Jiles-Atherton Model
Implementation to Edge Finite Element Method

by

Péter Kis
MSc in Electrical Engineering

supervisor

Prof. Amália Iványi

A thesis submitted to the
Budapest University of Technology and Economics
for the degree of
Doctor of Philosophy
in Electrical Engineering

Department of Broadband Infocommunications and Electromagnetic Theory
Budapest University of Technology and Economics
Budapest
2006
Acknowledgment

I would like to express my gratitude to my supervisor Prof. Amália Iványi whose support and guidance made my thesis work possible. I wish to thank the members of the Department of Broadband Infocommunications and Electromagnetic Theory for their guidance and encouragement.

Special thanks to dr. Tamás Barbarics for guiding my student research works. Thanks to dr. Imre Sebestyén and Prof. Oszkár Bró for giving me an introduction into the finite element analysis and for sharing their knowledge. I am very grateful to dr. Miklós Kuczmann for learning together and for the useful discussions. I wish to thank to Prof. Maurizio Repetto for providing computational data for the problem T.E.A.M. 13. I wish to thank to dr. István Vajk and Prof. István Nagy for employing me at the Department of Automation and Applied Informatics.

I wish to thank to the leaders of the Furukawa Electric Technology of Institute, Sándor Csec-sődy and dr. Gyula Besztercey for the encouragement and allowing me to use computational resources of the company.

Thanks go to Beatrix Tóth and dr. Gyula Besztercey for proof-reading. Particular thank to my parents and my fiancée Edina Kertész for their patience and daily support.

Péter Kis
Budapest, 22 August 2006
## Contents

1 Introduction and scope of research .......................... 1
   1.1 Proposed research activity .................................. 2

2 Literature overview ........................................... 4
   2.1 The Jiles-Atherton model of hysteresis ......................... 4
       2.1.1 Langevin theory for paramagnets ......................... 4
       2.1.2 Weiss correction for ferromagnetic materials ............... 6
       2.1.3 Rate-independent Jiles-Atherton hysteresis model ........... 6
       2.1.4 Determination of model parameters from measured data ........ 9
       Saturation magnetization $M_s$ .................................. 10
       Determination of the parameter $c$ which represents reversible wall motion 10
       Relationship between $a$ and $\alpha$ ........................... 11
       Determination of the parameter $k$ which determines the hysteresis loss  12
       Determination of $a$ and $\alpha$ .................................. 13
       Relationship between hysteresis parameters at the loop tip .......... 14
       Procedure for calculating parameters .......................... 14
   2.2 Review of Electromagnetics .................................. 15
       2.2.1 Constitutive relations ...................................... 15
   2.3 The finite element method ................................... 16
       2.3.1 Nodal finite element method ............................... 16
       2.3.2 Edge finite element method ............................... 18

3 Improvement of the Jiles-Atherton model ..................... 20
   3.1 Reformulation of the rate-independent Jiles-Atherton model ........ 20
   3.2 Reformulation of the rate-dependent Jiles-Atherton model .......... 25
   3.3 Inverse Jiles-Atherton model ................................ 28
   3.4 Hysteresis measurement in LabVIEW environment .............. 29
       3.4.1 Measurement of first order reversal curves ................. 33
       3.4.2 Sinusoidal $B$ ............................................. 33
   3.5 Parameter identification of the Jiles-Atherton model ............ 35
       3.5.1 Initial test .................................................. 36
       3.5.2 Fitting to measured curves at 0.2 Hz ........................ 37
3.5.3 Fitting to measured curves at 1Hz ........................................ 37
3.5.4 Fitting to measured curves at 5 Hz ................................. 39
3.6 Thesis 1 ................................................................. 40

4 Hysteresis operator in higher order edge elements ................. 41
  4.1 Formalisms ........................................................................................................ 41
  4.1.1 Magnetostatics with magnetic vector potential ......................... 42
  4.1.2 Eddy currents with magnetic vector and electric scalar potentials ..... 44
  4.1.3 Eddy currents with magnetic vector potential .............................. 46
  4.2 Polarization method ................................................................. 48
    4.2.1 Banach fixed point theorem ......................................................... 49
    Contraction mapping ................................................................. 49
    The theorem ........................................................................ 49
  4.2.2 Polarization method for inverse characteristics ..................... 50
  4.2.3 Polarization method for direct characteristics ....................... 53
  4.2.4 Fixed-point procedure ............................................................... 53
  4.2.5 Fixed-point procedure with relaxation .................................. 54
  4.2.6 Fixed-point procedure with relaxation and $\mu$ update .............................................. 56
  4.3 Higher order triangular elements ................................................. 56
    4.3.1 Higher order scalar shape function ........................................ 59
    4.3.2 Higher order vector shape function .................................... 61
  4.4 Higher order tetrahedral elements ............................................... 61
    4.4.1 Higher order scalar shape function ........................................ 62
    4.4.2 Higher order vector shape function .................................... 62
  4.5 Thesis 2 ................................................................. 63

5 Applications ............................................................................... 64
  5.1 1D example – Ferromagnetic half-space ......................... 64
    5.1.1 Solution of the linear problem ............................................. 65
    5.1.2 Solution of the nonlinear problem with Langevin characteristics 67
      Formalism ........................................................................ 67
      Solution with fixed point iteration .................................... 68
      Solution with fixed point iteration + relaxation ....................... 71
      Solution with fixed point iteration + relaxation and $\mu$ update .......... 71
    5.1.3 Solution of the nonlinear problem with Jiles-Atherton model of hysteresis 73
  5.2 2D example – Hysteresis measurement simulation ..................... 75
    5.2.1 Model ........................................................................ 75
    5.2.2 Mesh parameters ............................................................... 76
    5.2.3 Results ........................................................................ 76
  5.3 3D example – Solution of the problem T.E.A.M 13 .......... 81
    5.3.1 Introduction ........................................................................ 81
    5.3.2 Setup of the linear problem ..................................................... 82
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.3.3</td>
<td>Nonlinearity</td>
<td>83</td>
</tr>
<tr>
<td>5.3.4</td>
<td>Results</td>
<td>83</td>
</tr>
<tr>
<td>5.3.5</td>
<td>Solution with first order elements</td>
<td>85</td>
</tr>
<tr>
<td>5.3.6</td>
<td>Solution with second order elements</td>
<td>85</td>
</tr>
<tr>
<td>5.3.7</td>
<td>Comparison</td>
<td>88</td>
</tr>
<tr>
<td>5.4</td>
<td>Thesis 3</td>
<td>90</td>
</tr>
<tr>
<td>6</td>
<td>Summary of Theses</td>
<td>91</td>
</tr>
<tr>
<td>7</td>
<td>Conclusions and future work</td>
<td>93</td>
</tr>
<tr>
<td>A</td>
<td>Results of the half-space problem with Langevin function</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>Results of the half-space problem with modified JAM</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td>Hysteresis measurement simulation at 0.2 Hz</td>
<td>9</td>
</tr>
<tr>
<td>D</td>
<td>Hysteresis measurement simulation at 5 Hz</td>
<td>12</td>
</tr>
<tr>
<td>E</td>
<td>Problem T.E.A.M.</td>
<td>13</td>
</tr>
<tr>
<td>F</td>
<td>The geometric multigrid method</td>
<td>15</td>
</tr>
<tr>
<td>G</td>
<td>The Vanka Algorithm</td>
<td>30</td>
</tr>
</tbody>
</table>

iv
List of notation and abbreviations

\( \mathbf{v} = \mathbf{v}(x, y, z, t) \)  \hspace{1cm} Bold italics letters denote space vectors

\( \mathbf{M} = \{m\}_{i,j} \)  \hspace{1cm} Bold roman block letters denote matrices

\( \mathbf{a} = \{a\}_i \)  \hspace{1cm} Bold roman normal letters denote vectors

\( \mathbf{H}, \mathbf{H} \)  \hspace{1cm} Magnetic field intensity vector and absolute value [A/m]

\( \mathbf{B}, \mathbf{B} \)  \hspace{1cm} Magnetic flux density vector and absolute value [T]

\( \mathbf{E}, \mathbf{E} \)  \hspace{1cm} Electric field intensity vector and absolute value [V/m]

\( \mathbf{J}, \mathbf{J} \)  \hspace{1cm} Current density vector and absolute value [A/m²]

\( \mathbf{A}, (\mathbf{B} = \text{curl} \mathbf{A}) \)  \hspace{1cm} Magnetic vector potential [Vs/m]

\( \mathbf{V}, (\mathbf{E} = -\text{grad} V - \partial \mathbf{A}/\partial t) \)  \hspace{1cm} Electric scalar potential [V]

\( \mathbf{V}_m, (\mathbf{H} = -\text{grad} V_m) \)  \hspace{1cm} Magnetic scalar potential [A]

\( \mu_0 = 4\pi \cdot 10^{-7} [\text{H/m}] \)  \hspace{1cm} Magnetic permeability of vacuum

\( \varepsilon_0 = 8.85 \cdot 10^{-12} [\text{F/m}] \)  \hspace{1cm} Electric permittivity of vacuum

\( \sigma \)  \hspace{1cm} Electric conductivity [S/m]

\( \mathbf{M}, \mathbf{M} \)  \hspace{1cm} Magnetization vector [A/m]

\( \mathbf{M}_{an} \)  \hspace{1cm} Anhysteretic magnetization [A/m]

\( \mathbf{M}_{irr} \)  \hspace{1cm} Irreversible component of magnetization [A/m]

\( \mathbf{M}_{rev} \)  \hspace{1cm} Reversible component of magnetization [A/m]

\( \mathbf{H}_e = \mathbf{H} + \alpha \mathbf{M} \)  \hspace{1cm} Effective field according to the Weiss’ theory [A/m]

\( \delta W_{\text{bat}} \)  \hspace{1cm} Externally supplied battery energy density [J/m³]

\( \delta W_{\text{mag}} \)  \hspace{1cm} Internal magnetic energy density change [J/m³]

\( \delta L_{\text{mag}} \)  \hspace{1cm} Hysteresis loss density [J/m³]

\( \delta L_{\text{EC}} \)  \hspace{1cm} Eddy current loss density [J/m³]

\( \delta L_A \)  \hspace{1cm} Anomalous (excess) loss density due to the domain wall bending [J/m³]

\( \mathbf{M}_s \)  \hspace{1cm} Saturation magnetization [A/m] (JAM parameter)

\( \alpha \)  \hspace{1cm} Domain interaction parameter [dimensionless] (JAM parameter)

\( a \)  \hspace{1cm} Shape parameter of \( M_{an} \) [A/m] (JAM parameter)

\( c \)  \hspace{1cm} Reversibility coefficient [dimensionless] (JAM parameter)

\( k \)  \hspace{1cm} Parameter linked with the coercitive field [A/m] (JAM parameter)

\( \mathcal{L}(\lambda) = \coth(\lambda) - \lambda \)  \hspace{1cm} The Langevin function

\( n_e \)  \hspace{1cm} Number of edges in the finite element mesh
$n_n$  Number of nodes in the finite element mesh
$e_x, e_y, e_z$  Cartesian basis normal vectors
NN  Neural network
VI  Virtual Instrument in LabVIEW programming
AI  Analog Input in LabVIEW programming
AO  Analog Output in LabVIEW programming
DAQ  Data Acquisition
FEM  Finite Element Method
FP  Fixed Point Method
OR  Over-relaxation
JAM  Jiles-Atherton Model
GUI  Graphical User Interface
DOFs  Degrees of Freedoms
GMRES  General Mean Residual method
SOR  Successive Over-relaxation
SSOR  Symmetric successive Over-relaxation
LHS  Left Hand Side
RHS  Right Hand Side
List of Tables

3.1 Initial test of the JAM parameter identification .......................... 36

4.1 Legendre polynomials and its integrals ................................. 58

5.1 Number of required FP iteration steps with different $\mu_{FP}$ starting value .................. 69
5.2 Number of required FP+Rel. iteration steps with different $\mu_{FP}$ starting value .... 72
5.3 Number of required FP+Rel. iteration steps with $\mu$ update .................. 73
5.4 Number of required iteration steps in the case of JAM .................. 74
5.5 Computational details by different sphere radius ........................ 83
5.6 Results based on the half geometry with Neumann condition ............ 84
5.7 Chosen relative reluctivity versus required iteration steps ............. 85
5.8 Description of computer program ............................................ 87
5.8 Description of computer program ............................................ 88
5.9 Comparison between literature values and my results .................. 89
List of Figures

2.1 The non-physical solutions can be seen at the loop tips .............................. 9
2.2 Tetrahedral finite elements: the reference (simplex) element and the real element 16

3.1 $\int_{\text{desc.}} H \mathrm{d}M$ ........................................................................ 22
3.2 $\int_{\text{asc.}} H \mathrm{d}M$ ........................................................................ 22
3.3 $\frac{1}{\kappa} \int_{\text{desc.}} H \mathrm{d}M = \int_{\text{asc.}} H \mathrm{d}M + \int_{\text{asc.}} H \mathrm{d}M$ ........... 22
3.4 $\int_{\text{desc.}} M \mathrm{d}H$ ........................................................................ 22
3.5 $\int_{\text{asc.}} M \mathrm{d}H$ ........................................................................ 22
3.6 $\frac{1}{\kappa} \int_{\text{desc.}} M \mathrm{d}H = \int_{\text{asc.}} M \mathrm{d}H + \int_{\text{asc.}} M \mathrm{d}H$ ......................... 22
3.7 Magnetization curves of an ideal and ferromagnetic material ............................. 24
3.8 The $\kappa_d$ tends to one if the relative permeability is much higher than one .... 27
3.9 Hysteresis models with $H$ and $B$ input ......................................................... 28
3.10 The Kikusui PBX2020 power supply ............................................................. 30
3.11 PCI-6052e data acquisition cards ................................................................. 30
3.12 BNC-2090 patch panel .................................................................................. 30
3.13 The arrangement of the automated computer aided magnetic hysteresis arrange-
        ment ......................................................................................................... 31
3.14 Measured symmetrical minor loops on C19 structural steel at 1 Hz .................. 31
3.15 The graphical user interface of LabVIEW measurement software .................... 32
3.16 Realization of analog input reading from analog ports in LabVIEW .............. 33
3.17 Realization of analog output writing to analog ports in LabVIEW ................. 33
3.18 The measured first order reversal curves at $f = 0.2$ Hz, $n = 20$ ...................... 33
3.19 Measured hysteresis loop with sinusoidal $H$. Non-equidistant points .............. 34
3.20 Measured hysteresis loop with sinusoidal $B$. The points are distributed more
        evenly like in the case of sinusoidal $H$ ....................................................... 34
3.21 Block scheme of nonlinear iteration for achiving sinusoidal $B$ ......................... 34
3.22 The graphical user interface for parameter identification of the JAM ............... 35
3.23 The block scheme of the JAM parameter identification .................................... 36
3.24 Fitting to the self generated hysteresis loop ................................................. 37
3.25 Error of the self-fitting .................................................................................. 37
3.26 Result of parameter identification of quasi static J-A model at 0.2 Hz. Fitted curve is plotted solid line, and some points of the measured curve with ‘o’. . . 37
3.27 Result of parameter identification of quasi static J-A model at 0.2 Hz on a symmetrical minor loop. Fitted curve is plotted solid line, and some points of the measured curve with ‘o’. .......................................................... 37
3.28 Relative error between the simulated and measured hysteresis loops ........ 38
3.29 Relative error between the simulated and measured hysteresis loops ........ 38
3.30 Result of parameter identification of quasi static J-A model at 1 Hz. The fitted curve is represented by solid line, and some points of the measured curve with ‘o’. 38
3.31 Result of parameter identification of quasi static J-A model at 1 Hz on a symmetrical minor loop. Fitted curve is plotted solid line, and some points of the measured curve with ‘o’. ................................................. 38
3.32 Relative error between the simulated and measured hysteresis loops ........ 38
3.33 Relative error between the simulated and measured hysteresis loops ........ 38
3.34 Simulation of the measured hysteresis at 5Hz with quasi static J-A model . . 39
3.35 Simulation of the measured hysteresis at 5Hz with extended J-A model . . . 39

4.1 Magnetostatic problem ............................................................................ 42
4.2 Eddy current problem with voltage prescription ..................................... 44
4.3 Eddy current problem without voltage prescription .............................. 47
4.4 The Fixed Point Algorithm .................................................................. 54
4.5 The Fixed Point Algorithm with relaxation .......................................... 55
4.6 The Fixed Point Algorithm with relaxation and $\mu$ update .................. 57
4.7 Reference element for deriving of triangular shape functions from quadrilateral element ................................................................. 59

