Wrinkling behavior of highly stretched thin films

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BRIEF SUMMARY

In the literature of the wrinkling of ultrathin films, theoretical models were often used to investigate phenomena that fall outside the range of validity of their assumptions. Contradictory results were published regarding the wrinkling of axially stretched, clamped, rectangular films. Based on the Föppl-von Kármán plate theory (FvK) wrinkles appear on the surface and their amplitude scales with the macroscopic strain. A finite membrane strain extension of the FvK theory (eFvK) predicted that only a bounded region of the macroscopic strain and aspect ratio parameters exhibit wrinkling in the problem.

The first goal of the work is to verify these predictions experimentally. Secondly, the eFvK model was extended into directions, that are beneficial in the modeling of real-life structures. The effect of material and geometrical properties and nonlinearities on wrinkling was also examined in detail.

Orthotropy was incorporated into the model and its effect was investigated. By comparing the results of the orthotropic model and experiments carried out on prestressed polyurethane films, the predictions of the eFvK model were experimentally verified. Motivated by further experimental observations, a pseudoelastic model based on the Mullins effect was also proposed and it was shown, that it can explain the intriguing phenomenon arisen during the prestressing of polyurethane films.

The eFvK model was extended to curved surfaces, and it was shown, that the intrinsic curvature can reduce wrinkling. Finally, we carried out an investigation of the parameter space by letting the thickness of the film arbitrary small. We introduced the concept of disturbed zones near the clamps which gave a physical explanation of the aspect ratio dependency of the wrinkling in the model problem.
A vékony filmek ráncosodását különböző elméleti megközelítésekkel vizsgálják az irodalomban, ezeket azonban gyakran a modellek alkalmazhatósági határain kívül eső jelenségek leírására is felhasználták. Ennek eredményeképpen egymásnak el-lentmondó eredmények jelentek meg a két végén befogott, húzott, tégglap alakú filmek ráncosodásának témakörében. A Kármán-féle lemezelmélettel (FvK) lev-ezetett elméleti eredmények alapján a felületen ráncok jelennek meg, amelyek a megnyúlással arányosan nőnek. Ezzel szemben az FvK elmélet véges membrán nyúlásos kiterjesztése (eFvK) a megnyúlás és az oldalarány paramaméterek csak egy zárt tartományában jelez ráncosodást.


Az anyagi ortotrópia hatását az eFvK elmélet ortotróp kiterjesztésén keresztül vizsgáltuk. Előfeszített poliuretán filmeken végzett kísérletek eredményeinek és az ortotróp modell eredményeinek összevetése segítségével kísérletileg igazoltuk az eFvK elmélet előrejelzéseit. Bevezettünk továbbá egy pszeudoelasztikus modellt, amely tartalmazza az ún. Mullins hatást és magyarázza a poliuretán filmek előfeszítése során megfigyelt ráncosodási viselkedést.

Az eFvK elméletet kiterjesztettük gőrbült felületekre és rámutattunk, hogy a felület kezdeti gőrűlete csökkenti a ráncosodást. Végül pedig az eredeti probléma paraméterterét vizsgáltuk tetszőlegesen vékony film esetén. Bevezettük a peremek mentén kialakuló, zavart feszültségi zóna fogalmát, amely fizikai magyarázatot ad a két végén befogott, húzott filmek ráncosodásának oldalarányfüggésére.
NOMENCLATURE

Lower case Latin letters

\( a \) alternating tensor
\( b \) curvature tensor
\( c \) vector of unknowns in the Finite Element method
\( e \) linearized in-plane strain tensor
\( g_{\alpha\beta} \) metric tensor
\( g_1 \) axial unitvector
\( g_2 \) transversal unitvector
\( g_3 \) normal unitvector
\( h \) thickness of the film
\( k \) linearized bending strain tensor
\( l \) distance between two points of the plate
\( m \) number for counting
\( n \) normal vector
\( p_i \) parameters of the pseudoelastic model
\( q \) orthotropy parameter
\( r \) orthotropy parameter
\( s \) map between the parametric and global coordinate systems
\( t \) surface tangent
\( u \) displacement field of the mid-surface
\( \hat{u} = [u_1, u_2] \) in-plane displacements
\( v \) general three-dimensional displacement vector of a plate
$w$ out-of-plane displacement

$x$ axial direction

$y$ transversal direction

$z$ direction perpendicular to the surface

**Upper case Latin letters**

$A_i$ discretisation of the derivatives of the basis functions

$C$ right Cauchy-Green strain tensor

$E$ in-plane strain tensor

$F$ deformation gradient

$H^k$ Sobolev space

$I$ unit tensor

$I$ total potential energy

$J$ Jacobian

$K$ a triangular finite element

$L$ length of the rectangular film

$L^2$ space of square-integrable functions

$M$ bending moment

$N$ 2nd Piola-Kirchhoff stress tensor

$N$ number of iterations in the Newton-Raphson method

$P_i$ a general point of a plate

$R$ radius of a cylinder

$S_{xy}$ shear modulus

$S$ finite element boundary

$V$ finite element space
$W$  
width of the rectangular film

$Y$  
elastic modulus of an isotropic material

$Y_x$  
elastic modulus in the axial direction

$Y_y$  
elastic modulus in the transversal direction

**Lower case Greek letters**

$\alpha_i$  
material parameters

$\beta$  
aspect ratio

$\gamma$  
parametric domain of the mid-surface

$\delta$  
arbitrary positive number

$\varepsilon$  
Green-Lagrangian strain

$\varepsilon$  
macroscopic strain

$\varepsilon_{cr1}$  
critical stretch at the appearance of wrinkles if $\varepsilon$ is increased

$\varepsilon_{cr2}$  
critical stretch at the disappearance of wrinkles if $\varepsilon$ is increased

$\varepsilon_{cr3}$  
critical stretch at the appearance of wrinkles if $\varepsilon$ is decreased

$\zeta$  
test function a function from the finite element space

$\eta$  
dissipation field of the pseudoelastic model

$\theta$  
scalar

$\theta_j$  
basis function

$\kappa$  
intrinsic curvature of the surface

$\lambda$  
principal stretch

$\lambda_{\text{min}}$  
smallest eigenvalue of the Jacobian

$\mu$  
vector-valued test function from the finite element space

$\nu$  
Poisson’s ratio

$\zeta, \eta$  
parametric coordinate system
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>Cauchy stress tensor</td>
</tr>
<tr>
<td>$\sigma_{\text{eng}}$</td>
<td>engineering stress</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>penalty parameter</td>
</tr>
<tr>
<td>$\tau$</td>
<td>triangulation of the domain</td>
</tr>
<tr>
<td>$\varphi_j$</td>
<td>basis function</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Airy stress function</td>
</tr>
<tr>
<td>$\omega$</td>
<td>stepsize in the numerical continuation</td>
</tr>
</tbody>
</table>

**Upper case Greek letters**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>union of the interior edges of the finite elements</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>vector of basis functions</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>map between the curvilinear and global coordinate systems</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>a closed domain occupied by the plate</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>dissipation function</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>energy density</td>
</tr>
<tr>
<td>$\Psi_m$</td>
<td>membrane energy density</td>
</tr>
<tr>
<td>$\Psi_b$</td>
<td>bending energy density</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>reference domain in $\mathbb{R}^2 \times {0}$</td>
</tr>
<tr>
<td>$\partial \Omega$</td>
<td>boundary of the reference domain</td>
</tr>
</tbody>
</table>

**Other notation**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\det(.)$</td>
<td>determinant</td>
</tr>
<tr>
<td>$\text{tr}(.)$</td>
<td>trace</td>
</tr>
<tr>
<td>$\nabla(.)$</td>
<td>gradient</td>
</tr>
<tr>
<td>$\nabla \cdot (.)$</td>
<td>divergence</td>
</tr>
<tr>
<td>$\nabla^2(.) = \nabla \nabla$</td>
<td>second gradient</td>
</tr>
</tbody>
</table>
\[ \Delta(\cdot) = \nabla \cdot \nabla \quad \text{laplace operator} \]

\[ \Delta^2(\cdot) = \Delta \Delta \quad \text{biharmonic operator} \]

\[ \cdot \quad \text{scalar product} \]

\[ \otimes \quad \text{outer product} \]

\[ \int_{\Omega} \cdot d\Omega = \int_{-W/2}^{W/2} (\int_{-L/2}^{L/2} (\cdot) dx) dy \]

\[ [:\cdot:] \quad \text{jump operator of the Interior Penalty method} \]

\[ \{\cdot\} \quad \text{average operator of the Interior Penalty method} \]
CHAPTER 1

Introduction

1.1 The wrinkling phenomenon

Ultrathin films under compression due to their small bending stiffness prone to a buckling phenomenon, called wrinkling. Generally speaking, being lightweight and having low space requirements, they are easy to transport and can cover large areas cost-efficiently in the form of tensile membrane structures or pneumatic load-bearing structures. These advantageous properties also make them frequently employed in space technology; thus thin films are the main components of solar sails (Figure 1.1a), inflatable antennas, radars (Talley et al. 2002; Sakamoto & Park 2005) and thin film ballutes (Rohrschneider 2007).

![Figure 1.1: (a) Models of solar sails (NASA Langley Research Center). (b) Wrinkling of a tensile structure (The Nomad Concept, tensinet.com).](image)

However, uneven surfaces often cause an undesired alteration of the functionality of engineering structures. Accordingly, they are designed to stay wrinkle-free under the design loads. Nevertheless, in most of the cases, wrinkles are impossible to avoid entirely, and the discussion of their source and shape is inevitable. Besides in the engineering practice, wrinkle prevention is also necessary in industrial applications, e.g., in deep-drawing processes (L. C. Zhang & Yu 1986), in case of large sheets
transported by rolls (Jacques, Elias, et al. 2007; Yanabe, Nagasawa, & Kaneko 2018) (Figure 1.2a), metal coils (Figure 1.2b) and in biomedical problems such as the healed shape of a surgical wound (Cerda 2005; Hudson & Renshaw 2006).

Figure 1.2: (a) Wrinkling during strip conveying (Jacques, Elias, et al. 2007). (b) Edge waves of a stainless steel coil (Leveltek Processing, LLC).

In some cases, wrinkled patterns comprise information that can be resolved lying on the morphology of thin films. Analyzing the wrinkles on soft elastic substrates induced by cell motility (Harris, Wild, & Stopak 1980; Burton & D. L. Taylor 1997; Arocena et al. 2018) (Figure 1.3a) or the wrinkles on microcapsules in a flow (Figure 1.3b) (Walter, Rehage, & Leonhard 2000) turned out to carry information about the working forces.

Figure 1.3: (a) Wrinkling induced by cell motility (Burton & D. L. Taylor 1997). (b) Shear-induced wrinkles on vesicles (Walter, Rehage, & Leonhard 2001).

Most of the research studies related to wrinkling examine simple model problems. Instead of investigating complicated structures under various loads, composed of thin films having complex material properties the different factors affecting wrinkling aimed to be handled separately. In most of the cases, a model problem consists of a thin film with a simple geometry subjected to a well-defined load. Some of the
most popular problems are rectangular films subjected to shear forces (Jenkins, Hau- gen, & Spicher 1998; Wong & Pellegrino 2006), diagonal tension (Tomita & Shindo 1988), axial tension (Friedl, Rammerstorfer, & Fischer 2000; Cerda, Ravi-Chandar, & Mahadevan 2002) thermal loading (Attipou et al. 2015) and cases where circle symmetric wrinkles occur (Davidovitch et al. 2011; Vella et al. 2015; Wang et al. 2016). Although some of these problems are well-understood, the appropriate mechanical models for wrinkling under different conditions as well as the shape and distribution of the emerging wrinkle patterns are still under discussion.

1.2 Model problem

1.2.1 Mechanical background

Figure 1.4: The laterally contracted, stretched film under \( \varepsilon \) macroscopic strain. Two edges of the rectangular domain are clamped and the other two edges are free. The dashed lines represent the undeformed rectangular domain in the reference configuration.

Thin films clamped at two ends while the other two sides are free (Figure 1.4) wrinkle under axial tensile load. Buckling of plates and wrinkling of thin films under compression is a well-known phenomenon, but the wrinkling behavior of stretched sheets was first introduced by Friedl, Rammerstorfer, & Fischer 2000 and gained interest only after the publications of Cerda, Ravi-Chandar, & Mahadevan 2002. Benthem 1963 analyzed the stress state of a semi-infinite, stretched, clamped strip and found that the transverse stress is significantly affected by the boundary conditions. In the
vicinity of the boundaries, a zone of transversal compression develops, which can lead to the stability loss of the initially flat surface. At a critical value of the macroscopic strain, the surface buckles and the sheet becomes decorated with the emerging wrinkle patterns. The wrinkle crests are perpendicular to the direction of the maximum compression.

The mechanical background of the observed phenomenon is still under discussion. Friedl, Rammerstorfer, & Fischer 2000 considered the problem as a superposition of axial stretching and transversal contraction resulting from the Poisson effect. Consequently, the emerging compression was attributed to the restrained transversal displacement at the boundaries. However, Silvestre 2016 revealed, that warping shear also plays an essential role in the wrinkling phenomenon besides the Poisson effect. The wrinkle crests decorate only a part of the sheet, which is called mode localization and attributed to the localization of the compressive stresses by Jacques & Potier-Ferry 2005.

1.2.2 Approaches to examine the problem

Wrinkling, in general, can be investigated using the tension-field theory (Reissner 1938; Coman & Bassom 2007), which focuses on the in-plane stress state of the sheet and neglects the bending stiffness of the film. Consequently, it is incapable of determining the shape of the wrinkles, but depending on the $m$ number of the negative eigenvalues of the stress tensor, it distinguishes taut ($m = 0$), wrinkled ($m = 1$) and slack ($m = 2$) zones in the domain. Moreover, the direction of the positive principal stress indicates the alignment of the wrinkle crests. The theory is related to the zero limit in the thickness and works with the stress field. Since stresses are hard to measure in the laboratory, experimental comparisons are challenging to carry out.

Cerda & Mahadevan 2003 suggested to examine wrinkling using the Föppl-von Kármán plate theory (from now: FvK theory, von Kármán 1910) that is capable of modeling the wrinkled pattern. It assumes non-vanishing, but small thickness compared to the other dimensions of the plate and allows for deflections that are in the same order as the thickness. Nonetheless, the normals are assumed to stay perpendicular to the surface during the deformation. As a result, the displacements of the plate are represented as the displacements of its mid-surface. The in-plane strains are considered to be infinitesimal therefore the deformation tensor is linearized. The FvK theory was used to examine numerically and analytically the wrinkling of stretched rectangular films (Puntel, Deseri, & Fried 2011; Kim, Puntel, & Fried 2012; Q. Huang et al. 2015). The theoretical results were also compared with experiments (Cerda &
Mahadevan 2003; Jen & Wu 2015; Zhu, X. Zhang, & Wierzbicki 2018). However, as it was pointed out by Healey, Q. Li, & Cheng 2013, the small strain assumption did not hold for the examined parameter regions. In particular, the FvK theory was applied out of the range of its validity.

Healey, Q. Li, & Cheng 2013 abandoned the small strain assumptions of the FvK theory and extended the model to finite membrane strains by keeping the nonlinear terms of the deformation tensor. The need for finite membrane strains to analyze the problem was recognized by M. Taylor, Bertoldi, & Steigmann 2014 as well and modeled the phenomenon with the Koiter shell theory extended to finite strains.

Consequently, not only the mechanical background but also the appropriate theoretical model is under discussion. Although there exist analytical investigations, closed-form solutions of the governing equations are not known. The numerical computations are usually based on the Finite Element Method and accompanied with dynamic relaxation (Day A S. 1965; M. Taylor, Davidovitch, et al. 2015) or the numerical continuation of the solution set (Allgower & Georg 1990; Healey, Q. Li, & Cheng 2013). In particular, the commercial software ABAQUS is widely used to compute the shape of the wrinkles and its results are compared either to experimental results or analytical predictions (Friedl, Rammerstorfer, & Fischer 2000; Zheng 2008; Nayyar, Ravi-Chandar, & R. Huang 2011; Nayyar, Ravi-Chandar, & R. Huang 2014; Silvestre 2016).

The experimental investigation of a stretched, rectangular film requires an ultrathin, hyperelastic material being able to sustain large strains (∼ 30% – 70%) without failure. Typical materials, such as Kapton used to examine wrinkling under shear deformation (Wong & Pellegrino 2006) are not flexible enough for stretching. In the literature silicone (Zheng 2008), polyethylene (Cerda, Ravi-Chandar, & Mahadevan 2002; Nayyar, Ravi-Chandar, & R. Huang 2014; Jen & Wu 2015) and polypropylene (Zhu, X. Zhang, & Wierzbicki 2018) films were used to capture the wrinkles on clamped, stretched rectangular films. Unfortunately, not only the material behavior of these films is nonlinear, but they also have significant plastic deformations. Both of these factors make it hard to compare the experimental data with numerical computations or analytical results usually derived assuming linear elasticity. Moreover, most of the models in the literature tend to use simple, isotropic, hyperelastic constitutive models.

Different approaches often led to contrasting results. Surprisingly, there is controversy about whether a specific geometry and a fixed stretch of the film lead to wrinkling. The existence of wrinkles is a basic question and needs clarification before investigating more complex problems. Although the investigation of the exact shape of the wrinkles is out of the scope of this study, it is worth to mention that the shape
of the wrinkles emerging under different parameters and the number of wrinkles are also highly debated (Puntel, Deseri, & Fried 2011; Kim, Puntel, & Fried 2012; Cerda & Mahadevan 2003; Zhu, X. Zhang, & Wierzbicki 2018). The range of validity of the applied theories should be clarified to resolve these issues.

1.2.3 The parameter space

It is agreed from the earliest discussions of the problem, that having a sufficiently small, fixed thickness $h$ the two main parameters affecting the emerging wrinkle pattern are the applied macroscopic strain $\varepsilon$ and the aspect ratio $\beta$. They are straightforward to control in the experiments, and they turned out to have a qualitative effect on the shape of the wrinkles. However, conflicting numerical, analytical and experimental results appeared in the literature regarding the wrinkling behavior in the $\beta - \varepsilon$ plane.

![Figure 1.5: Wrinkling depending on the $\beta - \varepsilon$ parameter configurations according to the finite strain extension of the Föppl-von Kármán plate theory. For certain aspect ratios wrinkles appear and disappear as the macroscopic strain is increased, but for a fixed length sufficiently narrow or wide films do not wrinkle.](image)

According to the work of Cerda, Ravi-Chandar, & Mahadevan 2002; Cerda & Mahadevan 2003 based on the FvK theory, the amplitude of the wrinkles increase as the stretch is increased, which was proved to be only valid at small strains relying on computations with ABAQUUS (Nayyar, Ravi-Chandar, & R. Huang 2011) and experiments (Nayyar, Ravi-Chandar, & R. Huang 2014; Zheng 2008). After the wrinkles
emerged on the surface at a critical stretch $\varepsilon_{ct1}$, their amplitude first increases but then it decreases after the stretch exceeded a specific value. Furthermore, in the numerical simulations with ABAQUS, it eventually reached zero at large stretches. The experiments available in the literature were unsuitable to verify the results. Moreover, the nonlinear elastic material used in the computations made the interpretation of the behavior unclear and the effect of the aspect ratio was not investigated.

Healey, Q. Li, & Cheng 2013 carried out a precise and reliable bifurcation analysis based on a geometrically exact model, the finite strain extension of the Föppl-von Kármán equations (from now on: eFvK theory) and pointed to the error of the small strain assumption. The extended model predicted the disappearance of wrinkles at a second critical stretch $\varepsilon_{cr2}$, assuming linear elasticity. Furthermore, they showed that just a bounded regime of aspect ratios exhibits wrinkling (Figure 1.5). These findings pointed to an *isola-center bifurcation*. Nonetheless, independently of the thickness and the macroscopic strain, there exists a region of the aspect ratios, where no wrinkling occurs. In contrast, the FvK theory fails to catch the disappearance of wrinkles and exhibits wrinkling for a vast range of aspect ratios out of the bounded region determined by the eFvK.

The shape of the wrinkles is also strongly affected by the aspect ratio. Short, medium-sized and elongated sheets result in different wrinkle patterns (Friedl, Rammerstorfer, & Fischer 2000; Nayyar, Ravi-Chandar, & R. Huang 2014). It was observed, that if the sheets are elongated enough, the critical tensile stress is independent of the length parameter which was attributed to the localization of the compressive stress (Jacques & Potier-Ferry 2005).

### 1.3 Research goals

As summarized above, the wrinkling phenomenon is analyzed by several mechanical and material models, different numerical methods leaving open questions that could not be resolved based on the few experiments available in the literature. Moreover, comparability of experimental and theoretical results in the literature is often questionable due to the lack of understanding of the effect of material properties. Most of the works neglect the material nonlinearities and anisotropy, although polymers used in the experiments are often orthotropic as a result of the fabrication process. To be able to apply the findings to complex, real-life structures it is necessary to clarify the applicability of the theoretical models and the effect of material properties.

The main subject of this work is to explore the wrinkling behavior of the model problem in the $\beta - \varepsilon$ parameter plane both theoretically and experimentally. In previ-
ous experimental and numerical studies phenomena originating from mechanical and material properties were often handled together leaving it unclear what is the primary cause of the observed behavior. We aim to strictly separate material effects and carry out new experiments to address some of the controversial topics. Furthermore, we mostly concentrate only on the existence of wrinkles depending on various geometric and material parameters. The first and foremost goal of this study is to

- (1) Experimentally verify the predictions of the eFvK theory on the disappearance of wrinkles and the bounded stability region.

