(x_i\) (Figure 29) follows a given – Gaussian – distribution. Since the precise value of \(x_i\) is not known, \(x_i\) is given by its mean value \(\bar{x}_i\) and standard deviation \(\sigma_i\). Arising from the stochastic nature of the point of application of the load the topology optimization cannot be performed simply.
Instead of solving the problem described above an equivalent (adjoint) structural design problem (see Figure 30) is created where the vertical forces are substituted by vertical supports at the expected position of the point of application of the corresponding load $f_i$, and the vertical components of the supports substituted by the calculated reaction force components with stochastic magnitudes $(A_y, B_y)$.

Their mean values are:

$$\bar{A}_y = \sum_{i=1}^{n} f_i - \sum_{i=1}^{n} \frac{\bar{x}_i}{w} f_i, \quad \bar{B}_y = \sum_{i=1}^{n} \frac{\bar{x}_i}{w} f_i$$

and standard deviation can be calculated as

$$\bar{\sigma}_A = \bar{\sigma}_B = \sqrt{n} \left( \frac{f_i}{w} \right)^2 \sigma_i^2.$$  

(5.23)

Using these values, the design problem turns back into the problem Eq. (5.18) where the magnitude of the loads is probabilistic. It is convenient only the diagonal elements of the covariance matrix $K_{ov}$ are different from zero. This type of problem is solved by the formerly developed method for uncertain loading magnitudes.
The forces act in an interval \([-3\sigma_i, 3\sigma_i]\) and force \(f_{i1}\) is located at \(\bar{x}_i\). The magnitudes of the forces — \(f_{i1}, f_{i2}, f_{i3}, f_{i4}\) — are approximated using continuous beam theory. Depending on the support rigidities and location, the magnitudes of the “reaction” forces vary. The variations of these forces are shown in Figure 31a-c.

Applying these forces as load the stochastic design problem becomes a deterministic one. The solution procedure can be seen in [86].
5.7 Numerical Examples for Alternative Type Multiple Loads

5.7.1 Example 1

To demonstrate the method introduced above the following example problem is used to create the base problem (Figure 32). The rectangular shape ground structure – with 4 to 1 side ratio – is supported at two points: with hinge on the left, roller on the right. Total number of elements is 10000 using 2×2 sub-elements. The Poisson’s ratio is 0. The alternative loads are 50 units acting at the ¼ and ¾ of top side, respectively. The penalty parameter p was run from $p = 1$ to $p = 1.5$ with smooth increasing (increment is 0.1) and later to $p = 2.5$ with increment 0.25. The applied compliance limit is $C = 97000$. The entire problem is deterministic.

![Figure 32: Ground structure](image)

Two loads were used individually to overview the optimal topologies. One can see in Figure 33 the obtained optimal topology in the case of $f_1$. The second case contains only a single force at the ¾ of the top side. The corresponding optimal topology can be seen in Figure 34. The fictitious case of both forces acting simultaneously was also computed. The result is given in Figure 35. The optimal topology is given in Figure 36 when alternative loads are used.

![Figure 33: Optimal topology by the use of $f_1$](image)
5.7.2 Example 2

In this example a dimensionless 100×50-unit rectangular cantilever structure is the object of the design, see Figure 38a.

100×50 ground elements with 2×2 sub-elements are used. Total number of elements is 20000. The Poisson’s ratio is 0. The load (100 units) is placed at seven different positions distributed along the right edge (element indices from bottom to top: 16, 19, 22, 25, 28, 31, 34). During the simulation the penalty parameter \( p \) varied from \( p = 1 \) to \( p = 2 \) by steps 0.25 and then by steps 0.20 until \( p = 12 \). The applied compliance limit is \( C = 600000 \).

The optimal topologies obtained by the algorithm are shown in Figure 39a-g referring to load positions from bottom to top, respectively.
Figure 38: (a) The design domain (b) with the modified loadings

Figure 39: Optimal topologies with different positions of the applied loads (below)
As it was indicated earlier in the case of stochastic topology optimization the point of applications of the loads are random variables. They follow a normal distribution, see Figure 38b. The simulation is based on seven base points of the loads, which correspond to the load positions applied in the deterministic design.

The required probability of load bearing is defined by the parameter $q$. The same compliance limit is applied ($C = 600000$). The modifications and the termination criteria of the penalty parameter are the same as they are in the deterministic examples.

The penalty parameter $p$ was increased up to $p = 8$ with increment 0.10. The optimal topologies obtained by the simulation are shown in Figure 40a-g referring to values $q = 0.65$ to 0.95 by steps 0.05.

*Figure 40: Optimal topologies with load bearing probability $q = 0.65 - 0.95$. The number of such elements is limited to 1% of the total element number at most. (below)*
5.8 Conclusions

A numerical procedure was elaborated for topology optimization in the case of uncertain load positions and its magnitude. The parametric studies confirm that the method is suitable for numerical calculation. The computational times are not significant. Minimum three independent load cases need to model the uncertainty connected to an uncertain point of application of the original load. The surrogate loading system is problem dependent.

The uncertainties can modify significantly the deterministically obtained optimal topologies. In case of probabilistic loading the optimal layout can be statically indeterminate structure.

The resulted optimal structures have smaller volume than the deterministic one.

**Thesis 3**

I extended the topology optimization formulation of discretized plane disks based on Continuum-type Optimality Criteria (COC) to take into consideration the uncertainty of the position of the loads by introducing surrogate deterministic load cases. I confirmed via parametric studies that the method is suitable for numerical calculation with appropriate computational times without considerable numerical errors provided at least three surrogate load cases are used.
6 Statical Determinacy of Structures


6.1 Introduction

The loading uncertainties can be expressed by the stochastic nature of any part of the load definition. If any information is probabilistic among the three independent data of the load definition (the magnitude, the line of action and/or the point of application) a more precise design requires the elaboration of a probabilistic method. This goal can be achieved by the use of an appropriate approximation technique in which the original stochastic mathematical programming problem is substituted by a deterministic one. A similar technique can be followed in the case of a multiple loading. In addition to the numerical modelling a parametric study is given where the influence of the multiple loading is investigated to determine the optimal layout in terms of the initial layout. The application is illustrated by numerical examples.

The minimum weight design as an objective was a rather popular topic during “golden ages” of the optimization (e.g. during the 1950s to 1970s of the past century). The optimum design may be either statically determinate or indeterminate. The latter may be considered safer for probabilistic load cases.

This question is more difficult whenever the uncertainty of the loading is investigated. In this case the load can be considered as a quantity given in an interval with a certain possibility of location or/and direction, and/or magnitude. This uncertainty can be simplified by the use of surrogate loading where the original problem is considered as “infinite number” of loads in a given interval [87], [134].

In this study a comparison is made between minimum volume design of statically determinate and indeterminate structures. Cases of considering compliance and stress constraints are examined in the problem statement. The loading is considered probabilistic with variable direction and constant magnitude and point of application. In the analysis a limited variability of the domain is assumed, i.e. consider only certain points of supports and loads hence the optimum topology could only vary depending on its static determinacy.
The constrained mathematical programming problem is formulated with the idea that the unknowns are the cross-sectional areas of the members. The problem is solved numerically by a sequential quadratic programming algorithm of Mathcad 15, see in Appendix B.

In this chapter it is shown by the use of two simple examples that one can construct several (practically infinite number of) alternative statically indeterminate structures having the same volume and compliance or stress-constraint value if a statically determinate structure exists.