5.1 The ferromagnetic half-space. The $z > 0$ half-space is the ferromagnetic and electrically conductive material, the $z < 0$ can be considered as air or vacuum, where the sinusoidal magnetic field is coming from. It is prescribed as a boundary condition at $z = 0$ ................................................................. 64
5.2 Analytical and numerical solutions of the linear half-space problem in the first period ................................................................................. 65
5.3 Analytical and numerical solutions of the linear half-space problem in the fifth period ................................................................................. 66
5.4 The material nonlinearity follows the Langevin curve ......................... 67
5.5 The Fixed Point Algorithm .................................................................. 69
5.6 Number of iteration steps required at specified time steps. More iterations are required if we are far from the linear part. $H_{\text{amp}} = 100$ A/m at left, $H_{\text{amp}} = 1000$ A/m in the middle and $H_{\text{amp}} = 10000$ A/m at right .......................... 70
5.7 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=35$ ms .......................................................... 70
5.8 Number of iteration steps required at specified time steps in the case of fixed-point method with relaxation .......................... 71
5.9 Number of iteration steps required at specified time steps in the case of fixed-point method with relaxation and \( \mu \) update .................................................. 72
5.10 The Langevin function used in the previous section and the JAM. The JAM parameters are \( M_s = 8.1 \cdot 10^5 \text{ A/m}, \alpha = 10^{-3}, a = 300 \text{ A/m}, c = 0.5, k = 800 \text{ A/m} \) .................................................. 73
5.11 Number of iteration steps required at specified time steps in the case of JAM .......................... 74
5.12 Solutions with JAM, \( H_{\text{amp}} = 100 \text{ A/m}, 1000 \text{ A/m}, 10000 \text{ A/m} \) at \( t=35 \text{ ms} \) 74
5.13 The drawing and the photo about the specimen .................................................. 75
5.14 The geometry model of the ferromagnetic ring simulation .................................................. 75
5.15 The mesh of the 2D problem .................................................. 76
5.16 The measured and simulated hysteresis curves at 0.2 Hz .................................................. 76
5.17 The measured and simulated hysteresis curves are plotted at 1 Hz. From left to right the figures show simulation results with the rate-independent and the rate-dependent JAM .................................................. 77
5.18 The measured and simulated hysteresis curves are plotted at 5 Hz. From left to right the figures show simulation results with the rate-independent and the rate-dependent JAM .................................................. 77
5.19 Solutions with the rate-independent JAM at \( f = 0.2 \text{ Hz} \). The \( \varphi \) component of magnetic flux density \( B_\varphi \) is plotted by surface and the magnetic vector potential \( A \) by arrows .................................................. 78
5.20 Solutions with the rate-dependent JAM at \( f = 5 \text{ Hz} \). The \( \varphi \) component of magnetic flux density \( B_\varphi \) is plotted by surface and the magnetic vector potential \( A \) by arrows .................................................. 79
5.21 Sketch of the problem. The coil is surrounded by iron plates .................................................. 81
5.22 The full and the half geometry. Surfaces are added to the plates, where the magnetic flux density has been measured. The dimensions are not noted in this figure but they are agreed with Appendix E .................................................. 83
5.23 The experimental first magnetization curve is approximated by the Jiles-Atherton model with the parameters \( M_s = 1.4 \cdot 10^6 \text{ A/m}, \alpha = 7 \cdot 10^{-4}, a = 200 \text{ A/m}, c=0.1, k=175 \text{ A/m} \) .................................................. 84
5.24 The finite element mesh .................................................. 84
5.25 The measured [1] and simulated results with first order elements on a coarse finite element mesh .................................................. 85
5.26 Magnetic field lines .................................................. 86
5.27 The measured [1] and simulated results with second order elements .................................................. 86
5.28 Relative error of the measured and the simulated data .................................................. 86
5.29 Comparison with the literature .................................................. 89

A.1 Solutions with Langevin characteristics, \( H_{\text{amp}} = 100 \text{ A/m}, 1000 \text{ A/m}, 10000 \text{ A/m} \) at \( t=0.5 \text{ ms} \) .................................................. 1
A.2 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=5$ ms ................................................................. 1
A.3 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=10$ ms ................................................................. 2
A.4 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=15$ ms ................................................................. 2
A.5 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=20$ ms ................................................................. 2
A.6 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=25$ ms ................................................................. 3
A.7 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=30$ ms ................................................................. 3
A.8 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=35$ ms ................................................................. 3
A.9 Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=40$ ms ................................................................. 4

B.1 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=0.5$ ms ................................................................. 5
B.2 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=5$ ms ................................................................. 5
B.3 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=10$ ms ................................................................. 6
B.4 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=15$ ms ................................................................. 6
B.5 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=20$ ms ................................................................. 6
B.6 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=25$ ms ................................................................. 7
B.7 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=30$ ms ................................................................. 7
B.8 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=35$ ms ................................................................. 7
B.9 Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=40$ ms ................................................................. 8
Chapter 1

Introduction and scope of research

Several hysteresis models have arisen due to the evolution of the digital computers. The validity of the models is not restricted to the description of the ferromagnetic hysteresis phenomena. The hysteresis models are widely accepted in other areas of science (e.g. biology, economics, mechanics etc.), as these phenomena also play very significant role in several domains of scientific practice.

There are several classifications of the hysteresis models. One of them is according to the description level. Three classes can be distinguished: the microscopic, the macroscopic and the mezoscopic models. The microscopic models deal with the description of the hysteresis phenomena, they often use a sub-atomic range approach (e.g. Ising-model, Landau-Lifshitz equation). This type of models is not suitable for simulating the hysteresis effect in complicated situations, as modeling a real scale material would be a very time consuming task. The macroscopic models seek to eliminate these shortcomings of the microscopic model. In fact they can not be compared with the microscopic ones, since the macroscopic models can provide no information about the physical background of the phenomena, though they can be employed in real scale problems. The macroscopic models are usually function approximations of the hysteresis curves, where several types of analytical functions can be applied (e.g. transcendent functions, polynomial functions, ratio of two polynomial functions etc.).

We can realize that neither the micro- nor the macroscopic models meet engineering needs, as the microscopic models are too slow in computations and the macroscopic models do not take into account the physical background of the phenomena. As a result, the mezoscopic models are very popular in engineering. The mezoscopic models are ranked between the micro- and macroscopic models, because they are less accurate than the microscopic ones, but are more flexible than the macroscopic models. They include the Jiles-Atherton model, the Preisach model, the Chua model and the neural network based models and so on.

The Jiles-Atherton model is discussed in detail in this work. The Jiles-Atherton model was established in the last decades of the twentieth century. Despite being a mezoscopic model, its equations can be derived from the energy balance equation, which is very close to the physical representation of the hysteresis. It can easily be extended to describe the frequency dependence (rate dependence) and the influence of mechanical stress, due to the energy formulation. Though the model has some outstanding features, some drawbacks must be highlighted. Namely the
energy based model equations have non-physical solutions, which triggers the negative slope part of the hysteresis curve at the loop tips. Furthermore, the solution of the model differential equation is very sensitive to the step size. Totally different hysteresis curves can be generated by applying different step sizes.

The investigation of the magnetic field in the presence of ferromagnetic materials constitutes the focus of my research work. The Jiles-Atherton model is applied for describing ferromagnetic materials in the simulations. The partial differential equations are discretized by the finite element method (FEM). Two types of FEM is distinguished, the nodal and the edge elements. The magnetic field computations including the eddy-current investigations cannot be carried out by using the nodal FEM, because of the spurious modes.

The edge element formulations with respect to the electromagnetic field computations are summarized and combined with nonlinearity caused by the hysteresis effect.

Nonlinear system of equations are usually solved by applying the classical Newton method. However the Newton method does not converge at inflection points, which is usual in the case of magnetization curves. Another drawback is that the derivative must be determined analytically, otherwise numerical differentiation should be used. The analytical derivative is not available in most of the cases e.g. measured curves. The method is usually applied for magnetostatic problems by modifying the magnetization curve by removing inflection points. Of course in this case not the original problem is solved. Instead of the Newton method, the fixed-point method (polarization method) is applied in this work for solving nonlinear magnetic field equations, since it is convergent for any trial value and it is not sensitive to inflection points and the derivatives must not be known. Furthermore a piecewise linear approximation of measured points is also acceptable. A drawback of the fixed-point method is the slow convergence, which can be improved in several ways. These are discussed in the second thesis.

A test problem is going to be solved to prove the validity of the above mentioned computational procedure. The numerical solution of the test problem is compared with the given measured data.

### 1.1 Proposed research activity

The scope of my PhD dissertation is to develop a new computational procedure to analyse magnetic field based on the Jiles-Atherton model of hysteresis and combined with the higher order edge finite element method.

I intend to find a fast and accurate hysteresis model, which is required in electromagnetic field computations, since, in combination with the finite element method, a very huge number (several tens of thousand) of hysteresis models must be run simultaneously. The Jiles-Atherton model is based on a simple first order ordinary differential equation. Despite having a simple structure, the Jiles-Atherton model is based on physical considerations. I will reformulate the Jiles-Atherton model by means of the energy balance. As a result, the mathematical background will become clear. I intend to extend the original rate independent Jiles-Atherton model to a rate dependent one by considering the eddy currents and the excess losses. This is easily done by the energy balance equation introduced. I intend to build a scalar hysteresis measurement
system in LabVIEW environment to prove the validity of both the rate independent and the rate dependent models.

I want to introduce the edge finite element method as a numerical technique for determining electromagnetic fields. I intend to use higher order elements for obtaining better accuracy. I intend to fine tune the combination of hysteresis modeling and finite element method with the help of a simple one dimensional example of the well known half-space problem. I intend to investigate this problem by progressing from the simplest case to complicated ones. This means that I want to start the investigations with the linear case, I want continue with the Langevin characteristics and finally the Jiles-Atherton model will be used as material characteristics. I intend to introduce the fixed-point method to handle nonlinearity. I want to describe some speed up techniques for the fixed point method, which will be illustrated on the half-space problem.

The organization of the work is as follows.

Chapter 2 contains an overview of the corresponding literature. The basics of the Jiles-Atherton model are discussed based on the relevant papers and a short summary of the electromagnetics and the finite element method can be found in this chapter.

In Chapter 3, I present the reformulation and extension of the Jiles-Atherton model. Furthermore I discuss a scalar hysteresis measurement system for experimentally proving the validity of both the classical and the extended models, as well.

In Chapter 4, I discuss the insertion of hysteresis into the higher order edge element method by using the fixed point technique. Regarding magnetostatics, eddy current analysis and diffusion equation I derive potential formalisms for taking into account the nonlinear relationship between the magnetic field intensity and the magnetic flux density with respect to the magnetic vector potential.

In Chapter 5, I present applications for the developed method. I solve the half-space example by the one dimensional diffusion equation. I simulate the magnetic measurement carried out in Chapter 3 by two dimensional eddy current analysis. I present a three dimensional magnetostatic example proposed in the frame of the TEAM Workshop by the COMPUMAG society. I will prove that the proposed computational technique works for large problems, too.

The \LaTeX\ word processor has been used to produce this document. The figures are placed at the top of the current page where possible. The following notations are used in the following: space vectors are typed by bold italics, e.g. \( \mathbf{v} \), scalars are denoted by italics e.g. \( k \), matrices and arrays are denoted by bold block capital and lowercase symbols, e.g. \( \mathbf{M}, \mathbf{a} \).
Chapter 2

Literature overview

In this chapter, I briefly summarize the current state of the Jiles-Atherton model (JAM) development as found in the relevant literature. The theoretical background of the model is also presented including the Langevin and the Weiss theory for computing the anhysteretic curve of the JAM. The literature deals with the numerical determination of JAM parameters by means of experimentally measured hysteresis curves.

2.1 The Jiles-Atherton model of hysteresis

2.1.1 Langevin theory for paramagnets

The earliest model of magnetization characteristics based on microscopic structures of materials is the Langevin function approximation [2–7].

Let us denote by \( m \) the non-balanced magnetic dipole moment, including the moments of the spin and the orbital motion in an atom of a paramagnetic material. If a magnetic field \( H \) is applied, the corresponding potential energy has the form

\[
 w_m = -\mu_0 m H = -\mu_0 m H \cos \Theta, \tag{2.1}
\]

where \( \Theta \) is the angle between the magnetic moment and the applied magnetic field. Langevin assumed, that in paramagnetic materials, the moments do not interact. Consequently, the classical Maxwell-Boltzmann statistics can be used to express the probability of any electron occupying an energy state of \( w_m \):

\[
 p(w_m) = e^{-\frac{w_m}{kT}}, \tag{2.2}
\]

where \( k = 1.381 \times 10^{-23} \text{ J/K} \) is the Boltzmann constant and \( T \) is the absolute temperature. The number \( dn \) of magnetic particles per unit volume, lying between the angles \( \Theta \) and \( \Theta + d\Theta \) with respect to the applied field \( H \), is proportional to the solid angle \( 2\pi n_0 \sin(\Theta)d\Theta \):

\[
 dn = 2\pi n_0 \exp(\mu_0 m H \cos \Theta/kT) \sin(\Theta)d\Theta, \tag{2.3}
\]
where \( n_0 \) is a proportionality factor determined by the fact that the total volume density of magnetic particles per unit volume is \( N \). Evaluating over a sphere, \( n_0 \) can be determined as

\[
N = 2\pi n_0 \int_0^\pi \exp(\mu_0 m H \cos \Theta/kT) \sin \Theta d\Theta, \tag{2.4}
\]

\[
n_0 = \frac{N}{2\pi \int_0^\pi \exp(\mu_0 m H \cos \Theta/kT) \sin \Theta d\Theta}. \tag{2.5}
\]

The magnetization postulated as a vectorial sum of the magnetic moments per unit volume parallel to the applied field, can be formulated as

\[
M = \int_0^\pi m \cos \Theta d\Theta = N m \int_0^\pi \exp(\mu_0 m H \cos \Theta/kT) \cos \Theta \sin \Theta d\Theta. \tag{2.6}
\]

Introducing the quantities \( \lambda = \mu_0 m H/kT \), \( x = \cos \Theta \) and \( dx = -\sin \Theta d\Theta \), the above integrals can be evaluated as

\[
M = \frac{Nm \int_{-1}^1 x e^{\lambda x} dx}{\int_{-1}^1 e^{\lambda x} dx} = \frac{Nm \left( e^\lambda + e^{-\lambda} - \frac{1}{\lambda} \right)}{\left( e^\lambda - e^{-\lambda} \right) \lambda}. \tag{2.7}
\]

As a result, the equation for describing the magnetization of paramagnetic materials, with respect to the magnetic moments aligned parallel to the applied field can be formulated as

\[
M = Nm \left( \coth \lambda - \frac{1}{\lambda} \right). \tag{2.8}
\]

The function in parenthesis is called the Langevin function: \( \mathcal{L}(\lambda) = \coth \lambda - \frac{1}{\lambda} \). It is important to emphasize that the Langevin function has a discontinuity at \( \lambda = 0 \) therefore it is approximated by the first term of the Taylor series for small \( \lambda \) as

\[
\mathcal{L}(\lambda) = \begin{cases} 
\coth \lambda - \frac{1}{\lambda} & \text{for } |\lambda| > \lambda_0, \\
\frac{\lambda}{3} & \text{for } |\lambda| < \lambda_0.
\end{cases} \tag{2.9}
\]

where \( \lambda_0 \) is a sufficiently small positive value. If the temperature is above the Curie point, then
\( \lambda \) becomes very small, therefore the magnetization can be approximated by

\[
M = Nm \frac{\lambda}{3} = Nm \frac{\mu_0 m H}{3kT}.
\]

(2.10)

The resulting paramagnetic susceptibility \( \chi \) is the well-known Curie law \( \chi = C/T \), where \( C \) is a constant.

### 2.1.2 Weiss correction for ferromagnetic materials

In ferromagnetic materials, the neighboring moments interact with each other. The interaction between the magnetic moments gives an exchange field \( H_{ex} \) as it was pointed out by Weiss [8].

If, within the domain, any magnetic moment \( m_i \) due to its interaction with other moment \( m_j \) has an exchange field \( H_{ex,j} = \alpha_{i,j} m_j \), then this interaction with all moments within the domain can be described as

\[
H_{ex} = \sum_j \alpha_{i,j} m_j.
\]

(2.11)

If the interactions between all moments are assumed to be identical and hence independent of the displacement between the moments, then \( \alpha_{i,j} = \alpha, (\forall i, j) \) can be used:

\[
H_{ex} = \alpha \sum_j m_j = \alpha M.
\]

(2.12)

The effective field can be defined as

\[
H_e = H + \alpha M.
\]

(2.13)

Substituting this relation into (2.8) introducing the saturation magnetization \( M_s = Nm \) and using the shape parameter \( a = \frac{kT}{\mu_0 m} \), the magnetization can be determined by the relation

\[
M = M_s \left( \coth \frac{H + \alpha M}{a} - \frac{a}{H + \alpha M} \right).
\]

(2.14)

### 2.1.3 Rate-independent Jiles-Atherton hysteresis model

When the magnetic field intensity and the magnetization vectors are directed in the same direction, the scalar hysteresis model can describe the connection between them. Otherwise a vector hysteresis model is required. The eddy current effect depending on the excitation frequency is not negligible in electrically conductive materials (e.g., iron). If the eddy current effect and other frequency dependent effects (e.g., excess loss) play only a minor role then the rate-independent hysteresis model can be used for describing the phenomena, otherwise the
rate-dependent model is required.

D. C. Jiles and D. L. Atherton, in a series of papers [3, 9–12], present some physically based equations for magnetization in ferromagnetic materials. The mathematical model for ferromagnetic hysteresis introduced in 1983 is based on physical principles, rather than strictly mathematical arguments or experimental curve fitting. Following the first publication [3], several papers appeared which explained the basis of the model in more detailed way, modified the model to include a reversible component, and presented a method for solving the equations of the model.

The Jiles-Atherton model is widely used for modeling the nonlinear characteristics of magnetic hysteresis. It describes the B–H curve using five parameters. Numerous literatures deal with the identification of these parameters based on their physical meaning.

Another approach to this problem involves solving for five parameters to identify a simulated B–H curve based upon measured data. The goal of this mathematical problem is to find the best fitted simulation of hysteresis. The method must lead to the global minimum of the chosen error function.