Our subsequent investigations are based on the eFvK theory and motivated by experimental observations and the intention to gain a deeper understanding of how wrinkling is affected by different parameters. Furthermore, we extend the model into directions that can be beneficial in the modeling of real-life structures. Although in the literature the FvK theory was applied and extended in many directions, most of the results were formulated taking a special form of the equations with the so-called Airy-stress function. It can be showed, that in case of finite strains no such simple form can be derived, hence most of the results in the literature does not apply automatically for the eFvK theory.

Before the interpretation of goal (1) in Chapter 2, we introduce the FvK theory and its finite strain extension, then the appropriate numerical methods used to solve the arising equations. Subsequently, we carry out experiments to validate the predictions on two different elastomers. To make a quantitative validation of the predictions, we extend the model to orthotropic materials. Then we answer the following questions:

- (2) What is the effect of material orthotropy on wrinkling?
- (3) How inelastic material properties affect the wrinkling behavior?
- (4) What is the effect of intrinsic curvature?

Question (2) is answered based on numerical computations in Chapter 2. To answer question (3), we carry out experiments in Chapter 3 on thin sheets showing inelastic material properties and then introduce a pseudoelastic model to explain the observed behavior. We return to linear elasticity and investigate question (4) in Chapter 4 by incorporating small curvature in the model problem. Motivated by the findings, we extend the model to general, curved surfaces. Finally, we turn back to the original problem to investigate a more extended region of the parameter space by letting $h$ arbitrary small and by examining the $\varepsilon \to \infty$, $\beta \to \infty$ directions, we raise the following question:
• (5) What are the possible wrinkling configurations in the $\beta - \varepsilon$ plane and what is the physical explanation for the observed behavior?

We answer question (5) in Chapter 5 using the 2nd Piola-Kirchhoff stress tensor to explore the possible wrinkling parameter configurations in the $\beta - \varepsilon$ plane and introduce the concept of the disturbed zones at the boundaries that can give a physical explanation to the findings. We summarize the results in Chapter 6.
A finite deformation linear elastic model

In this chapter we summarize the Föppl-von Kármán equations and following (Healey, Q. Li, & Cheng 2013), we extend them to the finite in-plane strain regime. After we introduced the governing equations, the applied numerical techniques are also discussed. To verify the predictions of the eFvK model, we carry out experiments on two kind of elastomers. We quantitatively verify the predictions by comparing the wrinkling of prestressed polyurethane films to numerical results of the eFvK extended to orthotropic materials. Finally, we further investigate the orthotropic model numerically to examine the effect of orthotropy.

2.1 Mechanical model

2.1.1 Föppl-von Kármán equations

We investigate the static equilibrium configurations of clamped, axially stretched films. The film is assumed to have homogeneous, isotropic, linear elastic material properties with a constant \( h \) thickness. The length of the undeformed film is measured in the axial direction and denoted by \( L \), and \( W \) is the width in the transversal direction. The aspect ratio is a non-dimensional parameter of the undeformed sheet and defined as:

\[
\beta := \frac{L}{2W}. \tag{2.1}
\]

The transversal edges of the film are clamped, while the other two sides are free, and it is stretched in the axial direction by a \( \varepsilon = \Delta L/L \) macroscopic strain, where \( \Delta L \) is the change in the length (Figure 1.4). Ultrathin films ranging from a few to a few hundred \( \mu m \) thickness values can be considered as thin plates with low bending rigidity. Consequently, the Föppl-von Kármán plate theory (von Kármán 1910; von Kármán & Tsien 1941) allowing large deflections is appropriate to model the wrinkling.
We follow the derivation of (Wierzbicki 2013) and (Howell, Kozyreff, & Ockendon 2008).

We apply the Einstein summation convention, where repeated indices are summed over implicitly. Furthermore, we restrict a range convention for the indices as well: latin indices run over 1,2,3; while greek letters range only over 1,2. The reference coordinate system is \( x = [x_1, x_2, x_3] = [x, y, z] \) and the current coordinate system is \( \bar{x} = [\bar{x}, \bar{y}, \bar{z}] \). The axial, transversal and out-of-plane directions are denoted by \( x, y, z \) and \( \bar{x}, \bar{y}, \bar{z} \) respectively. The plate occupies a closed domain \( \Pi \in \mathbb{R}^3 \) and the displacement of its points is denoted by \( v : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), where \( v(x, y, z) = [v_1, v_2, v_3] \) in the axial, transversal and out-of-plane directions respectively. Let us consider two close points in the initial coordinate systems \( P_1 = x_0 \) and \( P_2 = x_0 + \delta x \), where \( |x| = 1 \) and \( \delta > 0 \) is arbitrary. After the deformation, their position is \( \bar{P}_1 = x_0 + v(x_0) \) and \( \bar{P}_2 = x_0 + \delta x + v(x_0 + \delta x) \). The distance between the points, infinitesimal lengths in the initial and current configurations are denoted by \( \bar{l} = |P_1 - P_2| \) and \( \bar{l} = |\bar{P}_1 - \bar{P}_2| \), respectively. The Green-Lagrangian strain \( \epsilon \) is defined using the square of the lengths:

\[
\frac{\bar{l}^2 - l^2}{2} = \delta x \epsilon \delta x \quad (2.2)
\]

The term \( \bar{l}^2 \) is approximated by

\[
\bar{l}^2 = |\delta x + (\delta x \cdot \nabla) v(x_0)|^2. \quad (2.3)
\]

The strain tensor can be formulated as

\[
\epsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i} + v_{k,i}v_{k,j}), \quad (2.4)
\]

where \( ,i \) denotes the derivatives with respect to \( x_i \). The out-of-plane and thickness wise elements of the strain tensor are negligible, since the corresponding stresses are small compared to the in-plane stress. The strain tensor therefore reduces to a 2D in-plane strain tensor:

\[
\epsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha} + v_{\kappa,\alpha}v_{\kappa,\beta}). \quad (2.5)
\]

Furthermore the out-of-plane displacement is considered to be independent of the \( z \) coordinate and according to the Kirchhoff-Love hypothesis the in-plane displacements are linear functions of the \( z \)-coordinate. Consequently, the displacements of the plate can be described as the displacements of its mid-surface.

Let us assume, that initially the mid-surface of the plate occupies a closed rectan-
gular domain $\Omega \subset \Pi$ and lies in the $z = 0$ plane:

$$\Omega(x, y) = \{(x, y) : -L/2 \leq x \leq L/2, -W/2 \leq y \leq W/2\}. \quad (2.6)$$

The boundary of the domain is denoted by

$$\partial\Omega(x, y) = \{(x, y) : (x = -L/2 \cup x = L/2, -W/2 \leq y \leq W/2) \cup (y = -W/2 \cup y = W/2, -L/2 \leq x \leq L/2)\}. \quad (2.7)$$

We use a fixed orthonormal basis $\{g_1, g_2, g_3\}$, where the unit vectors $g_1$ and $g_2$ span the plane of the mid-surface in the reference configuration (Figure 1.4). The displacement vector of the mid-surface is denoted by $u$, where $u : \Omega \rightarrow \mathbb{R}^3$. Furthermore, the in-plane displacements can be summarized in the $\hat{u}$ vector, where $\hat{u}(x, y) = [u_1(x, y), u_2(x, y)]$ and the out-of-plane displacement is denoted by $w(x, y) = u_3(x, y)$. Applying the Kirchhoff-Love hypothesis, it follows that

$$v_\alpha = u_\alpha - zw_\alpha. \quad (2.8)$$

Assuming small strains, the gradients of the in-plane displacements are small, therefore their product and square can be neglected. From Equation (2.5)

$$\epsilon_{\alpha\beta} \approx \frac{1}{2} (u_\alpha - zw_\alpha)_{,\beta} + (u_\beta - zw_\beta)_{,\alpha} + w_\alpha w_\beta) =$$

$$= \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + w_\alpha w_\beta) - zw_{\alpha\beta}. \quad (2.9)$$

Taking

$$\epsilon_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + w_\alpha w_\beta), \quad (2.10)$$

$$k_{\alpha\beta} = -w_{\alpha\beta}, \quad (2.11)$$

$\epsilon$ is the linearized in-plane strain tensor and $k$ is the linearized bending strain tensor (the curvature change tensor), the strain tensor can be written in a more compact form:

$$\epsilon \approx \epsilon + zw. \quad (2.12)$$

Note, that here we assume that the principal curvatures of the deformed surface are small enough and they can be approximated by the second derivatives of the out-of-plane deformation. Hooke’s law for the Cauchy stress tensor $\sigma$ assuming isotropic material reads:

$$\sigma = \frac{Y}{1 - \nu^2} [\text{tr}(\epsilon) I + (1 - \nu)\epsilon], \quad (2.13)$$
where $\nu$ is the Poisson’s ratio ($0 < \nu < 0.5$) and $Y$ is the Young modulus.

The Föppl-von Kármán theory can be interpreted as a two-dimensional approximation of a three-dimensional theory. It assumes that the total potential energy is the sum of the internal energies of the membrane and bending behavior. There are no external loads in the model problem, the film is in static equilibrium, and we prescribe the displacement on a part of the boundary of the domain as a Dirichlet boundary condition. Accordingly, the total potential energy is the internal energy from the bending and membrane behavior:

$$ I_s(u) = \int_{\Omega} \left( \Psi_m(e) + \Psi_b(k) \right) \, d\Omega, \quad (2.14) $$

where $\Psi_m$ and $\Psi_b$ are the membrane and bending energy densities respectively.

The bending energy density is related to the $M$ bending moment. We assume that the bending and the membrane energies are independent, therefore $e = 0$ is considered for calculating the bending moment. Hence

$$ M(k) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma z \, dz \approx \frac{Y}{1 - \nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\nu tr(k)I + (1 - \nu)k] \, z \, dz, \quad (2.15) $$

where $I$ is the 2-dimensional unit tensor. From the relation

$$ M := \frac{d\Psi_b(k)}{dk}, \quad (2.16) $$

the bending energy density is

$$ \Psi_b(k) = \frac{Yh^3}{12(1 - \nu^2)} [\nu tr(k)^2 + (1 - \nu)k \cdot k]. \quad (2.17) $$

Similarly $k = 0$ can be considered when computing the in-plane force $N$. As a result

$$ N(e) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma \, dz \approx \frac{Y}{1 - \nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\nu tr(e)I + (1 - \nu)e] \, dz. \quad (2.18) $$

From

$$ N := \frac{d\Psi_m(e)}{de}, \quad (2.19) $$

the membrane energy density is

$$ \Psi_m(e) = \frac{Yh}{1 - \nu^2} [\nu tr(e)^2 + (1 - \nu)e \cdot e]. \quad (2.20) $$
The problem can be solved by seeking the minimum of the total potential energy (Equation (2.14)) or the equilibrium equations can be delivered by computing its first variation. In the following we compute the first variation of Equation (2.14). The admissible variations are smooth fields satisfying the boundary conditions $\mu : \Omega \to \mathbb{R}^2$ and $\zeta : \Omega \to \mathbb{R}$. Consequently

$$\delta I_s = \int_{\Omega} \left( \frac{d}{d\theta} \Psi_m (\nabla \hat{u} + \theta \mu) |_{\theta=0} + \frac{d}{d\theta} \Psi_b (w + \theta \zeta) |_{\theta=0} \right) d\Omega, \quad (2.21)$$

for all admissible variations $\mu, \zeta$, where $\theta$ is a scalar. After evaluating at $\theta = 0$

$$\delta I_s = \int_{\Omega} \left( \mathbf{N} (\mathbf{e}) \cdot \nabla \mu + (\mathbf{N} (\mathbf{e}) \nabla w) \cdot \nabla \zeta - \mathbf{M} (\mathbf{k}) \cdot \nabla^2 \zeta \right) d\Omega = 0, \quad (2.22)$$

for all admissible variations $\mu, \zeta$. After partial integration and rescaling with $12(1 - \nu^2)/(Yh)$:

$$\nabla \cdot \mathbf{N}_s = 0, \quad (2.23)$$

$$h^2 \Delta^2 w - \nabla \cdot (\mathbf{N}_s \nabla w) = 0, \quad (2.24)$$

with

$$\mathbf{N}_s = 12[\nu \text{tr}(\mathbf{e}) \mathbf{I} + (1 - \nu) \mathbf{e}]. \quad (2.25)$$

On the clamped boundaries the Dirichlet conditions are

$$u_1 = \varepsilon L/2 \quad \text{on} \quad \partial \Omega |_{x=L/2}, \quad (2.26)$$

$$u_1 = -\varepsilon L/2 \quad \text{on} \quad \partial \Omega |_{x=-L/2}, \quad (2.27)$$

$$u_2 = 0 \quad \text{on} \quad \partial \Omega |_{x=L/2} \cup \partial \Omega |_{x=-L/2}, \quad (2.28)$$

$$w = 0 \quad \text{on} \quad \partial \Omega |_{x=L/2} \cup \partial \Omega |_{x=-L/2}, \quad (2.29)$$

and the Neumann conditions are

$$\frac{\partial w}{\partial x} = 0 \quad \text{on} \quad \partial \Omega |_{x=L/2} \cup \partial \Omega |_{x=-L/2}, \quad (2.30)$$

$$\frac{\partial w}{\partial y} = 0 \quad \text{on} \quad \partial \Omega |_{x=L/2} \cup \partial \Omega |_{x=-L/2}, \quad (2.31)$$

$$\frac{\partial w}{\partial xy} = 0 \quad \text{on} \quad \partial \Omega |_{x=L/2} \cup \partial \Omega |_{x=-L/2}. \quad (2.32)$$

Although Equations (2.31) and (2.32) are automatically satisfied if Equation (2.29) holds, if the derivatives of $w$ are also considered to be unknowns during the solu-
tion (for example using the Ritz-method described in Section 2.2.1) they need to be restricted. Following Healey, Q. Li, & Cheng 2013, the following Neumann boundary conditions are imposed on the free boundaries:

\[ n_b \cdot M_{n_b} = 0 \quad \text{on} \quad \partial \Omega |_{y=-W/2} \cup \partial \Omega |_{y=W/2}, \tag{2.33} \]

\[ N_s g_2 = 0 \quad \text{on} \quad \partial \Omega |_{y=-W/2} \cup \partial \Omega |_{y=W/2}, \tag{2.34} \]

\[ g_2 \cdot \left[ N_s \nabla w + \frac{\partial}{\partial x} (M g_1) + \nabla \cdot M \right] = 0 \quad \text{on} \quad \partial \Omega |_{y=-W/2} \cup \partial \Omega |_{y=W/2}, \tag{2.35} \]

where \( n_b \) is the outward normal of the boundary. Note, that these boundary conditions arise naturally from the integration by parts.

Most of the works in the literature formulate Equations (2.23) and (2.24) in a simpler form using the so-called Airy stress function

\[ \Delta^2 \chi + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 = 0, \tag{2.36} \]

\[ -h \left( \frac{\partial^2 \chi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 \chi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 \chi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right) = 0, \tag{2.37} \]

where \( \chi : \Omega \to \mathbb{R} \) is a \( C^4 \) function. The following relations hold for the Airy stress function

\[ \frac{\partial^2 \sigma_{11}}{\partial x^2} = \frac{\partial^2 \chi}{\partial y^2}, \tag{2.38} \]

\[ \frac{\partial^2 \sigma_{22}}{\partial y^2} = \frac{\partial^2 \chi}{\partial x^2}, \tag{2.39} \]

\[ \frac{\partial^2 \sigma_{12}}{\partial x \partial y} = -\frac{\partial^2 \chi}{\partial y \partial x}, \tag{2.40} \]

which can be derived by computing the first variation of the bending and membrane energies separately.

### 2.1.2 Extension to finite strains

In this subsection, we summarize the finite strain extension of the Föppl-von Kármán equations of Healey, Q. Li, & Cheng 2013.

The out-of-plane and thickness wise elements of Equation (2.4) are still negligible, but the nonlinear terms of the gradients of the in-plane displacements cannot be neglected anymore. Therefore instead of Equation (2.9) the Green-Lagrangian strain
takes the following form:

\[ \epsilon_{\alpha\beta} = \frac{1}{2} \left( v_{\alpha,\beta} + v_{\beta,\alpha} + v_{\gamma,\alpha} v_{\gamma,\beta} + w_{,\alpha} w_{,\beta} \right). \]  

(2.41)

Applying the Kirchhoff-Love hypotheses and assuming that the curvatures are still small leads to

\[ \epsilon_{\alpha\beta} \approx \frac{1}{2} \left( (u_{,\alpha} - z w_{,\alpha})_{,\beta} + (u_{,\beta} - z w_{,\beta})_{,\alpha} + (u_{,\gamma} - z w_{,\gamma})_{,\alpha} (u_{,\gamma} - z w_{,\gamma})_{,\beta} + w_{,\alpha} w_{,\beta} \right) = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\gamma,\alpha} u_{\gamma,\beta} + w_{,\alpha} w_{,\beta} - 2 z w_{,\alpha\beta} \right). \]  

(2.42)

Keeping the definition of the linearized bending strain tensor Equation (2.11) and introducing \( E \) the nonlinear in-plane strain tensor as

\[ E_{\alpha\beta} = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\gamma,\alpha} u_{\gamma,\beta} + w_{,\alpha} w_{,\beta} \right) = e + \frac{1}{2} u_{\gamma,\alpha} u_{\gamma,\beta}, \]  

(2.43)

the strain tensor can be written summarized in

\[ \epsilon \approx E + z k. \]  

(2.44)

The total potential energy reads:

\[ I_f(u) = \int_{\Omega} \left( \Psi_m(E) + \Psi_b(k) \right) d\Omega. \]  

(2.45)

The 2nd Piola-Kirchhoff tensor is

\[ N(E) = \frac{Y h}{1 - \nu^2} \left( \nu \text{tr}(E) I + (1 - \nu) E \right). \]  

(2.46)

Taking the first variation of the energy with respect to the components of the displacement-field \( u \) leads to

\[ \delta I_f = \int_{\Omega} \left( (I + \nabla \hat{u}) N(E) \cdot \nabla \mu + (N(E) \nabla w) \cdot \nabla \zeta - M \cdot \nabla^2 \zeta \right) d\Omega = 0, \]  

(2.47)

for all admissible variations \( \mu, \zeta \).

After integration by parts and rescaling by \( 12(1 - \nu^2)/(Y h) \), the Euler-Lagrange
equations take the following form:

\[ \nabla \cdot [FN_f] = 0, \]  
\[ h^2 \Delta^2 w - \nabla \cdot (N_f \nabla w) = 0, \]

with

\[ N_f = 12 [\nu \text{tr}(E) I + (1 - \nu) E] \]

and the deformation gradient

\[ F = I + \nabla \hat{u}. \]

On the clamped boundaries the Dirichlet conditions read

\[ u_1 = \varepsilon L/2 \quad \text{on} \quad \partial \Omega \mid_{x=L/2}, \]  
\[ u_1 = -\varepsilon L/2 \quad \text{on} \quad \partial \Omega \mid_{x=-L/2}, \]  
\[ u_2 = 0 \quad \text{on} \quad \partial \Omega \mid_{x=L/2} \cup \partial \Omega \mid_{x=-L/2}, \]  
\[ w = 0 \quad \text{on} \quad \partial \Omega \mid_{x=L/2} \cup \partial \Omega \mid_{x=-L/2}, \]

and the Neumann conditions are

\[ \frac{\partial w}{\partial x} = 0 \quad \text{on} \quad \partial \Omega \mid_{x=L/2} \cup \partial \Omega \mid_{x=-L/2}, \]  
\[ \frac{\partial w}{\partial y} = 0 \quad \text{on} \quad \partial \Omega \mid_{x=L/2} \cup \partial \Omega \mid_{x=-L/2}, \]  
\[ \frac{\partial w}{\partial xy} = 0 \quad \text{on} \quad \partial \Omega \mid_{x=L/2} \cup \partial \Omega \mid_{x=-L/2}. \]

Although Equations (2.57) and (2.58) are automatically satisfied if Equation (2.29) holds, if the derivatives of \( w \) are also considered to be unknowns during the solution (for example using the Ritz-method described in Section 2.2.1) they need to be restricted. The Neumann conditions on the free sides of the sheet arise naturally from the integration by parts:

\[ n_6 \cdot M n_6 = 0 \quad \text{on} \quad \partial \Omega \mid_{y=W/2} \cup \partial \Omega \mid_{y=-W/2}, \]  
\[ (I + \nabla \hat{u}) N_f g_2 = 0 \quad \text{on} \quad \partial \Omega \mid_{y=W/2} \cup \partial \Omega \mid_{y=-W/2}, \]  
\[ g_2 \cdot \left[ N_f \nabla w + \frac{\partial}{\partial x} (M g_1) + \nabla \cdot M \right] = 0 \quad \text{on} \quad \partial \Omega \mid_{y=W/2} \cup \partial \Omega \mid_{y=-W/2}. \]

Note that instead of Equations (2.57) to (2.58) Healey, Q. Li, & Cheng 2013 applied Equation (2.59) on \( \Omega \mid_{x=-L/2} \cup \partial \Omega \mid_{x=L/2} \) as well. This can be a reasonable choice if
there are only in-plane loads on the sheet. As a result the Neumann conditions on the edges $\Omega|_{x=-L/2}$ and $\partial \Omega|_{x=L/2}$ become natural boundary conditions, which simplifies the numerical solution of the equations.