6.2 Simple Example for Equivalent Determinate vs Indeterminate Structures

The first model is a 3-bar truss-as a base structure- with a vertical force at the top (Figure 41). The material is linearly elastic (for sake of simplicity the Young’s modulus $E = 210000 \text{ kN/m}^2$) and the vertical load is 100 unit. The members are supported by hinges at the bottom. The total compliance is calculated by

$$C = \sum_{i=1}^{n} \frac{F_i^2 L_i}{EA_i}$$

where $F_i$ is the elastically calculated force in member $i$, $L_i$ is the length of member $i$, $A_i$ is the cross-sectional area of member $i$.

Table 1 shows the mechanically identical (same volume and same compliance) structures which can be composed by simple modification of the number of the members, i.e. doubling the number of bars and halving the cross-sectional areas of the members. 6- and 12-bar structures were calculated as examples.
Table 1: Comparative values of the optimal 3-, 6- and 12-bar structures in the case of a vertical point load at the top

<table>
<thead>
<tr>
<th>Length</th>
<th>pc</th>
<th>Section</th>
<th>Volume</th>
<th>Normal force</th>
<th>Stress</th>
<th>Top displ.</th>
<th>Compliance (external pot. energ.)</th>
<th>Compliance of the bars (strain energy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,820</td>
<td>3</td>
<td>1,5700</td>
<td>1332,18</td>
<td>47,140</td>
<td>-30,0</td>
<td>0,0572</td>
<td>5,72</td>
<td>5,719083764</td>
</tr>
<tr>
<td>2,828</td>
<td>6</td>
<td>0,7850</td>
<td>1332,18</td>
<td>23,570</td>
<td>-30,0</td>
<td>0,0572</td>
<td>5,72</td>
<td>5,719083764</td>
</tr>
<tr>
<td>2,828</td>
<td>12</td>
<td>0,3925</td>
<td>1332,18</td>
<td>11,785</td>
<td>-30,0</td>
<td>0,0572</td>
<td>5,72</td>
<td>5,719083764</td>
</tr>
</tbody>
</table>

A very similar example can be calculated if the top vertical force (100 unit) is modified and a horizontal force (57.74 unit) is added (see Figure 42). All the other data and the way of the calculations are the same. The results of the calculations are shown in Table 1.

*Figure 42: Alternative truss problems in the case of two-point loads at the top*
As a conclusion of the calculation above one can state that if a statically determinate structure exists as a solution of a deterministic problem with a single load case, several (infinite number) statically equivalent indeterminate structures can be derived with the same weight and the compliance. In addition to the conclusion above the weight can be decreased by the different layouts.

### 6.3 Minimum Volume Design of Structures According to the Optimal Layout Theory

In the case of probabilistic loading the magnitude, the line of action, the direction and the point of application of the load can be uncertain. Here through a simple example it is proved that not only one type of layout can be optimal. There will be singular layout solutions for certain cases or the optimal layout can be changed if the magnitude of the horizontal load is uncertain. The model is shown in Figure 43.
Figure 43: Three-bar truss problem in the case of two-point loads at the top

Table 3 and Table 4 show the values of the optimal 3- and 2-bar structures in the case of two-point loads at the top.

<table>
<thead>
<tr>
<th>3-bar truss</th>
<th>2-bar truss</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>$H$</td>
</tr>
<tr>
<td>8,660</td>
<td>0</td>
</tr>
<tr>
<td>8,660</td>
<td>0.5</td>
</tr>
<tr>
<td>8,660</td>
<td>1.0</td>
</tr>
<tr>
<td>8,660</td>
<td>1.5</td>
</tr>
<tr>
<td>8,660</td>
<td>2.0</td>
</tr>
<tr>
<td>8,660</td>
<td>2.5</td>
</tr>
<tr>
<td>8,660</td>
<td>3.0</td>
</tr>
<tr>
<td>8,660</td>
<td>3.5</td>
</tr>
<tr>
<td>8,660</td>
<td>4.0</td>
</tr>
<tr>
<td>8,660</td>
<td>4.5</td>
</tr>
<tr>
<td>8,660</td>
<td>5.0</td>
</tr>
<tr>
<td>8,660</td>
<td>5.5</td>
</tr>
<tr>
<td>8,660</td>
<td>6.0</td>
</tr>
<tr>
<td>8,660</td>
<td>6.5</td>
</tr>
<tr>
<td>8,660</td>
<td>7.0</td>
</tr>
</tbody>
</table>

According to the papers of Rozvany and Maute [134] or Silva et al [140] the optimal layout is a two-leg structure with a well-defined inclination angle if the horizontal force is uncertain. Here a very special case is studied where the initial layout is based on the optimal layout coming from the above cited papers. In addition to the two-leg structure an additional vertical leg is considered forming a statically indeterminate structural layout. The problem is a minimum volume design of a three-leg structure with constrained compliance – the
formulation Eq. (5.1) is the same and smaller than a given bound. The member forces are calculated from the equilibrium equations taking into account the compatibility equations as well. The top load is deterministic with given value while the top horizontal force is probabilistic. It is modelled on the way that this force can be any value in a given interval (Figure 45) as it is indicated in the above cited papers [134], [140]. The optimality condition to determine the layout is that the horizontal force cannot exceed the expected bounds ($\pm 7.5$).

![Figure 44: Optimal cross-sectional areas and minimum volumes of 3- and 2-bar truss problems](image)

The constrained mathematical programming problem is form with the idea that the unknowns are the cross-sectional areas of the members and the two side legs are in 30-degree inclination angle. There are two load cases (the horizontal force in each case can change its direction). The problem is solved numerically by a sequential quadratic programming algorithm of MathCad 15, see in Appendix B.

One can see the numerical values of the optimal cross-sectional areas in the case of three-bar truss (Table 3) and the numerical values of the optimal cross-sectional areas in the case of two-bar structure (Table 4), respectively. Graphically these results are presented in Figure 44. One can see that in case of $H = 0$ one vertical bar is the optimal layout while for $-3.5 < H < 3.5$ the optimal layout is a three-bar structure. Otherwise the optimal layout is a two-leg structure.
6.4 Conclusions

It is known in engineering design that a statically determinate structure is not sensitive for kinematic loading but any change in static loading may produce an unexpected collapse. In this way statically indeterminate structures can be safer for the unexpected load cases.

These studies presented that as long as the inclination from vertical is small, which is probable in practice, 3-bar structure has less total volume.

A very similar suspicion was presented in the almost forgotten paper of Nagtegaal and Prager [100]. The question of the optimal layout in the case of two alternative loads with the same point of application is investigated. A necessary and sufficient condition for global optimality was derived for the plane truss where the loadings were created on the way that the load factors for plastic collapse under one or the other load were not to exceed a given value. The results were one, two- or three-bar trusses depending on the loading domains.

The optimal layout problem of a minimum weight truss design problem with a single vertical force load presented by Save [135] in the case of stress constraints. The conclusions of his results and optimal layouts coming from the results obtained from our examples are in great agreement.

Thesis 4

I investigated the statical determinacy and non-uniqueness of optimal truss layouts under elastic stress and compliance constraints.

(a) Through two simple spatial structural examples with fix loads I demonstrated that if a statically determinate least-weight optimal solution exists, then an infinite number of alternative statically indeterminate structures may be constructed with the same weight and compliance or stress-constraint.