In the Jiles-Atherton model two related mechanisms of magnetization are represented. One of them is the domain wall motion which is influenced by the applied field in a way that the domains favorably aligned with respect to the applied field grow at the expense of unfavorably aligned domains. The other mechanism represents the rotation of aligned moments within the domains toward the applied field direction.

The magnetization is composed of two terms in the Jiles-Atherton model of hysteresis, an irreversible and a reversible component:

\[ M = M_{\text{irr}} + M_{\text{rev}}. \]  

(2.15)

The irreversible component represents the irreversible domain wall motion. The reversible magnetization corresponds to the reversible domain wall bending.

The hysteresis characteristic can be described on the basis of the energy density balance in a material. The change of energy density supplied in the material is equal to the change of the magnetostatic energy density and the hysteresis loss density.

\[ w = w_{\text{mag}} + w_{\text{hys}}. \]  

(2.16)

The energy loss density, which is generated by the irreversible domain wall motion, can be expressed as

\[ dw_{\text{wall}} = \mu_0 k d M_{\text{irr}}, \]  

(2.17)

where \( k \) is the pinning coefficient.

In ferromagnetic materials the neighboring domains interact with each other. The interaction between the domains can be taken into account by introducing an effective field acting on the domains per unit volume, as in (2.13) [13, 14].

The magnetization is a function of the effective magnetic field intensity in the expression
taking into consideration that the energy loss density generated by the irreversible domain wall motion can be expressed by

\[ dw_{\text{wall}} = \mu_0 k \delta \frac{dM_{\text{irr}}}{dH_e} dH_e, \]  \tag{2.18}

where \( \delta = \text{sign}(dH_e/dt) \). The irreversible magnetization changes can be obtained from the energy equation, in which the supplied energy is equal to magnetic energy changes and the hysteresis loss according to [13]\(^1\)

\[ \mu_0 \int M_{\text{an}}(H) dH = \mu_0 \int M(H) dH + \mu_0 \int k \delta \frac{dM}{dH} dH. \]  \tag{2.19}

Consequently

\[ M_{\text{an}}(H) = M(H) + k \delta \frac{dM}{dH}. \]  \tag{2.20}

If during the magnetization process the anhysteretic magnetization shows a lower value than the irreversible magnetization it yields a non-physical solution, represented by a negative derivative of the irreversible magnetization \( dM_{\text{irr}}/dH_e < 0 \), see Fig. 2.1. In this case the domain walls actually remain in the previous defect size and \( dM_{\text{irr}}/dH_e = 0 \). The solution of the differential equation can be represented as [15]

\[ dM_{\text{irr}} = \frac{1}{k \delta} [(M_{\text{an}} - M_{\text{irr}}) dH_e]^+ \]  \tag{2.21}

with the symbol

\[ x^+ = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases} \]  \tag{2.22}

Reversible magnetization can be taken into account by introducing a dimensionless reversible coefficient \( c \), so that

\[ M_{\text{rev}} = c (M_{\text{an}} - M_{\text{irr}}). \]  \tag{2.23}

\(^1\)The relationship (2.19) has many different forms in Jiles’ papers. In [12], the following form can be found without explanation (this is the correct form, see the next section for explanation):

\[ \mu_0 \int M_{\text{an}}(H) dH_e = \mu_0 \int M_{\text{irr}}(H) dH_e + \mu_0 \int k \delta \frac{dM_{\text{irr}}}{dH_e} dH_e. \]

In [10], equation (2.19) has another form

\[ \mu_0 \int M_{\text{an}} dH_e = \mu_0 \int M dH_e + \mu_0 k \delta (1 - c) \int dM_{\text{irr}}. \]

Finally in [9] another version can be read

\[ \mu_0 \int M_{\text{an}}(H) dH_e = \mu_0 \int M(H) dH_e + \mu_0 \int k \delta \frac{dM_{\text{irr}}}{dH_e} dH_e. \]
Using equation (2.15) the total magnetization can be obtained as
\[ M = (1 - c)M_{irr} + cM_{an}. \] (2.24)

Substituting the differential expression (2.21) into (2.24) allows to derive the equation
\[ dM = \frac{1 - c}{k_0} [(M_{an} - M_{irr})dH_c]^+ + cdM_{an}. \] (2.25)

In the \((j + 1)\)-th step to the hysteresis characteristic can be evaluated from the \(j\)-th value of the magnetization
\[ dM_{j+1} = \frac{1}{k_0} [(M_{an} - M_j)dH_c]^+ + cdM_{an}. \] (2.26)

Equation (2.26) can be used for numerical simulation of the Jiles-Atherton model of hysteresis.

2.1.4 Determination of model parameters from measured data

The JAM parameter identification procedure according to the Jiles proposal is based on the knowledge of some points of the experimental hysteresis curve. These points are:
- the coercivity \((H_c)\);
- the remanence \((M_r)\);
- coordinates of the loop tip \((H_m, M_m)\);
- the differential susceptibility at the coercive point \((\chi_c = \frac{dM}{dH_{|H=H_c, M=0}})}\);
• the differential susceptibility at remanence \( (\chi_r = \frac{dM}{dH} \mid H=0, M=M_r, \delta=-1) \);
• the differential susceptibility of the initial magnetization curve at the loop tip \( (\chi_m \approx \frac{dM}{dH} \mid H=H_{\text{max}}, \delta=1) \);
• the initial anhysteretic susceptibility \( (\chi_{\text{an}} = \frac{dM}{dH} \mid H=0, M=0) \);
• the initial normal susceptibility \( (\chi_{\text{in}} = \frac{dM}{dH} \mid H=0, M=0) \);
• maximum differential susceptibility \( (\chi_{\text{max}} = \max (\frac{dM}{dH})) \).

Since this identification process based on only the points of the experimental hysteresis curve listed above it means that the procedure can not lead to an exactly identified curve.

The following parameter identification procedure is proposed by Jiles, Thoelke and Devine in [12]. It is a difficult procedure since it is not immediately clear which "fixed reference points" on a measured hysteresis curve should be used to calculate the parameters. Furthermore, the implicit nature of the hysteresis equations makes the problem intractable for an injudicious choice of these fixed points. It has been found that the simplest solution to the problem is obtained by using the parameters summarized above.

**Saturation magnetization \( M_s \)**

The parameter easiest to obtain is the saturation magnetization \( M_s \). It is often known with respect to a particular material, and can be extracted from data sheets or other references. It can also be measured as accurately as desired by subjecting the material to a field of sufficiently high strength, and then either measuring the flux density \( B \) with a coil or the magnetization \( M \) with a vibrating sample magnetometer, and then calculating \( M_s \) from these measurements.

**Determination of the parameter \( c \) which represents reversible wall motion**

The reversible component of magnetization due to reversible wall bending and reversible translation of wall is determined in the model by the coefficient \( c \). This can be calculated from the ratio of the initial normal susceptibility \( \chi_{\text{in}} \) to the initial anhysteretic susceptibility \( \chi_{\text{an}} \).

Using equation (9) from [12], which is

\[
\frac{dM}{dH} = (1 - c) \frac{M_{\text{an}} - M_{\text{irr}}}{k\delta - \alpha(M_{\text{an}} - M_{\text{irr}})} + c \frac{dM_{\text{an}}}{dH},
\]

the initial differential susceptibility is

\[
\chi_{\text{in}} = \frac{dM}{dH} \bigg\mid_{H=0, M=0} = \frac{(1 - c)M_{\text{an}}}{k\delta - \alpha M_{\text{an}}} + c \frac{dM_{\text{an}}}{dH}.
\]

Substituting (2.14) into the equation above, when \( M = 0 \), the following expression results:

\[
\chi_{\text{in}} = \frac{(1 - c)M_s}{k\delta - \alpha M_s} \left[ \coth \left( \frac{H}{a} \right) - \frac{a}{H} \right] + M_s \frac{c}{a} \left[ 1 - \coth^2 \left( \frac{H}{a} \right) + \frac{a^2}{H^2} \right].
\]
and taking the limit as $H \to 0$ of $\coth(H/a) - (a/H)$,

$$
\lim_{H \to 0} \left[ \coth \left( \frac{H}{a} \right) - \left( \frac{a}{H} \right) \right] = \lim_{H \to 0} \left\{ \frac{H}{3a} - \frac{H^3}{45a^3} + \frac{H^5}{945a^5} - \ldots \right\} = 0 \quad (2.30)
$$

so that

$$
\chi_{in} = \lim_{H \to 0} \frac{dM}{dH} = 0 + c \frac{dM_{an}}{dH} \quad (2.31)
$$

Since $M = 0$ at the origin of the magnetization curve,

$$
\chi_{in} = \frac{cM_s}{3a} \quad (2.32)
$$

This then gives

$$
c = \frac{3a\chi_{in}}{M_s} \quad (2.33)
$$

which is a relationship between $c$ and the initial susceptibility.

**Relationship between $a$ and $\alpha$**

The anhysteretic susceptibility itself constitutes a relationship between the model parameters $a$ and $\alpha$. This relationship depends on the form of the function chosen to model the anhysteretic magnetization curve. The modified Langevin function has been used successfully to model the anhysteretic magnetization, although it should be remembered that this choice of $M_{an}$ is very specific, and that other functions for $M_{an}$ exist for particular circumstances.

From the anhysteretic function given in (2.14), it is easily shown that the anhysteretic susceptibility at the origin is given by

$$
\chi_{an} = \lim_{H,M \to 0} \left\{ \frac{d}{dH}M_{an}(H) \right\} = \frac{M_s}{3a - \alpha M_s} \quad (2.34)
$$

and so

$$
a = \frac{M_s}{3} \left( \frac{1}{\chi_{an} + \alpha} \right) \quad (2.35)
$$

This equation can then be used as a constraint on the model parameters $a$ and $\alpha$, although a further condition is needed to determine the values of these parameters.
**Determination of the parameter \( k \) which determines the hysteresis loss**

The coercivity is determined by the amount of pinning, and hence by the parameter \( k \). For very soft magnetic materials, it is found that \( k \approx H_c \), provided \( k \) is defined in units of \([\text{Am}^{-1}]\) as given above. For this reason, the definition of the pinning parameter in units of \([\text{Am}^{-1}]\) is preferred since the pinning force acts like a field opposing the prevailing magnetic field \( H \).

The general relationship between \( k \) and \( H_c \) can be expressed most simply if the differential susceptibility at the coercive point \( \chi_c \) is known.

Again, we return to (2.27), and now consider the situation at the coercive point. Let \( \chi_c = \chi_{\text{max}} \) denote the differential susceptibility at the coercive point, which in the model is always the maximum value of differential susceptibility observed around the hysteresis loop

\[
\chi_{\text{max}} = \frac{1}{k\delta - \alpha[M_{\text{an}}(H_c) - M_{\text{irr}}]}[M_{\text{an}}(H_c) - M_{\text{irr}}] \\
+ c \left( \frac{dM_{\text{an}}(H_c)}{dH} - \frac{dM_{\text{irr}}}{dH} \right) \tag{2.36}
\]

at the coercive point \( \delta = 1, H = H_c, M = 0 \) and rearranging the equation leads to

\[
k = \frac{1}{\chi_{\text{max}} - c \left( \frac{dM_{\text{an}}(H_c)}{dH} - \frac{dM_{\text{irr}}}{dH} \right)}[M_{\text{an}}(H_c) - M_{\text{irr}}] \\
+ \alpha[M_{\text{an}}(H_c) - M_{\text{irr}}] \tag{2.37}
\]

Explicit expression for \( M_{\text{irr}} \) and \( dM_{\text{irr}}/dH \) at the coercive point can be obtained in terms of \( M_{\text{an}}(H_c) \), \( \chi_{\text{max}} \), and \( dM_{\text{an}}(H_c)/dH \) since it has already been postulated that

\[
M = M_{\text{rev}} + M_{\text{irr}}, \tag{2.38}
\]

and since \( M_{\text{rev}} = c(M_{\text{an}} - M_{\text{irr}}) \), it follows that

\[
M = cM_{\text{an}} + (1 - c)M_{\text{irr}}, \tag{2.39}
\]

and rearranging gives

\[
M_{\text{irr}} = \frac{1}{1 - c} (M - cM_{\text{an}}). \tag{2.40}
\]

Since \( M = 0 \) at the coercive point, (2.40) yields

\[
M_{\text{irr}} = -\frac{c}{1 - c} M_{\text{an}}(H_c), \tag{2.41}
\]

while differentiating (2.40) with respect to \( H \) and considering the values at the coercive point
gives
\[ \frac{dM_{irr}(H_c)}{dH} = \frac{1}{1-c} \chi_{\text{max}} - c \frac{dM_{an}(H_c)}{dH}. \] (2.42)

Substituting these expressions into (2.37) gives the following equation for \( k \)
\[ k = \frac{M_{an}(H_c)}{\chi_{\text{max}} - c \frac{dM_{an}(H_c)}{dH}} \] (2.43)

which can be used to calculate \( k \) provided all of the other parameters are known.

**Determination of \( a \) and \( \alpha \)**

The remanence point \( M_r \) is dependent on \( \alpha \) and other parameters. If the other parameters \( a \), \( k \) and \( c \) are known, the remanence can be used to calculate \( \alpha \). However, in this case, it is not possible to obtain an explicit expression for \( \alpha \).

Using the remanence \( M_r \) and the differential susceptibility at remanence \( \chi_r \), the parameter \( a \) can be determined if the other parameters are already known. Starting from (2.27), with \( \delta = -1 \), \( H = 0 \) and \( M = M_r \),
\[ \chi_r = \frac{M_{an}(M_r) - M_{irr}}{-k - \alpha(M_{an}(M_r) - M_{irr}) + c \left( \frac{dM_{an}(M_r)}{dH} - \frac{dM_{irr}}{dH} \right)} \] (2.44)
and since \( M_r = M_{\text{rev}} + M_{irr} \) and \( M_{\text{rev}} = c(M_{an} - M_{irr}) \), it can be shown that, at remanence
\[ M_{irr} = \frac{M_r - cM_{an}(M_r)}{1 - c} \] (2.45)
and
\[ \frac{dM_{irr}}{dH} = \frac{1}{1-c} \frac{dM_r}{dH} - \frac{c}{1-c} \frac{dM_{an}(M_r)}{dH}. \] (2.46)

Substituting these results back into (2.44) gives
\[ \chi_r = \frac{M_{an}(M_r) - M_r}{-(1-c)k - \alpha(M_{an}(M_r) - M_r)} + \frac{c}{1-c} \left( \frac{dM_{an}(M_r)}{dH} - \frac{dM_r}{dH} \right) \] (2.47)

This equation can be used to give an explicit expression for \( M_r \), which is
\[ M_r = M_{an}(M_r) + \frac{k}{1 - c + \frac{1}{\chi_r - c \frac{dM_{an}(M_r)}{dH}}}. \] (2.48)
Relationship between hysteresis parameters at the loop tip

Finally, in calculating $\alpha$ and $a$, it has been found useful to include some redundancy by incorporating the coordinates of the loop tip $M_m$, $H_m$ and the slope of the initial magnetization curve at the loop tip $\chi_m$.

We start again from (2.27) and consider the differential susceptibility along the initial magnetization curve at the loop tip with $\delta = 1$. If the loop tip is sufficiently close to saturation, then the differential susceptibility of the initial magnetization curve at the loop tip will approach the differential susceptibility of the anhysteretic $\frac{dM}{dH} \approx \frac{dM_{an}}{dH}$. This can be used as an approximation to obtain an equation relating the hysteresis parameters. Using the general result $M_{irr} = \frac{M - cM_{an}}{1 - c}$, it can easily be seen that the above approximation also implies that $\frac{dM_{irr}}{dH} = \frac{dM}{dH} = \frac{dM_{an}(H_m)}{dH}$. In addition, $M_{irr} = M_m$ under these conditions.

Replacing $M_{irr}$ with $M_m$ leads to

$$\chi_m = \frac{M_{an}(H_m) - M_m}{k\delta - \alpha[M_{an}(H_m) - M_m]} + c \left( \frac{dM_{an}(H_m)}{dH} - \frac{dM_{irr}(H_m)}{dH} \right). \tag{2.49}$$

These approximations also allow the second term on the right-hand side of (2.49) to be eliminated

$$\chi_m = \frac{M_{an}(H_m) - M_m}{k\delta - \alpha[M_{an}(H_m) - M_m]}, \tag{2.50}$$

and rearranging this leads to

$$M_m = M_{an}(H_m) - \frac{(1 - c)k\chi_m}{\alpha\chi_m + 1}. \tag{2.51}$$

In principle, the incorporation of this equation in the parameter calculation algorithm is not entirely necessary, but it has been found that numerical solutions show faster convergence when this condition is included.

Procedure for calculating parameters

Since some of the equations needed for determining the parameters can only be expressed implicitly in terms of these and other parameters, a numerical method has been devised for calculating the values by using successive iteration. The reversible coefficient $c$ is obtained directly from the initial slope of the normal magnetization curve using (2.32). The values of $a$, $\alpha$ and $k$ are then obtained by using (2.43), (2.48) and (2.51) successively in an iterative procedure. A seed value of $\alpha$ is used and by means of (2.35) a first estimate of $a$ is found. Then $k$ is calculated from (2.43). Using the current values of $k$, $\alpha$ and $a$ are then calculated from (2.48), and then using the current values of $\alpha$ and $k$, $a$ is calculated from (2.51). The procedure for calculating $k$, $\alpha$ and $a$ is then repeated.
2.2 Review of Electromagnetics

The problem of electromagnetic analysis on a macroscopic level is the problem of solving Maxwell’s equations subject to certain boundary conditions. Maxwell’s equations are a set of equations, written in differential or integral form, stating the relationships between the fundamental electromagnetic quantities. These quantities are the electric field intensity $\mathbf{E}$, the electric displacement or electric flux density $\mathbf{D}$, the magnetic field intensity $\mathbf{H}$, the magnetic flux density $\mathbf{B}$, the current density $\mathbf{J}$ and the electric charge density $\rho$.