An essential property of Equations (2.48) and (2.49) is the fact that it does not have a more straightforward form such as Equations (2.36) and (2.37) for the FvK theory. In case of finite strains, the first variation of the membrane energy has nonlinear terms, therefore there is no such stress function $\chi$ such that Equations (2.38) to (2.40) hold. Following the derivation of Dost & Glockner 1983, the symmetry of the left-hand side of Equation (2.48) would be required to express Equations (2.48) and (2.49) in terms of only two field variables. Given that $\mathbf{FN}_f$ is nonsymmetric, two stress functions would be necessary. In conclusion, the equations of the eFvK theory lack a simple two-field form such as Equations (2.36) and (2.37) in the FvK theory.

By investigating the wrinkling of stretched thin films, Healey, Q. Li, & Cheng 2013 pointed on a qualitative error of the small strain assumptions of the original FvK theory. It successfully catches the appearance of wrinkles at a $\varepsilon_{cr1}$ stretch, but it predicts the stability of wrinkled solution for $\varepsilon > \varepsilon_{cr1}$. On the contrary, the eFvK model predicted the disappearance of wrinkles in a second bifurcation point. Furthermore, for a fixed thickness $h$, wrinkling appeared only for a bounded regime of the aspect ratios. Conversely, the FvK theory predicted wrinkling for a vast range of aspect ratios outside this region. Examination of the stability boundary of the wrinkled configurations in the $\beta - \varepsilon$ plane pointed to an isola-center bifurcation in the model eFvK. The small strain assumption embodied in FvK turned out to be appropriate only for $\varepsilon < 0.1$ stretch, which is usually exceeded in the literature of the wrinkling of stretched rectangular films.

The eFvK model incorporates the Saint Venant-Kirchhoff material for which $\varepsilon$ is known to be bounded above for uniaxial stretching. Q. Li & Healey 2016 determined the stability boundary for more realistic materials such as for the Neo-Hookean and the Mooney-Rivlin hyperelastic models. In Chapters 2 and 4 we use the Saint Venant-Kirchhoff material and in Chapters 3 and 5 more realistic material models are introduced in some parts of the work.
2.2 Numerical method

2.2.1 Finite Element discretisation

Possible approaches

The resulting PDEs for both the FvK (Equations (2.23) and (2.24)) and eFvK theories (Equations (2.48) and (2.49)) are nonlinear, fourth order and no closed form of their solution is known. Using the Finite Element method the biharmonic term (containing the fourth order derivatives of the out-of-plane displacement) needs extra care. Following the standard Finite Element approach, at least \( C^1 \) continuity of the basis functions should be granted. There are many conforming and non-conforming element types with this property, such as the non-conforming rectangular element with 24 DoFs or the conforming Argyris triangular element. Another option is to use a Discontinuous Galerkin formulation with \( C^0 \) continuous elements. In the following we describe two finite element approaches emphasizing the possible ways of handling the fourth order term.

Ritz method

Taking into consideration, that the total potential energy is known for both theories, the Ritz-method is a reasonable choice to solve the problem (Bojtár & Gáspár 2003; Braess 2007; Reddy 2006). The method aims to minimize the total potential energy (Equation (2.14) and Equation (2.45) for the FvK and eFvK theories, respectively) considering the boundary boundary conditions defined by Equations (2.26) to (2.35) and Equations (2.52) to (2.61).

On \( \partial \Omega_{y=\pm W/2} \) edges of the film the natural boundary conditions (Equations (2.33) to (2.35), and Equations (2.59) to (2.61)) are automatically satisfied during the solution, due to the variational nature of the method. Neumann conditions can be prescribed during the compilation of the system of equations while to prescribe Dirichlet boundary conditions numerous techniques are possible (e.g. with Lagrange multipliers). Here we use a homogenization technique by replacing variables:

\[
    u'_1 = u_1 - \varepsilon x. \tag{2.62}
\]

We drop the \( . \)' notation in the following to simplify the equations. As a result the
following conditions have to be prescribed during the compilation for both theories:

\[ u = 0 \quad \text{on} \quad \partial \Omega |_{x=\pm L/2} \]
\[ \frac{\partial w}{\partial x} = 0 \quad \text{on} \quad \partial \Omega |_{x=\pm L/2} \]
\[ \frac{\partial w}{\partial y} = 0 \quad \text{on} \quad \partial \Omega |_{x=\pm L/2} \]
\[ \frac{\partial w}{\partial xy} = 0 \quad \text{on} \quad \partial \Omega |_{x=\pm L/2} \]

Subsequently the homogenised strain tensors are

\[ e_\varepsilon = \frac{1}{2} \left[ \varepsilon^2 + 2\varepsilon + (\varepsilon + 1)u_{1,x} + w_{x}^2 - 1 \begin{array}{c} u_{1,y} + u_{2,y} + w_{x}w_{y} \\ 2u_{2,y} + w_{y}^2 \end{array} \right] \]

for small strains and

\[ E_\varepsilon = \frac{1}{2} \left[ \left( u_{1,x} + \varepsilon \right)^2 + u_{2,x}^2 + w_{x}^2 - 1 \begin{array}{c} u_{1,y} + \varepsilon u_{1,y} + u_{2,y} + u_{1,x}u_{1,y} + u_{2,x}u_{2,y} + w_{x}w_{y} \\ 2u_{2,y} + u_{1,y}^2 + u_{2,y}^2 + w_{y}^2 \end{array} \right] \]

for finite strains.

Using the classical Finite Element Method \( u_1(x,y) \) and \( u_2(x,y) \) can be approximated using Lagrange polynomials (\( C^0 \) continuity), but \( w(x,y) \) has to be approximated using Hermite polynomials (\( C^1 \) continuity). Therefore we discretise the domain with rectangular elements, having 6 degrees of freedom at each nodes. We discretise \( \Omega \) with \( M = m_x \cdot m_y \) rectangular elements Figure 2.1 (\( m_x \) in the \( x \) direction and \( m_y \) in the \( y \) direction) with \( l_x, l_y \) sides. This element is a non-conform finite element with 24 DoF, but it results in a simple implementation for our problem defined on a rectangular domain. The displacement vector of the \( i \)th finite element is summarised in the following vector:

\[ v^i = \left[ u_1^{iI}, u_2^{iI}, w^{iI}, w_{,x}^{iI}, w_{,y}^{iI}, w_{,xy}^{iI}, u_1^{iII}, u_2^{iII}, w^{iII}, ..., w_{,xy}^{iII}, u_1^{iIII}, ..., w_{,xy}^{iIV}, u_1^{iIV}, ..., w_{,xy}^{iIV} \right]^T \]

(2.69)

where \( I, II, III, IV \) suffices denote the appropriate element nodes. The discretised displacement components:

\[ u_1^i = \Theta_1 v^i, \]
\[ u_2^i = \Theta_2 v^i, \]
\[ w^i = \Theta_3 v^i, \]

(2.70)  (2.71)  (2.72)
where $\Theta_1, \Theta_2, \Theta_3$ are row vectors containing the $\varphi_j$ basis functions.

$$\Theta_1 = [\varphi_1, 0, 0, 0, 0, 0, \varphi_2, 0, 0, 0, 0, \varphi_3, 0, 0, 0, 0, \varphi_4, 0, 0, 0, 0], \quad (2.73)$$

$$\Theta_2 = [0, \varphi_1, 0, 0, 0, 0, 0, \varphi_2, 0, 0, 0, 0, \varphi_3, 0, 0, 0, 0, \varphi_4, 0, 0, 0], \quad (2.74)$$

where in the $[\zeta, \eta]$ parametric coordinate system, the linear basis functions read

$$\varphi_j = \frac{1}{4}(1 + \zeta \xi_j)(1 + \eta \eta_j). \quad (2.75)$$

$\Theta_3$ contains the basis functions having $C^1$ continuity. They are created as products of $\vartheta_j$ Hermite-polynomials:

$$\Theta_3 = [0, 0, \vartheta_1(\zeta) \vartheta_3(\eta), \vartheta_2(\zeta) \vartheta_3(\eta), \vartheta_1(\zeta) \vartheta_4(\eta), \vartheta_2(\zeta) \vartheta_4(\eta),$$

$$0, 0, \vartheta_3(\zeta) \vartheta_3(\eta), \vartheta_4(\zeta) \vartheta_3(\eta), \vartheta_3(\zeta) \vartheta_4(\eta), \vartheta_4(\zeta) \vartheta_4(\eta),$$

$$0, 0, \vartheta_3(\zeta) \vartheta_1(\eta), \vartheta_4(\zeta) \vartheta_1(\eta), \vartheta_3(\zeta) \vartheta_2(\eta), \vartheta_4(\zeta) \vartheta_2(\eta),$$

$$0, 0, \vartheta_1(\zeta) \vartheta_3(\eta), \vartheta_2(\zeta) \vartheta_3(\eta), \vartheta_1(\zeta) \vartheta_4(\eta), \vartheta_2(\zeta) \vartheta_4(\eta)], \quad (2.76)$$

where for the $i$th element having $l_x, l_y$ sides:

$$\vartheta_1(\zeta) = \frac{1}{2} - \frac{3}{4} \zeta + \frac{1}{2} \zeta^3, \quad \vartheta_1(\eta) = \frac{1}{2} - \frac{3}{4} \eta + \frac{1}{2} \eta^3,$$

$$\vartheta_2(\zeta) = \frac{l_x}{8}(1 - \zeta - \zeta^2 + \zeta^3), \quad \vartheta_2(\eta) = \frac{l_y}{8}(1 - \eta - \eta^2 + \eta^3),$$

$$\vartheta_3(\zeta) = \frac{1}{2} + \frac{3}{4} \zeta - \frac{1}{2} \zeta^3, \quad \vartheta_3(\eta) = \frac{1}{2} + \frac{3}{4} \eta - \frac{1}{2} \eta^3,$$

$$\vartheta_4(\zeta) = \frac{l_x}{8}(-1 - \zeta + \zeta^2 + \zeta^3), \quad \vartheta_4(\eta) = \frac{l_y}{8}(-1 - \eta + \eta^2 + \eta^3). \quad (2.77)$$
To compute the derivatives of the displacement components, we have to express the derivatives of the basis functions with respect to the global coordinate systems. Taking into account that \( s_j(\xi, \eta), \ j = 1, \ldots, 4 \) are the linear functions describing the map between the parametric and the global coordinate systems, the Jacobian takes the following form:

\[
J_{xy} = \begin{bmatrix}
\frac{\partial s_1}{\partial \xi} & \frac{\partial s_2}{\partial \xi} & \frac{\partial s_3}{\partial \xi} & \frac{\partial s_4}{\partial \xi} \\
\frac{\partial s_1}{\partial \eta} & \frac{\partial s_2}{\partial \eta} & \frac{\partial s_3}{\partial \eta} & \frac{\partial s_4}{\partial \eta}
\end{bmatrix} \cdot \begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
x_4 & y_4
\end{bmatrix},
\]

(2.78)

where \( x_j, y_j \) are the global coordinates of the \( j \)th node of the element. The inverse of the Jacobian determines the relationship between the parametric and the global derivatives:

\[
J_{xy}^{-1} = \begin{bmatrix}
\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\
\frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y}
\end{bmatrix} = \begin{bmatrix}
\frac{2}{l_x} & 0 \\
0 & \frac{2}{l_y}
\end{bmatrix}.
\]

(2.79)

Accordingly, using the chain rule, derivatives of the \( k \)th vector of basis functions:

\[
\begin{bmatrix}
\frac{\partial \Theta_k}{\partial x} \\
\frac{\partial \Theta_k}{\partial y}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \Theta_k}{\partial \xi} & \frac{\partial \Theta_k}{\partial \eta}
\end{bmatrix} J_{xy}^{-1} \begin{bmatrix}
\frac{\partial \Theta_k}{\partial \xi} \\
\frac{\partial \Theta_k}{\partial \eta}
\end{bmatrix}.
\]

(2.80)

For integration, the map between the global and the parametric coordinates reads

\[
d\Omega = dx dy = |J_{xy}|d\xi d\eta.
\]

(2.81)

The derivatives of the discretised displacement functions can be computed using the derivatives of the \( \Theta_k \) vectors \( k = 1, 2, 3 \):

\[
\begin{align*}
    u_{1,x}^i &= \Theta_{i,1,x}^j v^i = A_1, & w_{,x}^i &= \Theta_{i,3,x}^j v^i = A_5, \\
    u_{1,y}^i &= \Theta_{i,1,y}^j v^i = A_2, & w_{,y}^i &= \Theta_{i,3,y}^j v^i = A_6, \\
    u_{2,x}^i &= \Theta_{i,2,x}^j v^i = A_3, & w_{,xx}^i &= \Theta_{i,3,xx}^j v^i = A_7, \\
    u_{2,y}^i &= \Theta_{i,2,y}^j v^i = A_4, & w_{,yy}^i &= \Theta_{i,3,yy}^j v^i = A_8, \\
    u_{,xy}^i &= \Theta_{i,3,xy}^j v^i = A_9.
\end{align*}
\]

(2.82)
The discretised energy functional for small strains (Equation (2.14)) reads

\[
I_s(u) = \sum_{i=1}^{M} \int_{\Omega^*} \left( \frac{12}{k^2} \left( \frac{1}{2} \left( \varepsilon^2 + \varepsilon + (\varepsilon + 1)A_1 + \frac{1}{2}A_3^2 + A_4 + \frac{1}{2}A_6^2 \right)^2 +
\right. 
+ (1 - \nu) \left( \frac{1}{2} \varepsilon^2 + \varepsilon + (\varepsilon + 1)A_1 + \frac{1}{2}A_3^2 \right)^2 + \frac{1}{2}((\varepsilon + 1)A_2 + A_3 + A_5A_6)^2 + \\
\left. + (A_4 + \frac{1}{2}A_6^2)^2 \right) \right) d\Omega
\]

and for finite strains (Equation (2.45)):

\[
I_f(u) = \sum_{i=1}^{M} \int_{\Omega^*} \left( \frac{12}{k^2} \left( \frac{1}{4} \left( (A_1 + 1 + \varepsilon)^2 + A_3^2 + A_5^2 - 2 + A_2^2 + (A_4 + 1)^2 + A_6^2 \right)^2 +
\right. 
+ (1 - \nu) \left( \frac{1}{4}((A_1 + 1 + \varepsilon)^2 + A_3^2 + A_5^2 - 1)^2 + \frac{1}{4}(A_4^2 + (A_4 + 1)^2 + A_6^2 - 1)^2 + \\
\left. + \frac{1}{2}((A_1 + 1 + \varepsilon)A_2 + A_3(A_4 + 1) + A_5A_6)^2 \right) \right) + v(A_7 + A_8)^2 + \\
+ (1 - \nu)(A_7^2 + A_8^2 + 2A_6^2) d\Omega.
\]

The integration should be carried out over the finite elements \(\Omega^*\) using Gauss-Legendre formulas (Reddy 2006). The number of finite elements is \(M = m_xm_y\). The equilibrium equations can be computed by differentiating the discretised functionals:

\[
f_s(u) = \frac{\partial I_s(u)}{\partial \nu} = \sum_{i=1}^{M} \int_{\Omega^*} \left( \frac{\partial I_s(u)}{\partial A_1} \frac{\partial A_1}{\partial \nu} + \frac{\partial I_s(u)}{\partial A_2} \frac{\partial A_2}{\partial \nu} + \ldots 
\right.
\]

\[
\left. + \frac{\partial I_s(u)}{\partial A_8} \frac{\partial A_8}{\partial \nu} + \frac{\partial I_s(u)}{\partial A_9} \frac{\partial A_9}{\partial \nu} \right) d\Omega = 0
\]

and

\[
f_f(u) = \frac{\partial I_f(u)}{\partial \nu} = \sum_{i=1}^{M} \int_{\Omega^*} \left( \frac{\partial I_f(u)}{\partial A_1} \frac{\partial A_1}{\partial \nu} + \frac{\partial I_f(u)}{\partial A_2} \frac{\partial A_2}{\partial \nu} + \ldots 
\right.
\]

\[
\left. + \frac{\partial I_f(u)}{\partial A_8} \frac{\partial A_8}{\partial \nu} + \frac{\partial I_f(u)}{\partial A_9} \frac{\partial A_9}{\partial \nu} \right) d\Omega = 0,
\]

which leads for one finite element to a system of 24 equations. After the compilation it results in \(d = 6(m_x + 1)(m_y + 1) - b\) equations, where \(b\) is the number of fixed displacement components originating from the boundary conditions. Equations (2.85) and (2.86) can be solved using the Newton-Raphson iteration. For the Newton-Raphson iteration, the Jacobian, namely the tangent stiffness matrix has to
be computed. It can be expressed as the second derivative of the functional leading to

\[ J_s(u) = \frac{\partial f_s(u)}{\partial v} = \sum_{i=1}^{M} \int_{\Omega^*} \left( \frac{\partial f_s(u)}{\partial A_1} \frac{\partial A_1}{\partial v} + \frac{\partial f_s(u)}{\partial A_2} \frac{\partial A_2}{\partial v} + \ldots + \frac{\partial f_s(u)}{\partial A_8} \frac{\partial A_8}{\partial v} + \frac{\partial f_s(u)}{\partial A_9} \frac{\partial A_9}{\partial v} \right) d\Omega = 0 \]  

(2.87)

and

\[ J_f(u) = \frac{\partial f_f(u)}{\partial v} = \sum_{i=1}^{M} \int_{\Omega^*} \left( \frac{\partial f_f(u)}{\partial A_1} \frac{\partial A_1}{\partial v} + \frac{\partial f_f(u)}{\partial A_2} \frac{\partial A_2}{\partial v} + \ldots + \frac{\partial f_f(u)}{\partial A_8} \frac{\partial A_8}{\partial v} + \frac{\partial f_f(u)}{\partial A_9} \frac{\partial A_9}{\partial v} \right) d\Omega = 0. \]  

(2.88)

During the compilation of Equations (2.87) and (2.88) the boundary conditions have to be prescribed.

The stability of the trivial, unwrinkled solution can be checked using the smallest eigenvalue \( \lambda_{min} \) of the Jacobian. If \( \lambda_{min} < 0 \), the trivial solution is unstable, otherwise it is stable.

**Discontinuous Galerkin method**

To handle the required continuity for the fourth order term, using the *Discontinuous Galerkin method* (Rivièrè 2008) it is enough to discretize the unknown field with Lagrangian elements (having only \( C^0 \) continuity). The Galerkin method (Ciarlet 2002) solves the weak form of the equations. In case of discontinuous elements, additional penalty terms prescribed on the element boundaries in the weak form enforce weakly the required continuity. For fourth order problems, the method was derived by Engel et al. 2002; Brenner & Sung 2005; Brenner, Monk, & Sun 2015. In the literature for biharmonic problems it is called *Interior Penalty method* or *Continuous/Discontinuous Galerkin method* indicating, that the elements are not completely discontinuous (they have \( C^0 \) continuity), but the required continuity is not fulfilled without additional penalty terms. Brenner, Neilan, et al. 2017 also derived the method for the FvK theory expressed in terms of the Airy function (Equations (2.36) to (2.40)). We aimed to derive the appropriate weak forms for both FvK and eFvK models. Accordingly, we based our derivations on the form suggested in Brenner & Sung 2005 for the biharmonic term and used the standard Galerkin formulation for the other terms following Rivièrè 2008. We implemented the solution of the resulting equations using *FEniCS* software (Logg & Wells 2012), which is an open-source collection of finite element
based tools for the automated solution of differential equations.

We follow the standard notation of the literature regarding the Discontinuous Galerkin method to describe our implementation. First, we define the function spaces necessary for the derivations. The space for square-integrable functions for a \( \Omega \in \mathbb{R}^2 \) bounded region:

\[
L^2(\Omega) := \{ u : \Omega \to \mathbb{R}, \int_\Omega |u|^2 \, d\Omega < \infty \}. \tag{2.89}
\]

We use Sobolev spaces,

\[
H^k(\Omega) := \{ u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega), \forall \alpha \text{ such that } |\alpha| \leq k \}, \tag{2.90}
\]

where \( C_0^\infty \) is the space of \( C^\infty \) functions with compact support in \( \Omega \), and its closure

\[
H^k_0(\Omega) := \{ u \in H^k(\Omega), u|_{\partial \Omega} = 0 \}. \tag{2.91}
\]

Apart from being square integrable, these function spaces require the existence of the derivatives in a weak sense.

We derive the weak form of the equations of the eFvK theory and then present the resulting equations for the FvK theory as well. As it was mentioned earlier, for this particular problem, Equations (2.33) and (2.59) can be prescribed on the ends of the sheet instead of Equations (2.30) to (2.32) and (2.56) to (2.58). As a result, all of the Neumann boundary conditions become natural boundary conditions leaving only Equations (2.52) to (2.55) to handle. Moreover, using the same homogenisation technique as previously (Equation (2.62)) the boundary conditions reduce to

\[
\mathbf{u} = 0 \text{ on } \partial \Omega|_{x = -L/2} \cup \partial \Omega|_{x = L/2}. \tag{2.92}
\]

The 2nd Piola-Kirchhoff stress tensor depending on the homogenised strain tensor (Equation (2.68)) is denoted by

\[
N^f_\varepsilon = N_f(\varepsilon). \tag{2.93}
\]

In the standard Galerkin formulation the weak form of Equations (2.48) and (2.49) reads: find \( u_1 \in H^1_0(\Omega), u_2 \in H^1_0(\Omega), w \in H^2_0(\Omega) \):

\[
\int_\Omega \nabla \cdot (F \mathbf{N}^f_\varepsilon) \cdot \hat{\mathbf{\mu}} \, d\Omega = 0 \quad \forall \hat{\mathbf{\mu}} \in H^1_0(\Omega)^2, \tag{2.94}
\]

\[
h^2 \int_\Omega \Delta^2 w \zeta \, d\Omega - \int_\Omega \nabla \cdot (N^f_\varepsilon \nabla w) \zeta \, d\Omega = 0 \quad \forall \zeta \in H^2_0(\Omega) \tag{2.95}
\]

considering the boundary condition Equation (2.92).