(b) I determined the least-weight optimal solution of a planar structural parametric problem with a load of uncertain direction and found a certain range of the load direction within which the optimal solution is statically indeterminate whereas it is determinate outside. It is concluded that the optimal least-weight structure is not necessarily determinate if variable load direction is considered.
7 Stress Constraint Design with Loading Uncertainty

Related research papers: [EP6], [EP5], [EP4], [EP3], [EP2]

7.1 Introduction

This chapter deals with a fundamental optimization problem setting of loading uncertainties, varying combinations of loading. This phenomenon is modelled by the application of two mutually excluding (i.e. alternating) forces such that the magnitudes and directions are varied parametrically in a range. The optimization problem is stated as to find the minimum volume (i.e. the minimum weight) load-bearing elastic truss structure that transfers such loads acting at a fix point of application to a given line of support provided that stress limits are set.

This basic example features a truss-like structure with rigid connections based on the results of the truss optimization and analyses it both as a bar structure (frame model) and a planar continuum (disk) structure to compare with the truss model.

The aim has been to numerically determine the layout, size, and volume of the optimal truss and to support the numerical results by appropriate analytical derivations. The validity of the optimal truss and truss-like structures under stochastic loading has been analysed and compared through the example of a fundamental and popular optimization problem which was stated e.g. by Nagtegaal and Prager [100]. This analysis illustrates the non-unique feature of optimum solution, which affects the statical determinacy of the structure as well.

7.2 General Problem Statement

The fundamental question arises whether the optimum structure is statically determinate or indeterminate. Consider for example the case of a horizontal beam supported at three points (Figure 45). Depending on whether the beam is continuous or divided into two separate simply supported parts the mechanical behaviour changes. It is also an important question to consider whether the design domain should be a solid continuum (e.g. a solid beam) or a ground structure containing the potential structural elements to choose from, see Figure 46, and how it affects the optimal solution.
A typical structural problem to investigate [100] is to define a line support and a point of application of the loads and the task is to design a structure (truss, frame, etc.) to transfer the loads to the supports, see Figure 47.

This study deals with a problem setting with a straight horizontal line of support and a fix point of application of the loads at a certain height above the line based on the above mentioned example taken from the literature. (The details are given in Section 7.3.1)

A skeleton structure has been designed opposed to continuum structures to connect the point of application to points on the support line. The joints of the framework are considered frictionless in accordance with problems usually analysed in the literature thus the optimization model is a truss.
The number of bars applied in the model is three in order to account for both statically indeterminate and determinate structures.

Uncertainties of loading may relate to the magnitude, direction or position of the loads of which here the former two are taken into considerations while the latter one is obviously not as a fix point of application is set. In the problem these uncertainties are included to provide a more realistic description of real loading behaviour.

The optimization task in this problem is to find a truss which is able to carry any of the alternating loads such that the stresses in the structure do not exceed a prescribed stress limit and is optimal with respect to the total volume that is with respect to the total weight which is an equivalent condition. A numerical computational algorithm (elaborated in Section 7.3) has been applied to determine the optimal layout and size of the structural elements. The task also deals with special cases and performs analytical investigations to support numerical calculations (Section 7.3.4).

In order to extend earlier studies (see [EP3],[EP5]) in the continuum analysis of the framework and to have conclusions regarding the optimality and applicability of the results, one of the investigated cases is chosen and remodelled with rigid connections to form a truss-like structure which now has shearing and bending further to the axial forces (Section 7.4). Both a bar structure (frame) and a planar continuum model (disk) are analysed. These two models have been examined to compare the behaviour with that of the optimal truss. Its relevance lies in the fact that realistic construction technologies do not typically build perfectly moment-free joints. The aim is to demonstrate that optimal trusses might fail requirements in such cases and to establish the significance of the choice of models on the optimal solution.

7.3 Truss Optimization Problem

7.3.1 Truss Layout

A point and a straight line are given in the plane. In this point two forces with given directions and given magnitudes can be applied alternately. The aim is to determine the minimum volume elastic structure which can carry the loads, its supports are in the given line and its tension and compression stresses do not exceed the prescribed limit.

In the case of truss number \( n \) bars connect the point of application and the support line with hinged connections. Figure 48 shows a schematic sketch of the problem.
Notations used hereafter are as follows:

$(x; y)$ coordinate system; the origin is in the point of application of the forces

$H$ distance of the point of application and the support line (support line: $y = -H$)

$F_j$ the $j.$ loading force ($j = 1; 2$)

$\beta_j$ the angle of the $j.$ load vector (measured clockwise from $+y$ axis)

$\alpha_i$ the inclination of the $i.$ bar (measured clockwise from $-y$ axis)

$A_i$ the cross-section area of the $i.$ bar

$\sigma_{ij}$ the stress in the $i.$ bar due to the $j.$ load

$\sigma_L$ stress limit (the same absolute value for tension and compression)

$E$ elastic modulus

$V$ volume

![Figure 48: Sketch of the optimality problem in the case of truss model](image)

7.3.2 Method of Calculation

The optimization is performed on a statically indeterminate three-bar structure, where $-\pi/2 < \alpha_3 < \alpha_2 < \alpha_1 < \pi/2$ and $A_1, A_2, \alpha_3 > 0$ without limiting the generality. The lengths of the bars are $L_i = H / \cos \alpha_i$ with the notations mentioned above. In the case of given loads, the response of the structure is obtained with matrix analysis as follows.

The unit direction vectors of the bars are $e_i = [-\sin \alpha_i \quad -\cos \alpha_i]^T$, from which the equilibrium matrix $G = [e_1 \quad e_2 \quad e_3]$ is constructed as well as the diagonal flexibility matrix $F = \langle L_i / EA_i \rangle$.

For the two loads the load vectors of the structure are $q_j = [F_j \sin \beta_j \quad F_j \cos \beta_j]^T$ ($j = 1; 2$) and the total stiffness matrix is $K = GF^{-1}G^T$. 
By the solution of the equation system of the structure the displacement vector can be expressed as \( \mathbf{v}_j = K^{-1}\mathbf{q}_j \) then the vectors of bar forces from the compatibility equations as \( \mathbf{s}_j = -F^{-1}G^T\mathbf{v}_j \). The normal stresses are \( \sigma_{ij} = S_{ij}/A_i \) where \( S_{ij} \) is the force in bar \( i \) due to load \( j \). Note that from the calculation above it follows that the bar forces and stresses do not depend on the elastic modulus (provided that the material is homogeneous). The volume of the structure is \( V = \sum_i L_i A_i \).

With stress constraints the optimization problem can be stated as follows:

Let \( V = \sum_i L_i A_i = \min! \) subject to

\[
\left\{ \begin{array}{l}
\mathbf{s}_j = -F^{-1}G^T K^{-1} \mathbf{q}_j, \\
-\sigma_L < \frac{S_{ij}}{A_i} < \sigma_L.
\end{array} \right.
\]

The problem can be solved for various load conditions (regarding magnitude and direction) using an approximate numerical iterative procedure. At fixed directions of the bars the values of the cross-section areas in the optimal structure are obtained on the condition that the stress has to be \( \pm \sigma_L \) in each bar due to at least one load. The minimum of the volume is found by varying the inclination angles of the bars in the range given above. During the process the cross-sectional area of a bar or bars may converge to zero; in this case a small minimum value is used to avoid numerical singularity. Furthermore, during the procedure, the inclination angles of any two bars may converge to the same value; in this case the two bars are going to be merged and the process continues as a determinate structure.