The equations can be formulated in differential or integral form. The differential form is presented here as it leads to differential equations that the finite element method can handle. For general time-varying fields, Maxwell’s equations can be written as

\begin{align*}
\text{curl } \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \\
\text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\text{div } \mathbf{B} &= 0, \\
\text{div } \mathbf{D} &= \rho.
\end{align*}

The first two equations are also referred to as the Maxwell-Ampere’s law and the Faraday’s law, respectively. Equation three and four are two forms of the Gauss’ law, the electric and magnetic form, respectively [16].

Another fundamental equation is the equation of continuity, which can be written as

\begin{equation}
\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t}.
\end{equation}

Of the five equations mentioned above, only four are independent. The first three combined with either the electric form of the Gauss’ law or the equation of continuity form such an independent system.

2.2.1 Constitutive relations

To obtain a close system, the constitutive relations describing the macroscopic properties of the medium, are included. They are given as

\begin{align*}
\mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P}, \\
\mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}), \\
\mathbf{J} &= \sigma \mathbf{E}.
\end{align*}

Here $\varepsilon_0$ is the permittivity of vacuum, $\mu_0$ is the permeability of vacuum, and $\sigma$ is the electrical conductivity.

The electric polarization vector $\mathbf{P}$ describes how the material is polarized when an electric
Field $E$ is present. It can be interpreted as the volume density of electric dipole moments. $P$ is generally a function of $E$. Some materials can have a nonzero $P$ also when there is no electric field present.

The magnetization vector $M$ similarly describes how the material is magnetized when a magnetic field $H$ is present. It can be interpreted as the volume density of magnetic dipole moments. $M$ is generally a function of $H$. Permanent magnets, for instance, have a nonzero $M$ also when there is no magnetic field present.

2.3 The finite element method

The FEM is the most popular and flexible numerical technique to determine approximate solution of partial differential equations in engineering. The fundamental idea of the FEM is to divide the problem region to be analyzed into smaller finite elements with given shape (e.g. triangles in 2D or tetrahedra is 3D). The scalar potential functions can be approximated by nodal shape functions, and vector potential functions can be approximated by either nodal or vector shape functions. A shape function is a simple continuous polynomial function defined in a finite element. Applying first-order nodal and edge shape functions, the unknown potentials can be associated with the nodes as well as edges of the finite element mesh. In the case of nodal finite elements the unknowns are connected to the element vertices, nodes [17–21].

2.3.1 Nodal finite element method

Approximation of a scalar function $\varphi(\mathbf{r}, t)$ on a first order tetrahedral finite element can be formulated as

$$\varphi(\mathbf{r}, t) \approx \sum_{k=0}^{4} \varphi_k(t) N_k(\mathbf{r}), \quad (2.60)$$

where $N_k(\mathbf{r})$ are scalar basis functions on the tetrahedral element and $\varphi_k(t)$ are the scalar
values at the corresponding vertices. The definition of the nodal basis function is that it is equal to one at the \( k \)\textsuperscript{th} node and zero at the other nodes

\[
N_k(r) = \begin{cases} 
1, & \text{at } k\text{th node}, \\
0, & \text{at other nodes}.
\end{cases}
\] (2.61)

A point \( P \) within the tetrahedron (Fig. 2.2) subdivides the tetrahedron into four sub-tetrahedra having the volumes \( V_1, V_2, V_3 \) and \( V_4 \). The physical nature of the volume coordinates can be identified as

\[
N_k(r) = \frac{V_k}{V},
\] (2.62)

where \( V \) is the total volume of the tetrahedron \((\sum_{k=1}^{4} V_k = V)\). The volumes can be computed as follows

\[
\begin{align*}
V &= \frac{1}{6} \begin{vmatrix}
1 & x_1 & y_1 & z_1 \\
1 & x_2 & y_2 & z_2 \\
1 & x_3 & y_3 & z_3 \\
1 & x_4 & y_4 & z_4
\end{vmatrix}, \\
V_1 &= \frac{1}{6} \begin{vmatrix}
1 & x & y & z \\
1 & x_2 & y_2 & z_2 \\
1 & x_3 & y_3 & z_3 \\
1 & x_4 & y_4 & z_4
\end{vmatrix}, \\
V_2 &= \frac{1}{6} \begin{vmatrix}
1 & x_1 & y_1 & z_1 \\
1 & x & y & z \\
1 & x_3 & y_3 & z_3 \\
1 & x_4 & y_4 & z_4
\end{vmatrix}, \\
V_3 &= \frac{1}{6} \begin{vmatrix}
1 & x_1 & y_1 & z_1 \\
1 & x_2 & y_2 & z_2 \\
1 & x & y & z \\
1 & x_4 & y_4 & z_4
\end{vmatrix}, \\
V_4 &= \frac{1}{6} \begin{vmatrix}
1 & x & y & z \\
1 & x_1 & y_1 & z_1 \\
1 & x_2 & y_2 & z_2 \\
1 & x_3 & y_3 & z_3
\end{vmatrix},
\end{align*}
\] (2.63)

where \((x_1, y_i, z_i)\) are the Cartesian coordinates of the tetrahedral vertices and \((x, y, z)\) are the coordinates of the point \( P \) inside of the tetrahedral finite element. The correspondence between the volume coordinates and the \( \xi-\eta-\zeta \) coordinate system is

\[
\begin{align*}
n_1 &= 1 - \xi - \eta - \zeta, \\
n_2 &= \xi, \\
n_3 &= \eta, \\
n_4 &= \zeta.
\end{align*}
\] (2.64)
The Jacobian matrix of the linear coordinate transformation is

$$
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}
$$

(2.65)

The Jacobian determinant is correspondingly $|J| = 6V$.

It is also possible to represent the vector functions $\mathbf{v}(\mathbf{r}, t) = v_x(\mathbf{r}, t)\mathbf{e}_x + v_y(\mathbf{r}, t)\mathbf{e}_y + v_z(\mathbf{r}, t)\mathbf{e}_z$ by using nodal basis function approximation for each vector component separately

$$
\mathbf{v}(\mathbf{r}, t) \approx \sum_{k=1}^{4} \left[ v_{x,k}(t)\mathbf{e}_x + v_{y,k}(t)\mathbf{e}_y + v_{z,k}(t)\mathbf{e}_z \right] \mathbf{N}_k(\mathbf{r}),
$$

(2.66)

where $v_{x,k}(t)$, $v_{y,k}(t)$ and $v_{z,k}(t)$ are vector components at the element nodes. Several serious problems were identified when the ordinary nodal based finite elements were employed to compute vector electric or magnetic fields:

- lack of adequate gauge conditions for vector magnetostatic analysis;
- satisfaction of the appropriate boundary conditions at material interfaces;
- difficulty in treating the conducting and dielectric edges and corners due to field singularities associated with these structures;
- occurrence of nonphysical or so-called spurious solutions, especially in waveguide and scattering problems.

### 2.3.2 Edge finite element method

A revolutionary approach has been developed to address the above problems. The approach uses the so-called vector basis or vector finite elements which assign degrees of freedom to the edges rather that to the nodes of the finite elements. For this reason they are also called edge finite elements.

Vector finite elements were for the first time described by Whitney [22] as a family of differential forms. Their usefulness and importance in electromagnetic field analysis has not been realized until recently thanks to the work of Nedelec [23].

The edge basis functions over a tetrahedron are denoted by $\mathbf{W}_i(\mathbf{r})$, $i = 1, \ldots, 6$. A vector function can be represented on a finite element by the linear combination of the vector basis functions

$$
\mathbf{v}(\mathbf{r}, t) \approx \sum_{k=1}^{6} v_k(t)\mathbf{W}_k(\mathbf{r}),
$$

(2.67)
where \( v_k(t) \) are the line integrals of the vector field on the \( k \text{th} \) edge.

The line integral of a vector basis function on the corresponding edge is equal to one and the integral on the other edges are zeros

\[
\int_{k \text{th edge}} W_i dt = \begin{cases} 
1, & \text{if } k = i, \\
0, & \text{if } k \neq i.
\end{cases}
\]

The vector basis functions can be defined by the nodal basis functions

\[
W_{\text{from,to}} = N_{\text{from}} \text{grad} N_{\text{to}} - N_{\text{to}} \text{grad} N_{\text{from}},
\]

where the "from" and "to" mean the node numbers of the starting and the ending points of the edge vector, therefore the six linear vector basis functions can be formulated as

\[
W_1 = N_1 \text{grad} N_2 - N_2 \text{grad} N_1, \\
W_2 = N_1 \text{grad} N_3 - N_3 \text{grad} N_1, \\
W_3 = N_1 \text{grad} N_4 - N_4 \text{grad} N_1, \\
W_4 = N_2 \text{grad} N_3 - N_3 \text{grad} N_2, \\
W_5 = N_2 \text{grad} N_4 - N_4 \text{grad} N_2, \\
W_6 = N_3 \text{grad} N_4 - N_4 \text{grad} N_3.
\]

Vector basis functions have some important properties such as:

- the tangential component of vector field \( W_i \) is continuous across facets;
- \( \text{div } W_i \equiv 0 \), for \( \forall i \).

The edge finite elements also exhibit several disadvantages such as:

- difficulty in prescription of source current values which satisfy the solenoidal character \( \text{div } J_s = 0 \) exactly;
- discrete character of the magnetic flux density distribution obtained using first order edge finite elements.
Chapter 3

Improvement of the Jiles-Atherton model

The Jiles-Atherton model equations are derived from the energy balance equation, which makes clear the physical background of the model. Utilizing this approach, the rate-independent model can be extended to a rate-dependent model with minimal effort, due to the introduction of energy balance equation. On the other hand model parameter fitting is discussed in this chapter with the model scalar parameters determined to simulate the magnetization curve of realistic ferromagnetic materials. I propose here a novel method to determine these model parameters. I have developed a scalar hysteresis measurement method to obtain experimental data to verify the Jiles-Atherton model.

3.1 Reformulation of the rate-independent Jiles-Atherton model

The starting point for Jiles and Atherton’s theory is the "energy" function \( W_m = \int M dB_e \), where \( B_e \) is the effective magnetic field density \( B_e = \mu_0 (H + \alpha M) \). No explanation is provided in the literature [3, 11, 12] for choosing the energy function defined above. The relationship between the chosen energy function and the classical hysteresis loss \( \oint H dB \) is also unclear. The clarification is of paramount importance for obvious reasons of validity of the theory but also for providing further insights into extending the model to cover the phenomenon of magnetostriction. According to my approach detailed below, this gap can be bridged by starting with W. F. Brown’s expression for the work done by a battery in charging the magnetization of a rigid ferromagnetic body [24].

The equations below are expressed in scalar form although we know that the variables are vector quantities. We suppose these scalars are the absolute values of the real vector quantities and all of the vectors point in the same direction [25,26]. The relationship between the magnetic field intensity \( H \) and the magnetic field density \( B \) is represented by the scalar JAM which is detailed in the further sections. The energy balance equation declares, that the externally supplied battery energy density \( \delta W_{bat} \) covers the inner magnetic state of the ferromagnetic
material $\delta W_{\text{mag}}$ and dissipating as a hysteresis loss density $\delta L_{\text{mag}}$

$$\delta W_{\text{bat}} = \delta W_{\text{mag}} + \delta L_{\text{mag}}.$$  \hfill (3.1)

The battery energy density can be expressed as

$$\delta W_{\text{bat}} = \oint H \, dB,$$  \hfill (3.2)

where the integral is calculated over one cycle. Substituting the relationship $B = \mu_0(H + M)$ in the above integral we have

$$\delta W_{\text{bat}} = \oint H \, dB = \oint \mu_0 H \, d(H + M) = - \oint \mu_0 M \, dH,$$  \hfill (3.3)

because $\oint \mu_0 H \, dH = 0$ and

$$\oint H \, dM + \oint M \, dH = 0,$$  \hfill (3.4)

as the figures (Fig. 3.1 – Fig. 3.6) below explain. The integrals in (3.4) are evaluated separately. First, $\oint H \, dM$ is considered in Figs. 3.1, 3.2 and 3.3. The integrals on the ascending ($dM > 0$) and the descending ($dM < 0$) parts of the hysteresis loop are plotted in different figures and the positive and negative areas are denoted by $\bigcirc$ and $\bigotimes$ signs. The result of the $\oint H \, dM$ expression is the area of the hysteresis loop with positive sign, as it was expected. Now consider the integral $\oint M \, dH$, which is also integrated along the descending and the ascending branches separately as it can be seen in the Figs. 3.4, 3.5 and 3.6. The result of the integral is also the area of the hysteresis loop, but with negative sign. Therefore the relationship (3.4) is valid. The above form of $\delta W_{\text{bat}}$ is almost suitable, but some remarks are desired before I continue derivation. The average magnetic moment per unit volume $M$ is decomposed into an irreversible component $M_{\text{irr}}$ and a reversible component $M_{\text{rev}}$ [27]. $M_{\text{rev}}$ is related to the anhysteretic or ideal magnetization by the following expressions

$$M = M_{\text{rev}} + M_{\text{irr}},$$

$$M_{\text{rev}} = c(M_{\text{an}} - M_{\text{irr}}),$$  \hfill (3.5)

where $M_{\text{an}}$ is the anhysteretic curve, which is the average of the descending and ascending parts of the major hysteresis loop. The choice of the function $M_{\text{an}}$ depends on the hysteresis type (e.g. pinning type characteristics). Usually analytical models provide this function e.g. Frölich
function. The Langevin theory also proposes an anhysteretic curve, which has the form

\[ M_{an}(H_e) = M_s \mathcal{L} \left( \frac{H_e}{a} \right) \]

\[ = M_s \left( \coth \frac{H_e}{a} - \frac{a}{H_e} \right), \] (3.6)

where \( \mathcal{L}() \) denotes the Langevin function, \( H_e \) is the effective field (2.13) according to the Weiss-theory [2, 13], \( \alpha \) is a parameter representing the interaction between the magnetic domains and \( M_s \) is the maximum value of the magnetization called saturation magnetization.

The final form of \( \delta W_{bat} \) can be obtained by substituting (3.5) into (3.3)

\[ \delta W_{bat} = - \oint \mu_0 (M_{rev} + M_{irr}) dH. \] (3.7)

Since the above equation is valid only if the orbits of \( M(t) \) and \( H(t) \) are periodic in the \( M-H \) plane, therefore the integral of \( M_{rev} \) vanishes [25]. The relationship of the externally supplied battery energy density has the form

\[ \delta W_{bat} = - \oint \mu_0 M_{irr} dH. \] (3.8)
The energy representing the internal magnetic state $\delta W_{\text{mag}}$ can be derived from (3.8) in the ideal case

$$\delta W_{\text{mag}} = \delta W_{\text{bat}|\text{ideal}} = -\int \mu_0 M_{\text{an}} dH. \tag{3.9}$$

The hysteresis loss $\delta L_{\text{mag}}$ is proportional to a scalar parameter $k$, which is linked with the hysteresis loss and it can be determined from experimental curves [9].

$$\delta L_{\text{mag}} = \int \mu_0 k \delta dM_{\text{irr}}, \tag{3.10}$$

where $\delta$ is defined as $\delta = \text{sign}(dH/dt)$ [28].

Now we can substitute the (3.8), (3.9) and (3.10) into the energy balance (3.1) resulting the following relationship:

$$\int \mu_0 \left( M_{\text{irr}} - M_{\text{an}} + k\delta \frac{dM_{\text{irr}}}{dH} \right) dH = 0. \tag{3.11}$$

I have proposed the hypothesis that the following equation is valid when we go from any point $P_1$ on the periodic orbit to another point $P_2$ on the periodic orbit

$$\int_{P_1}^{P_2} \mu_0 \left( M_{\text{irr}} - M_{\text{an}} + k\delta \frac{dM_{\text{irr}}}{dH} \right) dH = 0. \tag{3.12}$$

The above equation holds only on the periodic orbit. Therefore on the periodic orbit, the integrand must be equal to zero

$$M_{\text{irr}} - M_{\text{an}} + k\delta \frac{dM_{\text{irr}}}{dH} = 0. \tag{3.13}$$

Therefore the differential equation of the rate-independent scalar JAM has the form

$$\frac{dM_{\text{irr}}}{dH} = \frac{M_{\text{an}} - M_{\text{irr}}}{k\delta}, \tag{3.14}$$

where from the $M_{\text{irr}}$ can be determined and the total magnetization $M$ can be calculated by using (3.5) and a new differential equation can be derived with respect to the total magnetization $M$ - instead of the irreversible part $M_{\text{irr}}$, which can be formed as follows

$$\frac{dM}{dH} = (1 - c) \frac{dM_{\text{irr}}}{dH} + c \frac{dM_{\text{an}}}{dH}. \tag{3.15}$$

Jiles and Atherton made the assumption [11] that if the actual magnetization is less than the anhysteretic value and the magnetic field intensity $H$ is lowered, then until the value of $M$ becomes equal to the anhysteretic value $M_{\text{an}}$, the change in magnetization is reversible. That
Fig. 3.7. Magnetization curves of an ideal and ferromagnetic material

is,

\[
\frac{dM_{\text{irr}}}{dH} = 0, \quad \text{if} \quad \begin{cases} dH/dt < 0 \text{ and } M_{\text{an}} - M > 0, \\ dH/dt > 0 \text{ and } M_{\text{an}} - M < 0. \end{cases} \quad (3.16)
\]

The motivation for the assumptions is mathematical. Without (3.16), the incremental susceptibility at the reversal points $dM/dH$ can become negative. This can be checked by numerical simulations. Experimental observations suggest that the quasi-static incremental susceptibility is a non-negative quantity.