Consider a \( \tau \) triangulation of the \( \Omega \) domain. The finite element space is the stan-
dard continuous Lagrangian function space:

\[ V^k(\Omega) = \{ u \in H^1(\Omega) : u|_K \in H^k(K) \forall K \in \tau \} \]  (2.96)

and its closure is

\[ V^k_o(\Omega) = \{ u \in H^1_o(\Omega) : u|_K \in H^k(K) \forall K \in \tau \}. \]  (2.97)

Applying the interior penalty method on the fourth order term (Brenner & Sung 2005; Engel et al. 2002), the discretised form of the equations reads: find \( \hat{u} \in V^1_o(\Omega)^2, w \in V^2_o(\Omega) \), such that:

\[
- \sum_{K \in \tau} \int_K F N^f_\varepsilon \cdot \nabla \hat{\mu} \, d\Omega + \]
\[
h^2 \left( \sum_{K \in \tau} \int_K \Delta w \Delta \zeta \, d\Omega - \int_\Gamma \{\Delta w\} [\nabla \zeta] \, dS - \int_\Gamma \{\Delta \zeta\} [\nabla w] \, dS - \frac{\varepsilon}{h_{avg}} \int_\Gamma [\nabla w] [\nabla \zeta] \, dS \right) + \]
\[
+ \sum_{K \in \tau} \int_K N^f_s \cdot \nabla w \cdot \nabla \zeta \, d\Omega = 0 \quad \forall \hat{\mu} \in V^1_o(\Omega)^2, \zeta \in V^2_o(\Omega), \]

(2.98)

where \( \Gamma \) is the union of the interior edges and \( dS \) is the integration on the element boundaries, \( h_{avg} \) is the average of the size of the neighboring cells, \( \varepsilon \geq 0 \) is a penalty parameter, \( \{w\} = \frac{1}{2}(w_+ + w_-) \), \( [\nabla w] = w_+ n_+ + w_- n_- \) are the average and jump operators, where \( n \) is the outward pointing normal on the element boundary. Subscripts ‘+’ and ‘-’ indicate the evaluation of the functions on the opposite sides of the edges. The integral terms on the element boundaries are needed for consistence, symmetry and stability (Brenner & Sung 2005). It can be shown, that the scheme is stable for a sufficiently large penalty parameter. In our computations we chose \( \varepsilon = 8 \).

Considering the FvK theory, the discretised form of the equations reads: find \( \hat{u} \in V^1_o(\Omega)^2, w \in V^2_o(\Omega) \), such that:

\[
- \sum_{K \in \tau} \int_K N^f_s \cdot \nabla \hat{\mu} \, d\Omega + \]
\[
h^2 \left( \sum_{K \in \tau} \int_K \Delta w \Delta \zeta \, d\Omega - \int_\Gamma \{\Delta w\} [\nabla \zeta] \, dS - \int_\Gamma \{\Delta \zeta\} [\nabla w] \, dS - \frac{\varepsilon}{h_{avg}} \int_\Gamma [\nabla w] [\nabla \zeta] \, dS \right) + \]
\[
+ \sum_{K \in \tau} \int_K N^f_s \cdot \nabla w \cdot \nabla \zeta \, d\Omega = 0 \quad \forall \hat{\mu} \in V^1_o(\Omega)^2, \zeta \in V^2_o(\Omega), \]

(2.99)

where \( N^f_s = N_s(e_\varepsilon) \).
The Jacobian can be computed by taking the derivative of Equations (2.98) and (2.99) with respect to \( \hat{u} \) and \( w \). FEniCS supports the derivation of functionals and the smallest real eigenvalue \( \lambda_{\text{min}} \) of the Jacobian can be computed using the PETSc linear algebra library (Balay, Gropp, et al. 1997; Balay, Abhyankar, et al. 2018a; Balay, Abhyankar, et al. 2018b) to check the stability of the solution.

Comparison

Our first implementation based on the Ritz method in Matlab uses a non-conform rectangular element with 24 degrees of freedom. Due to the \( C^1 \) continuity of the elements it is computationally expensive. Moreover, the rectangular elements restrict the geometry of the domain, and the high-level programming language makes the computation significantly slow. FEniCS supports discontinuous elements and the implementation of user-defined equations over arbitrary meshes, which comes in handy in case of extending the theories and modifying the equations. Furthermore, its components are written in C++ and Python, which makes the software considerably faster than our Matlab implementation. Since we aimed to compute the wrinkling of films having arbitrary geometry with fine discretization, we implemented our numerical continuation algorithm based on the FEniCS solver.

2.2.2 Numerical continuation

The problem is nonlinear and can be solved using numerical continuation (Allgower & Georg 1990; Seydel 2010). We implemented a classical predictor-corrector numerical continuation algorithm in C++ in FEniCS. The aim of the implementation is to compute the shape of a stretched, initially flat sheet depending on the applied macroscopic strain. We start from the unwrinkled solution and carry out the continuation in \( \varepsilon \). A general, more compact form of Equations (2.98) and (2.99) can be formulated as a bilinear form:

\[
a(u_h, \mu) = 0 \quad \forall \mu \in V_h,
\]

where \( u_h = (\hat{u}, w) \), \( \mu = (\hat{\mu}, \zeta) \) and \( V_h = V^1_0(\Omega)^2 \times V^2_0(\Omega) \). Taking \( v_i \in V_h \) the components of the \( c \in \mathbb{R}^m \) unknown vector of the problem are defined as

\[
u_h = \sum_{i=1}^{m} c_i v_i,
\]

where \( m \) is the number of degrees of freedom.

Solution of Equation (2.100) following the standard Galerkin formulation (Ciar-
let 2002) leads to a system of nonlinear algebraic equations depending on the \( m \)-dimensional vector of unknowns \( c \) and the stretch parameter:

\[
f(c, \varepsilon) = 0. \tag{2.102}
\]

Equation (2.102) can be solved for a fixed \( \varepsilon \) with the Newton-Raphson method from some initial guess \( c_0 \). We use a tangent predictor (Seydel 2010) to compute the initial guess for the next step. We differentiate Equation (2.102) with respect to the continuation parameter:

\[
f_\varepsilon = \frac{\partial f}{\partial \varepsilon} \tag{2.103}
\]

and with respect to the unknown vector

\[
f_c = \frac{\partial f}{\partial c} = J, \tag{2.104}
\]

which is the Jacobian matrix (or the tangent stiffness matrix). The tangent \( z \) with a dimension \( m + 1 \) can be computed by solving the following algebraic system of equations:

\[
\begin{pmatrix} f_c & f_\varepsilon \\ e_k^T & 0 \end{pmatrix} z = e_{m+1}, \tag{2.105}
\]

where \( e_i \) is the \( i \)th unit vector and \((.)^T\) denotes the transpose of a vector. In Equation (2.105) \( k \) has to be chosen such that the matrix on the left-hand side of Equation (2.105) is of rank \( m + 1 \). The predictor step in the \( j \)th continuation step is the calculation of:

\[
(\bar{c}_j^{i+1}, \bar{\varepsilon}_j^{i+1}) := (c_j^i, \varepsilon_j^i) + \omega_j z, \tag{2.106}
\]

where \( \omega_j \) is an appropriate step size and \( (\cdot) \) marks the predicted values. We use a stepcontrol suggested by (Seydel 2010). An adjustable factor is defined based on the number of iteration steps of the corrector iterations \( N_j \) in the \( j \)th continuation step:

\[
\rho := \begin{cases} 
0.5, & \text{if } \frac{N_{opt}}{N_j} < 0.5 \\
\frac{N_{opt}}{N_j}, & \text{if } 0.5 \leq \frac{N_{opt}}{N_j} \leq 2 \\
2, & \text{if } \frac{N_{opt}}{N_j} > 2
\end{cases}, \tag{2.107}
\]

where \( N_{opt} \) is the optimal number of iterations. Then the new step size is chosen according to the following law:

\[
\omega_{j+1} := \rho \omega_j. \tag{2.108}
\]
In our computations $N_{opt} = 5$ was a reasonable choice and led to a stable continuation. As a corrector step in the $j$th continuation step, Equation (2.102) is solved by taking $c_o = \bar{c}^{j+1}$ as the initial value for the parameter $\varepsilon_{j+1} = \varepsilon_j + \omega_j$.

The bifurcation points of the system and the stability of the solutions are monitored during each step of the continuation using the smallest real eigenvalue $\lambda_{min}$ of the Jacobian as a test function. If $\lambda_{min} < 0$, the solution is unstable. After the detection of a bifurcation point, $\lambda_{min} = 0$ is approximated by a step size control suggested by (Allgower & Georg 1990):

$$\omega_{j+1} := -\frac{\lambda^j_{min}}{\lambda^j_{min} - \lambda^{j-1}_{min}} \omega^j$$  (2.109)

A perturbation technique is used to join the stable solution branch according to the following formula

$$u'_h = u_h + sg_1,$$  (2.110)

where $s$ is a scalar field of random numbers uniformly distributed between 0 and $s_0$. Furthermore, $s_0$ is chosen to be the order of the expected wrinkle amplitudes. To find a stable branch $u'_h$ is used as an initial value of the subsequent corrector step. Since we aim to compare the models with experimental results, we are mainly interested in the stable solutions. If the second variation of the total potential energy is positive definite for an equilibrium solution ($\lambda_{min} > 0$), it is considered as stable.

2.3 Experimental validation of the eFvK model

2.3.1 Motivational experiments on polypropylene films

We aimed to experimentally verify the disappearance of wrinkles on rectangular biaxially oriented, hot-melt polypropylene films (Polymer Plus Kft) having $L = 100$ mm length, $W = 50$ mm width and $h = 28$ $\mu$m thickness, covered with a 20 $\mu$m adhesive layer on one side. The typical Poisson’s ratio of polypropylene films fluctuates between $\nu = 0.40$ and $\nu = 0.45$. The ultimate strength of the film is around $\varepsilon_u = 1.65$, which is far above the relevant region ($\varepsilon = 0.1...0.25$). Technical data of the applied film provided by the distributor is listed in Appendix A. We performed displacement controlled pull tests using a Zwick Z150 tensile testing machine equipped with Zwick 9103 10 grips developed for technical membranes at the Czakó Adolf Solid Mechanics Laboratory of the BME Department of Mechanics, Materials and Structures. We argued that the adhesive layer covering one side of the strip has a negligible effect on the wrinkling behavior, however, it causes some uncertainty in the thickness of
Table 2.1: Measurements carried out on rectangular polypropylene sheets having $L = 100\,\text{mm}$ length, $W = 50\,\text{mm}$ width and $h = 28\,\mu\text{m}$ thickness. $\Delta L$ denotes the displacement between the clamps, and $\Delta L_p$ is the plastic deformation determined after unloading the sheet.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\Delta L$ [mm]</th>
<th>$\Delta L_p$ [mm]</th>
<th>observed shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0</td>
<td>0.0</td>
<td>flat</td>
</tr>
<tr>
<td>0.05</td>
<td>5.0</td>
<td>0.0</td>
<td>wrinkled</td>
</tr>
<tr>
<td>0.10</td>
<td>10.0</td>
<td>2.8</td>
<td>wrinkled</td>
</tr>
<tr>
<td>0.20</td>
<td>20.0</td>
<td>12.0</td>
<td>wrinkled</td>
</tr>
<tr>
<td>0.30</td>
<td>30.0</td>
<td>18.0</td>
<td>wrinkled</td>
</tr>
<tr>
<td>0.40</td>
<td>40.0</td>
<td>25.5</td>
<td>wrinkled</td>
</tr>
<tr>
<td>0.70</td>
<td>70.0</td>
<td>51.2</td>
<td>wrinkled</td>
</tr>
<tr>
<td>0.80</td>
<td>80.0</td>
<td>59.2</td>
<td>flat</td>
</tr>
<tr>
<td>0.90</td>
<td>90.0</td>
<td>67.1</td>
<td>flat</td>
</tr>
</tbody>
</table>

Figure 2.2: Appearance and disappearance of wrinkles on a polypropylene sheet recorded at different macroscopic strains ($L = 100\,\text{mm}, W = 50\,\text{mm}, h = 28\,\mu\text{m}$).

The critical macroscopic strains for the appearance and disappearance of wrinkles were determined (Table 2.1). Repeated tests yielded the same results, and the initial imperfections did not affect the outcome of the tests. Subsequently, the experiments were robust and reproducible. However, unloading the sheets revealed residual strains depending on the maximal applied macroscopic strain. Consequently, we unloaded the specimens for each $\varepsilon$ values to determine the plastic deformation. Despite the residual strain, in agreement with the predictions of the eFvK model the wrinkles disappeared in the experiments. In detail, as the macroscopic strain was increased, wrinkles first appeared on the initially flat surface, their amplitude increased then
decreased, and finally, they disappeared (Figure 2.2). This observation qualitatively verified the first prediction of the eFvK theory and the error of the FvK theory. However, the polypropylene film was not suitable for quantitative comparison due to its large plastic deformations and material nonlinearities.

2.3.2 Experiments on prestressed polyurethane films

We tested many different films from various manufacturers and selected polyurethane for the second set of experiments. Polyurethane films have many advantages including excellent tensile strength, elongation and small residual strains (Farkas 2004). Due to their microbial resistance, semi-permeabilic polyurethane films have many medical applications. We carried out the experiments on a wound-dressing health-care product under the name Hydrofilm Roll manufactured by Paul Hartmann AG (Appendix A). According to our measurements the thickness of the film is 20 µm and it is covered with a 40 µm adhesive layer. Here we also considered only the uncertainty of the thickness and neglected the asymmetry caused by presence of the adhesive layer. It is slightly orthotropic, which can be a consequence of the manufacturing technique (Ward 1997). Moreover, such films might be slightly prestretched as a result of the assembly process of the final product. Analysing the stress-strain diagram of the film under cyclic loading revealed, that the film behaves hyperelastic at the first loading but almost linear elastic during the subsequent loading cycles (Figure 2.3). By examining the wrinkling of prestressed films, it is possible to eliminate material nonlinearities in the model problem. However, after the prestress, the initially small orthotropy of the film becomes significant.

We examined the material properties and the wrinkled patterns separately, in particular, two series of experiments were carried out:

- Traditional displacement controlled pull-tests to obtain the force-displacement diagrams of the prestressed films.

- Displacement controlled pull-tests on prestressed rectangular films having fixed length but varying width to determine the critical stretch for the disappearance of wrinkles.

We determined the material properties using the Zwick Z150 testing machine equipped with Zwick 9103 10 grips. During the displacement-controlled test, the machine detected the force-displacement diagram in 0.01 mm steps. To eliminate the effect of the strain rate on the behavior we determined the diagram for various rates. Since the resulting diagrams were close to each other, we fixed the strain rate at
Figure 2.3: Applied strain $\varepsilon$ vs. engineering stress $\sigma_{\text{eng}}$ measured for $W \in \{18, 20, 25, 30, 35, 40\}$ mm, and fixed $L = 50$ mm long sheets. Blue lines denote the trend of average stress and standard deviation during the first loading and the unloading, and the red line is the reloading. Altogether 28 series of measurements were carried out to determine the diagram.

120 mm/min. After the prestress, the following loading cycles turned out to have a negligible effect on the material properties. We fixed the prestress at $\varepsilon_0 = 0.66$ in all experiments and measurements since we expected the wrinkles to appear and disappear below this threshold. The specimens had $L = 30$ mm length and $W = 25$ mm width and were cut out parallel, in 45 degrees and perpendicular to the machine direction. The results of these measurements provided the material parameters, the elastic moduli in three directions $Y_x, Y_y, Y_{45}$ and the Poisson ratio $\nu_{xy}$ (Cho et al. 2001; Lempriere & B. M. 1968). The shear modulus $S_{xy}$ can be determined from the Young moduli following Yokoyama & Nakai 2007; Daniel & Ishai 2006 using the relation:

$$\frac{1}{Y_{45}} = \frac{1-\nu}{4Y_x} + \frac{1-\nu}{4Y_y} + \frac{1}{4S_{xy}}.$$  \hspace{1cm} (2.111)

We carried out control tests using a Zwick Z020 material testing machine and a video extensometer Messphysik ME 46 and determined the force-displacement diagram in 0.002 mm intervals. All the testing series resulted in close outcomes, and the variation of the material parameters was also low. The change in the Poisson ratio was negligible $\nu = \nu_{xy}$ (Table 2.2).

After determining the material properties, we carried out wrinkling experiments. We focused on the disappearance of wrinkles and determined the $\varepsilon_{cr2}$ critical
Table 2.2: Material parameters measured before and after prestressing determined as the average of at least five measurements in each direction.

<table>
<thead>
<tr>
<th>Before prestressing</th>
<th>After prestressing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_x$ $48.45 \text{ N/mm}^2$</td>
<td>$Y_x$ $15.36 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>$Y_y$ $37.75 \text{ N/mm}^2$</td>
<td>$Y_y$ $27.69 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>$Y_{45}$ $44.23 \text{ N/mm}^2$</td>
<td>$Y_{45}$ $28.66 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>$S_{xy}$ $17.41 \text{ N/mm}^2$</td>
<td>$S_{xy}$ $14.55 \text{ N/mm}^2$</td>
</tr>
<tr>
<td>$\nu$ $0.30 \text{ N/mm}^2$</td>
<td>$\nu$ $0.30 \text{ N/mm}^2$</td>
</tr>
</tbody>
</table>

stretches. The unloaded length of the specimens was $L_0 = 50 \text{ mm}$. After a $\varepsilon_0 = 0.66$ prestretch, we unloaded the sheets and waited a few seconds for the sheets to settle. For such prestretch 3 mm residual strain was observable regardless of the width of the film. We carried pull tests on these prestressed films with a machine equipped with servo engine. The critical stretch for the appearance of wrinkles ($\varepsilon_{cr1}$) was very close to $\varepsilon = 0$ thus hard to measure such as it was reported by Zheng 2008. As a result, we determined only $\varepsilon_{cr2}$ for different aspect ratios.

Figure 2.4: Experimental results for determining the disappearance of wrinkles. Figures (a)-(d) show a $L = 53 \text{ mm, } W = 25 \text{ mm}$ rectangular sheet at different macroscopic strain values. The subscript $r$ marks, that the experiment was carried out on a prestressed film. The average values and the standard deviations are determined at least from ten measurements for each aspect ratio.
2.3.3 Orthotropic extension of the model

To compare the experimental results with numerical computations, we incorporated orthotropy in the eFvK model. The potential energy functional for orthotropic films is derived similarly to the isotropic case (Libai & Simmonds 1998). Assuming $x$ and $y$ to be the main material directions, the eFvK model can be extended to orthotropic materials with the introduction of nondimensional orthotropy parameters $r = \frac{Y_x}{Y_y}$ and $q = \frac{S_{xy}}{Y_x}$, where $Y_x, Y_y, S_{xy}$ are the elastic moduli and shear modulus respectively. Symmetry conditions on the elastic stiffness tensor imply that the Poisson ratios in the two principal directions fulfill $\nu_{yx} = r \nu_{xy}$ (Howell, Kozyreff, & Ockendon 2008). The stress-strain relationship (Equation (2.13)) has the following form assuming anisotropy:

$$\sigma_{ij} = Q_{ijkl} \varepsilon_{kl},$$  \hspace{1cm} (2.112)

where

$$Q_{ijkl} = \frac{Y_x}{1 - rv_{yx}^2} \tilde{Q}_{ijkl}$$ \hspace{1cm} (2.113)

is the material stiffness tensor. Assuming an orthotropic medium, the nonzero elements of the tensor are $\tilde{Q}_{1111} = 1, \tilde{Q}_{1122} = \tilde{Q}_{2211} = rv_{yx}, \tilde{Q}_{1212} = q(rv_{yx}^2 - 1)$. The elastic energy densities are given by integrating through the thickness of the film:

$$\Psi_m(E) = \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} E \cdot Q \cdot E \, dz = \frac{Y_x h}{1 - rv_{yx}^2} E \cdot \tilde{Q} \cdot E,$$ \hspace{1cm} (2.114)

$$\Psi_b(k) = \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} k \cdot Q \cdot kz \, dz = \frac{1}{12} \frac{Y_x h^3}{1 - rv_{yx}^2} k \cdot \tilde{Q} \cdot k.$$ \hspace{1cm} (2.115)

After substituting, rescaling by $12(1 - rv_{yx}^2)/(Y_x h)$ the total potential energy becomes

$$I_f(u) = \int_{\Omega} \left( \Psi_m(E) + \Psi_b(k) \right) \, d\Omega = \int_{\Omega} 12 \left( E \cdot \tilde{Q} \cdot E + h^2 k \cdot \tilde{Q} \cdot k \right) \, d\Omega.$$ \hspace{1cm} (2.116)

The first variation of the energy Equation (2.116) leads to the Euler-Lagrange equations:

$$\nabla \cdot (FN^o_f) = 0,$$ \hspace{1cm} (2.117)

$$h^2 \Delta^2 w - \nabla \cdot (N^o_f \nabla w) = 0,$$ \hspace{1cm} (2.118)
where

\[
N^0_f = 12 \begin{bmatrix}
E_{11} + r v_{xy} E_{22} & 2q(1 - r v_{xy}) E_{12} \\
2q(1 - r v_{xy}) E_{12} & r E_{22} + r v_{xy} E_{11}
\end{bmatrix}.
\]

(2.119)

Note that taking \( r = 1 \) and \( q = 0.5(1 + \nu)^{-1} \) is identical to the isotropic case.