7.3.3 Numerical Calculations

7.3.3.1 Parameter Domain

The geometrical \((H)\) and material \((E, \pm \sigma_L)\) parameters of the problem have no qualitative effect on the optimal topology due to the linearity. In the calculation \( H = 100 \) [cm] and \( \sigma_L = 20 \) [kN/cm²] values are used. Value of one of the loads was fixed at \( F_1 = 100 \) [kN] and the other load value is chosen as \( F_2 = 25; 50; 75; 100; 133,33; 200; 400 \) [kN]. The inclination of the first force \((F_1)\) is varied between \((\beta_1) \pm \pi/2\) with increments \(\pi/12\) and the inclination of the other force \((F_2)\) is varied between 0 and \(\pi/2\) with the same increments. In the case of \(\beta_1\) it defines the half-plane, which does not cross the support line. (The other half-plane is not necessary because central symmetry leads to results of the same magnitude and opposite sign.) In the case of \(\beta_2\) it is sufficient to consider the positive part of this half-plane because appropriate mirroring of the results provides solutions for the missing domain.
7.3.3.2 Results

In the optimal truss design obtained with the method bars are eliminated with cross-section area equal to the numerical minimum value and thus regarded as zero. Among optimal designs obtained for different load cases there are structures consisting of one, two, and three bars. These cases correspond to statically over-determinate, determinate and indeterminate trusses, respectively. In the investigated domain of the three-dimensional parameter space the optimal truss typically is a statically determinate two-bar truss except for some null subsets. The over-determinate and indeterminate structures can be optimal only in special cases.

In a two-dimensional subspace of the three-dimensional parameter space a statically over-determinate degenerate one-bar truss is obtained; this occurs when the two alternating loads have equal angles and the direction of the common angle is between $\pm \pi/4$. A statically indeterminate three-bar structure is obtained in a one-dimensional subspace when the two loads have the same magnitude, their directions are symmetric to the line which is perpendicular to the support line and the angle is less than a certain value. In these two special cases the numerical results are verified analytically as well.

7.3.4 Analytical Calculations

7.3.4.1 Degenerate Case

If the two forces have the same direction the optimal solution is the one which is obtained by the greater force. Consider a two-bar structure in which the inclinations of the bars are $\alpha_1$ and $\alpha_2$ and the inclination of the load $F$ is $\beta$. The bar forces can be calculated from the equilibrium of the hinge and then the stresses are:

$$\sigma_1 = \frac{F \cdot c_1 \sin(\alpha_2 - \beta)}{A_1 \sin(\alpha_2 - \alpha_1)}$$
$$\sigma_2 = \frac{F \cdot c_1 \sin(\alpha_1 - \beta)}{A_2 \sin(\alpha_2 - \alpha_1)}$$

(7.3)

where $c_1$ and $c_2$ are constants with the value $\pm 1$ depending on the sign of angles in the numerator. In the case of total utilization of the structure the stresses are equal to the limit stress from which the cross-sectional areas can be calculated.

The volume of the structure in terms of the inclination of the bars is $V(\alpha_1, \alpha_2) = \sum_i L_i A_i$. 
The optimal topology is obtained from the solution of the equation system \( \partial V / \partial \alpha_i = 0 \) \( (i = 1; 2) \). A number of cases need to be examined depending on the load and the relative value of the inclinations of the bars. If \( \beta \) is in the \((\alpha_2, \alpha_1)\) interval, the system of equations does not have any solution. If \( \beta \) is outside the \((\alpha_2, \alpha_1)\) interval, the solution will be either the \((\alpha_1 = \pi/4, \alpha_2 = -\pi/4)\) two-bar structure or a degenerated case in which one of the bars is parallel to the direction of the load and the other bar has zero force. This latter solution is valid in the case in which \( \beta \) is on the boundary of the interval.

From the solution set of the equation system it follows that the optimum is a one-bar structure with inclination \( \alpha = \beta \) in the case of \(-\pi/4 \leq \beta \leq \pi/4\) whereas in the case of load directions outside this domain it is the two-bar structure with inclination angles \( \pm \pi/4 \) because in this case the one-bar solution corresponds to larger volume.

Note that it is recommended to apply a mathematical program using symbolic description (i.e. Maple, etc.) to solve the equation system analytically.

7.3.4.2 Indeterminate Solution

In the case of two loads of equal magnitude and symmetrical layout \((F_1 = F_2 = F, \beta_1 = -\beta_2 = \beta > 0)\) the optimal structure can be a symmetrical three-bar statically indeterminate structure \((\alpha_1 = -\alpha_3 = \alpha, \alpha_2 = 0, A_1 = A_3)\), see Figure 49a.

The stresses from the \( F_1 \) force are

\[
\begin{align*}
\sigma_1 &= \frac{F \sin \beta}{2 A_1 \sin \alpha} + \frac{F \cos^2 \alpha \cos \beta}{2 A_1 \cos^3 \alpha + A_2}, \\
\sigma_2 &= \frac{F \cos \beta}{2 A_1 \cos^3 \alpha + A_2}, \\
\sigma_3 &= -\frac{F \sin \beta}{2 A_1 \sin \alpha} + \frac{F \cos^2 \alpha \cos \beta}{2 A_1 \cos^3 \alpha + A_2}.
\end{align*}
\]

(7.4)

In the optimal structure \( \sigma_1 = \sigma_L \) and \( \sigma_2 = \sigma_L \) (and \( \sigma_3 < \sigma_L \)) from which \( A_1 \) and \( A_2 \) can be calculated.

The volume of the structure is

\[
V_3(\alpha) = \frac{HF \cos \beta}{\sigma_L \sin^3 \alpha} \cdot \left( \frac{\tan \beta}{\cos \alpha} + \cos \alpha \tan \beta \sin^2 \alpha - \cos \alpha \tan \beta + \sin^3 \alpha \right).
\]

(7.5)

The solution of the \( \partial V_3(\alpha) / \partial \alpha = 0 \) equation is independent from \( \beta : \alpha_0 = \arctan \sqrt{2} \). In the optimal structure the cross-section areas of the bars are

\[
\begin{align*}
A_1 &= A_3 = \frac{F}{\sigma_L} \left( \frac{3 \sqrt{2}}{8} \sin \beta \right), \\
A_2 &= \frac{F}{\sigma_L} \left( \cos \beta - \frac{\sqrt{2}}{4} \sin \beta \right).
\end{align*}
\]

(7.6)
thus the volume of the structure is
\[ V_{\text{opt},3} = \frac{HF}{\sigma L} \left( 2\sqrt{2} \sin \beta + \cos \beta \right). \] (7.7)

It is remarkable to note that the numerical calculation also resulted in a statically determinate two-bar solution which is equivalent with the three-bar solution regarding volume (Figure 49b). In this case \( \alpha_1 > 0 \) and \( \alpha_2 = 0 \). After calculating the bar forces, stresses and cross-sections, the volume can be written in the form
\[ V_2(\alpha) = \frac{HF}{\sigma L} \left( \sin \beta \frac{\sin \alpha \cos \alpha}{\sin \alpha \cos \alpha} + \cos \beta + \frac{\sin \beta}{\tan \alpha} \right). \] (7.8)

The solution of the \( \partial V_3(\alpha) / \partial \alpha = 0 \) equation is the same as the angle obtained in the case of the three-bar structure: \( \alpha_0 = \arctan \sqrt{2} \). In the optimal structure the cross-section areas of the bars are
\[ A_1 = \frac{F}{\sigma L} \left( \frac{\sqrt{6}}{2} \sin \beta \right), \]
\[ A_2 = \frac{F}{\sigma L} \left( \cos \beta + \frac{\sqrt{2}}{2} \sin \beta \right), \] (7.9)

then after substituting back for the volume of the structure an expression is obtained which is identical to the previous case:
\[ V_{\text{opt},2} = \frac{HF}{\sigma L} \left( 2\sqrt{2} \sin \beta + \cos \beta \right), \] (7.10)

that is, the two optimum cases, which are topologically different, are equivalent.