Taking into account (3.16), I have introduced the parameter $\delta_M$ as follows [25, 29–33]

\[
\delta_M = \begin{cases} 0, \quad \text{if } dH/dt < 0 \text{ and } M_{\text{an}} - M > 0, \\ 0, \quad \text{if } dH/dt > 0 \text{ and } M_{\text{an}} - M < 0, \\ 1, \quad \text{otherwise}. \end{cases} \quad (3.17)
\]

for handling non-physical solutions of (3.15). It is shown in Fig. 3.7. For this reason (3.15) has to be modified as appropriate

\[
\frac{dM}{dH} = (1 - c)\delta_M \frac{dM_{\text{irr}}}{dH} + c \frac{dM_{\text{an}}}{dH}. \quad (3.18)
\]
which has an other more useful form,
\[
\frac{dM}{dH} = \frac{\delta M}{kB}(M_{an} - M) + c\frac{dM_{an}}{dH}.
\] (3.19)

### 3.2 Reformulation of the rate-dependent Jiles-Atherton model

The effects of classical eddy current losses can be used to extend the rate-independent JAM to account for the frequency dependence of hysteresis in electrically conductive media. In this section I deal with the extension of the classical hysteresis model to include both the eddy current losses and anomalous or excess losses. The instantaneous power loss due to eddy currents then consists of two terms, one of which depends on \((dB/dt)^2\) and the other one on \(|dB/dt|^{1.5}\). The first term is the classical power loss and the second term according to Bertotti [34], is the excess power loss. It is shown how these terms are incorporated into a time dependent hysteresis model [10]. I take into account the eddy current and the anomalous losses as an extension of (3.1) [31, 35]

\[
\delta W_{bat} = \delta W_{mag} + \delta L_{mag} + \delta L_{(hyst)} + \delta L_{(dyn)},
\] (3.20)

\(\delta L_{(hyst)}\) is proportional to the area of the quasi-static hysteresis loop. The physical reason for such a decomposition is that \(\delta L_{(hyst)}\) originates from discontinuous character of the magnetization process at microscopic scale, whereas \(\delta L_{(dyn)}\) is associated with the macroscopic large-scale behavior of the magnetic domain structure. A proper statistical treatment of the loss problem shows that (3.20) is naturally obtained whenever the characteristic time scales of these two processes do not overlap.

Beside the dynamic losses \(\delta L_{(dyn)}\) in (3.20) two different kinds of losses can be distinguished. One is coming from the classical eddy current loss \(\delta L_{EC}\), which can be computed form the Maxwell’ equations. On the other hand, as a consequence of domain effects the dynamic loss is generally found to be larger than the classical eddy current loss \(\delta L_{(dyn)} > \delta L_{EC}\) [8]. The difference between them, called excess or anomalous loss, is denoted by \(\delta L_A\). By substituting the eddy current and anomalous loss terms into the (3.20) the energy balance equation has the form

\[
\delta W_{bat} = \delta W_{mag} + \delta L_{mag} + \delta L_{EC} + \delta L_{A},
\] (3.21)

where \(\delta L_{EC}\) the eddy current loss density and \(\delta L_A\) the energy loss per unit volume for the domain wall motion (anomalous/excess loss). The classical eddy current loss is obtained by solving the Maxwell equation \(\text{curl} E = -dB/dt\) for a given geometry, assuming that the magnetic field penetrates uniformly throughout the material.

In the frequency domain, loss separation is widely used with problems involving magnetic laminations. Loss separation breaks the total core loss into static hysteresis loss \(\delta L_{(hyst)}\), clas-
Mechanical eddy current loss $\delta L_{EC}$, and anomalous loss $\delta L_A$ as the equations (3.20), (3.21) above show [34]

$$P/f = k_0 + k_{EC}f + k_A\sqrt{f},$$

where $f$ is the frequency and $P$ is the input power. The eddy current loss and the anomalous loss can be expressed from the time domain form of (3.22)

$$\frac{d\delta L_{EC}}{dt} = k_{EC}\left(\frac{dB}{dt}\right)^2,$$

(3.23)

and

$$\frac{d\delta L_A}{dt} = k_A\left|\frac{dB}{dt}\right|^{1.5}.$$

(3.24)

Reformulation of expressions (3.23) and (3.24) is required to obtain a differential equation for the magnetization similar to (3.18).

The eddy current loss density can be derived by time integration of (3.23)

$$\delta L_{EC} = \oint k_{EC}\left(\frac{dB}{dt}\right)^2 dt$$

$$= \oint k_{EC}\mu_0^2\left(\frac{dH}{dt} + \frac{dM}{dt}\right)^2 dt$$

$$= \oint k_{EC}\mu_0^2\left(\frac{dM}{dt}\right)^2\left(1 + \frac{dH}{dM}\right)^2 dt$$

$$= \oint k_{EC}\mu_0^2\left(\frac{dM}{dt}\right)^2\left(1 + \frac{1}{\kappa_d}\right)^2 dt$$

$$\approx \oint k_{EC}\mu_0^2\frac{dM_{irr}}{dt} dM_{irr}.$$

The relationship $B = \mu_0(H + M)$ has been used to obtain the second row of (3.25). The definition of differential susceptibility $\kappa_d$ is $\kappa_d = dM/dH$. In the case of ferromagnetic material $\kappa_d$ is always much larger than unity, therefore the term $|1 + 1/\kappa_d|^2$ can be neglected because it tends to one as well. The loop integral of the reversible term of the magnetization is also equal to zero as it has already been explained on page 22.
The excess loss density can be derived by time integration of (3.24).

\[ \delta L_A = \int k_A \left| \frac{dB}{dt} \right|^{1.5} dt \]

\[ = \int k_A \left| \frac{dB}{dt} \right| \sqrt{\left| \frac{dM}{dt} \right| dt} \]

\[ = \int k_A \mu_0 \sqrt{\mu_0} \left| \frac{dM}{dt} \right| + \frac{dM}{dt} \sqrt{\left| \frac{dM}{dt} \right|} \left| 1 + \frac{dH}{dM} \right| dt \]

\[ = \int k_A \mu_0 \sqrt{\mu_0} \left| \frac{dM}{dt} \right| \frac{1 + \frac{dM}{dt}}{\kappa_d} \left| 1 + \frac{dH}{dM} \right|^{1.5} \left| 1 - \kappa_d \right| dt \]

\[ \approx \int k_A \mu_0 \sqrt{\mu_0} \left| \frac{dM_{irr}}{dt} \right| \left| dM_{irr} \right|, \]

the term \(| 1 + 1/\kappa_d |^{1.5}\) in (3.26) tends to one as explained above. If a ferromagnetic hysteresis is considered, then this condition is always satisfied (Fig. 3.8).

\[ 0 \quad 100 \quad 200 \quad 300 \quad 400 \quad 500 \]

\[ 1.0 \quad 1.2 \quad 1.4 \quad 1.6 \quad 1.8 \]

\[ \begin{array}{c}
\kappa \\
\mu_r
\end{array} \]

**Fig. 3.8.** The \( \kappa_d \) tends to one if the relative permeability is much higher than one

Substituting the energy terms (3.8), (3.9), (3.10), (3.25) and (3.26) into the energy balance
equation (3.21) the following relationship is established

\[ \oint \mu_0 \left( M_{irr} - M_{an} + k\delta \frac{dM_{irr}}{dH} + \mu_0 k_{EC} \frac{dM_{irr}}{dt} \frac{dM_{irr}}{dH} + k_A \sqrt{\mu_0} \frac{dM_{irr}}{dH} \right) \, dH = 0. \] (3.27)

Assuming that the magnetic field intensity varies periodically, the differential equation of the rate dependent Jiles-Atherton model of hysteresis with respect to \( \frac{dM_{irr}}{dH} \) has the form

\[ \frac{dM_{irr}}{dH} = \frac{M_{an} - M_{irr}}{k\delta + \mu_0 k_{EC} \frac{dM_{irr}}{dt} + k_A \sqrt{\mu_0} \left| \frac{dM_{irr}}{dt} \right|}. \] (3.28)

Two new terms have been added to (3.14) which are responsible for representing the eddy current and anomalous losses respectively. Furthermore two new parameters have been introduced to the Jiles-Atherton model, \( k_{EC} \) and \( k_A \), which depend on the material parameters (e.g. electrical conductivity) and the geometrical shape and dimensions of the conductive magnetic material. However, we can ignore them as the identification procedure developed (see in section 3.5 on page 35) requires no a priori knowledge to determine these parameters. They are determined automatically based on simple measurements.

The condition (3.16) must be fulfilled for (3.28) too, because of the same reasons as in the case of the rate independent model. Consequently, it is still necessary to introduce the parameter \( \delta_M \). The magnetic susceptibility can be derived from (3.28) using

\[ \frac{dM}{dH} = (1 - c) \delta_M \frac{dM_{irr}}{dH} + c \frac{dM_{an}}{dH}. \] (3.29)

### 3.3 Inverse Jiles-Atherton model

Both the rate-dependent and the rate independent hysteresis models described above are based on the magnetic field intensity \( H \) as an independent variable. However, when working with the magnetic vector potential formulation, the magnetic induction vector \( B \) is directly obtained and a hysteresis model based on \( B \) could be obtained directly as in Fig. 3.9. In this section I have introduced a method to obtain the inverse Jiles-Atherton model.

![Fig. 3.9. Hysteresis models with \( H \) and \( B \) input](image)

Both \( H \) based rate-dependent and rate-independent models can be transformed to \( B \) based
models in same way. The magnetic field intensity the output of the inverse model is determined from the direct model by using the *regula falsi method* [36]. The procedure of the inverse JAM is as follows:

1. The interval within the input $B$ is determined by stepping $\Delta H$ starting from the previous value of magnetic field intensity $H_{\text{prev}}$
2. The new value of magnetic field intensity is calculated by
   \[ H_{\text{new}} = H_{\text{left}} + \frac{H_{\text{right}} - H_{\text{left}}}{B_{\text{right}} - B_{\text{left}}}(B_{\text{input}} - B_{\text{left}}), \]
   where the coordinates of left and right points of the interval are $(H_{\text{left}}; B_{\text{left}})$ and $(H_{\text{right}}; B_{\text{right}})$.
3. The new value of magnetic flux density is computed by the direct model
   \[ B_{\text{new}} = \mathcal{H}\{H_{\text{new}}\}. \]
4. The left point of interval is moved to the coordinates $(H_{\text{new}}; B_{\text{new}})$ if $\delta(B_{\text{input}} - B_{\text{new}}) > 0$ or the right point of interval is moved to the coordinates $(H_{\text{new}}; B_{\text{new}})$ if $\delta(B_{\text{input}} - B_{\text{new}}) < 0$. So the interval become smaller and smaller in each iteration step.
5. The process is continued from 2) until the difference between $B_{\text{input}}$ and $B_{\text{new}}$ becomes sufficiently small. Than $H_{\text{new}}$ is the output of the inverse model.

Naturally (3.29) can be derived for obtaining $dM/dB$ directly and the above described regula falsi method can be eliminated, but due to numerical reasons the regula falsi method is faster despite of the required iteration procedure.

### 3.4 Hysteresis measurement in LabVIEW environment

This section deals with computer aided automated magnetic scalar hysteresis measurement of a specific ferromagnetic toroidal shape specimen. The measurement has been developed in LabVIEW environment using a National Instrument Data Acquisition Card. The measurement technique of symmetric minor loops and first order reversal curves are presented in this section. It is necessary to measure the magnetic hysteresis curves of ferromagnetic materials to describe the material from a magnetic point of view. The measured scalar hysteresis characteristics taking into account the nonlinearity of the material can be used in further simulations in numerical field computations [37–40].

A procedure of automated computer aided ferromagnetic hysteresis measurement is presented in this section. Within the frame of the measurement exercise, the computer is task with the management of all phases of the measurement process, e.g. controlling data acquisition cards, saving measured data to file and post processing of experimental data. The voltage of the measuring coil and the current of the excitation coil are measured and the magnetic field intensity $H$ and the magnetic flux density $B$ are calculated according to Ampere’s and
Faraday’s laws. The main advantage of digital measurement consists in measurement driving and post processing, as digitalized data can easily be manipulated. The scanning rate must be chosen carefully as the down-sampling analog signals are inaccurate. At the same time, in case of over-sampling the size of data files can become unnecessarily large.

The hysteresis curves, minor loops and reversal curves describe the magnetic properties of ferromagnetic materials. My examination focuses on a certain type of scalar hysteresis measurement of toroidal shape ferromagnetic steel. The measurement requires two coils (excitation and measurement) on the material under examination. A magnetic measurement system has been set up in our Magnetic Lab. The components of the measurement include a personal computer with measuring cards (Fig. 3.11), software and power supply (Fig. 3.10) the automated magnetic hysteresis measurement is controlled by a personal computer. The excitation signal can be generated by means of the computer and the required signals can also be measured. The applied KIKUSUI PBX 2020 bipolar power supply can amplify analog input signal ensuing the required power for excitation. The current of the primary coil is generated by the power supply in current controlled operation. The data acquisition and the generation of excitation current can be performed simultaneously, but some common problems may arise during the magnetic hysteresis measurement in the digital signal processing part e.g. the issue of correct sampling rate, noise and so on [41, 42].

The measuring arrangement of magnetic hysteresis can be seen in Fig. 3.13. The ferromagnetic material under investigation has a toroidal shape. The magnetic field inside the toroid (outer/inner diameters are 60/40 mm, height is 16 mm) is approximately uniform with the scalar value of \( H \). The magnetization \( M \) can be considered as a scalar valued quantity and it is parallel to \( H \).

The primary coil is controlled by the current of the Kikusui power supply. The secondary coil is used for measuring the induced voltage at the open circuit pick-up coil. The amplitude of excitation current can be measured as a voltage on the resistance \( R \). The value of resistance \( R \) is regarded to be constant, which is independent of the ambient temperature and the voltage.

The computer communicates with the measuring environment through NI-DAQ BNC - 2090 panel (Fig. 3.12), which is wired directly to the DAQ cards. PBX 2020 power supply is applied to generate the excitation for the magnetic specimen. This device can generate 20 V and 20 A bipolar signals with arbitrary signal-shapes. It can be controlled by voltage (CV mode) or current (CC mode) depending on the task. In the case of magnetic hysteresis loop
measurement the CC mode has been applied, because the current is proportional to the magnetic field intensity. The excitation signal is generated by LabVIEW. The LabVIEW generates analog output on DAQ (Data Acquisition) card through the card-driver.

DAQ measuring card is connected to the patch panel, which contains 16 analog input and 2 analog output channels. There are some digital channels as well, but these are not important here, because only the analog channels have been used for this measurement. One analog output channel controls the power supply and two analog input channels are used for measuring voltage and current of the coils. Measured symmetrical minor loops can be seen in Fig. 3.14. The characteristics were measured on C19 structural steel at 1 Hz.

The controlling of the measurement and the post-processing of measured data are carried out in LabVIEW environment. Magnetic flux density and magnetic field intensity are calculated from measured data during the post-processing phase with the following relationships

\[
H(t) = \frac{N_e \cdot i(t)}{l}, \quad (3.32)
\]

\[
B(t) = B_0 + \frac{1}{A \cdot N_m} \int_0^t u(\tau) d\tau, \quad (3.33)
\]

where \(l\) is the equivalent magnetic length of the toroidal shaped material, \(N_e\) and \(N_m\) are the number of turns of the excitation and the measuring coils respectively, \(i(t)\) is the excitation current, \(B_0 = B(t = 0)\) is integration constant, \(A\) is the cross-section of the material and \(u(t)\) is the voltage of the measuring coil.

The graphical user interface (GUI) of a magnetic hysteresis measurement software can be seen in Fig. 3.15. There are five plots in the figure. Four plots can be seen on the left. In this group, the calculated magnetic flux density (3.33) can be seen at the top left. The calculated magnetic field intensity (3.32) is plotted at the left bottom. All of the measured signals displayed at top right and the measured induced voltage can be observed after the noise compensation at bottom right. Finally the fifth plot shows the measured hysteresis. The most
important input parameters can be manipulated through the graphical user interface. These parameters are the amplitude and frequency of excitation, the scanning and sampling rates, the parameters of demagnetization process and reversal curves.