### 2.3.4 Discussion

![Figure 2.5: Comparison of computed and measured value of the critical stretch \( \varepsilon_{cr2} \), where the wrinkles disappeared on prestressed polyurethane films. The dot-dashed line is the border of the possible wrinkled configurations. The grey area is the region of parameter configurations leading to stable wrinkled solutions in the numerical computations.](image)

We used the Ritz-method described in Section 2.2.1 to implement the orthotropic model. The material behavior of prestressed polyurethane films can be modeled assuming linear elasticity and orthotropy. Consequently, the results of the wrinkling tests can be compared to the results of the eFvK model extended to orthotropic materials. From Table 2.2 the best fit for the unloaded material was \( r = 0.78 \) and \( q = 0.36 \), while for the prestressed film we measured \( r = 1.80 \) and \( q = 0.94 \) and \( v_{xy} = \nu = 0.3 \). Because of the rescaling of the energy (Equation (2.116)), \( \gamma_s \) is not necessary for the computations. We computed and checked the stability of the trivial solution to determine the stability boundary in the \( \beta - \varepsilon \) plane. To incorporate the change in the length after the prestress, we considered a fixed \( L = 53 \) mm unloaded length in the computations. For the measured orthotropy parameters \( h = 40 \mu \text{m} \) resulted in the best approximation of the experimental results, which can be explained with the un-
uncertainty of the exact thickness of the film, due to the adhesive layer. In agreement with the predictions of the eFvK theory, wrinkles emerged only for a bounded range of aspect ratios, rectangles wide or narrow enough exhibited no wrinkling independent of the applied macroscopic strain in both the computations and the experiments (Figure 2.4). Comparison of the experimental data and the numerical calculations led to the quantitative validation of both predictions of the eFvK model (Figure 2.5).

Figure 2.6: The effect of orthotropy. (a) The grey area represents the possible wrinkled parameter configurations, outside that $N_f^0 > 0$. (b) Points on the sheet where $N_f^0 < 0$ are marked with dark color. (c) The continuous lines represent the stability boundary for different thicknesses, while dash-dot lines indicate the border of the possible wrinkled configurations.

To further understand the wrinkling behavior affected by the material parameters, we examine the effect of orthotropy depending on the aspect ratio and the macroscopic strain. According to the predictions of the finite strain theory, for a fixed thickness, the parameter pairs with a stable wrinkled solution form a closed region in the $\beta - \varepsilon$ plane. Healey, Q. Li, & Cheng 2013 showed in a lemma that the negativity of the 2nd Piola-Kirchhoff stress tensor is a necessary condition for wrinkling. Accordingly if $vN_f^0v^T \geq 0, \forall v = [a, b]$ no wrinkling occurs independently of the thickness of the film. Therefore, $N_f^0 < 0$ determines the region containing the possibly wrinkled parameter configurations. The effect of orthotropy is compared to the isotropic case.
Numerical analysis showed that, when the transverse elastic modulus dominates the axial modulus ($r > 1$), the bifurcation points for a fixed thickness occupy a more extended region. Furthermore, the region of the possible wrinkled parameter configurations is also larger, otherwise ($r < 1$), they are both reduced. In other words, increasing the transverse elastic modulus induce and amplify wrinkling (Figure 2.6).

### 2.4 Conclusions and principal results

Using the orthotropic model we made a quantitative comparison between experiments carried out on prestressed polyurethane films and computations relying on the Saint Venant-Kirchhoff constitutive model to investigate the behavior. We reported about successful experiments to validate the predictions of the eFvK theory. Wrinkles emerged and disappeared on clamped films under uniform tension as the applied macroscopic strain was increased. Our work not only provided experimental proof for the existence of $\varepsilon_{cr2}$ critical stretch but also confirmed, that the aspect ratio affects the location of the bifurcation points. Our findings support the rigorously predicted isola-center bifurcation in the eFvK model in Healey, Q. Li, & Cheng 2013.

**PRINCIPAL RESULT 1.** (Fehér and Sipos 2014, Sipos and Fehér 2016)

Geometrically exact models of finite-deformation nonlinear elasticity are needed to understand the mechanics of highly stretched thin films. The classical Föppl-von Kármán plate theory (FvK) has been extended to the finite membrane strain regime recently. The model, called extended Föppl-von Kármán (eFvK) theory, applies some hyperelastic constitutive relation for the in-plane behavior.

1.1. The extended Föppl-von Kármán model with the Saint Venant-Kirchhoff material assumes isotropic material. Motivated by experimental results, I further developed the eFvK theory to accommodate orthotropic materials, at which the main material directions are parallel to the $x$ and $y$ directions of the reference configuration. In the model I used two, nondimensional parameters, $r$ and $q$ to describe orthotropy.

1.2. I carried out numerical computations to investigate the effect of geometric and material parameters on wrinkling. I found, that at a fixed thickness $h$ the aspect ratio of the domain $\beta$, the macroscopic strain $\varepsilon$ and the ratio of the modulii of elasticity $r$ strongly influence the wrinkling behavior. Compared to the isotropic case the stability boundary, that separates wrinkled and flat configuration in the $\beta - \varepsilon$ plane, significantly extends as long as the transverse elastic modulus dominates the axial modulus ($r > 1$). For ($r < 1$) the stability boundary shrinks.
We incorporated orthotropic material in the eFvK theory by introducing the $r$ and $q$ orthotropy parameters and pointed on the effect of orthotropy on wrinkling.

**PRINCIPAL RESULT 2.** (Sipos and Fehér 2016)  
To validate the theoretical predictions of the eFvK model, I carried out experiments on previously prestressed, clamped, rectangular, orthotropic, 32 $\mu$m thick polyurethane films.

2.1. I experimentally verified the first prediction of the eFvK model. It predicted, that if wrinkles appear on the initially flat surface as the macroscopic stretch is increased, then there is a maximal amplitude for the wrinkles. Further stretch leads to decrease in the amplitude and at a second bifurcation point the wrinkles eventually disappear: The sheet becomes flat again. The experiments clearly demonstrated the disappearance of wrinkles, and as polyurethane is dominantly elastic, this validated the prediction of the model. Using the orthotropic model I computed the location of the bifurcation points in the parameter space and found good quantitative agreement with the experimental data.

2.2. I experimentally verified the second prediction of the eFvK model. It predicted, that wrinkling should appear only for a bounded interval of aspect ratios. In agreement with the prediction, the experiments clearly demonstrated that for fixed length and thickness sufficiently narrow or wide sheets do no exhibit wrinkling. The computed and measured stability boundaries are in good agreement.

A more accurate hyperelastic model for elastomers is expected to tighten the gap in the quantitative comparison. Furthermore, our work also suggests other questions for further research, including the adjustment of the material during prestressing via some well-chosen damage propagation approach.
A pseudoelastic model

In this chapter, we demonstrate how inelastic behavior affects the wrinkling of stretched, rectangular films. We experimentally examine polyurethane films under loading and unloading called prestress in Section 2.3.2 and implement a material model suitable to represent the wrinkling behavior of the sheet. Finally, we fit material parameters of the model to the measured stress-strain diagrams and compare the experimentally observed wrinkling behavior to the numerical computations.

3.1 Experiments

Cyclic loading of axially stretched, rectangular polyurethane films revealed an intriguing phenomenon. For some aspect ratios, the sheet remains flat after the first loading, but it wrinkles during the unloading and the subsequent loading cycles. We used the same polyurethane sheet and equipment as in Section 2.3.2 to determine the material properties and the wrinkling behavior. The measurements were carried out on L = 50 mm long sheets at a speed of 120 mm/min with no pause between the loading and unloading parts. Two series of experiments were carried out on polyurethane sheets:

- Traditional displacement controlled pull-tests to obtain a force-displacement diagram for the material.

- A series of loading-unloading cycles of specimens with different aspect ratios to determine the disappearance of wrinkles during loading $\varepsilon_{cr2}$ and the appearance of wrinkles during unloading $\varepsilon_{cr3}$ visually.

Firstly, altogether 28 series of measurements were performed on a Zwick Z150 testing machine to produce the force-displacement diagram. The stress-strain diagram
was determined using

$$\sigma_{\text{eng}} = \frac{F^{\text{exp}}}{hW},$$

(3.1)

$$\varepsilon = \frac{\Delta L}{L_0},$$

(3.2)

where $F^{\text{exp}}$ is the measured force, $\Delta L = L - L_0$, $L$ is the actual distance between the clamps, and $L_0$ is the initial distance between the clamps. We approximated the thickness of the sheet with 32 $\mu$m and fixed the $L_0 = 50$ mm. According to the stress-strain diagram, the stress-softening of the material is significant, and about 8% of residual strain is observable (Figure 3.1).

![Applied strain $\varepsilon$ vs. engineering stress $\sigma_{\text{eng}}$](image)

Figure 3.1: Applied strain $\varepsilon$ vs. engineering stress $\sigma_{\text{eng}}$ (measured force per unit reference area) measured for $W \in \{18, 20, 25, 30, 35, 40\}$ mm, and fixed $L = 50$ mm long sheets. Blue lines denote the trend and standard deviation of the stress during the first loading and the unloading (the prestress), and the red line is the reloading. Altogether 28 series of measurements were carried out to determine the diagram.

Both the unrecoverable deformations and the stress-softening of the applied polyurethane films point to the so-called Mullins effect (Dorfmann & Ogden 2004; Ogden & Roxburgh 1999). It distinguishes a primary loading path and damaged sub-paths, where the stresses depend on the previously occurred maximal stresses in the material. The fact, that we found no significant dependence of the measured constitutive law on speed or aspect ratio strongly suggested the choice for a pseudoelastic model. Although the problem is two-dimensional, it can be considered as a dominantly uniaxial stretching, since the transverse stresses are small (except the vicinity
of the clamped boundaries) compared to the axial tension. We suggested that the incorporation of the stress-softening of the axial stress-strain diagram is enough to explain the observed wrinkling behavior. Therefore we used these uniaxial measurements to fit the material parameters to the numerical simulations.

Secondly, the wrinkling behavior of ten to twenty specimens with $L_o = 50$ mm and $W \in \{15...40\}$ mm were examined. Depending on the aspect ratio three different scenarios were observed:

- The specimen stayed flat during both the loading and the unloading part.
- Wrinkles appeared ($\varepsilon_{cr1}$) and disappeared ($\varepsilon_{cr2}$) during the loading and reappeared ($\varepsilon_{cr3}$) during unloading.
- The specimen stayed flat during the loading but became wrinkled ($\varepsilon_{cr3}$) during the unloading.

![First loading vs cyclic loading](image)

Figure 3.2: Experimental results $W = 35$ mm, $L_o = 50$ mm.

We determined $\varepsilon_{cr2}$ and $\varepsilon_{cr3}$ for each specimen (Figure 3.3). In case wrinkles appeared during the unloading, the sheet stayed wrinkled until $\Delta L/L_o$ reached the residual strain $\sim 8\%$. Then the film became slack, and keeping $\Delta L = 0$ for a few seconds the material healed and the residual strain reduced to $\sim 6\%$. Note that this
state of the sheet was the starting point of the experiments in Section 2.3.2. The collected data, including the standard deviations of the measured values of the critical stretches, are given in Table 3.1.

![Graph](image1.png)

Figure 3.3: Experimental results of $L_o = 50\text{ mm}$ rectangular specimens, taking $\beta_o = \frac{L_o}{(2W)}$ to determine the bifurcation points: (a) $\epsilon_{cr2}$ during the loading, (b) $\epsilon_{cr3}$ during the unloading. Dotted lines mark the examined aspect ratios.

<table>
<thead>
<tr>
<th>$W$ (mm)</th>
<th>$\beta_o$</th>
<th>$\epsilon_{cr2}$</th>
<th>$\epsilon_{cr3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1.67</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>1.25</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>23</td>
<td>1.09</td>
<td>0.16</td>
<td>0.15</td>
</tr>
<tr>
<td>25</td>
<td>1.00</td>
<td>0.27</td>
<td>0.13</td>
</tr>
<tr>
<td>28</td>
<td>0.91</td>
<td>0.30</td>
<td>0.28</td>
</tr>
<tr>
<td>30</td>
<td>0.83</td>
<td>0.24</td>
<td>0.15</td>
</tr>
<tr>
<td>35</td>
<td>0.71</td>
<td>-</td>
<td>0.18</td>
</tr>
<tr>
<td>40</td>
<td>0.63</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3.1: Measured values of $\epsilon_{cr2}$ and $\epsilon_{cr3}$ with standard deviations (calculated from at least ten measurements for each aspect ratio) for the loading and unloading part respectively, measured for $L_o = 50\text{ mm}$ specimens.

### 3.2 Qualitative explanation

A qualitative explanation for the appearance of wrinkles upon unloading is
• the developing orthotropy
• and the residual strains

measured in Section 2.3.2. In Figure 2.6 we presented how orthotropy extends the wrinkled zone in the $\beta - \varepsilon$ plane for fixed length and thickness. Here we show, that residual strains also contribute to the phenomenon. Considering finite deformation linear elasticity (Section 2.1.2), we fixed the thickness and the width of the sheet and computed the bifurcation points for different lengths. For a fixed $W = 35 \text{ mm}$ the sheet stays flat for $L = 50 \text{ mm}$ but wrinkles for $L = 54 \text{ mm}$. Note that the observed residual strain in the experiments was $\sim 8\%$ which changed the stress-free length from 50 to 54 mm.

![Figure 3.4: Numerical computation of a stretched rectangular sheet taking $W = 35 \text{ mm}$, $h = 20 \mu\text{m}$, $\nu = 0.5$. The shaded area corresponds to the wrinkled configurations. The dashed lines illustrate, that the change in the length of the sheet without changing any other material parameters contributes to the observed behavior.](image)

3.3 A pseudoelastic model

Our goal is to present a simple pseudoelastic material model, that captures the stress-softening and the axial residual strain and accurately predicts the three different kinds of wrinkling behavior observed experimentally, in particular, it reproduces the case when the sheet stayed flat upon loading but wrinkled during the unloading process. We argue that our problem is dominantly uni-axial and the transversal stress is around
one percent of the axial stress (except at the boundaries). Subsequently, the material changes take place mostly in the axial direction. In our model, the loading part is considered damage free, and stress-softening is incorporated during unloading. In contrary to the Mullins model proposed by Ogden & Roxburgh 1999, here we propose a single dissipation field acting on the axial and transversal components of right Cauchy-Green strain tensor \( \mathbf{C}(\mathbf{u}) = \mathbf{F}^T \mathbf{F} \), where \( \mathbf{F}(\mathbf{u}) \) is defined by Equation (2.51). Moreover, the dissipation only affects the membrane part of the energy. Following the terminology of Ogden & Roxburgh 1999; Dorfmann & Ogden 2004, the energy density has the following form:

\[
\bar{\Psi} = \bar{\Psi}(\mathbf{C}, \mathbf{k}, \eta) = \bar{\Psi}_m(\mathbf{C}, \eta) + \Psi_b(\mathbf{k}) + \Phi(\eta),
\]

(3.3)

where \( 0 \leq \eta \leq 1 \) is a state variable. In the \( \Phi(\eta) \) dissipation function, \( \eta \) is responsible to incorporate dissipation into the model and depends on the maximal applied strain in the loading history in such a way, that the variable is inactive during the first loading and active during the unloading. Here \( \eta = 1 \) is associated with the undamaged material, while \( \eta = 0 \) represents an entirely damaged state.

To accurately model the material upon loading, we incorporated a hyperelastic material model, namely the incompressible Mooney-Rivlin model (Q. Li & Healey 2016) in the membrane energy density part of Equation (3.3). We found that it results in an acceptable agreement with the experimental data regarding the undamaged material. It is a variation of the polynomial hyperelastic models and has the following form in two-dimensions:

\[
\bar{\Psi}_m(\mathbf{C}) = \alpha_1 [\text{tr} \mathbf{C} + \frac{1}{\det \mathbf{C}} - 3] + \alpha_2 [\frac{\text{tr} \mathbf{C}}{\det \mathbf{C}} + \det \mathbf{C} - 3],
\]

(3.4)

where \( \alpha_1 \) and \( \alpha_2 \) are material parameters. To incorporate the damage of the material into the energy density, we propose the following form:

\[
\bar{\Psi}_m(\mathbf{C}, \eta) = h p_1 \left[ ((1 + p_3) \eta - p_3)(C_{11} - 1) + \eta(C_{22} - 1) + \frac{1}{\det \mathbf{C}} - 1 \right] +
+h p_2 \eta \left[ \frac{\text{tr} \mathbf{C}}{\det \mathbf{C}} + \det \mathbf{C} - 3 \right],
\]

(3.5)

where \( p_1 [\text{N/mm}^2], p_2 [\text{N/mm}^2] \) are material constants and \( p_3 \) is a fixed nondimensional scalar parameter representing the ratio of the damage associated with the axial and transversal directions. As it was mentioned previously, anisotropic damage is expected and \( p_3 > 0 \) characterizes a higher axial damage. When \( \eta = 1 \) (the material is undamaged) Equation (3.4) and Equation (3.5) are formally the same.
As for bending, we keep the isotropic, linear elastic, linearized bending strain energy density $\Psi_b$ defined by Equation (2.17). To keep consistency with the incompressible membrane model, we take $Y = 3(p_1 + p_2)$ and $\nu = 0.5:

$$
\Psi_b(k) = \frac{(p_1 + p_2)h^3}{4(1 - \nu^2)} \left[ v(trk)^2 + (1 - \nu)k^2 \right].
$$

(3.6)

A more accurate model would account for the Mullins effect in bending by introducing a state variable into the bending energy density or incorporate the emerging orthotropy as in Equation (2.115). However, given that the bending energy density is $O(h^3)$ and the membrane energy density is $O(h)$, none of these refinements are necessary.

The last term of Equation (3.3) is the dissipation function, which is implicitly defined by

$$
\frac{\partial \Psi(C, k, \eta)}{\partial \eta} = \frac{\partial \Psi_m(C, k, \eta)}{\partial \eta} + \frac{d\Phi(\eta)}{d\eta} = 0.
$$

(3.7)

We require $\eta = 1$ at the point where the first unloading was initiated and $\Phi$ need to satisfy $\Phi(1) = 0$ along the primary loading part (here it is called loading) and $\Phi''(\eta) < 0$ to any admissible value of $\eta$ (Dorffmann & Ogden 2004). We define the evolution law of the state variable field as follows:

$$
\eta := 1 - p_4 \tanh(p_5(\Xi_{\text{max}} - \Xi_i)),
$$

(3.8)

where $p_4 > 0, p_5 > 0$ are fixed nondimensional material parameters and based on Equation (3.7):

$$
\Xi_i = -\frac{d\Phi(\eta)}{d\eta} = \frac{\partial \Psi_m(C, \eta)}{\partial \eta},
$$

(3.9)

$$
\Xi_{\text{max}} = -\left. \frac{d\Phi(\eta)}{d\eta} \right|_{\epsilon = \epsilon_{\text{max}}} = \left. \frac{\partial \Psi_m(\partial C, \eta)}{\partial \eta} \right|_{\epsilon = \epsilon_{\text{max}}}.
$$

(3.10)

The state variable $\eta$ defined by Equation (3.8) corresponds to a $\Phi(\eta)$ dissipation function, that fulfills the above-mentioned requirements.

Finally, the total potential energy is summarized as:

$$
\tilde{I}(u, \eta) = \int_\Omega \Psi_m(C, \eta) \, d\Omega + \int_\Omega \Psi_b(k) \, d\Omega + \int_\Omega \Phi(\eta) \, d\Omega.
$$

(3.11)

Computing the first variation of Equation (3.11) with respect to the $u$ displacement
field leads to:

$$\nabla \cdot (\mathbf{F} \mathbf{N}) = 0, \quad (3.12)$$

$$\frac{(p_1 + p_2) h^3}{4(1 - v^2)} \Delta^2 w - \nabla \cdot (\mathbf{N} \nabla w) = 0. \quad (3.13)$$

Here the 2nd Piola-Kirchhoff stress is given by:

$$\mathbf{N} = 2 \frac{\partial \Psi_m}{\partial \mathbf{C}}. \quad (3.14)$$

Observe that the variation of the energy respect to the $\eta$ field is identically zero, while the evolution of the $\eta$ field is defined via Equations (3.8) to (3.10). In summary, the model depends on five material parameters, $p_1 \text{[N/mm}^2], p_2 \text{[N/mm}^2] \text{and nondimensional} p_3, p_4, p_5$, all of them considered to be a time-invariant, fixed scalars.

## 3.4 Discussion

Since the problem is essentially uniaxial, all material parameters ($p_1, p_2, p_3, p_4, p_5$) can be determined from the uniaxial tensile tests. Assuming uniaxial stretching the right Cauchy-Green tensor has the following form:

$$\mathbf{C} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \cong \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad (3.15)$$

where the principal stretch in the axial direction in case of uniaxial tension:

$$\lambda = 1 + \varepsilon. \quad (3.16)$$

Using these simplifications the engineering stress (force per unit reference area) in the axial direction $\sigma_{\text{eng}}$ is found to be

$$\sigma_{\text{eng}} = \frac{\partial \Psi}{\partial \lambda} = \frac{\partial \Psi_m}{\partial \lambda} = \frac{2p_1(p_3 + \eta - p_3 \eta)\lambda^4 \eta - p_1(1 + \eta)\lambda + 2p_2\lambda^3\eta - 2p_2\eta}{\lambda^3}. \quad (3.17)$$

Since $\eta = 1$ during loading, parameters $p_1, p_2$ can be fitted for the stress strain curve for the primary loading, then $p_4, p_5$ can be determined based on the unloading part. The parameters can be fitted using minimal least-squares error between the measured and the computed data (Figure 3.5). The model predicts 8% residual strain and the state variable field reduces to $\eta = 0.88$ upon unloading. The ratio of the dissipation
Figure 3.5: Applied strain $\varepsilon$ vs. engineering stress $\sigma_{\text{eng}}$. Dashed lines and error bars correspond to the measured data, solid lines depict the model prediction taking $p_1 = 2.00 \text{ N/mm}^2$, $p_2 = 0.45 \text{ N/mm}^2$, $p_3 = 1.25$, $p_4 = 0.12$, $p_5 = 0.8$.