The numerical calculations show that this double optimum is valid in a certain domain of angle \( \beta \) (\( \beta < 0.569612851 \ldots \)) and above this limit value a statically determinate symmetric structure is the optimal solution (Figure 49c).

\[ \text{Figure 49: Optimal topologies in the special cases of two symmetrical alternative loads: (a) statically indeterminate symmetrical structure, (b) statically determinate asymmetrical structure which is equivalent with the previous one, (c) statically determinate symmetrical structure. Sketch of the optimality problem in the case of truss model} \]
7.4 Truss-Like Design

The main difference between optimization of structures modelled as a truss or a truss-like frame or disk is that in the first case uniaxial stress state is formed in all members while in the other cases the stress state is two-dimensional. The real design of the connections in the optimal structure determines the stress state and it necessarily affects the optimality.

In the following a comparative analysis of the two systems is performed through the example of the structure shown in Figure 49a. Due to the symmetry of the structure and the load it is sufficient to consider only one of the loads. Figure 50a shows the frame structure in which the cross-sectional dimensions can be determined by forming rectangular cross-sections such that they have unit (1 cm) thickness and areas equal to the values calculated by the algorithm (see in Appendix C) for a chosen value of angle $\beta$. Typical normal force and moment diagrams are shown in Figure 50b and Figure 50c. Although the bending moment values are less than those of beams designed for bending but their effect is significant because the structure is optimal with respect to normal forces. The maximum bending moment and the maximum normal stress (in the bottom cross-section of the central bar) are affected by the length of the bars. Figure 51 shows the ratio ($\sigma/\sigma_L$) of the maximum stress and the stress limit in terms of the ratio of the beam length and height (in the case of $\beta = \pi/6$).

![Figure 50](image)

**Figure 50:** Analysis of a three-bar structure: (a) model, (b) normal force- (c) bending moment-diagram

![Figure 51](image)

**Figure 51:** The maximum normal stresses scaled by the stress limit in terms of the length-to-height ratio of the frame structure
It can be seen in Figure 51 that in the case of ratio 10:1 the maximum normal stress is higher than the allowable limit stress by nearly 25 %. Increasing the ratio the excess stress is reduced but it is still more than 12 % in the case of ratio 20:1 and it is 5 % even at ratio 50:1. Therefore the optimal structures designed as trusses will be undersized significantly when the joints are not perfectly moment-free hinges. Thus, a procedure using continuum model is necessary for optimization of structures with moment bearing connections.

Stresses calculated in the frame model can be verified by finite element stress analysis. Figure 52 shows the distribution of the major principal stresses in the structure. Dark orange and red colours in bars 1 and 2 illustrate well the maximum stresses corresponding to bending moments shown in Figure 50c. Figure 53 shows the larger principal stresses at the bottom part of the central bar where the stresses are maximal. The difference between the maximum stresses calculated in the frame structure and in the continuum model is within 1 %. The finite element model shown in Figure 52 is highlighted in red in Figure 51.

*Figure 52: The larger principal stresses of the continuum model scaled by the stress limit. Load and layout as in Figure 50a. (Note that that around the point of application of the concentrated force high stress peak is generated thus the values above 1.5 are not marked with separate colours for better visualization only.)*

*Figure 53: The larger principal stresses of the continuum model focus on the bottom cross-section of the central bar. Large stresses develop on the tension side.*
Note that, around the point of application of the concentrated force high stress peak is generated which is not to be taken into consideration in the comparison (Figure 54). Also note that its local effect is limited to a narrow domain and the stresses of the bars show a very good agreement with the results of the frame model.

7.5 Conclusions

In the case of topology optimization on a structure where the loading is defined by stochastic variables one possible way to create the equivalent mechanical model is to use a number of mutually excluding (alternating) loads. In this study optimal truss structure topologies were determined in the case of two alternative loads subject to stress limit and elastic behaviour in a wide range of described system parameters. The topology and size of the optimal structure are determined which minimize the volume (that is the amount of material) for various force ratios and force vector directions using a numerical iterative procedure. It is found that the optimum topology is typically statically determinate, however, in a certain domain of parameters a statically over-determinate degenerate structure represents the minimum volume and in another domain the solution can be statically indeterminate. In these special cases the numerical results were confirmed by analytical calculations as well. Our numerical and analytical investigations have proved that the solution in certain cases is non-unique.

Figure 54: The larger principal stresses of the continuum model focus on the point of application of the load. Large stresses develop on the tension side of the central bar and of the left bar.

Furthermore, it is found that optimal designs calculated by the truss model (that is structures with hinges) are valid only for this type of structures. In structures where the joints are not
constructed perfectly free of moments even the small bending moments occurring simultaneously with the dominant normal forces lead to significant excess normal stresses which involve the undersizing of the structure. To verify this, an analysis of a chosen structure was performed modelled as a frame construction and a two-dimensional (plane stress state) continuum model by finite elements methods. The two analyses had the same results (with negligible difference) both showing the significant excess stresses. The investigated example demonstrates that for the determination of optimum frame structures an optimization procedure using continuum model is required. The model chosen has to be justified sufficiently. The comparative investigation assesses the validity of computational models and proves that the choice affects design negatively since rigidity of connections resulted by usual construction technologies imply extra stresses leading to significant undersizing.

**Thesis 5**

(a) I determined numerical solution for the topology, shape and size of the least-weight optimal truss structure consisting of at most three bars in the case of a given line of support and two alternative loads in a given point of application subject to elastic stress limit for a wide range of system parameters.

I found that the typical optimal solution is statically determinate (2 bars) while in a certain parameter domain it degenerates to a single bar and in another domain statically indeterminate (3 bars) solution co-exists with a determinate solution of the same weight, i.e. the solution is non-unique. I proved the numerical results of the degenerate case and the indeterminate case by analytical derivations.

(b) I created both a frame construction with rigid node and a two-dimensional (plane stress state) continuum structure with the same shape and size as a chosen 3-bar truss and performed stress analysis in order to determine the influence of the choice of the model on the optimal structure.

I found that the frame structure and the continuum structure have the same maximum stresses, which due to the extra bending effect significantly exceed those of the truss leading to the undersizing of the structure if modelled as a truss. It is concluded that for the determination of optimum structures with rigid connections the truss model could be inadequate.
8 Further Research

One of the important further aims is to obtain multi-load benchmark solutions for more complicated boundary conditions because almost all practical design problems have to consider several load conditions. The analytical solutions are derived on continuum-based models, but the final solutions are usually interpreted as trusses. A larger comparative study is needed to clarify the situation. The exact analytical solution for many popular benchmark problems is still not known, although existing exact benchmarks were summarized by Rozvany for single load condition and new benchmark solutions were derived by Lewinski and Rozvany (e.g. [130], [131]). Further new benchmark solutions are necessary for the investigation. The area of analytically determined optimal topologies under stochastic boundary conditions is almost unexplored. Rozvany (e.g. [79]) has derived exact analytical benchmarks for probabilistic design and these are being compared with numerical solutions for the same problems. Still some modelling verification is needed because the continuum problem is modelled as a truss.