The most important part of the LabVIEW realization is the generation of excitation signals and the acquisition of measured signals. The LabVIEW programming means no command line programming. The software programming is based on virtual instruments (VI). The VIs can be placed and wired together depending on the task. The block scheme of the analog input ports can be seen in Fig. 3.16, which is responsible for data acquisition. Four VIs can be found in the analog input program Fig. 3.16. Data acquisition can be started (AI Start VI) after port configuration (AI Config VI). The "AI Read" is placed in a while loop and collects data from the ports as long as the loop is running. The acquisition is stopped as soon as an error occurs or the user clicks the stop button on the interface. AI Clear VI clears port configuration and the measurement stops. The block scheme of the analog output port generating excitation can be observed in Fig. 3.17. The operation of the port writing is very similar to the port reading detailed above.
3.4.1 Measurement of first order reversal curves

The first order reversal curves can also be measured by applying a special excitation signal shape. This type of curves is needed for the identification of some hysteresis models e.g. Preisach model. The required first order reversal curves can be obtained with the input function

\[
H(t) = H_s \left[ \frac{\alpha - 1}{2} + \frac{\alpha + 1}{2} \sin(\omega t + \pi/2) \right],
\]

where \(H_s\) is the magnetic field intensity in saturation state, \(\omega\) is the frequency of excitation, \(\alpha = k/n\) and \(k \in [-n, n]\) is an integer, and \(n\) denotes the number of reversal curves [13, 43]. Measurement results can be observed in Fig. 3.18. The measurement was performed at 1 Hz excitation frequency and \(n = 20\) reversal curves are plotted [37].

3.4.2 Sinusoidal \(B\)

As it was mentioned before our hysteresis measurement was driven by sinusoidal current, it means that the resulted magnetic field intensity \(H\) is sinusoidal. However there are some disadvantages of this configuration, namely that measured points are not distributed uniformly

**Fig. 3.16.** Realization of analog input reading from analog ports in LabVIEW

**Fig. 3.17.** Realization of analog output writing to analog ports in LabVIEW

**Fig. 3.18.** The measured first order reversal curves at \(f = 0.2\) Hz, \(n = 20\)
Fig. 3.19. Measured hysteresis loop with sinusoidal $H$. Non-equidistant points

Fig. 3.20. Measured hysteresis loop with sinusoidal $B$. The points are distributed more evenly like in the case of sinusoidal $H$

Fig. 3.21. Block scheme of nonlinear iteration for achieving sinusoidal $B$

along the hysteresis loop (Fig. 3.19). This means that a lot of points are located at the saturation part of the curve and few points are at the high slope part of the characteristics. This problem arises in hysteresis model identification. If $B$ is sinusoidal, the distribution of the points becomes favorable (Fig. 3.20). The sinusoidal shape of magnetic flux density can be generated by modifying the signal shape of the excitation current. A nonlinear iteration method has been applied in the measurement, which modifies the excitation through a feedback. The block scheme of the measurement can be seen in Fig. 3.21. The iteration starts with sinusoidal excitation and the differences between the prescribed reference sinusoidal signal $B_{\text{ref}}$ and the measured quantities are calculated ($\Delta B = B - B_{\text{ref}}$), according to the difference $\Delta B$ the signal shape of excitation is modified in each time step. The iteration stops when the maximum value of $\Delta B$ is small enough. The convergence of the method can be declared fast, approx. 10 cycles are needed for reaching less than 1% maximal difference between the sinusoidal reference signal $B_{\text{ref}}$ and the measured one [41].
3.5 Parameter identification of the Jiles-Atherton model

The parameters of JAM are determined by nonlinear least square method. The block scheme of the parameter identification can be seen in Fig. 3.23. The measured experimental data and initial parameters \( x_0 = [M_s, \alpha_0, a_0, c_0, k_{EC0}, k_{A0}] \) are given. The JAM generates a hysteresis curve based on the initial parameters. The experimental and modeled hysteresis is compared by a criterion function, which calculates the error. The nonlinear least square method modifies the input parameters of the JAM depending on the error. The cycle is running until the error becomes small enough. The convergence speed of the algorithm depends on the initial parameters. In general, the method requires approximately 50 steps for identifying the model parameters.

I have developed a method to facilitate the parameter identification procedure of JAM. The experimental data can be read from a file into the program as it can be seen at the right hand side in Fig. 3.22. The program immediately plots the measured curve in the figure entitled "Measured Data". After the initial parameter guessing, at the left hand side, parameter fitting can be started by pushing the "Run" button. Both the simulated and the experimental curves are plotted in the figure entitled "Approximated curve" and the difference between them is also plotted in the "Error" figure. The program is able to identify the parameters of both the
rate-independent and the rate-dependent JAM.

3.5.1 Initial test

The parameter identification program is verified by a JAM generated hysteresis loop. Known parameters are used in this test shown in Table 3.1 as the first column, then they are considered as measured data to be fitted by the identification program. The initial values can be seen in the second column of Table 3.1. The third column contains the result of the identification. The Figs. 3.24 and 3.25 shows the result of the initial test checking the efficiency of the parameter identification program.

Table 3.1. Initial test of the JAM parameter identification

<table>
<thead>
<tr>
<th></th>
<th>Original parameters</th>
<th>Initial parameters</th>
<th>Fitted parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_s$</td>
<td>$1.5 \cdot 10^6$</td>
<td>$2.1 \cdot 10^6$</td>
<td>$1.51 \cdot 10^6$</td>
</tr>
<tr>
<td>$a$</td>
<td>$10^{-6}$</td>
<td>$1.5 \cdot 10^{-6}$</td>
<td>$1.01 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1500</td>
<td>2000</td>
<td>1531.44</td>
</tr>
<tr>
<td>$c$</td>
<td>0.1</td>
<td>0.5</td>
<td>0.09688</td>
</tr>
<tr>
<td>$k$</td>
<td>1000</td>
<td>1200</td>
<td>989.26</td>
</tr>
</tbody>
</table>
3.5.2 Fitting to measured curves at 0.2 Hz

The program was used to extract fitted parameters from a measured hysteresis curve obtained from measurements of C19 structural steel ring core (outer/inner diameters are 60/40 mm, and the height is 16 mm). The applied measurement system consists of a PC with data acquisition A/D and D/A cards [37]. The measurements can be regarded as DC measurements at this frequency. The effect of eddy currents and other additional losses can be neglected. The Figs. 3.26 and 3.27 show the result of the parameter fitting, 'o' denotes some measured points and the simulated hysteresis curves are plotted by solid lines. The following parameters are resulted $M_s = 1.33 \cdot 10^6$, $\alpha = 4.7 \cdot 10^{-4}$, $a = 326.01$, $c = 0.00565$, $k = 468.985$. The relative error can also be observed in Figs. 3.28 and 3.29. The relative error is less than 10% everywhere it means the simulated and measured curves are good agreement.

3.5.3 Fitting to measured curves at 1 Hz

The measurement is performed on same material as before. Only the excitation frequency is different, 1 Hz. This measurement results can be modeled with the rate-independent JAM. The
same material is measured therefore the same model parameters has been applied like before. The results can be seen in Figs. 3.30, 3.31, 3.32 and 3.33.

**Fig. 3.30.** Result of parameter identification of quasi static J-A model at 1 Hz. The fitted curve is represented by solid line, and some points of the measured curve with 'o'.

**Fig. 3.31.** Result of parameter identification of quasi static J-A model at 1 Hz on a symmetrical minor loop. Fitted curve is plotted solid line, and some points of the measured curve with 'o'.

**Fig. 3.32.** Relative error between the simulated and measured hysteresis loops

**Fig. 3.33.** Relative error between the simulated and measured hysteresis loops
3.5.4 Fitting to measured curves at 5 Hz

The cross-section area of the measured ring is relatively large and not laminated, therefore the eddy current plays a major role at this frequency. In this case the experimental curve can not be modeled by the rate-independent JAM as it can be observed in Fig. 3.34. The extended JAM (see in section 3.2) must be introduced taking into account the effect of eddy currents and the domain wall motion. The extended model has been identified in the same way as the classical one. The resulting additional parameter values are $k_{EC} = 1.8 \cdot 10^{-6}$ and $k_A = 0.01$ and the rate-dependent JAM curve can be seen in Fig. 3.35. The top and the bottom of the curve is not fitted exactly because of the $\delta_M$ parameter, but simulation without this parameter leads to a nonphysical solution. That is the slope of the hysteresis curve at the peaks becomes negative so it is not recommended to use. For instance, using the model with negative slope may lead to numerical instability in field computation depending on the applied numerical iteration technique. The approximation with the extended model (Fig. 3.35.) is much better than the quasi static approach (Fig. 3.34) but the accuracy of the fitting is worse than in the case of lower frequency simulations (Figs. 3.26-3.33).

![Fig. 3.34. Simulation of the measured hysteresis at 5Hz with quasi static J-A model](image1)

![Fig. 3.35. Simulation of the measured hysteresis at 5Hz with extended J-A model](image2)
3.6 Thesis 1

I have extended the JAM of hysteresis to avoid the non-physical solutions and to represent the rate-dependent variation of the response with the input for the direct H-B and for the inverse B-H characteristics as well.

a) I have developed an engineering approach based on an energy balance equation for obtaining the differential equation of the rate-independent Jiles-Atherton model. I have introduced a parameter $\delta_M$ to avoid the non-physical solution (negative slope) of the model differential equation. I have validated the developed rate-independent model by experiments.

b) I have extended the energy balance equation of the rate-independent Jiles-Atherton model by two additional energy (eddy current and anomalous loss) terms to take into account the time variation of excitation magnetic field. I have validated the developed rate-dependent model by experiments.

c) I have built a magnetic measurement system for measuring magnetic hysteresis in LabVIEW environment. I have proposed measurement method for measuring hysteresis loops, minor loops, first order reversal curves and I have worked out an under-relaxation method to obtain sinusoidal waveform in $B$.

d) I have worked out a procedure for parameter identification of both the rate-dependent and rate independent Jiles-Atherton model of hysteresis based on the nonlinear least square method.
Chapter 4

Hysteresis operator in higher order edge elements

A method for the inclusion of the hysteresis operator into the electromagnetic field equations is presented in this chapter. The electromagnetic (EM) field equations are discretized by finite element method (FEM). After the discretization, a nonlinear system of equations is obtained, where both the stiffness matrix and the left-hand-side (LHS) vector depend on the permeability (or reluctivity), which is a nonlinear function of the magnetic field intensity (or the magnetic flux density) due to the hysteresis. This is called material nonlinearity. The material nonlinearity can be handled by many nonlinear iteration methods. The Newton method is one of the most famous one due to its fast convergence speed, just a few iteration steps are required to reach the correct solution. In the case of FEM, the Newton method has several drawbacks. Namely the stiffness matrix must be reassembled in each iteration steps, which is a time consuming task. Instead of the Newton method I apply a different method to solve the nonlinear problem. The polarization or fixed-point (FP) method is discussed. The polarization method determines a permeability $\mu_{FP}$ (or reluctivity $\nu_{FP}$) similar to the bias point in nonlinear circuit theory and these values are constant for all of the iterations. The constant permeability (or reluctivity) allows to assemble the stiffness matrix once and for all and only the LHS excitation vector is varied during the iteration process. So, reassembling can be avoided by using the polarization method. However the convergence speed of the method is very far from the Newton method as the contraction factor of the iteration is very close to one. Therefore some technique is required to speed up the convergence. Several methods can be found in the literature [31,44,45], but I propose a different one, which is more efficient than the previous ones making the field computation procedure faster.

4.1 Formalisms

In this section formalisms are introduced with the help of the weighted residual method for magnetostatic and eddy current fields. The relationship between the magnetic field intensity and the magnetic flux density is represented by by the operators $\mathcal{H}$ and $\mathcal{H}^{-1}$. The polarization method for handling the nonlinearity of these operators are introduced in section 4.2.
4.1.1 Magnetostatics with magnetic vector potential

In magnetostatic case the quantities of magnetic field \((H, B, J_e)\) are constant in time \((\partial/\partial t = 0)\) and the magnetic field is generated by currents \((J_e)\) and permanent magnets.

Let \(\Omega\) be a domain and \(\partial \Omega\) its boundary (Fig. 4.1). The boundary \(\partial \Omega\) is divided into two distinct parts \(\Gamma_B\) and \(\Gamma_H\) (\(\partial \Omega = \Gamma_B + \Gamma_H\)). Surface currents \(n \times H = K\) can be prescribed on \(\Gamma_H\) and the normal component the magnetic flux density \(n \cdot B = -b\) can be prescribed on \(\Gamma_B\).

The Maxwell’s equations for magnetostatics in the region \(\Omega\) have the form

\[
\begin{align*}
\operatorname{curl} H &= J_e, & \text{in } \Omega \\
\operatorname{div} B &= 0, & \text{in } \Omega \\
H &= \mathcal{H}^{-1}\{B\}, & \text{in } \Omega \\
n \cdot B &= -b, & \text{on } \Gamma_B \\
n \times H &= K, & \text{on } \Gamma_H
\end{align*}
\]  

(4.1)

where the equation in the third row is the constitutive equations of the magnetic field and \(J_e\) is the current density generated by the external DC solenoidal currents. The operator \(\mathcal{H}^{-1}\{\}\) represents the linear, nonlinear or hysteretic relationship between the magnetic field intensity \(H\) and the magnetic flux density \(B\). The boundary conditions can be prescribed in the last two rows, where \(K\) denotes surface current.

![Fig. 4.1. Magnetostatic problem](image)

Substituting the constitutive relationship into the first equation of (4.1) the following equation is resulted

\[
\operatorname{curl} \mathcal{H}^{-1}\{B\} = J_e.
\]  

(4.2)

By introducing the magnetic vector potential \(B = \operatorname{curl} A\) the relationship above has the form

\[
\operatorname{curl} \mathcal{H}^{-1}\{\operatorname{curl} A\} = J_e.
\]  

(4.3)
The equation above can be represented by the weighted residual

\[
\int_{\Omega} \tilde{A} \cdot \text{curl} \mathcal{H}^{-1} \{\text{curl} A\} \, d\Omega = \int_{\Omega} \tilde{A} \cdot J_e \, d\Omega,
\]

(4.4)

where \(\tilde{A}\) is a weighting function in \(\Omega\). If \(A^* = A + \text{grad} \phi\) is substituted into (4.4), where \(\phi\) is a scalar function, then same equation is obtained since \(\text{curl} \text{grad} \phi \equiv 0\). Therefore the solution of (4.4) is not unique, but the magnetic flux density is always unique \(B = \text{curl} A = \text{curl} A^*\).

The resulting linear system of equations after the discretization and linearization of (4.4) is singular, because \(\text{div} A\) is not fixed. Despite of the singularity of the system it can be solved by Krylov subspace solvers (e.g. CG, BiCG) if the system is consistent [46, 47].

On the other hand, the Coulomb gauge \(\text{div} A = 0\) can be ensured by the Lagrange multiplier method [36] as

\[
\int_{\Omega} \tilde{A} \cdot \text{curl} \mathcal{H}^{-1} \{\text{curl} A\} \, d\Omega + \int_{\Omega} \lambda \text{div} \tilde{A} \, d\Omega = \int_{\Omega} \tilde{A} \cdot J_e \, d\Omega,
\]

\[
\int_{\Omega} \tilde{\lambda} \text{div} A \, d\Omega = 0,
\]

(4.5)

where the Lagrange multipliers and its weighting functions are denoted by \(\lambda\) and \(\tilde{\lambda}\). Using the mathematical identities

\[
\text{div} \left( \tilde{A} \times \mathcal{H}^{-1} \{\text{curl} A\} \right) = \mathcal{H}^{-1} \{\text{curl} A\} \cdot \text{curl} \tilde{A} - \tilde{A} \cdot \text{curl} \mathcal{H}^{-1} \{\text{curl} A\},
\]

\[
\text{div} \left( \lambda \tilde{A} \right) = \tilde{A} \text{grad} \lambda + \lambda \text{div} \tilde{A},
\]

(4.6)

the weak form of (4.5) can be written as

\[
\int_{\Omega} \mathcal{H}^{-1} \{\text{curl} A\} \cdot \text{curl} \tilde{A} \, d\Omega - \oint_{\partial\Omega = \Gamma_B + \Gamma_H} \tilde{A} \cdot \mathcal{H}^{-1} \{\text{curl} A\} \times n \, d\Gamma
\]

\[
- \int_{\Omega} \tilde{A} \cdot \text{grad} \lambda \, d\Omega + \oint_{\partial\Omega = \Gamma_B + \Gamma_H} \lambda \tilde{A} n \, d\Gamma = \int_{\Omega} \tilde{A} \cdot J_e \, d\Omega,
\]

\[
- \int_{\Omega} \text{grad} \tilde{\lambda} \cdot A \, d\Omega + \oint_{\partial\Omega = \Gamma_B + \Gamma_H} A \cdot \tilde{\lambda} n \, d\Gamma = 0.
\]

(4.7)

Usually the integrals on the boundaries can be neglected. The boundary conditions in (4.1) can be fulfilled by

\[
\int_{\Gamma_B} \tilde{b} \cdot (n \cdot B + b) \, d\Gamma = 0 \quad \rightarrow \quad \int_{\Gamma_B} \tilde{b} \cdot (n \cdot \text{curl} A + b) \, d\Gamma = 0,
\]

\[
\int_{\Gamma_H} \tilde{H} \cdot (n \times H - K) \, d\Gamma = 0 \quad \rightarrow \quad \int_{\Gamma_H} \tilde{H} \cdot (n \times \mathcal{H}^{-1} \{\text{curl} A\} - K) \, d\Gamma = 0,
\]

(4.8)

where \(\tilde{b}\) and \(\tilde{H}\) are scalar and vector weighting functions on \(\Gamma_B\) and \(\Gamma_H\) respectively.
4.1.2 Eddy currents with magnetic vector and electric scalar potentials

In case of eddy currents the quantities of electromagnetic field \((E, H, B, J_e)\) vary in time \((\partial/\partial t \neq 0)\), but the displacement current \(\partial D/\partial t\) is neglected.