Figure 3.6: Comparison of experimental and numerical results depending on the aspect ratio and the applied strain. Taking $L_o = 50 \text{ mm}$, $h = 32 \mu \text{m}$, $p_1 = 2.00 \text{ N/mm}^2$, $p_2 = 0.45 \text{ N/mm}^2$, $p_3 = 1.25$, $p_4 = 0.12$, $p_5 = 0.8$.

described by $p_3$ can be obtained by computing $\varepsilon_{cr2}$ and $\varepsilon_{cr3}$. Using Figure 3.3 the best match for the ratio was $p_3 = 1.25$ (Figure 3.6).

To obtain the wrinkling patterns of the model we carried out numerical computations using the continuation method mentioned in Section 2.2.2. Numerical results for the critical stretches $\varepsilon_{cr2}$ and $\varepsilon_{cr3}$ are compared with the mean value of the experimental data in Figure 3.7. The model accurately predicts no wrinkles during loading
Figure 3.7: Comparison of the experimental and numerical results. Wrinkling appears only during the unloading for $\beta_o = 0.71$.

(a) $\beta_o = 1.00$

(b) $\beta_o = 0.71$

Figure 3.8: Maximal amplitude of wrinkles for sheets with $L_0 = 50$ mm, $h = 32 \mu$m, $\beta_o = 1.00$ and $\beta_o = 0.71$. For $\beta_o = 0.71$, there is no wrinkling during the loading but wrinkles appear during the unloading. Observe, that the model predicts wrinkles at the residual strain upon unloading.

and wrinkling upon unloading for $\beta_o = 0.71$. Moreover, by computing the maximal wrinkle amplitudes (Figure 3.8), we showed, that the model rightly recovers the phenomenon observed on Figure 3.2: the sheet is still wrinkled at the residual strain.
3.5 Conclusions and principal results

Loading and unloading of highly stretched, rectangular polyurethane films resulted in an exceptional wrinkling behavior. For some aspect ratios, the sheet stayed flat during the loading process but wrinkled upon unloading. The stress-strain diagram of the film pointed to the Mullins effect. To model the whole process, we incorporated hyperelasticity and a finite-deformation pseudoelastic material model accounting for the Mullins effect in the mechanical model. We argued that the problem is dominantly uniaxial and suggested a significantly simplified model. In contrary to the classical pseudoelastic model with two dissipation fields, our model is characterized by a single state variable. The material parameters of the model are first fitted to the measured stress-strain diagrams to capture the stress-softening and the residual strain. Next, we demonstrated that the pseudoelastic model accurately predicts the experimentally observed wrinkling behavior. We found good agreement between the measurements and computations. In summary, inelastic behavior can qualitatively affect the wrinkling of highly stretched thin films.

PRINCIPAL RESULT 3. (Fehér, Healey and Sipos 2018)
The obtained experimental data clearly showed, that inelastic material properties significantly affect wrinkling. In specific, stress softening, emerging residual strain and significant change in the orthotropy parameters were recorded.

3.1. In the experiments with polyurethane sheets I demonstrated, that for specific aspect ratios no wrinkling appears during the first loading of the virgin sheet, but wrinkles emerged during the unloading and the subsequent cyclic loading. The measured data clearly hints that the Mullins effect causes the observed phenomenon.

3.2. I introduced a pseudoelastic extension of the eFvK model to take the Mullins effect into account. This model describes the whole loading program of the experiments. The model uses a single state variable field (damage field) and it has only 5 material parameters. Yet, it predicts stress-softening, residual strain and emerging orthotropy, in accordance with the measured data. The parameter values at wrinkles appear/disappear, are in fair agreement.
Outlook on wrinkling of curved surfaces

In this chapter, we extend our investigation to slightly curved surfaces through surface analogies of two-dimensional problems. After some motivational experiments, we investigate the effect of curvature on an axially stretched, rectangular but slightly curved sheet by mapping the surface geometry back to a two-dimensional plane. Finally, we formulate the model to general, curved surfaces.

4.1 Examinations on the effect of curvature

4.1.1 Motivational experiments

In the experiments, we aimed to examine problems related to engineering structures. They are usually designed to have a wrinkle-free shape, and wrinkle formation as a result of a change in the load distribution or failure of a structural component is often undesirable. We focused on membrane structures usually constructed from a coated fabric.

First of all, we examined flexible fabric material in hydrostatic stress state by stretching it over a fixed, planar frame. We constructed unsupported edges by removing the support of the frame along a line (Figure 4.1a). As a result of the asymmetric stresses, the fabric rolled up along the unsupported edge. However, shaping the edges curved resulted in less rolling up. The same phenomenon was observable in case of cut holes on the fabric under hydrostatic stress. A straight cut caused rolling up, but a circular hole resulted in no out-of-plane deformations. Moreover, the direction of the rolling up was also affected by the direction of the straight cut due to the orthotropy of the fabric. Defects in the material have a significant effect on the wrinkling behavior, and it can also be used to suppress wrinkling (Flores-Johnson et al. 2015; M. Li et al. 2017; Luo et al. 2017).

The problem of the unsupported edge occurs in the engineering practice in case of tents. To create a surface analogy of the planar problem we built and examined
physical tent models. The first model is created using a regular dodecagon shape cut out a flat flexible fabric material. The fabric is stretched by a central pole and one column stabilized by two ropes per corner. The tent stands on a plywood board, and all structural elements connect to the board with pin-joint connections. In particular, bending moments are not transferred by the supporting elements. The initially straight and unsupported edges between the corners rolled up as a result of the tension (Figure 4.1b). We were able to suppress these rolling ups by shaping the edges of the initial polygon curved.

Finally, motivated by the solutions to avoid the edge rolling up applied in the engineering practice, we built a tent model with stiffened and curved edges having a cutting pattern generated from a Pelikan-membrane shape designed with the software provided by Dezső Hegyi (Hegyi 2006). A central pole and six corners supported by a column and two cables ensured the appropriate stretch in the fabric. We tested three textiles: cotton, polyester, polyester coated with PVC. The forces in the cables were measurable and they were attached to tuners to ensure the even stress distribution in the surface. No rolling up was observable on the free edges of the tent as
a result of the stiffening and the curved edge shaping (Figure 4.2a). The tent model is a perfect candidate to examine the wrinkles on curved surfaces. Due to its initial shape, it was unwrinkled without external loads but became wrinkled in failure situations. We removed some of the support cables and the support columns to disturb the stress state (Figure 4.2b). As a result, disturbed corners moved to the side and the stress distribution between the cables changed remarkably. The emerging wrinkles created straight lines connecting the highly loaded, disturbed corners. The wrinkles highlighted the directions of tension on the surface as we observed in case of the stretched, two-dimensional rectangular sheet.

4.1.2 Computation of a curved rectangle

The interpretation of wrinkles on curved surfaces is not trivial since deformation in the normal direction is not always considered as wrinkling. For instance, in case of axially stretched cylindrical surfaces the Poisson’s effect causes deformation in the normal direction. Consequently, we define wrinkling on thin film surfaces as a state when the bending energy becomes nonzero. The surface analogy of the model problem is the wrinkling of a slightly curved, stretched rectangle. We aim to investigate the effect of curvature in the eFvK model assuming Saint Venant-Kirchhoff material described in Section 2.1.2 by mapping the surface back to a two-dimensional rectangle. For a cylindrical surface, this can be easily carried out. Keeping the definition of the strain tensor $\mathbf{E}$ for the planar problem in Equation (2.43), the Green-Lagrangian strain tensor for a cylindrical surface has the following form:

$$\tilde{\mathbf{E}}^m = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} + \frac{w}{R} \end{bmatrix}, \quad (4.1)$$

where $R$ is the radius of the cylinder. The additional term $w/R$ is a result of the strains in the circumferential direction caused by a change in the radial direction (Donnell 1933):

$$\frac{R + w}{R} - 1 = \frac{w}{R}. \quad (4.2)$$

The total potential energy is formally the same as in the planar problem:

$$\tilde{J}_f(u) = \int_{\Omega} \left( \Psi(\tilde{\mathbf{E}}^m) + \Psi(k) \right) \Omega. \quad (4.3)$$
After computing the first variation of the energy, integration by parts leads to:

\[ \nabla \cdot [F_{\tilde{N}}^m] = 0, \quad (4.4) \]

\[ h^2 \Delta^2 w - \nabla \cdot (\tilde{N}_f^m \nabla w) + \tilde{N}_f^m \cdot \left[ \frac{1}{R} (g_2 \otimes g_2) \right] = 0, \quad (4.5) \]

where

\[ \tilde{N}_f^m = N_f(\hat{E}^m). \quad (4.6) \]

Figure 4.3: Axially compressed open cylinder.

<table>
<thead>
<tr>
<th>$2\alpha$</th>
<th>$h$</th>
<th>$\sigma_{cr}$</th>
<th>$\sigma_{cr}^{comp}$</th>
<th>$40\times80$ mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>0.02</td>
<td>0.00021</td>
<td>0.00021</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.00052</td>
<td>0.00054</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>0.00103</td>
<td>0.00100</td>
<td></td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>0.02</td>
<td>0.00031</td>
<td>0.00030</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.00075</td>
<td>0.00073</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>0.00155</td>
<td>0.00150</td>
<td></td>
</tr>
<tr>
<td>$9\pi/5$</td>
<td>0.02</td>
<td>0.00037</td>
<td>0.00035</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.00093</td>
<td>0.00085</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>0.00186</td>
<td>0.00155</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Nondimensional numerical critical loads $\sigma_{cr}^{comp}$ of a compressed cylindrical shell with $W = 25$, $L = 50$, $Y = 1$, $\nu = 0.30$ compared with an analytical solution $\sigma_{cr}$ from the literature.

The model is validated by computing the critical stress for buckling under axial
compression (Magnucka-Blandzi & Magnucki 2004). We prescribed the displacement at the ends as Dirichlet boundary conditions and computed the axial stress using

\[ \sigma_x = \frac{\bar{N}^{m}_{f_{11}}}{h} = \frac{Y(\bar{E}^{m}_{11} + \nu \bar{E}^{m}_{22})}{1 - \nu^2}. \]  

(4.7)

Figure 4.4: Disappearance of wrinkles for a stretched open cylinder and the effect of curvature on the critical stretches. a) A cylindrical shell stretched along its curved edges (the other two edges are free). b) Critical stretches, where wrinkles appear/disappear for thickness \( h = 0.01 \) and \( h = 0.02 \). In the shaded area the wrinkled configuration is stable. The emerging patterns for several cases are plotted in the c)-e) subfigures.

According to (Magnucka-Blandzi & Magnucki 2004), the critical axial compressive stress \((\sigma_x)\) along the curved, hinged edges (while the other two, straight edges are free) is

\[ \sigma_{cr} = \frac{1}{8.11} \left( 1 - 0.0146 \frac{\alpha}{\pi} \right) \frac{Yh}{R \sqrt{3(1 - \nu^2)}} \]  

(4.8)

where \( 0.5\pi \leq \alpha \leq \pi \) is the half of the sectorial angle. In Table 4.1 we compare the stress at the numerical bifurcation of the trivial, unbuckled state to this analyt-
ical value. The differences between the analytical and numerical results are higher for larger $\alpha$ values which can be accounted for the fact, that the model is valid for moderate curvatures.

This model is used for investigating the effect of curvature on wrinkling under tensile loads. A cylindrical shell with fixed $L$ and $W$ is subjected to axial tension. The curvature of the undeformed surface is $\kappa_0 = 1/r$, where $r \geq 0$. For zero curvature, the model reproduces critical stretches associated with the appearance and disappearance of wrinkles of the flat rectangle. We computed the stability boundary at a fixed aspect ratio. Increase in the curvature reduces the distance of the critical points in the stretch direction. For a given thickness of the film, there is a critical curvature value at which the lower and upper critical stretches coincide. Above that curvature, no wrinkles appear regardless of the magnitude of the macroscopic strain (Figure 4.4). Subsequently, the curvature indeed affects the wrinkling behavior.

### 4.2 Model extension to general curved surfaces

#### 4.2.1 Introduction

Motivated by the experimental and numerical results, we aim to extend our model to general curved surfaces. There are many extensions of the FvK equations incorporating curvature, yet they are based on the small strain assumption and often restricted to a particular geometry (Donnell 1933; von Kármán & Tsien 1941; Ciarlet & Gratie 2006). We extended the eFvK model to general curved surfaces incorporating the curvature in the membrane part of the energy. The dependence on the curvature tensor of the reference configuration is introduced into the model. Traditional shell theories treat cases when the parametrization of the surface is known, for example, using cylindrical or polar coordinate systems (Donnell 1933; von Kármán & Tsien 1941; Southwell 1914) the versions of the FvK model are well-known. Although the literature of shells is rich in this scope, the generalization of the formalisms to arbitrary (but sufficiently smooth) surfaces is not straightforward.

Here we follow the approximations of the Donnell-Mushtari-Vlasov theory (Ventsel & Krauthammer 2001; Sanders Jr. 1963). It is natural to interpret the mechanical model in local coordinates using a coordinate system aligned with the $\mathbf{n}$ outward pointing unit normal of the surface when using the finite element method for the computations. In some previous works, the FvK theory was derived using local coordinates to model surfaces having specific geometries and loads such as the buckling of a closed cylinder (von Kármán & Tsien 1941). Although the resulting
governing equations compound a system of eighth order PDE-s, they have an appealing form due to the application of the Airy stress function. As it was mentioned in Section 2.1.2, there is no such convenient way for the eFvK theory.

4.2.2 Derivation

![Figure 4.5: The reference configuration of the shell.](image)

We follow the derivation of Budiansky 2015; Sanders Jr. 1963. The shell (Figure 4.5) occupies a closed domain $\Pi$ in the Euclidean space $\mathbb{E}^3$ with a fixed orthotonormal coordinate system $\{E^1, E^2, E^3\}$. The two-dimensional parametric domain of the mid-surface is denoted by $\gamma(\xi^1, \xi^2) \in \mathbb{R}^2$ and mapped to the Euclidean space by $\Lambda : \mathbb{R}^2 \rightarrow \mathbb{E}^3$. The local curvilinear coordinate system attached to the mid-surface of the shell is denoted by $\xi = \Lambda(\xi^1, \xi^2) = [\xi^1, \xi^2]$. We use the summation notation as in Section 2.1, where the upper indices denote contravariant and lower indices denote covariant components. Here $,\alpha$ denotes partial differentiation with respect to the surface coordinates. The derivatives lie in the tangent plane of the surface. The coordinates of the surface in the reference and current configuration are denoted by $\tilde{x}(\xi^1, \xi^2) = [\tilde{x}^1, \tilde{x}^2, \tilde{x}^3]$ and $\bar{x}(\xi^1, \xi^2) = [\bar{x}^1, \bar{x}^2, \bar{x}^3]$ respectively. The unit tangent vectors of the undeformed and deformed surface are $t_{\alpha} = \tilde{x}_{,\alpha}$ and $\bar{t}_{\alpha} = \bar{x}_{,\alpha}$ and the unit normal vectors are $n$ and $\bar{n}$, respectively. Let us denote the deformations of the mid-surface in the in-plane and out of plane directions with $\tilde{u}^1, \tilde{u}^2, \tilde{w}$ respectively. Then the deformation $\tilde{v}$ of the mid-surface is:

$$\tilde{v}(\xi^1, \xi^2) = \bar{x} - \tilde{x}. \quad (4.9)$$
In the following we leave the \( \tilde{\cdot} \) notation for simplicity, hence
\[
\bar{x} = x + v(\xi^1, \xi^2).
\] (4.10)

Subsequently,
\[
\bar{x}^i = x^i + u^i x^i_{\alpha} + wn^i. \tag{4.11}
\]

The metric tensor (first fundamental form) and the curvature tensor (second fundamental form) define the surface and they are denoted by \( g_{\alpha\beta} \) and \( b_{\alpha\beta} \) in the undeformed state and by \( \bar{g}_{\alpha\beta} \) and \( \bar{b}_{\alpha\beta} \) in the deformed state:
\[
g_{\alpha\beta} = x^i_{,\alpha} x^i_{,\beta}, \tag{4.12}
\]
\[
\bar{g}_{\alpha\beta} = \bar{x}^i_{,\alpha} \bar{x}^i_{,\beta}, \tag{4.13}
\]
\[
b_{\alpha\beta} = n^i_{,\alpha} x^i_{,\beta}, \tag{4.14}
\]
\[
\bar{b}_{\alpha\beta} = \bar{n}^i_{,\alpha} \bar{x}^i_{,\beta}. \tag{4.15}
\]

From Equation (4.11) it follows that
\[
\bar{x}^i_{,\alpha} = x^i_{,\alpha} + u^\beta x^i_{,\alpha} + u^i x^i_{,\beta} + w_{,\alpha} n^i + wn^i_{,\alpha}. \tag{4.16}
\]

Using the Gauss and Weingarten equations (Todd & Struik 1951):
\[
n^i_{,\alpha} = b^\gamma_{\alpha} x^i_{,\gamma} \tag{4.17}
\]
and
\[
x^i_{,\alpha\beta} = -b_{\alpha\beta} n^i \tag{4.18}
\]
the derivatives become
\[
\bar{x}^i_{,\alpha} = x^i_{,\alpha} + d^\gamma_{\alpha} x^i_{,\gamma} - \varphi_{,\alpha} n^i, \tag{4.19}
\]
where
\[
d^\gamma_{\alpha} = u^\gamma_{,\alpha} + b_{\gamma\alpha} w, \tag{4.20}
\]
\[
\varphi_{,\alpha} = -w_{,\alpha} + b_{\alpha\beta} u^\beta. \tag{4.21}
\]

**Membrane strain tensor**

The Green-Lagrangian membrane strain tensor of the curved surface can be defined as
\[
\bar{E}_{\alpha\beta} := \frac{1}{2}(\bar{g}_{\alpha\beta} - g_{\alpha\beta}). \tag{4.22}
\]
The deformed metric using Equation (4.19) takes the following form:
\[
\bar{g}_{\alpha\beta} = \bar{t}_{\alpha} \cdot \bar{t}_{\beta} = g_{\alpha\beta} + g_{\gamma\beta} d^\gamma_{\alpha} + g_{\gamma\alpha} d^\gamma_{\beta} + g_{\gamma\mu} d^\gamma_{\alpha} d^\mu_{\beta} + \varphi_{\alpha} \varphi_{\beta}.
\] (4.23)

The linear part of Equation (4.23) determines the linearized membrane strain tensor, assuming small out-of-plane deformations \( \hat{e}_{\alpha\beta} \) in curvilinear coordinates:
\[
g_{\gamma\beta} d^\gamma_{\alpha} + g_{\gamma\alpha} d^\gamma_{\beta} = d_{\beta\alpha} + d_{\alpha\beta} = u_{\beta,\alpha} + u_{\alpha,\beta} + 2b_{\alpha\beta} w = 2\hat{e}_{\alpha\beta}.
\] (4.24)

Hence,
\[
\tilde{E}_{\alpha\beta} = \hat{e}_{\alpha\beta} + \frac{1}{2} g_{\gamma\mu} d^\gamma_{\alpha} d^\mu_{\beta} + \frac{1}{2} \varphi_{\alpha} \varphi_{\beta}.
\] (4.25)

Here the following assumptions can be made: \( \varphi \) is the rotation about the normal vector \( n \) and \( \varphi_{\alpha} \) are the out-of-plane rotations. Therefore we can define \( \varphi \) as
\[
\varphi := \frac{1}{2} a^{\alpha\beta} u_{\beta,\alpha}
\] (4.26)
leading to
\[
a^{\alpha\beta} \varphi = \frac{1}{2} (u_{\beta,\alpha} - u_{\alpha,\beta})
\] (4.27)
where \( a^{\alpha\beta} \) is the alternating tensor with only \( a_{12} = a_{21} = \sqrt{g} \) nonzero elements, where \( g = |g_{\alpha\beta}| \). Then
\[
d_{\alpha\beta} = \hat{e}_{\alpha\beta} - a_{\alpha\beta} \varphi.
\] (4.28)

Additionally, we assume short wavelength deformations and we approximate Equation (4.21) using
\[
\varphi_{\alpha} = -w_{\alpha}.
\] (4.29)

Note, that
\[
d^\gamma_{\alpha} = g^{\gamma\mu} d_{\mu\alpha},
\] (4.30)
leading to
\[
\tilde{E}_{\alpha\beta} = \hat{e}_{\alpha\beta} + \frac{1}{2} g^{\gamma\mu} d_{\mu\alpha} d_{\gamma\beta} + \frac{1}{2} w_{\alpha} w_{\beta},
\] (4.31)

where
\[
\frac{1}{2} g^{\gamma\mu} d_{\mu\alpha} d_{\gamma\beta} = \frac{1}{2} g^{\gamma\mu} (u_{\mu,\alpha} + b_{\mu\alpha} w)(u_{\gamma,\beta} + b_{\gamma\beta} w).
\] (4.32)

Taking that \( w_{\alpha} \sim O(\varrho) \), where \( \varrho < 1 \), there are four possibilities.