Other reasonably difficult problems are numerical topology optimization for stress constraints and the practical topology optimization problems may have many hundred thousand local stress constraints. According to previous experience the applied constitutive equations affect the optimal solution significantly including its reliability and risk of collapse.

One of the weaknesses of numerical topology optimization is the fact that most papers discussing new methods and solutions do not validate their results by comparison with reliable benchmarks or they use only a subjective visual comparison. Further new benchmark solutions are necessary for the investigation discussed above to validate the optimal solutions in deterministic and stochastic cases as well.

In Hungary the reliability analysis and design as well as the risk analyses have only been investigated recently. The optimization work for trusses and panels can be extended to folded curved plated structures, thin-walled structures, inflated membranes, morphology of polyhedra in connection to historical objects and natural (receptacle of lotus) formations.
9 Thesis Statements

This section and in Hungarian Appendix D present a summary of new scientific results achieved within the framework of this research period and presented in this dissertation.

Thesis 1

I derived, independently, analytical solution for plastic stress-based optimization of truss-like structures with two, alternative loads of different magnitude and direction satisfying the least-weight optimality conditions. It is found that for certain load cases the optimal plastic design became statically determinate, therefore it is also the optimal elastic design and the obtained result is in accordance with the Nagtegaal-Prager solutions, the conclusion is in agreement with Sved.

Connected publications: [EP10], [EP9], [EP7]

Thesis 2

Based on a large number of computer simulations I investigated the effect of two discretization parameters (ground element size and Poisson’s ratio) on the optimal topologies of four plane stress disk benchmark models for the minimization of the weight of the structure subjected to a compliance condition using an iterative algorithm.

(a) I found that the optimal choice for a ground element size is three while the finite element mesh dimension is the same, when four-node quadrilateral elements are involved, to achieve the best efficiency on the computing time in terms of ground element size.

(b) I found that while the finite element mesh dimension is the same, greater size of the ground element causes faster topology optimization convergence but decreases resolution and the boundary constraints result in a numerically induced, artificially high stiffness. Furthermore, different optimal solution patterns are obtained until the result becomes too coarse to be appropriate for design.

Connected publication: [EP1]
Thesis 3

I extended the topology optimization formulation of discretized plane disks based on Continuum-type Optimality Criteria (COC) to take into consideration the uncertainty of the position of the loads by introducing surrogate deterministic load cases. I confirmed via parametric studies that the method is suitable for numerical calculation with appropriate computational times without considerable numerical errors provided at least three surrogate load cases are used.


Thesis 4

I investigated the statical determinacy and non-uniqueness of optimal truss layouts under elastic stress and compliance constraints.

(a) Through two simple spatial structural examples with fix loads I demonstrated that if a statically determinate least-weight optimal solution exists, then an infinite number of alternative statically indeterminate structures may be constructed with the same weight and compliance or stress-constraint.

(b) I determined the least-weight optimal solution of a planar structural parametric problem with a load of uncertain direction and found a certain range of the load direction within which the optimal solution is statically indeterminate whereas it is determinate outside. It is concluded that the optimal least-weight structure is not necessarily determinate if variable load direction is considered.


Thesis 5

(a) I determined numerical solution for the topology, shape and size of the least-weight optimal truss structure consisting of at most three bars in the case of a given line of support and two alternative loads in a given point of application subject to elastic stress limit for a wide range of system parameters.

I found that the typical optimal solution is statically determinate (2 bars) while in a certain parameter domain it degenerates to a single bar and in another domain statically indeterminate (3 bars) solution co-exists with a determinate solution of the same weight, i.e. the solution is non-unique. I proved the numerical results of the degenerate case and the indeterminate case by analytical derivations.
(b) I created both a frame construction with rigid node and a two-dimensional (plane stress state) continuum structure with the same shape and size as a chosen 3-bar truss and performed stress analysis in order to determine the influence of the choice of the model on the optimal structure.

I found that the frame structure and the continuum structure have the same maximum stresses, which due to the extra bending effect significantly exceed those of the truss leading to the undersizing of the structure if modelled as a truss. It is concluded that for the determination of optimum structures with rigid connections the truss model could be inadequate.

Connected publications: [EP6], [EP5], [EP4], [EP3], [EP2]
Publications of the Author Related to Thesis


References


[48] Hemp WS. Notes on the problem of the optimum design of structures. College Aeronaut Note. 1958. 73.


[77] Lévy M. La statique graphique et ses applications aux constructions. 1873.


Appendix A

Table A 1: Optimal topologies as a result of the parametric study of beam presented in Figure 24

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Table A.2: Topology optimization results for Figure 25a

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<td><img src="G_10_0.4" alt="Image" /></td>
</tr>
<tr>
<td>12</td>
<td><img src="G_12" alt="Image" /></td>
<td><img src="G_12_0.1" alt="Image" /></td>
<td><img src="G_12_0.2" alt="Image" /></td>
<td><img src="G_12_0.3" alt="Image" /></td>
<td><img src="G_12_0.4" alt="Image" /></td>
</tr>
<tr>
<td>24</td>
<td><img src="G_24" alt="Image" /></td>
<td><img src="G_24_0.1" alt="Image" /></td>
<td><img src="G_24_0.2" alt="Image" /></td>
<td><img src="G_24_0.3" alt="Image" /></td>
<td><img src="G_24_0.4" alt="Image" /></td>
</tr>
</tbody>
</table>
Table A 4: Topology optimization results for Figure 25c

<table>
<thead>
<tr>
<th>$G_e$</th>
<th>$v = 0$</th>
<th>$v = 0.1$</th>
<th>$v = 0.2$</th>
<th>$v = 0.3$</th>
<th>$v = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image6.png" alt="Image" /></td>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
<td><img src="image10.png" alt="Image" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
<td><img src="image13.png" alt="Image" /></td>
<td><img src="image14.png" alt="Image" /></td>
<td><img src="image15.png" alt="Image" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image16.png" alt="Image" /></td>
<td><img src="image17.png" alt="Image" /></td>
<td><img src="image18.png" alt="Image" /></td>
<td><img src="image19.png" alt="Image" /></td>
<td><img src="image20.png" alt="Image" /></td>
</tr>
<tr>
<td>5</td>
<td><img src="image21.png" alt="Image" /></td>
<td><img src="image22.png" alt="Image" /></td>
<td><img src="image23.png" alt="Image" /></td>
<td><img src="image24.png" alt="Image" /></td>
<td><img src="image25.png" alt="Image" /></td>
</tr>
<tr>
<td>6</td>
<td><img src="image26.png" alt="Image" /></td>
<td><img src="image27.png" alt="Image" /></td>
<td><img src="image28.png" alt="Image" /></td>
<td><img src="image29.png" alt="Image" /></td>
<td><img src="image30.png" alt="Image" /></td>
</tr>
<tr>
<td>8</td>
<td><img src="image31.png" alt="Image" /></td>
<td><img src="image32.png" alt="Image" /></td>
<td><img src="image33.png" alt="Image" /></td>
<td><img src="image34.png" alt="Image" /></td>
<td><img src="image35.png" alt="Image" /></td>
</tr>
<tr>
<td>10</td>
<td><img src="image36.png" alt="Image" /></td>
<td><img src="image37.png" alt="Image" /></td>
<td><img src="image38.png" alt="Image" /></td>
<td><img src="image39.png" alt="Image" /></td>
<td><img src="image40.png" alt="Image" /></td>
</tr>
<tr>
<td>12</td>
<td><img src="image41.png" alt="Image" /></td>
<td><img src="image42.png" alt="Image" /></td>
<td><img src="image43.png" alt="Image" /></td>
<td><img src="image44.png" alt="Image" /></td>
<td><img src="image45.png" alt="Image" /></td>
</tr>
<tr>
<td>24</td>
<td><img src="image46.png" alt="Image" /></td>
<td><img src="image47.png" alt="Image" /></td>
<td><img src="image48.png" alt="Image" /></td>
<td><img src="image49.png" alt="Image" /></td>
<td><img src="image50.png" alt="Image" /></td>
</tr>
</tbody>
</table>
Appendix B

The calculation method of the statically determination problem was written with MathCad 15 program.