Let \(\Omega\) be a domain and \(\partial \Omega\) its boundary. Boundary conditions can be prescribed for the magnetic field and the electric fields separately. The boundary for the magnetic field \((\partial \Omega = \Gamma_B + \Gamma_H)\) is drawn by blue in Fig. 4.2 and the boundary for the electric field \((\partial \Omega = \Gamma_V + \Gamma_J)\) is drawn by red. The electric scalar potential (introduced later) can be prescribed on \(\Gamma_V\), where the tangential component of electric field intensity is prescribed as zero. The Maxwell’s equations for eddy currents in the region \(\Omega\) and the boundary conditions have the form

\[
\begin{align*}
\text{curl } H &= J_e + \sigma E, \quad \text{in } \Omega \\
\text{curl } E &= -\frac{\partial B}{\partial t}, \quad \text{in } \Omega \\
\text{div } B &= 0, \quad \text{in } \Omega \\
\text{div } (J_e + \sigma E) &= 0, \quad \text{in } \Omega \\
H &= \mathcal{H}^{-1}\{B\}, \quad \text{in } \Omega \\
n \cdot B &= -b, \quad \text{on } \Gamma_B \\
n \times H &= K, \quad \text{on } \Gamma_H \\
n \times E &= 0, \quad \text{on } \Gamma_V \\
n \cdot J &= -j, \quad \text{on } \Gamma_J
\end{align*}
\]

(4.9)

where \(E\) is the electric field intensity and \(J = J_e + \sigma E\). The boundary conditions can be prescribed in the last four rows. The \(A-V\) formalism with respect to the eddy currents field

\[\text{curl } H = J_e + \sigma E, \quad \text{in } \Omega\]

\[\text{curl } E = -\frac{\partial B}{\partial t}, \quad \text{in } \Omega\]

\[\text{div } B = 0, \quad \text{in } \Omega\]

\[\text{div } (J_e + \sigma E) = 0, \quad \text{in } \Omega\]

\[H = \mathcal{H}^{-1}\{B\}, \quad \text{in } \Omega\]

\[n \cdot B = -b, \quad \text{on } \Gamma_B\]

\[n \times H = K, \quad \text{on } \Gamma_H\]

\[n \times E = 0, \quad \text{on } \Gamma_V\]

\[n \cdot J = -j, \quad \text{on } \Gamma_J\]

\[\partial \Omega = \Gamma_B + \Gamma_H \text{ or } \partial \Omega = \Gamma_J + \Gamma_V\]

\[V = V_o\]

\[\text{Fig. 4.2. Eddy current problem with voltage prescription}\]

\[\text{can be derived by introducing the magnetic vector and electric scalar potentials } (B = \text{curl } A \text{ and } E = -\text{grad } V - \frac{\partial A}{\partial t}) \text{ from the third and fourth equations of (4.9). The resulting eddy}\]
current equations are

\[
\text{curl} \mathcal{H}^{-1} \{\text{curl} \, \mathbf{A}\} = \mathbf{J}_e - \sigma \left( \text{grad} \, V + \frac{\partial \mathbf{A}}{\partial t} \right),
\]

\[
\text{div} \left( \mathbf{J}_e - \sigma \text{grad} \, V - \sigma \frac{\partial \mathbf{A}}{\partial t} \right) = 0,
\]

(4.10)

where \( \mathbf{A} \) and \( V \) denote the magnetic vector and electric scalar potentials. The weighted residual of (4.10) can be derived by multiplying scalar and vector weighting functions \( \tilde{V} \), \( \tilde{A} \) respectively and integrating over the region \( \Omega \)

\[
\int_{\Omega} \tilde{A} \cdot \text{curl} \mathcal{H}^{-1} \{\text{curl} \, \mathbf{A}\} \, d\Omega + \int_{\Omega} \tilde{A} \cdot \sigma \left( \text{grad} \, V + \frac{\partial \mathbf{A}}{\partial t} \right) \, d\Omega = \int_{\Omega} \tilde{A} \cdot \mathbf{J}_e \, d\Omega,
\]

\[
\int_{\Omega} \tilde{V} \cdot \text{div} \left( \mathbf{J}_e - \sigma \text{grad} \, V - \sigma \frac{\partial \mathbf{A}}{\partial t} \right) \, d\Omega = 0.
\]

(4.11)

If \( \mathbf{A}^* = \mathbf{A} + \text{grad} \, \phi \) and \( V^* = V - \partial \phi / \partial t \) are substituted into (4.11), then same equations are obtained since

\[
\text{grad} \, V^* + \frac{\partial \mathbf{A}^*}{\partial t} = \text{grad} \left( V - \frac{\partial \phi}{\partial t} \right) + \frac{\partial (\mathbf{A} + \text{grad} \, \phi)}{\partial t} = \text{grad} \, V + \frac{\partial \mathbf{A}}{\partial t},
\]

(4.12)

and \( \text{curl} \, \mathbf{A}^* = \text{curl} \, \mathbf{A} \). Therefore the solution of (4.11) is not unique, but the magnetic flux density \( \mathbf{B} = \text{curl} \, \mathbf{A}^* = \text{curl} \, \mathbf{A} \) and the electric field intensity \( \mathbf{E} = \text{grad} \, V^* + \partial \mathbf{A}^*/\partial t = \text{grad} \, V + \partial \mathbf{A}/\partial t \) are unique. The resulting linear system of equations after the discretization and linearization of (4.11) is singular, because \( \text{div} \, \mathbf{A} \) is not fixed. Despite of the singularity of the system it can be solved by Krylov subspace solvers (e.g. CG, BiCG) if the system is consistent [46, 47]. On the other hand, the Coulomb gauge \( \text{div} \, \mathbf{A} = 0 \) can be ensured by the Lagrange multiplier method [36] as

\[
\int_{\Omega} \tilde{A} \cdot \text{curl} \mathcal{H}^{-1} \{\text{curl} \, \mathbf{A}\} \, d\Omega + \int_{\Omega} \tilde{A} \cdot \sigma \left( \text{grad} \, V + \frac{\partial \mathbf{A}}{\partial t} \right) \, d\Omega + \int_{\Omega} \lambda \text{div} \tilde{A} \, d\Omega
\]

\[
= \int_{\Omega} \tilde{A} \cdot \mathbf{J}_e \, d\Omega,
\]

(4.13)

\[
\int_{\Omega} \tilde{V} \cdot \text{div} \left( \mathbf{J}_e - \sigma \text{grad} \, V - \sigma \frac{\partial \mathbf{A}}{\partial t} \right) \, d\Omega = 0,
\]

\[
\int_{\Omega} \lambda \text{div} \mathbf{A} \, d\Omega = 0.
\]

Using the mathematical identities (4.6) and

\[
\text{div} \left( \tilde{V} \cdot \text{grad} \, V \right) = \text{grad} \, V \cdot \text{grad} \, \tilde{V} + \tilde{V} \cdot \text{div} \, \text{grad} \, V,
\]

(4.14)
the weak form of (4.13) can be written as

\[
\int_\Omega \text{curl} \tilde{A} \cdot \mathcal{K}^{-1} \{\text{curl} A\} \, d\Omega - \oint_{\partial\Omega = \Gamma_B + \Gamma_H} \tilde{A} \cdot \mathcal{K}^{-1} \{\text{curl} A\} \times n \, d\Gamma \\
+ \int_\Omega \tilde{A} \cdot \sigma \left( \text{grad} V + \frac{\partial A}{\partial t} \right) \, d\Omega - \int_\Omega \tilde{A} \cdot \text{grad} \lambda \, d\Omega - \oint_{\partial\Omega = \Gamma_B + \Gamma_H} \lambda \tilde{A} n \, d\Gamma \\
= \int_\Omega \tilde{A} \cdot J_e \, d\Omega, \\
- \int_\Omega \sigma \text{grad} \tilde{V} \cdot \text{grad} V \, d\Omega - \oint_{\partial\Omega = \Gamma_E + \Gamma_J} \sigma \tilde{V} \cdot \text{grad} V n \, d\Gamma + \int_\Omega \tilde{V} \cdot \text{div} \left( J_e - \sigma \frac{\partial A}{\partial t} \right) \, d\Omega = 0, \\
- \int_\Omega \text{grad} \tilde{\lambda} \cdot A \, d\Omega - \oint_{\partial\Omega = \Gamma_B + \Gamma_H} A \cdot \tilde{\lambda} n \, d\Gamma = 0.
\]

(4.15)

Usually the integrals on the boundaries can be neglected. The boundary conditions in (4.9) can be fulfilled by

\[
\int_{\Gamma_B} \tilde{b} \cdot (n \cdot B + b) \, d\Gamma = 0 \quad \rightarrow \quad \int_{\Gamma_B} \tilde{b} \cdot (n \cdot \text{curl} A + b) \, d\Gamma = 0, \\
\int_{\Gamma_H} \tilde{H} \cdot (n \times H - K) \, d\Gamma = 0 \quad \rightarrow \quad \int_{\Gamma_H} \tilde{H} \cdot (n \times \mathcal{K}^{-1} \{\text{curl} A\} - K) \, d\Gamma = 0, \\
\int_{\Gamma_J} \tilde{j} \cdot (n \cdot J + j) \, d\Gamma = 0 \quad \rightarrow \quad \int_{\Gamma_J} \tilde{j} \cdot \left[ -\sigma n \cdot \left( \text{grad} V + \frac{\partial A}{\partial t} \right) + j \right] \, d\Gamma = 0, \\
\int_{\Gamma_E} \tilde{E} \cdot (n \times E) \, d\Gamma = 0 \quad \rightarrow \quad -\int_{\Gamma_E} \tilde{E} \cdot n \times \left( \text{grad} V + \frac{\partial A}{\partial t} \right) \, d\Gamma = 0,
\]

(4.16)

where \( \tilde{b}, \tilde{H}, \tilde{j} \) and \( \tilde{E} \) are scalar and vector weighting functions on \( \Gamma_B, \Gamma_H, \Gamma_J \) and \( \Gamma_E \) respectively.

4.1.3 Eddy currents with magnetic vector potential

Eddy current fields can be described by the magnetic vector potential \( A \) formalism without the electric scalar potential \( V \), if no voltage is prescribed in the arrangement (e.g. voltage of coil). In this case no extra electric field is involved (\(-\text{grad} V\)) since no voltage is prescribed only the time varying magnetic field induces electric field \( E = -\partial A/\partial t \). The arrangement for the vector potential formalism can be seen in Fig. 4.3. The governing equations of the \( A \) formalism are the same as the \( A-V \) formalism (4.9) without the last two boundary conditions.
The eddy current equation with respect to the magnetic vector potential can be formed as

\[ \text{curl} \mathcal{K}^{-1} \{ \text{curl} \mathbf{A} \} + \sigma \frac{\partial \mathbf{A}}{\partial t} = \mathbf{J}_e, \]  

(4.17)

based on (4.10). The weighted residual of (4.17) can be derived by multiplying the vector weighting function \( \tilde{\mathbf{A}} \) and integrating over the region \( \Omega \)

\[ \int_{\Omega} \mathbf{A} \cdot \text{curl} \mathcal{K}^{-1} \{ \text{curl} \mathbf{A} \} d\Omega + \int_{\Omega} \sigma \mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial t} d\Omega = \int_{\Omega} \tilde{\mathbf{A}} \cdot \mathbf{J}_e d\Omega. \]  

(4.18)

The solution of (4.18) is unique. No gauge fixing is required since by substituting \( \mathbf{A}^* = \mathbf{A} + \text{grad} \phi \) into (4.18) a different residual is resulted

\[ \int_{\Omega} \tilde{\mathbf{A}} \cdot \text{curl} \mathcal{K}^{-1} \{ \text{curl} \mathbf{A} \} d\Omega + \int_{\Omega} \sigma \frac{\partial \mathbf{A} + \text{grad} \phi}{\partial t} d\Omega = \int_{\Omega} \tilde{\mathbf{A}} \cdot \mathbf{J}_e d\Omega. \]  

(4.19)

The weak form of (4.18) can be derived by using mathematical identities (4.6)

\[ \int_{\Omega} \text{curl} \tilde{\mathbf{A}} \cdot \mathcal{K}^{-1} \{ \text{curl} \mathbf{A} \} d\Omega - \int_{\partial \Omega = \Gamma_h + \Gamma_H} \tilde{\mathbf{A}} \cdot \mathcal{K}^{-1} \{ \text{curl} \mathbf{A} \} \times \mathbf{n} d\Gamma \]

\[ + \int_{\Omega} \tilde{\mathbf{A}} \cdot \frac{\partial \mathbf{A}}{\partial t} d\Omega = \int_{\Omega} \tilde{\mathbf{A}} \cdot \mathbf{J}_e d\Omega. \]  

(4.20)

Usually the integral on the boundaries can be neglected. The boundary conditions can be
fulfilled by
\[
\int_{\Gamma_B} \tilde{b} \cdot (\mathbf{n} \cdot \mathbf{B} + b) d\Gamma = 0 \quad \rightarrow \quad \int_{\Gamma_B} \tilde{b} \cdot (\mathbf{n} \cdot \nabla \mathbf{A} + b) d\Gamma = 0,
\]
\[
\int_{\Gamma_H} \tilde{H} \cdot (\mathbf{n} \times \mathbf{H} - \mathbf{K}) d\Gamma = 0 \quad \rightarrow \quad \int_{\Gamma_H} \tilde{H} \cdot (\mathbf{n} \times \mathcal{H}^{-1} \{\nabla \mathbf{A}\} - \mathbf{K}) d\Gamma = 0,
\]
where \(\tilde{b}\) and \(\tilde{H}\) are scalar and vector weighting functions on \(\Gamma_B\) and \(\Gamma_H\) respectively.

4.2 Polarization method

I derived the formalisms with respect to the magnetostatic and eddy current fields in (4.7), (4.15) and (4.20) for general material nonlinearity between \(\mathbf{H}\) and \(\mathbf{B}\) by using the operator \(\mathcal{H}^{-1}\). I handle this material nonlinearity with the polarization method, which is detailed in this section.

A fixed-point technique was proposed for the first time in electrical engineering for solving nonlinear electric circuits. The method based on Picard-Banach theorem requires a contractive operator of the FP mapping. Its main disadvantage was slow convergence. An improved version was proposed for stationary magnetic field problems taking advantage of the magnetic polarization vector. Its convergence was improved combining the method with a relaxation procedure. A great benefit of contractive mapping is that uniqueness and existence of a solution are guaranteed by Banach’s theorem [48–51].

However, it seems impossible to formulate a FP mapping, which is contractive in general case for different problems. This is why it is important to look for a numerical algorithm, which also enables to find a FP for non-contractive operators. Such a procedure is presented here based on Brouwer’s FP theorem referred as the material nonlinear reluctivity. The FP procedure based on Brouwer’s theorem was successfully applied to solve systems of nonlinear and ill-posed equations arising in identification problems.

The polarization method is becoming more and more attractive in FEM analysis due to its advantages:

- it is robust since it is proved to be convergent with any kind of monotonic, Lipschitz continuous, nonlinear relationship, also in the presence of inflection points;
- preliminary operations, in order to define a suitable starting value of residual term, are not required because the process is proved to be convergent for any trial value;
- at each iteration the updates are performed only on the right-hand side of the linearized system, without modifications of the stiffness matrix which can be computed once and for all;
- it requires no constraint on the smoothness of the magnetic characteristic so a simple piecewise linear representation can be assumed.
4.2.1 Banach fixed point theorem

The Banach fixed point theorem (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892-1945), and was first stated by Banach in 1922.

Contraction mapping

In mathematics, a contraction mapping, or contraction, on a metric space \((M, d)\) is a function \(f\) from \(M\) to itself, with the property that there is some real number \(0 < q < 1\) such that, for all \(x\) and \(y\) in \(M\),

\[
d(f(x), f(y)) \leq q d(x, y).
\]

(4.22)

The smallest such value of \(q\) is called the Lipschitz constant of \(f\). Contractive maps are sometimes called Lipschitzian maps. If the above condition is satisfied for \(0 < q \leq 1\), then the mapping is said to be non-expansive.

More generally, the idea of a contractive mapping can be defined for maps between metric spaces. Thus, if \((M, d)\) and \((N, g)\) are two metric spaces, and \(f : M \to N\), then one looks for the constant \(q\) such that \(g(f(x), f(y)) \leq q d(x, y)\) for all \(x\) and \(y\) in \(M\).

Every contraction mapping is Lipschitz continuous and hence uniformly continuous.

A contraction mapping has at most one fixed point. Moreover, the Banach fixed point theorem states that every contraction mapping on a nonempty complete metric space has a unique fixed point, and that for any \(x\) in \(M\) the iterated function sequence \(x, f(x), f(f(x)), f(f(f(x))), \ldots\) converges to the fixed point. This concept is very useful for iterated function systems where contraction mappings are often used. The Banach fixed point theorem is also applied in proving the existence of solutions of ordinary differential equations.