1. \( u_{\mu,\alpha} \sim O(\varrho^2) \) and \( b_{\mu,\alpha} w \sim O(\varrho^2) \) Then we neglect the \( O(\varrho^3) \) terms:
\[
\tilde{E}_{\alpha\beta} = \hat{e}_{\alpha\beta} + \frac{1}{2} w_{\alpha} w_{\beta}.
\] (4.33)
This is the FvK theory for small curvatures.

2. \( u_{\mu,\alpha} \sim O(\phi) \) and \( b_{\mu,\alpha} w \sim O(\phi^2) \). Then

\[
\tilde{E}_{\alpha\beta} = \hat{e}_{\alpha\beta} + \frac{1}{2} w_{,\alpha} w_{,\beta} + \frac{1}{2} \delta^{\gamma\mu} u_{\mu,\alpha} u_{\gamma,\beta} \tag{4.34}
\]

This is the eFvK theory for small curvatures. Note, that for small curvatures the metric tensor is the identity tensor leading to Equation (4.1). For planar problems, this model gives back the eFvK theory.

3. \( u_{\mu,\alpha} \sim O(\rho^2) \) and \( b_{\mu,\alpha} w \sim O(\rho) \). Then

\[
\tilde{E}_{\alpha\beta} = \hat{e}_{\alpha\beta} + b_{\alpha\beta} w + \frac{1}{2} w_{,\alpha} w_{,\beta}. \tag{4.35}
\]

This is the Donnel-Mushtari-Vlasov Approximation, or the FvK theory for finite curvatures.

4. \( u_{\mu,\alpha} \sim O(\rho) \) and \( b_{\mu,\alpha} w \sim O(\rho) \). Then

\[
\tilde{E}_{\alpha\beta} = \hat{e}_{\alpha\beta} + \frac{1}{2} \delta^{\gamma\mu} (u_{\mu,\alpha} + b_{\mu,\alpha} w) (u_{\gamma,\beta} + b_{\gamma,\beta} w) + \frac{1}{2} w_{,\alpha} w_{,\beta}. \tag{4.36}
\]

This is the eFvK theory for finite curvatures. Note that in Equation (4.1) the term \( b_{\alpha\beta} w \) was expressed as \( w/R \) assuming a cylindrical surface.

**Bending strain tensor**

The bending strain tensor can be defined as:

\[
\tilde{K}_{\alpha\beta} = \tilde{b}_{\alpha\beta} - b_{\alpha\beta}. \tag{4.37}
\]

Using

\[
\tilde{n} = \frac{\tilde{x}_1 \times \tilde{x}_2}{|\tilde{x}_1 \times \tilde{x}_2|}, \tag{4.38}
\]

leads to

\[
\tilde{n}^i = \sqrt{\frac{\delta}{g}} [(\phi^\gamma + R^\gamma) x^i_{,\gamma} + (1 + \delta_{\omega}^\omega + H) n^i], \tag{4.39}
\]

where

\[
R^\gamma = \phi^\gamma d^\omega_{,\omega} - \phi^\omega d^\omega_{,\gamma}, \tag{4.40}
\]

\[
H = \frac{|\tilde{e}_{\alpha\beta}|}{g} + \phi^2. \tag{4.41}
\]
From Equation (4.19) and using the Gauss Weingarten equations

\[ \ddot{x}_{\alpha\beta} = x_{\alpha\beta} + (d^\gamma_{\alpha\beta} - \varphi_{\alpha\beta} \gamma^\gamma) x_{\gamma\gamma} - (b_{\gamma\beta} d^\gamma_{\alpha} - \varphi_{\alpha\beta}) n. \]  

The curvature tensor for the deformed surface is

\[ \ddot{b}_{\alpha\beta} = \ddot{n}_{\alpha} \cdot \ddot{x}_{\beta} = -\ddot{n} \cdot \ddot{x}_{\alpha\beta} = \sqrt{g} (1 + \ddot{e}_\omega^\omega + H) (b_{\alpha\beta} + \varphi_{\alpha\beta} + b^\gamma_{\beta} d_{\gamma\alpha}) - (\varphi^\gamma + R^\gamma) (d_{\gamma\alpha\beta} - b_{\gamma\beta} \varphi_{\alpha}). \]  

Linearization of the curvature tensor leads to

\[ \ddot{b}_{\alpha\beta} = b_{\alpha\beta} + \varphi_{\alpha\beta} + b^\gamma_{\beta} d_{\gamma\alpha} = b_{\alpha\beta} + \frac{1}{2} (b^\gamma_{\beta} u_{\gamma\alpha} + b^\gamma_{\alpha} u_{\gamma\beta}) + b^\gamma_{\beta} b_{\alpha\gamma} w. \]  

Then, the linearized bending strain tensor

\[ \ddot{K}_{\alpha\beta} = \ddot{b}_{\alpha\beta} - b_{\alpha\beta} \approx \ddot{k}_{\alpha\beta} = \frac{1}{2} (b^\gamma_{\beta} \varphi_{\alpha\gamma} + b^\gamma_{\alpha} \varphi_{\beta\gamma}) = \frac{1}{2} (b^\gamma_{\beta} \varphi_{\alpha\gamma} + b^\gamma_{\alpha} \varphi_{\beta\gamma}) \varphi = -w_{\alpha\beta} + b_{\alpha\gamma} u^\gamma_{\beta} + b_{\beta\gamma} u^\gamma_{\alpha} + \frac{1}{2} (b_{\alpha\gamma\beta} + b_{\beta\gamma\alpha}) u^\gamma + b_{\beta}^\beta b_{\alpha\gamma} w. \]  

Assuming moderate curvatures, the terms involving nonlinear terms in the curvature tensor can be neglected:

\[ \ddot{k}_{\alpha\beta} = \frac{1}{2} (b^\gamma_{\beta} \varphi_{\alpha\gamma} + b^\gamma_{\alpha} \varphi_{\beta\gamma}) \varphi. \]  

As in the derivation of the membrane strain tensor, we assume moderate out-of-plane rotations, small rotations about the normal and short wavelength deformations. Taking that \( w_{\alpha} \sim O(\varrho) \), where \( \varrho < 1 \), \( u_{\mu,\alpha} \sim O(\varrho^2) \) and \( b_{\mu,\alpha} w \sim O(\varrho^2) \) are assumed. Using Equation (4.29):

\[ \ddot{k}_{\alpha\beta} = -w_{\alpha\beta}. \]  

Note that, Equations (4.35) and (4.47) together comprise the Donnell-Mushtrai-Vlasov theory (Ventsel & Krauthammer 2001).

**Governing equations**

Now we derive the equations in terms of \( \ddot{v} \) and we assume that the outward pointing unit vector field \( n \) is known. The in-plane and out-out-plane components of the
The curvature tensor is the surface gradient of the normal vector field
\[ \mathbf{b} = \nabla n, \] (4.50)
where \( \nabla (,.) \) denotes the surface gradient operator. We based our implementation on the Green-Lagrangian strain defined by Equation (4.36), but neglected the nonlinear terms of the curvature tensor and the metric tensor was approximated by the identity. The total potential energy can be defined as previously as the sum of the membrane and bending energies. Assuming no external loads, the total potential energy is
\[ \tilde{I}_f (\tilde{\mathbf{v}}) = \int_{\Omega} \left( \Psi_m (\tilde{\mathbf{E}}) + \Psi_b (\tilde{\mathbf{k}}) \right) \, d\Omega \] (4.51)
for the finite strain theory. Computing the first variation of the energies leads to:
\[ \tilde{\nabla} \cdot \left[ (\mathbf{I} + \tilde{\mathbf{v}} s) \mathbf{N} (\tilde{\mathbf{E}}) \right] = 0, \] (4.52)
\[ h^2 \tilde{\Lambda}^2 w - \tilde{\nabla} \cdot (\mathbf{N} (\tilde{\mathbf{E}}) \tilde{\nabla} w) + \mathbf{N}(\tilde{\mathbf{E}}) \cdot \mathbf{b} = 0, \] (4.53)
assuming finite membrane strains, where \( \tilde{\Lambda} \) denotes the surface Laplacian. Note that in case of a flat surface \( \mathbf{b} = 0 \) which reproduces Equations (2.48) and (2.49). Furthermore, by eliminating the components of \( \mathbf{u}_s \) the equations take the form of a eight-order PDE for \( w \) published in Kármán’s works.

### 4.2.3 Implementation and validation

For the implementation, the geometry of the surface and its unit normal vector field is necessary to be known in the reference configuration. Supposing that it is not known apriori, it can be approximated using the coordinates of the finite element nodes. Provided we can calculate the surface gradient for arbitrary (smooth) fields given by local coordinates; there is no need to establish a map to define the local coordinate system on the surface.

FEniCS naturally computes the surface gradient lying in the tangent plane (Rognes et al. 2013). Therefore we can use the same numerical methodology as previously, to solve the equations. The weak form of the equations can be derived as previously and
can be solved using the numerical continuation algorithm. Using the same function spaces as before Equations (2.96) and (2.97), the discretised form of the equations for the finite strain theory reads: find \( \hat{u} \in V^1_o(\Omega)^2, w \in V^2_o(\Omega) \), such that:

\[
- \sum_{K \in \tau} \int_K \mathbf{F} \mathbf{N}_f \cdot \nabla \hat{\mu} \, d\Omega - \sum_{K \in \tau} \int_K \mathbf{N}_f^\varepsilon \cdot \nabla \mathbf{n} \zeta \, d\Omega + h^2 \left( \sum_{K \in \tau} \int_K \Delta w \Delta \zeta \, d\Omega - \int_{\Gamma} \{ \Delta w \} \| \nabla \zeta \| \, dS - \int_{\Gamma} \{ \Delta \zeta \} \| \nabla w \| \, dS \right) + \sum_{K \in \tau} \int_K \mathbf{N}_f^\varepsilon \cdot \nabla w \cdot \nabla \zeta \, d\Omega = 0 \quad \forall \hat{\mu} \in V^1_o(\Omega)^2, \zeta \in V^2_o(\Omega),
\]

where \( \mathbf{N}_f^\varepsilon = \mathbf{N}(\tilde{E}) \) using the homogenisation technique described by Equation (2.62).

A suitable problem to validate the model is the buckling of thin-walled cylinders under axial load. Analytical and numerical results are available in the literature for the critical stress. We considered the critical buckling stress of a closed, axially compressed cylinder Figure 4.6. Using numerical continuation in the axial macroscopic strain \( \varepsilon \), we determined the bifurcation points where the initially undeformed surface buckles. The analytical solution for the critical stress of an axially compressed cylinder was derived by (Timoshenko & Woinowsky-Krieger 1959). For a buckling mode
determined by the $n,m$ wave numbers circumferentially and axially, respectively is:

$$\sigma_{nm}^{cr} = \frac{S_1}{S_2 (1 - v^2)}, \quad (4.55)$$

where

$$S_1 = (1 - v^2)\lambda^4 + \alpha \left( (n^2 + \lambda^2) - (2 + \nu)(3 - \nu)\lambda^4 n^2 + 2\lambda^4 (1 - v^2) \right)$$

$$- \lambda^2 n^4 (7 + \nu) + \lambda^2 n^2 (3 + \nu) + n^4 - 2n^6], \quad (4.56)$$

$$S_2 = \lambda^2 \left( (n^2 + \lambda^2)^2 + \frac{2}{1 - \nu} \left( \lambda^2 + \frac{1 - \nu}{2} n^2 \right) \left[ 1 + \alpha (n^2 + \lambda^2)^2 \right] \right.$$  

$$\left. - \frac{2\nu^2}{1 - \nu} \left( \lambda^2 + \frac{1 - \nu}{2} n^2 \right) \left[ n^2 + (1 - \nu)\lambda^2 \right] \right) \quad (4.57)$$

$$\alpha = \frac{h^2}{12R^2} \quad (4.58)$$

$$\lambda = \frac{mR\pi}{H}, \quad (4.59)$$

where $R$ is the radius and $L$ is the length of the cylinder. The minimum of the Equation (4.55) for $n$ and $m$ determines the absolute critical value of the compressive stress.

The axial force $N_x$ and the axial stress $\sigma_x$ can be computed numerically from the strain tensor:

$$\sigma_x = \frac{N_x}{h} = \tilde{E}_{11} + \nu (g_2 \cdot (\tilde{E}g_2)) \quad (4.60)$$

In Table 4.2 we compare the stress at the numerical bifurcation of the trivial, unbuckled state with the analytical results. Although we neglected the nonlinear terms in the curvature tensor and approximated the metric tensor with the identity tensor, we got a reasonable agreement between the numerical computations and the analytical solution.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\sigma_{cr}$</th>
<th>$\sigma_{cr}^{comp}$ 35x40 mesh</th>
<th>$\sigma_{cr}^{comp}$ 60x90 mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>0.00144</td>
<td>0.00188</td>
<td>0.00167</td>
</tr>
<tr>
<td>0.05</td>
<td>0.00288</td>
<td>0.00333</td>
<td>0.00293</td>
</tr>
<tr>
<td>0.1</td>
<td>0.00549</td>
<td>0.00061</td>
<td>0.00568</td>
</tr>
</tbody>
</table>

Table 4.2: Nondimensional numerical critical loads $\sigma_{cr}^{comp}$ of the axially compressed, closed cylinder compared to solution of the literature for two different meshes and different thickness values compared to the analytical solution $\sigma_{cr}$.  

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4.3 Conclusions and principal results

We computationally demonstrated, that the disappearance of wrinkles observed in the case of flat rectangular domains is affected by the curvature of the reference configuration and that for a given thickness, there exists a critical value of the curvature, above which there is no wrinkling regardless of the macroscopic strain. We extended the eFvK theory to general curved surfaces incorporating the intrinsic curvature and the contravariant metric tensor in the model. Our model does not depend on a specific parameterization of the surface, thus non-conventional curved surfaces can be analyzed in the future as well.

PRINCIPAL RESULT 4. (Fehér and Sipos 2017; Fehér, Hegyi et. al. 2011)
The eFoK model assuming Saint Venant-Kirchhoff material model was used to investigate the effect of curvature.

4.1. I extended the eFoK model to moderately curved surfaces to investigate it's effect on wrinkling. The model incorporates the intrinsic curvature of the surface in the membrane part of the potential energy.

4.2. I demonstrated numerically, that intrinsic curvature of the reference domain reduces the amplitude of wrinkles, and over a critical value of the curvature the clamped, stretched cylindrical surface does not wrinkle at all.

4.3. I extended the eFoK theory to general curved surfaces. The model incorporates finite strains, the intrinsic curvature of the surface and the contravariant metric tensor.
Investigation of the parameter space

In this chapter, we use the eFvK model to carry out a numerical and analytical analysis of the 2nd Piola-Kirchhoff stress tensor to investigate the wrinkling behavior depending on the aspect ratio and stretch parameters independently of the thickness of the film.

5.1 2nd Piola-Kirchhoff stress tensor

Figure 5.1: Transversal stress $N_{22}(x,y)$ on a stretched rectangular sheet with $\beta = 1$, $\nu = 0.3$, under $\epsilon = 0.16$ macroscopic stretch. Blue colour marks the compressed region in the sheet.

Nayyar, Ravi-Chandar, & R. Huang 2011; Friedl, Rammerstorfer, & Fischer 2000 examined the wrinkling patterns and the minimum of the transversal stress numerically depending on the macroscopic strain and the aspect ratio. They pointed out that the aspect ratio plays an essential role in the wrinkling behavior. Here we aim to examine a broader range of the parameters and extend the analysis in the $\epsilon \to \infty$ and $\beta \to \infty$ directions. As it was mentioned in Section 2.3.4, according to a lemma of
Healey, Q. Li, & Cheng 2013, the negativity of the 2nd Piola-Kirchhoff stress tensor is necessary for wrinkling. Subsequently, $N_f$ defined by Equation (2.50) can be used to determine the point-set of the possible wrinkled configurations in the $\beta - \epsilon$ plane. We have already examined the border of this region and its dependence on the orthotropy in Figure 2.6. We also marked the points of the film, where the transversal component of $N_f$ was negative for the trivial solution in Figure 2.6. Here, we also take the transversal component of $N_f$ for the trivial branch in each point of the film, which results in the $N_{22}^{\beta \epsilon}(x,y)$ function for fixed $\beta$ and $\epsilon$ parameter pairs (Figure 5.1).

Figure 5.2: Numerical computation for a fixed $\beta = 1$ aspect ratio and $\nu = 0.45$. a) $\min(N_{22}^{\beta \epsilon}(x,0))$ on the trivial branch depending on the stretch, b) wrinkle amplitudes for different thickness values.

We compute its minimum in the symmetry axis ($y = 0$) in such way, that

$$N_{22}^{\min}(\beta, \epsilon) = \begin{cases} 0, & \text{if } \min(N_{22}^{\beta \epsilon}(x,0)) \geq 0 \\ \min(N_{22}^{\beta \epsilon}(x,0)) & \text{otherwise.} \end{cases} \quad (5.1)$$

The minimum of the transversal stress is related to the wrinkle amplitudes as it is presented in Figure 5.2. For a fixed aspect ratio, the amplitude of the wrinkles is increasing as the stretch is increased and after reaching a maximum they decrease and reach zero. $N_{22}^{\min}$ follows the same tendency for the trivial branch. As a result, we examine the $N_{22}^{\min}$ independent of the thickness, on the trivial branch instead of computing the wrinkle amplitudes for different thickness values.

From the numerical computation of $N_{22}^{\min}$ in the $\beta - \epsilon$ plane represented by Figure 5.3 it is suspected, that $N_{22}^{\min}$ always reaches zero for increasing macroscopic strain.
Figure 5.3: Numerical computation of minima of the transversal stress in the symmetry axis ($y = 0$) depending on the stretch and aspect ratio parameters. The Poisson’s ratio is $\nu = 0.3$.

This numerical result suggests, that independent of the thickness, if wrinkles appear on the surface there always exists a critical stretch, where they disappear. Furthermore, Figure 5.3 also hints, that $N_{22}^{\text{min}}$ is independent of the aspect ratio for large $\beta$ values. In the following, we address these conjectures and examine the $\varepsilon \to \infty$ and $\beta \to \infty$ directions. Additionally, we seek physical explanation to the behavior of $N_{22}^{\text{min}}$ in the $\beta - \varepsilon$ plane Figure 5.3.

### 5.2 Stretch dependency

We use the lemma of Healey et al. to prove, that wrinkles always disappear from the surface independent of the thickness. We show, that there always exists a critical stretch, for which the 2nd Piola-Kirchhoff tensor is positive definite, in particular, for all $\mathbf{v} = [a, b] \neq 0$:

$$\mathbf{vNv}^T > 0.$$  \hspace{1cm} (5.2)

Although Figure 5.3 suggests, that wrinkles disappear as $\varepsilon \to \infty$, the Saint Venant-Kirchhoff material model limits the $\varepsilon$ strain. In the classic uniaxial stretching problem, the transversal deformation at the ends of the film is unconstrained. Therefore we require the transversal stress to be zero. In the eFvK model, the transversal component of the nondimensional 2nd Piola-Kirchhoff stress takes the following form (Equation (2.50)):

$$N_{22} = 6(\nu(2u_{1,x} + u_{1,x}^2 + u_{2,x}^2) + 2u_{2,y} + u_{1,y}^2 + u_{2,y}^2).$$  \hspace{1cm} (5.3)
Supposed that \( u_{1,x} = \varepsilon, u_{1,y} = u_{2,x} = 0 \) (this assumption is justified by the free lateral contraction of the sheet under the Possion-effect, when the assumption is obviously holds), the transversal stress is:

\[
N_{22} \approx \nu (2\varepsilon + \varepsilon^2) + (u_{2,y} + 1)^2 - 1 = 0. \tag{5.4}
\]

Hence \( \varepsilon \) is constrained by the inequality

\[
\nu \varepsilon^2 + 2\nu \varepsilon - 1 \leq 0, \tag{5.5}
\]

leading to

\[
\varepsilon_{\text{max}} = \frac{-\nu + \sqrt{\nu^2 + \nu}}{\nu}. \tag{5.6}
\]

Let us assume, that in the model problem

\[
u_{1}(x,y) = \varepsilon x + \hat{u}_1(y) \tag{5.7}
\]

and

\[
u_{2}(x,y) = (-1 + (\sqrt{-\nu \varepsilon^2 - 2\nu \varepsilon + 1})y + \hat{u}_2(x). \tag{5.8}
\]

From Equations (5.7) and (5.8)

\[
\lim_{\varepsilon \to \varepsilon_{\text{max}}} (v\mathbf{Nv}^T) = \frac{6}{\nu} \left( a^2 \nu \hat{u}_{1,y}^2 - a^2 \nu^2 + a^2 + a^2 \hat{u}_{2,x}^2 \nu + \\
+ 2ab \hat{u}_{1,y} \sqrt{\nu (\nu + 1)} - 2ab \hat{u}_{1,y} \sqrt{\nu (\nu + 1)} \nu + \\
+ b^2 \nu^2 \hat{u}_{2,x}^2 + b^2 \hat{u}_{1,y}^2 \nu \right). \tag{5.9}
\]

Using that

\[
(a \sqrt{\nu + 1} (1 - \nu) - b \hat{u}_{1,y} \sqrt{\nu})^2 = \\
= 2ab \hat{u}_{1,y} (\sqrt{\nu + 1} \nu) (1 - \nu) + b^2 \hat{u}_{1,y}^2 \nu + a^2 (\nu + 1) (1 - \nu)^2 \tag{5.10}
\]

and

\[ - (\nu + 1) (1 - \nu)^2 - \nu^2 + 1 \geq 0 \text{ if } \nu \in (0, 0.5], \tag{5.11}\]

the 2nd Piola-Kirchhoff tensor is positive definite:

\[
\lim_{\varepsilon \to \varepsilon_{\text{max}}} (v\mathbf{Nv}^T) > 0 \text{ if } \nu \in (0, 0.5]. \tag{5.12}
\]

Note, that the model based on the Saint Venant-Kirchhoff material cannot be used
for arbitrary large macroscopic strain. In fact, for an incompressible matrial $\varepsilon_{\text{max}}$ is about 0.73. Although our results on the higher critical macroscopic strain $\varepsilon_{\text{cr}2}$ are definitely smaller than the above-mentioned limit, it is worthy to investigate the more realistic Neo-Hookean model introduced in Section 3.3. Recall, that the 2nd Piola-Kirchhoff tensor is defined as

$$N = \frac{\partial \Psi_m}{\partial C},$$

where the strain energy density assuming Neo-Hookean material is

$$\Psi_m = \text{tr}C + \frac{1}{\det C} - 3,$$

$C$ is the right Cauchy-Green tensor

$$C(u) = F^T F,$$

and $F$ is the deformation gradient

$$F = I + \nabla \hat{u}.$$

To approximate the components of the $\hat{u}$ deformation, we consider the simple uniaxial stretching problem with unconstrained ends. As previously, we assume that $u_{1,x} = \varepsilon, u_{1,y} = u_{2,x} = 0$ and require the transversal stress to be zero, leading to

$$N_{22} = 2 - \frac{2}{(\varepsilon + 1)^2(u_{2,y} + 1)^4} = 0.$$  

From Equation (5.17) the transversal deformation is

$$u_2(x, y) = -\frac{(\sqrt{\varepsilon + 1} - 1)y}{\sqrt{\varepsilon + 1}} + C,$$

where the constant $C$ vanishes, due to symmetry considerations.