<table>
<thead>
<tr>
<th>Geometry of the three-bar truss:</th>
</tr>
</thead>
</table>
| $a := 1$  
$b := \sqrt{3}$  
$cn := \frac{b}{\sqrt{a^2 + b^2}}$  
$sn := \frac{a}{\sqrt{a^2 + b^2}}$ |
| $L_1 := \frac{b}{cn}$  
$L_2 := b$  
$L_3 := \frac{b}{cn}$ |
| $E := 1$ |
| Load cases:  
$V := 10$  
$H := 5$ |
| Compliance limit:  
$K := 50$ |
| Objective function:  
$f(x_1, x_2, x_3, x_4) := x_1 \frac{b}{cn} + x_2 b + x_3 \frac{b}{cn}$ |
| $x_1 := 0.5$  
$x_2 := 1.1$  
$x_3 := 0.5$  
$x_4 := 0$ |
| Given:  
$K = \left[ L_1 \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})^2}{Ex_1} + L_3 \left( \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})^2}{Ex_3} + L_2 \frac{x_4^2}{Ex_2} \right) \right] \geq 0$  
$K = \left[ L_1 \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})^2}{Ex_1} + L_3 \left( \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})^2}{Ex_3} + L_2 \frac{x_4^2}{Ex_2} \right) \right] \geq 0$ |
| Cross-section: |
| $0,0001 \leq x_1 \leq 200$  
$0,0001 \leq x_2 \leq 200$  
$0,0001 \leq x_3 \leq 200$ |
| Statically indeterminate force:  
$-100 \leq x_4 \leq 100$ |
| Compatibility:  
$2 \left( L_2 \frac{x_4}{Ex_2} \right) cn - L_1 \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})^2}{Ex_1} = \left( \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})}{Ex_3} \right)^2$ |
| $P := \text{Minimize}(f, x_1, x_2, x_3, x_4)$ |
| $f(P_0, P_1, P_2, P_3) = 18,669$ |
| $f_1(x_1, x_2, x_3, x_4) := L_1 \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})^2}{Ex_1} + L_3 \left( \frac{(-x_4 \frac{V}{2cn} + \frac{H}{2im})^2}{Ex_3} + L_2 \frac{x_4^2}{Ex_2} \right)$ |
| $f_1(P_0, P_1, P_2, P_3) = 50$ |
| $f_{11}(x_1, x_2, x_3, x_4) := L_1 \frac{\left( \frac{0}{2cn} + \frac{V}{2im} \right)^2}{Ex_1} + L_3 \left( \frac{\left( \frac{0}{2cn} + \frac{V}{2im} \right)^2}{Ex_3} + 0 \right)$ |
| $f_{11}(P_0, P_1, P_2, P_3) = 50,021$ |
\[ f_2(x_1, x_2, x_3, x_4) := L_1 \left( \frac{-x_4}{2cn} \frac{V}{2cn} \frac{H}{2sn} \right)^2 + L_3 \left( \frac{-x_4}{2cn} \frac{V}{2cn} \frac{H}{2sn} \right)^2 + L_2 \frac{x_4^2}{E s} \]

\[ f_2(P_0, P_1, P_2, P_3) = 50 \]

\[ f_{22}(x_1, x_2, x_3, x_4) := L_1 \left( \frac{0}{2cn} \frac{V}{2cn} \frac{H}{2sn} \right)^2 + L_3 \left( \frac{0}{2cn} \frac{V}{2cn} \frac{H}{2sn} \right)^2 + 0 \]

\[ f_{22}(P_0, P_1, P_2, P_3) = 50,021 \]

\[ DL_1 := \frac{1}{E P_0} \left( \frac{-P_3}{2cn} \frac{V}{2cn} \frac{H}{2sn} \right)^{-1} \quad DL_1 = 0.165 \]

\[ DL_2 := \frac{1}{E P_1} (P_2)^2 \quad DL_2 = 0.88 \]

\[ DL_3 := \frac{1}{E P_2} \left( \frac{-P_3}{2cn} \frac{V}{2cn} \frac{H}{2sn} \right)^{-1} \]

\[ f(P_0, P_1, P_2, P_3) = 18,669 \]

\[ P_0 = 4,665 \]

\[ P_1 = 5,583 \times 10^{-3} \]

\[ P_2 = 4,665 \]

\[ P_3 = 4,911 \times 10^{-3} \]
Appendix C

The calculation method of the three-bar frame structure problem was written with Matlab R2016b program.

```
clearvars;
E = 20000;
F = 100;
beta = pi/6;
sgmL = 20;
alpha = atan(sqrt(2));
A1 = F/sgmL*(3*sqrt(6)/8*sin(beta));
A2 = F/sgmL*(cos(beta)-sqrt(2)/4*sin(beta));
A3 = A1;
b1 = 1; h1 = A1;
b2 = 1; h2 = A2;
b3 = b1; h3 = h1;
l1 = 1/12*b1*h1^3;
W1 = l1/h1*2;
l2 = 1/12*b2*h2^3;
W2 = l2/h2*2;
l3 = l1;
W3 = W1;
fi1 = 3/2*pi-alpha;
fi2 = 3/2*pi;
fi3 = 3/2*pi+alpha;
T1 = [cos(fi1),-sin(fi1),0;sin(fi1),cos(fi1),0;0,0,1];
T2 = [cos(fi2),-sin(fi2),0;sin(fi2),cos(fi2),0;0,0,1];
T3 = [cos(fi3),-sin(fi3),0;sin(fi3),cos(fi3),0;0,0,1];
Z = zeros(3,3);
z3 = zeros(3,1);
rv = (10:100);
```
sgmv = zeros(size(rv));
incv = zeros(size(rv));
sgmratv = zeros(size(rv));
for i = 1:size(rv,2)
    r = rv(i);
    H = r*A2;
    L1 = H/cos(alpha);
    L2 = H;
    L3 = L1;
    B1 = [1,0,0;0,1,0;0,L1,1];
    B2 = [1,0,0;0,1,0;0,L2,1];
    B3 = [1,0,0;0,1,0;0,L3,1];
    Ai = A1; li = l1; Li = L1;
    K1lok = E*[Ai/Li,0,0,-Ai/Li,0,0;
             0,12*li/Li^3,6*li/Li^2,0,-12*li/Li^3,6*li/Li^2;
             0,6*li/Li^2,4*li/Li,0,-6*li/Li^2,2*li/Li;
            -Ai/Li,0,0,Ai/Li,0,0;
            0,-12*li/Li^3,-6*li/Li^2,0,12*li/Li^3,-6*li/Li^2;
            0,6*li/Li^2,2*li/Li,0,-6*li/Li^2,4*li/Li];
    K3lok = K1lok;
    Ai = A2; li = l2; Li = L2;
    K2lok = E*[Ai/Li,0,0,-Ai/Li,0,0;
             0,12*li/Li^3,6*li/Li^2,0,-12*li/Li^3,6*li/Li^2;
             0,6*li/Li^2,4*li/Li,0,-6*li/Li^2,2*li/Li;
            -Ai/Li,0,0,Ai/Li,0,0;
            0,-12*li/Li^3,-6*li/Li^2,0,12*li/Li^3,-6*li/Li^2;
            0,6*li/Li^2,2*li/Li,0,-6*li/Li^2,4*li/Li];
K1 = T1*K1lok(1:3,1:3)*T1';
K2 = T2*K2lok(1:3,1:3)*T2';
K3 = T3*K3lok(1:3,1:3)*T3';
K = K1+K2+K3;
q = F*[sin(beta);cos(beta);0];
v = K\q; 
s1end = K1lok(4:6,:)*[T1'*v;z3];
s1st = B1*s1end;
sgm1end = s1end(1)/A1+abs(s1end(3))/W1;
sgm1st = s1st(1)/A1+abs(s1st(3))/W1;
s2end = K2lok(4:6,:)*[T2'*v;z3];
s2st = B2*s2end;
sgm2end = s2end(1)/A2+abs(s2end(3))/W2;
sgm2st = s2st(1)/A2+abs(s2st(3))/W2;
s3end = K3lok(4:6,:)*[T3'*v;z3];
s3st = B3*s3end;
sgm3end = s3end(1)/A3+abs(s3end(3))/W3;
sgm3st = s3st(1)/A3+abs(s3st(3))/W3;
sgmv(i) = sgm2end;
incv(i) = 100*(sgm2end-sgmL)/sgmL;
sgmratv(i) = sgm2end/sgmL;
fprintf(1,'r = %8.3f  H = %8.3f   sgm = %9.6f  inc = %6.3f%%\n',
); 
end
hf = figure;
hold on;
hp = plot(rv,sgmratv);
set(hp,'Color',[0,0,0],'LineWidth',2);
grid on;
xlabel('length / depth','FontSize',12);
ylabel('max stress / stress limit','FontSize',12);
Appendix D