We suppose, that for the original problem $u_1(x, y) = \varepsilon x + \hat{u}_1(x, y)$ and $u_2(x, y) = -\frac{(\sqrt{\varepsilon + 1} - 1)y}{\sqrt{\varepsilon + 1}} + \hat{u}_2(x, y)$ and the order of $\hat{u}_1$ and $\hat{u}_2$ is smaller than $\varepsilon$. From these mild assumptions it readily follows, that for any vector $v$

$$\lim_{\varepsilon \to \infty}(vNv^T) = 2a^2 + 2b^2 > 0.$$  

As a result, independently of the material thickness and the aspect ratio, there always exists $\varepsilon_{\text{cr}2}$ above which no wrinkling occurs.
5.3 Aspect ratio dependency

Figure 5.4: Possible wrinkled patterns and compressed zones of the $N_{22}(x,y)$ stress for a $h = 10 \, \mu m$ thick film under $\varepsilon = 0.11$ macroscopic stretch. a), c) $\beta = 1$; b), d) $\beta = 3.5$.

For a fixed thickness, the effect of aspect ratio was examined both numerically and experimentally in Section 2.3, where we stated that wrinkles appeared only for a bounded region of aspect ratios. On a $h = 32 \, \mu m$ thick sheet, no wrinkles appeared for elongated geometries. However, lowering the thickness extends the region of parameter pairs leading to wrinkling in the $\beta - \varepsilon$ plane. For $h = 10 \, \mu m$, wrinkles appear on an elongated sheet with $\beta = 3.5$ geometry. For $\beta = 1$ the maximal wrinkle amplitude and the compressed zone on the trivial branch are at the center of the rectangle (Figures 5.4a and 5.4c). In contrary, for $\beta = 3.5$, there are two maxima of the wrinkle amplitudes near the clamped boundaries, and similarly, there are two separated compressed zones on the trivial branch (Figures 5.4b and 5.4d).

Figure 5.5: Wrinkling of an elongated polyurethane sheet numerically and experimentally $\beta = 3.5$, $\varepsilon = 0.1$, $\nu = 0.3$, $h = 10 \, \mu m$.

Although the compressed zones are disconnected, the wrinkles are connected in
our computations. This observation is supported by the fact, that the behavior of elongated sheets far from the clamped boundaries can be considered as uniaxial stretching, in which case the transversal stresses are zero. Since tension would be needed to remove wrinkles, they stay connected on long sheets. In our experiments on a $h = 10 \mu m$ polyurethane film, two maxima of the wrinkle amplitudes emerged, and they were joined by shallow wrinkle crests in agreement with the numerical simulations.

In the following, we give a physical explanation on the wrinkling behavior depending on the aspect ratio and investigate the $\beta \to \infty$ direction. Motivated by the numerical computation of the $N_{22}^{\text{min}}$ diagram (Figure 5.3) and the observations for elongated sheets (Figure 5.5), we introduce the concept of disturbed zones of the stress state forming near the boundaries (Figure 5.6a). In these zones, there is a considerable tension close to the boundary and small compression slightly far away.

![Image of disturbed zones](image)

Figure 5.6: a) Disturbed zones of the stress state forming near the boundaries on an elongated sheet. b) Illustration of the overlap of the disturbed zones for a shorter geometry.

We assume that for fixed width and stretch, these zones are independent of the length of the film. Subsequently, above a $\beta_b$ aspect ratio these disturbed zones are entirely separated. For $\beta < \beta_b$ they overlap and the superposition of their effect is superposed (Figure 5.6b). These assumptions are in agreement with Saint-Venant’s principle (Sternberg 1954), meaning that far from the boundaries, our model problem can be considered as simple uniaxial stretching.

We supposed, that $\beta_b < 5$ and computed $N_{22}(x,0)$ for fixed $\beta = 5$ aspect ratio at different $\epsilon$ stretches on the trivial branch. The resulting function is zero at the center
Figure 5.7: Numerical result of the transversal stress in the symmetry axis ($y = 0$) for $\beta = 5$, $\nu = 0.3$, $\varepsilon = 0.1$. The shifting of the diagram is interpreted as moving the nonzero parts of the diagram towards each other.

Figure 5.8: Superposition of the disturbed zones in case of overlap using the numerical result of a $\beta = 5$, $\nu = 0.3$, $\varepsilon = 0.1$ film. Diagrams (b) and (d) show a closer look at diagrams (a) and (c) respectively.

of the film for each $\varepsilon$ values; therefore the disturbed zones do not overlap (Figure 5.7).

Using these numerically determined functions, we computed the $N_{22}^{\text{min}}$ diagram by shifting the disturbed zones towards each other (Figures 5.7 and 5.8) and calculating
the minima of their superposition. $N_{22}^{\min}$ produced by shifting the results of a $\beta = 5$ sheet turned out to be a good approximation of the numerical computations (Figures 5.3 and 5.9). The differences between the diagrams are accounted to numerical errors and inaccuracies. Note, that $\beta_{\min}$ under that no wrinkling occurs independent of the thickness can also be determined for $\varepsilon = 0.1$ from Figure 5.8d. It is around $\beta_{\min} \approx 0.8$ which is in good agreement with Figure 5.3.

$$N_{22}^{\min}, \text{ superposition of } \beta = 5$$

Figure 5.9: Minima of the transversal stress determined using the superposition of the numerical computation of $\beta = 5$.

The concept of the disturbed zones gives a physical explanation to the experimental and numerical observations. If the sheet is too short, the compressed parts of the disturbed zones are dominated by the tensional part or the clamps are too close to each other and only the tensional part of of the disturbed zone occur. Secondly, the maximal compressive stress for a fixed stretch is exactly occurs for a $\beta$ in case the maxima of the compressed regions of the disturbed zones overlap Figures 5.8c and 5.8d. It also explains why wrinkles disappear for large beta values. Nevertheless, it is supposed, that if a sheet wrinkles for a fixed $\varepsilon$ at $\beta > \beta_b$, the wrinkles amplitude is constant and they do not disappear as $\beta \to \infty$. It is important to emphasize, that although the problem is nonlinear, linear superposition applies for the stresses.

### 5.4 Conclusions and principal results

We investigated the $\beta - \varepsilon$ parameter space and gave a physical explanation to the observations. We found general, thickness-independent wrinkling behavior depending on the stretch and aspect ratio of the film and also investigated the $\beta \to \infty$
and the $\epsilon \to \infty$ directions. Our results correspond to Saint-Venant’s principle and supported by the previous experimental results.

**PRINCIPAL RESULT 5.** (Sipos and Fehér 2016)
Although the model problem is dominantly tensiled, wrinkling is caused by (slight) in-plane compression. To understand some of the observations, the 2nd Piola-Kirchhoff stress tensor $\mathbf{N}$ in the eFoK model was examined analytically and numerically.

5.1. I showed analytically, that increasing stretch makes the wrinkles disappear independently of the aspect ratio or the thickness of the film assuming Saint Venant-Kirchhoff material, because if $\epsilon$ is at its limit, the 2nd Piola-Kirchhoff stress tensor is positive definite. I further showed, that for a Neo-Hookean material if $\epsilon \to \infty$, then the 2nd Piola-Kirchhoff stress tensor is positive definite.

5.2. I introduced the concept of the disturbed zones of the stress state forming near the clamped boundaries. It is assumed, that the effect of the boundaries for a fixed width depends only on the applied stretch. In case of overlap of the disturbed zones, their superposition is assumed. I showed, that this concept has a negligible error and it provides a physical explanation for the numerical and experimental results.
6.1 Summary

Although thin structures have many advantages, such as the economical covering of large spans or the small space requirements, it is often unavoidable to keep them compression-free leading to the wrinkling of the surface. For this reason, the functionality of the structure might be negatively affected as well. Conversely, wrinkles can also be considered as visual indications of the working forces and stresses constituting rich information resources. In either case, examination of the wrinkles emerging on thin films is undoubtedly essential.

Being a complex, nonlinear behavior, the studies regarding wrinkling are often focusing on a simple model problem. In this work, we examined an axially stretched rectangular film clamped at its ends, while the other two edges are free. Under increasing stretch, wrinkles appear on the surface as a result of compressive transversal stresses emerging due to the constrained displacement at the clamped edges. Arguments and contradictory statements related to this problem in the literature illustrate that it is still an unsettled topic. Not only the wrinkling behavior depending on different material or geometrical parameters, but also the theories appropriate to model wrinkling are under discussion. Healey et al. incorporated finite strains in the celebrated and widely used Föppl-von Kármán plate theory and pointed out some of the inaccuracies and errors of the original FvK theory. In particular, based on a rigorous bifurcation analysis, they pointed out, that the amplitude of wrinkles decrease after reaching a maximum and they eventually disappear at a critical stretch. Moreover, for a fixed length, sufficiently narrow or wide geometries do not wrinkle at all under axial stretch. Experimental results in the literature were insufficient to support these predictions. Thin films having the required thickness, flexible enough for stretching are usually polymers. Polymers are not only orthotropic due to the fabrication process, but they also have material nonlinearities making it hard to compare the experiments to theoretical results based on models assuming isotropy and linear elasticity. One
possible workaround is to incorporate material nonlinearities in the models, but they might affect the wrinkling behavior as well leading to ambiguities around the mechanical background of the problem. Unless simple model problems are thoroughly understood it is hard to model the wrinkling of complex real-life structures.

In this work, we aimed to experimentally verify the predictions of the eFvK model assuming linear elasticity. Then we extended the model into directions that can be beneficial in the engineering practice. Besides material properties, such as orthotropy and pseudo-elasticity, we also examined the effect of the geometry by incorporating curvature in the model and by extending the investigation of the parameter space. Moreover, by applying Saint-Venant’s principle, we gave a physical explanation to the wrinkling behavior in the model problem depending on the aspect ratio and the macroscopic stretch.

We successfully eliminated material nonlinearities of polyurethane films using a prestressing technique. The process resulted in an almost linear elastic but significantly orthotropic film. After we incorporated orthotropy in the eFvK model, we compared numerical and experimental results carried out on prestressed films and verified the predictions. We also took a closer look on the effect of orthotropy on wrinkling and concluded, that the extent of the region of the $\beta - \varepsilon$ parameter pairs resulting in wrinkling depends on the ratio of the transversal and axial Young-moduli.

Inspired by an intriguing phenomenon observed during the prestressing process, we extended the eFvK model with a pseudo-elastic material model based on the Neo-Hookean model and the Mullins effect. The resulting model accurately gives back the witnessed wrinkling behavior: for some aspect ratios no wrinkling occur during the first loading, but they wrinkle during the unloading and the subsequent loading cycles. In fact, pseudo-elasticity significantly affects the wrinkling behavior.

Although these two-dimensional results are directly applicable for some structures, such as solar sails composed of thin films stretched over a frame, most of the wrinkling phenomena form on at least slightly curved surfaces. As a small step towards complex structures, we took a surface analogy of the model problem, by examining axially stretched, clamped, slightly curved rectangular films. We carried out a numerical computation incorporating the small curvature in our two-dimensional equations as an approximation. In particular, we showed, that curvature suppresses the wrinkle amplitudes and above a critical curvature no wrinkles appear on the surface. Motivated by the results, we generalized the eFvK model to curved surfaces.

Finally, we examined the influence of the two main geometric parameters of the model problem independent of the thickness by examining the stresses. Numerical and analytical results pointed out, that if wrinkles appear on the surface, there always
exists a critical stretch, where they disappear. Furthermore, following Saint-Venant’s principle, we introduced the concept of the disturbed zones and assumed linear superposition in case of their overlap. We illustrated by numerical computations that this approximation leads to negligible errors. Accordingly, for elongated sheets, where the disturbed zones do not overlap, the wrinkling behavior is independent of the aspect ratio, meaning that they appear and disappear at the same critical macroscopic stretch.

Generally speaking, this work clarified some of the arguments in the literature related to wrinkling and pointed out factors significantly influencing the wrinkling behavior. Additionally, the concept of disturbed zones gave a physical explanation to the experimental and numerical observations of the wrinkling of rectangular films.

### 6.2 Principal results

**PRINCIPAL RESULT 1.**

* (Sipos and Fehér 2016)

Geometrically exact models of finite-deformation nonlinear elasticity are needed to understand the mechanics of highly stretched thin films. The classical Föppl-von Kármán plate theory (FvK) has been extended to the finite membrane strain regime recently. The model, called extended Föppl-von Kármán (eFvK) theory, applies some hyperelastic constitutive relation for the in-plane behavior.

1.1. The extended Föppl-von Kármán model with the Saint Venant-Kirchhoff material assumes isotropic material. Motivated by experimental results, I further developed the eFvK theory to accommodate orthotropic materials, at which the main material directions are parallel to the $x$ and $y$ directions of the reference configuration. In the model I used two, nondimensional parameters, $\mathbf{r}$ and $\mathbf{q}$ to describe orthotropy.

1.2. I carried out numerical computations to investigate the effect of geometric and material parameters on wrinkling. I found, that at a fixed thickness $h$ the aspect ratio of the domain $\beta$, the macroscopic strain $\epsilon$ and the ratio of the modulii of elasticity $\mathbf{r}$ strongly influence the wrinkling behavior. Compared to the isotropic case the stability boundary, that separates wrinkled and flat configuration in the $\beta - \epsilon$ plane, significantly extends as long as the transverse elastic modulus dominates the axial modulus ($\mathbf{r} > 1$). For ($\mathbf{r} < 1$) the stability boundary shrinks.
PRINCIPAL RESULT 2.
(Fehér and Sipos 2014, Sipos and Fehér 2016)

To validate the theoretical predictions of the eFvK model, I carried out experiments on previously prestressed, clamped, rectangular, orthotropic, 32 \( \mu m \) thick polyurethane films.

2.1. I experimentally verified the first prediction of the eFvK model. It predicted, that if wrinkles appear on the initially flat surface as the macroscopic stretch is increased, then there is a maximal amplitude for the wrinkles. Further stretch leads to decrease in the amplitude and at a second bifurcation point the wrinkles eventually disappear: The sheet becomes flat again. The experiments clearly demonstrated the disappearance of wrinkles, and as polyurethane is dominantly elastic, this validated the prediction of the model. Using the orthotropic model I computed the location of the bifurcation points in the parameter space and found good quantitative agreement with the experimental data.

2.2. I experimentally verified the second prediction of the eFvK model. It predicted, that wrinkling should appear only for a bounded interval of aspect ratios. In agreement with the prediction, the experiments clearly demonstrated that for fixed length and thickness sufficiently narrow or wide sheets do no exhibit wrinkling. The computed and measured stability boundaries are in good agreement.

PRINCIPAL RESULT 3.
(Fehér, Healey and Sipos 2018)

The obtained experimental data clearly showed, that inelastic material properties significantly affect wrinkling. In specific, stress softening, emerging residual strain and significant change in the orthotropy parameters were recorded.

3.1. In the experiments with polyurethane sheets I demonstrated, that for specific aspect ratios no wrinkling appears during the first loading of the virgin sheet, but wrinkles emerged during the unloading and the subsequent cyclic loading. The measured data clearly hints that the Mullins effect causes the observed phenomenon.
3.2. I introduced a pseudoelastic extension of the eFvK model to take the Mullins effect into account based on the Mooney-Rivlin material model. This model describes the whole loading program of the experiments. The model uses a single state variable field (damage field) and it has only 5 material parameters. Yet, it predicts stress-softening, residual strain and emerging orthotropy, in accordance with the measured data. The parameter values at wrinkles appear/disappear, are in fair agreement.

**PRINCIPAL RESULT 4.**

*(Fehér and Sipos 2017; Fehér, Hegyi et. al. 2011)*

The eFvK model assuming Saint Venant-Kirchhoff material model was used to investigate the effect of curvature.

4.1. I extended the eFvK model to moderately curved surfaces to investigate it’s effect on wrinkling. The model incorporates the intrinsic curvature of the surface in the membrane part of the potential energy.

4.2. I demonstrated numerically, that intrinsic curvature of the reference domain reduces the amplitude of wrinkles, and over a critical value of the curvature the clamped, stretched cylindrical surface does not wrinkle at all.

4.3. I extended the eFvK theory to general curved surfaces. The model incorporates finite strains, the intrinsic curvature of the surface and the contravariant metric tensor.

**PRINCIPAL RESULT 5.**

*(Sipos and Fehér 2016)*

Although the model problem is dominantly tensiled, wrinkling is caused by (slight) in-plane compression. To understand some of the observations, the 2nd Piola-Kirchhoff stress tensor $\mathbf{N}$ in the eFvK model was examined analytically and numerically.

5.1. I showed analytically, that increasing stretch makes the wrinkles disappear independently of the aspect ratio or the thickness of the film assuming Saint Venant-Kirchhoff material, because if $\varepsilon$ is at its limit, the 2nd Piola-Kirchhoff stress tensor is positive definite. I further showed, that for a Neo-Hookean material if $\varepsilon \to \infty$, then the 2nd Piola-Kirchhoff stress tensor is positive definite.
5.2. I introduced the concept of the disturbed zones of the stress state forming near the clamped boundaries. It is assumed, that the effect of the boundaries for a fixed width depends only on the applied stretch. In case of overlap of the disturbed zones, their superposition is assumed. I showed, that this concept has a negligible error and it provides a physical explanation for the numerical and experimental results.
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- New National Excellence Program (ÚNKP) of the Ministry of Human Capacities;
- EGT Norway Grant;
- TÁMOP 4.2.1/B-09/1/KMR-2010-0002 providing the ZWICK Z150 testing machine.
PUBLICATIONS CONNECTED TO THE PRINCIPAL RESULTS


OTHER PUBLICATIONS

REFERENCES


The listed technical data sheets were provided by the distributor and manufacturer of the films used in the experiments.

### POLYPROPYLENE

<table>
<thead>
<tr>
<th>Name</th>
<th>Printed packaging polypropylene tape</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>hot-melt</td>
</tr>
<tr>
<td>Product description</td>
<td>Biaxially oriented, environment friendly polypropylene packaging tape for cardboard boxes</td>
</tr>
<tr>
<td>Dimension</td>
<td>66 m x 55 mm</td>
</tr>
<tr>
<td>Thickness</td>
<td>0.028 mm</td>
</tr>
<tr>
<td>Overall thickness</td>
<td>0.048 mm</td>
</tr>
<tr>
<td>Adhesive power to steel</td>
<td>4.6 g/cm</td>
</tr>
<tr>
<td>Elongation at brake</td>
<td>165 %</td>
</tr>
<tr>
<td>Break force</td>
<td>4.6 kg/cm</td>
</tr>
<tr>
<td>Inside diameter of the roll</td>
<td>76 mm</td>
</tr>
<tr>
<td>Distributor</td>
<td>Polimer Plus Kft (<a href="http://www.polimerplus.hu">http://www.polimerplus.hu</a>)</td>
</tr>
</tbody>
</table>

### POLYURETHANE

<table>
<thead>
<tr>
<th>Name</th>
<th>Hydrofilm Roll</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product description</td>
<td>self-adhesive film dressing for fixation of primary wound dressings it is made of semi-permeable, waterproof polyurethane film, transparent</td>
</tr>
<tr>
<td>Dimension</td>
<td>10 m x 15 cm</td>
</tr>
<tr>
<td>Adhesive power</td>
<td>2.5 - 7.5 N/in</td>
</tr>
<tr>
<td>Backing</td>
<td>semi-permeable, transparent polyurethane film</td>
</tr>
<tr>
<td>Adhesive</td>
<td>acrylic adhesive, hypoallergenic, colophonium-free</td>
</tr>
<tr>
<td>Moisture vapour transition rate</td>
<td>700 - 800 g/m² / 24h</td>
</tr>
<tr>
<td>Manufacturer</td>
<td>PAUL HARTMANN AG (<a href="https://hartmann.info">https://hartmann.info</a>)</td>
</tr>
</tbody>
</table>