Az értekezés tézisei

Az alábbiakban került összefoglalásra magyar nyelven a doktori képzés során elérterutatási eredmények, melyek jelen disszertáció részét képezik.

1. Tézis

Függetlenül levezettem a legkisebb súlyra történő tervezésnek megfelelő analitikus megoldásokat rácsos-jellegű tartók feszültségalapú képlékeny optimálásának esetében két különböző nagyságú és irányú alternáló erőt figyelembe véve. Megállapítottam, hogy bizonyos teheresetekben az optimális képlékeny megoldás statikailag határozott, így egyben az az optimális rugalmas megoldás is, és a kapott eredmények összhangban vannak a Nagtegaal-Prager levezetéseivel, a következtetések megfelelnek Sved megállapításainak.

Kapcsolódó publikációk: [EP10], [EP9], [EP7]

2. Tézis

Nagyszámú számítógépes simuláció segítségével megvizsgáltam két diszkretizációs paraméter (az alapelemméret és a Poisson-tényező) optimális topológiára gyakorolt hatását négy síkbeli tárcsa-alapfeladaton iterációs algoritmussal engedékenységi feltétel mellett legkisebb súlyra való optimális tervezés esetében.

(a) Megállapítottam, hogy négycsomópontú téglalap-elemek alkalmazása esetén, változatlan végeselemméret mellett, az optimális választás a hármas alapelem méret a leghatékonyabb számítási idő elérése érdekében.

(b) Megállapítottam, hogy míg a végeselem-hálózat mérete állandó, az alapelem-méret növelésével a topológia-optimálás konvergenciája gyorsabb lesz, de a felbontás csökken és a kényszerek numerikusan gerjesztett, mesterségesen nagy merevülését eredményeznek. Továbbá különböző optimális megoldási alakzatok keletkeznek, míg végül az eredmények durvasága miatt a megoldások nem lesznek alkalmasak tervezésre.

Kapcsolódó publikáció: [EP1]
3. Tézis
Kiterjesztettem diszkretizált síkbeli tárcsák Kontinuum-típusú Optimalitási Feltételen (COC) alapuló topológia-optimálási feladat-megfogalmazását a teherhelyzet bizonytalanságának figyelembevételére helyettesítő determinisztikus teheresetek alkalmazásával. Paraméteres vizsgálatok segítségével igazoltam, hogy a módszer alkalmas numerikus számítások elvégzésére elfogadható számítási idő alatt számottevő hibák nélkül, amennyiben legalább három helyettesítő erőt alkalmazunk.


4. Tézis
Megvizsgáltam az optimális rácsos-tartó kialakítások statikai határozottságát és egyediségét rugalmas feszültségi és engedékenységi feltételek mellett.
(a) Két egyszerű, fix terhelésű térbeli szerkezeti példa segítségével igazoltam, hogy ha létezik statikailag határozott legkisebb súlyt adó optimális megoldás, akkor végzetlen számú alternatív statikailag határozatlan szerkezet hozható létre azonos súlyval, azonos engedékenységi vagy feszültségi feltételek mellett.
(b) Meghatároztam egy síkbeli paraméteres szerkezeti probléma legkisebb súlyú optimális megoldását a teher irányának bizonytalanságát figyelembe véve és megállapítottam a teher irányának azon tartományát, amelyen belül az optimális megoldás statikailag határozatlan, és amelyen kívül statikailag határozott. Megállapítható, hogy a legkisebb súlyt adó optimális szerkezet nem feltétlenül statikailag határozott, amennyiben a teher iránya változó.


5. Tézis
(a) Meghatároztam egy legfeljebb három rúdból álló rácsos tartó legkisebb súlyra tervezett kialakításának megfelelő optimális topológiát, alakot és méretet, adott támaszegyenes és adott támadáspontban működő két alternáló erő esetén rugalmas feszültségkorlát mellett a szerkezetet leíró paraméterek széles tartományára.

Megállapítottam, hogy az optimális megoldás tipikusan statikailag határozott (2 rúd), míg egy bizonyos paraméter-tartományban egyetlen rúddá fajul el, illetve egy másik tartományban a határozott megoldással azonos súlyt adó statikailag határozatlan (3 rúd) kialakítás egyidejűleg létezik, vagyis a megoldás nem egyértelmű. Az elfajuló és a határozatlan esetek numerikus eredményeit analitikus levezetésekkel igazoltam.
(b) Létrehoztam egy merev kapcsolatú keretmodellt és egy (síkbeli feszültségállapotú) kétdimenziós kontinuum-szerkezetet egy kiválasztott háromrudás rácsostartó-megoldással azonos alakkal és mérettel, amelyeken feszültségszámítással meghatároztam a modellválasztásnak az optimális szerkezetre gyakorolt hatását. Megállapítottam, hogy a keretszerkezet és a kontinuum-szerkezet feszültségmaximuma megegyezik, és a hajlítóhatás-többlet miatt jelentősen túllépik a rácsos tartó feszültségeit, amely a szerkezet alultervezettségéhez vezet, amennyiben rácsos tartó modellt alkalmazunk. Következtetésképpen, a merev kapcsolatú optimális szerkezetek meghatározására a rácsostartó-modell nem alkalmas.

Kapcsolódó publikációk: [EP6], [EP5], [EP4], [EP3], [EP2]