The theorem

Let \((X, d)\) be a non-empty complete metric space. Let \(T : X \to X\) be a contraction mapping on \(X\), i.e: there is a nonnegative real number \(q < 1\) such that:

\[
d(Tx, Ty) \leq q \cdot d(x, y)
\]

(4.23)

for all \(x, y\) in \(X\). Then the map \(T\) admits one and only one fixed point \(x^*\) in \(X\) (this means \(Tx^* = x^*\)). Furthermore, this fixed point can be found as follows: start with an arbitrary element \(x_0\) in \(X\) and define an iterative sequence by \(x_n = Tx_{n-1}\) for \(n = 1, 2, 3, \ldots\). This sequence converges, and its limit is \(x^*\). The following inequality describes the speed of convergence:

\[
d(x^*, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0).
\]

(4.24)
Equivalently,

\[ d(x^*, x_{n+1}) \leq \frac{q}{1-q}d(x_{n+1}, x_n) \]  \hspace{1cm} (4.25)

and

\[ d(x^*, x_{n+1}) \leq qd(x_n, x^*) \]  \hspace{1cm} (4.26)

The smallest such value of \( q \) is sometimes called the \textit{Lipschitz constant}.

Note that the requirement \( d(Tx, Ty) < d(x, y) \) for all unequal \( x \) and \( y \) is in general not enough to ensure the existence of a fixed point, as is shown by the map \( T : [1, \infty) \rightarrow [1, \infty) \) with \( T(x) = x + 1/x \), which lacks a fixed point. However, if the space \( X \) is compact, then this weaker assumption does imply all the statements of the theorem.

When using the theorem in practice, the most difficult part is typically to define \( X \) properly so that \( T \) actually maps elements from \( X \) to \( X \), i.e. that \( Tx \) is always an element of \( X \).

### 4.2.2 Polarization method for inverse characteristics

The formalisms for eddy current fields and magnetostatics are summarized in the sections 4.1.1, 4.1.2 and 4.1.3 in general since the nonlinear relationship between \( B \) and \( H \) is represented by the inverse hysteresis operator \( \mathcal{H}^{-1} \{ \} \) and the way how nonlinearity should be considered has not been discussed. In this section I summarize the polarization method for handling this material nonlinearity [52]. The linearization of constitutive relation \( H = \mathcal{H}^{-1} \{ B \} \) can be formulated by the polarization method as

\[ H = \mathcal{H}^{-1} \{ B \} = \nu_{FP} B + R, \]  \hspace{1cm} (4.27)

which includes linear term \( \nu_{FP} B \), where \( \nu_{FP} \) being a constant reluctivity, and the residual nonlinearity \( R \). Common notation can be applied for the weak forms (4.7), (4.15) and (4.20) by substituting the relation (4.27) into them

\[ B = \phi_2(R), \]  \hspace{1cm} (4.28)

where the operator \( \phi_2 \) represents the solution of differential equation with respect to the input vector \( R \). The residual vector according to the fixed-point method can be computed by

\[ R = \mathcal{H}^{-1} \{ B \} - \nu_{FP} B \rightarrow R = \phi_1(B), \]  \hspace{1cm} (4.29)

where the operator \( \phi_1 \) represents the constitutive relation with respect to the input quantity.

The nonlinear EM field equation receives the residual vector \( R \) from the previous iteration or from the trial value as an input vector and it provides the magnetic flux density by solving the linearized equations as an output. This relation is denoted by \( B = \phi_2(R) \). On the other hand (4.29) for \( R \) receives \( B \) as an input and it provides a new residual value as output. It is
denoted by $R = \phi_1(B)$. Utilizing the benefits of this operator notation these two steps can be written as one

$$R = \phi_1(\phi_2(R)), \quad (4.30)$$

which is the fixed-point equation of magnetostatics. If the function $\phi_1$ is a contraction, then (4.30) can be solved by substituting back the residual, starting from any trial value of $R$. Although this procedure is straightforward and very robust, but unfortunately the convergence is slow.

The choice of $\nu_{FP}$ is essential from the convergence point of view. It is shown that an incorrect choice of $\nu_{FP}$ may lead to divergent iteration, since the criterion for contraction is not fulfilled. A criterion is discussed below which must be fulfilled for the contractive mapping.

The material characteristics $\mathcal{H}^{-1}\{\}$ can not be arbitrary. Two conditions must be fulfilled for ensuring the contraction. The operator $\mathcal{H}^{-1}\{\}$ must be Lipschitzian,

$$|\mathcal{H}^{-1}\{B'\} - \mathcal{H}^{-1}\{B''\}| < \Lambda(P) \left|B' - B''\right|, \quad \forall(B', B''), \quad (4.31)$$

and uniformly monotone

$$(\mathcal{H}^{-1}\{B'\} - \mathcal{H}^{-1}\{B''\}) (B' - B'') > \lambda(P) \left(B' - B''\right)^2, \quad \forall(B', B''), \quad (4.32)$$

where $P$ denotes any point in the region $\Omega$. The maximum and the minimum values of $\Lambda(P)$ and $\lambda(P)$ can be determined and denoted by

$$\Lambda(P) = \max_{B', B''} \frac{|\mathcal{H}^{-1}\{B'\} - \mathcal{H}^{-1}\{B''\}|}{|B' - B''|} = \nu_{\text{max}}(P),$$

$$\lambda(P) = \min_{B', B''} \frac{|\mathcal{H}^{-1}\{B'\} - \mathcal{H}^{-1}\{B''\}|}{|B' - B''|} = \nu_{\text{min}}(P), \quad (4.33)$$

where $\nu_{\text{max}}(P), \nu_{\text{min}}(P)$ are the maximum and minimum values of the local reluctivities. According to the contraction mapping theorem (4.22) the fixed-point of (4.30) must be approached

$$||\phi_1(\phi_2(R')) - \phi_1(\phi_2(R''))||_\mu \leq q||R' - R''||_\mu, \quad 0 < q < 1, \quad (4.34)$$

where $q$ is the contraction factor and the definition of the norm is $||x||_\mu = \sqrt{\int \mu(P)x^2d\Omega}$. This means that if the iteration procedure is contractive, then the error $||\phi_1(\phi_2(R')) - \phi_1(\phi_2(R''))||_\mu$ is getting smaller in each iteration step.

By substituting (4.29) into (4.34) and introducing $B' = \phi_2(R'), B'' = \phi_2(R'')$ the following results:

$$\sqrt{\int_\Omega \mu(P) \left|\mathcal{H}^{-1}\{B'\} - \mathcal{H}^{-1}\{B''\}\right|d\Omega - \nu_{FP}(B' - B'')}^2 d\Omega \leq q \sqrt{\int_\Omega \mu(P)(R' - R'')^2d\Omega}, \quad (4.35)$$
which is
\[
\int_\Omega \mu(P) (\mathcal{H}^{-1} \{B'\} - \mathcal{H}^{-1} \{B''\})^2 d\Omega - \\
2 \int_\Omega \mu(P) \nu_{FP} (\mathcal{H}^{-1} \{B'\} - \mathcal{H}^{-1} \{B''\}) (B' - B'') d\Omega + \\
\int_\Omega \mu(P) \nu_{FP}^2 (B' - B'')^2 d\Omega \leq q^2 \int_\Omega \mu(P) (R' - R'')^2 d\Omega.
\] (4.36)

Integration of the square of inequality (4.31) over the domain \(\Omega\) has the form
\[
\int_\Omega (\mathcal{H}^{-1} \{B'\} - \mathcal{H}^{-1} \{B''\})^2 d\Omega < \int_\Omega \Lambda^2(P) (B' - B'')^2 d\Omega.
\] (4.37)

By manipulating (4.32) the following inequality can be obtained
\[
\int_\Omega (\mathcal{H}^{-1} \{B'\} - \mathcal{H}^{-1} \{B''\}) (B' - B'') d\Omega > \int_\Omega \mu(P) (B' - B'')^2 d\Omega.
\] (4.38)

By substituting the inequalities (4.37), (4.38) and (4.33) above into (4.36), the inequality can be formulated as
\[
\int_\Omega \left[ \nu_{\text{max}}^2(P) - 2 \nu_{FP}(P) \nu_{\text{min}}(P) + \nu_{FP}^2(P) \right] (B' - B'')^2 d\Omega \leq q^2 \int_\Omega (R' - R'')^2 d\Omega.
\] (4.39)

For fulfilling contraction condition, the relationship below must be valid for all of the points \(P\) in the region \(\Omega\)
\[
\frac{\nu_{\text{max}}^2(P)}{\nu_{FP}^2(P)} - 2 \frac{\nu_{\text{min}}(P)}{\nu_{FP}(P)} + 1 \leq q^2,
\] (4.40)

where \(\nu_{FP} = 1/\mu_{FP}\). The minimum value of the contraction factor \(q\) is \(q_{\text{min}} = 1 - \nu_{\text{min}}^2(P)/\nu_{\text{max}}^2(P)\) and it is obtained for \(\nu_{FP}(P) = \nu_{\text{max}}^2(P)/\nu_{\text{min}}(P)\). The problem is that the contraction factor is close to one. Consider the ferromagnetic material, where \(\nu_{\text{min}}(P) = 1/(4000 \mu_0)\) and \(\nu_{\text{max}}(P) = 1/\mu_0\), after that the smallest contraction factor is \(1 - (1/4000)^2 \approx 0.9999999375\). This is the reason for the slow convergence rate.

In order to ensure the strict contractiveness of the function \(\phi_1\), the function \(\nu_{FP}(P)\) must satisfy the relation
\[
\nu_{FP}(P) > \frac{\nu_{\text{max}}(P)}{2},
\] (4.41)

where \(\nu_{\text{max}}(P)\) is the upper bound of the operator \(\mathcal{H}^{-1} \{\}\).
The convergence rate depends on the value assumed for \( \nu_{FP}(P) \), the optimal value is

\[
\nu_{FP}^{\text{opt}}(P) = \frac{\nu_{\text{max}}(P) + \nu_{\text{min}}(P)}{2},
\]

with \( \nu_{\text{min}}(P) \) being the lowest slope of \( \mathcal{H}^{-1} \{ \} \).

### 4.2.3 Polarization method for direct characteristics

In some cases the nonlinear relationship between \( B \) and \( H \) is represented by the direct hysteresis operator \( \mathcal{H} \{ \} \) (e.g. the half-space problem in section 5.1). In this case the hysteretic relationship can be formed by the polarization method as

\[
B = \mathcal{H} \{ H \} = \mu_{FP} H + R,
\]

where \( \mu_{FP} \) is a constant permeability and \( R \) is the residual vector, where the nonlinearity is hidden. The hysteresis operator in the differential equation can be replaced by (4.43). The linearized system can be represented by the operator

\[
H = \phi_2(R),
\]

which results magnetic field intensity as output (e.g. the diffusion equation of half-space), which allows to use the direct hysteresis characteristics to determine the next value of the residual

\[
R = \mathcal{H} \{ H \} - \mu_{FP} H \rightarrow R = \phi_1(H).
\]

The operator \( \phi_1() \) must be uniform monotone and Lipschitzian similar to (4.32) and (4.31). Same procedure can be carried out as in section 4.2.2 for finding the condition of contraction, which is

\[
\mu_{FP}(P) > \frac{\mu_{\text{max}}(P)}{2},
\]

where \( \mu_{\text{max}}(P) \) is the upper bound of the operator \( \mathcal{H} \{ \} \).

### 4.2.4 Fixed-point procedure

The flow chart of fixed point method can be seen in Fig. 4.4. This is based on the diffusion equation where the output variable is the magnetic field intensity \( H \). The same chart can be drawn for problems, where the output variable is the magnetic flux density \( B \) by replacing the variable from \( H \) to \( B \).

The FP algorithm starts with initializing the trial value of the residual \( R_0 \), usually as zero. The permeability \( \mu_{FP} \) or reluctivity \( \nu_{FP} \) at the fixed point is determined by (4.42), where \( \mu_{\text{max}} \) and \( \mu_{\text{min}} \) are the maximum and minimum possible slopes of the \( B = \mathcal{H} \{ H \} \) characteristics. The chosen value of \( \mu_{FP} \) will not change in the further process. It will remain constant once and for all.
The next box in the flow chart (Fig. 4.4) is the FP algorithm which consists of two steps. The solution of the linearized differential equations based on the current value of the residual and computing the new value of residual for the next FP iteration. If \( \phi_1 \) is contractive, then the new value of \( H \) will be closer to the correct solution each iteration step.

The error – based on the norm defined in section 4.2.2 – is computed after each FP iteration step to decide whether the FP algorithm must be performed again or it is not required.

If the FP algorithm has found the solution, then the next time step \( t = t + \Delta t \) is considered, if \( t < t_{\text{max}} \). Choice of \( \Delta t \) has influence on the accuracy. It is suggested to take at least ten points from a period \( \Delta t \leq \frac{T}{10} \).

This is the basic version of fixed-point method with very slow convergence rate. In the further sections two methods or combination of the these two methods are presented.

### 4.2.5 Fixed-point procedure with relaxation

I propose the relaxation method for speed up the original fixed-point technique in this section by introducing a relaxation parameter \( \omega > 0 \). The FP algorithm with relaxation method can be seen in Fig. 4.5. The purpose of the method is to reduce the distance \( d = ||\phi_1(\phi_2(R)) - R||_\nu \). The method uses a suitable relaxation factor \( \omega \) in order to force a reduction of the distance \( d \) and accelerate the convergence toward the solution.

The procedure has the following steps:

- the first FP algorithm solves the linearized problem with respect to \( R_0 \) and it computes
Fig. 4.5. The Fixed Point Algorithm with relaxation

the new residual value $R_{\text{prev}}$. Furthermore the second FP algorithm solves the linearized problem again based on the residual $R_{\text{prev}}$ and after that the new residual $R$ is determined;

• a relaxation factor $\omega$ is introduced and the new value of the residual is computed by

$$R' = R_0 + \omega(R_{\text{prev}} - R_0),$$

(4.47)

due to the linearity of $\phi_2$ the magnetic field intensity can be computed from the new value of the residual $R'$

$$H' = H_{\text{prev}} + \omega(H - H_{\text{prev}});$$

(4.48)

• the residual for the next iteration can be computed by

$$R'_{\text{next}} = \phi_1(H');$$

(4.49)
\[ d(\omega) = ||R'_{\text{next}} - R'||_\nu. \] (4.50)

The distance is a function of the chosen relaxation factor. The problem becomes then to look for the optimal \( \omega \) value which minimizes \( d(\omega) \). In my applications the optimal value of \( \omega \) is determined by the combination of the golden search and the parabolic search algorithms.

The relaxation method significantly accelerates the convergence, but its efficiency reduces as the iterative process approaches to the solution. This can be avoided by turning back to the original FP algorithm near to the solution.

### 4.2.6 Fixed-point procedure with relaxation and \( \mu \) update

I propose an additional speed up technique for the fixed-point method for achieving less non-linear iterations by allowing the local permeability value update in each time step.

The permeability \( \mu_{\text{FP}} \) regarding the fixed-point method is determined by (4.42). A constant permeability value is supposed for all points \( P \) of the region \( \Theta \) based on the major hysteresis curve disregarding that the magnetic field distribution is different from one point to another. However the procedure in time domain allows to update the fixed-point permeability in each time step according to some estimation based on the magnetic field distribution from the previous time steps. This may lead to different permeability values point by point according to the magnetic field distribution. In fact, this procedure can not be called fixed-point method any more since \( \mu_{\text{FP}} \) is not the same for all time steps. If only a certain time step is considered, then the algorithm is identical to the original fixed-point procedure since the \( \mu_{\text{FP}} \) is unchanged. A possible choice of \( \mu_{\text{FP}} \) is based on the slope of the magnetization characteristics \( \mathcal{H} \{ H \} \) at the previous time step

\[
\mu_{\text{FP}}(P) = \frac{\partial \mathcal{H} \{ H(P) \}}{\partial H} \bigg|_{H(P) = H_{\text{prev}}(P)}, \tag{4.51}
\]

where the permeability value may vary by the location \( P \). Since the contraction must be kept therefore the choice of \( \mu_{\text{FP}} \) is not arbitrary. The condition (4.41) must be considered to ensure the contractiveness of \( \phi_1 \circ \phi_2 \) like in Fig. 4.6. Where \( \mu_{\text{FP}} \) is less then \( \mu_{\text{max}}/2 \) according to (4.51), then \( \mu_{\text{FP}}(P) = \mu_{\text{max}}(P)/2 \) is supposed. It is important to note that if \( \omega < 1 \) then an under-relaxation procedure results, which is able to correct the problem of contractiveness. Despite of this, keeping contractiveness of \( \phi_1 \circ \phi_2 \) is recommended.

### 4.3 Higher order triangular elements

I extend the the nonlinear finite element computation summarized above by introducing higher order finite elements for obtaining better accuracy. Both scalar and vector finite elements are proposed here for the higher order approximations.

Higher order \textit{Schöberl-type Nedelec} basis functions are constructed on triangular elements [53]. These shape functions provide not only the global complete sequence property [22, 23],
but also local complete sequence properties for each edge-, face-, element-block. This local property allows an arbitrary variable choice of the polynomial degree for each edge, face, and element. A second advantage of this construction is that simple block-diagonal preconditioning gets efficient. The higher order shape functions contain gradient shape functions explicitly. In the case of a magnetostatic boundary value problem, the gradient basis functions can be skipped, which reduces the problem size, and improves the condition number.

I begin the construction of nodal elements and after that the corresponding edge elements are derived. Hierarchical finite elements are usually based on orthogonal polynomials. Let $l_i$ denote the Legendre polynomials up to order $p$, where $i = 0 \ldots p$. This is a basis on the interval $[-1, 1]$. Furthermore, the integrated Legendre polynomials are introduced denoted by $L_i$ and
defined by the integral
\[ L_i(x) = \int_{-1}^{1} l_{i-1}(\xi) \, d\xi. \] (4.52)

Some Legendre polynomials and its integrals are summarized in Table 4.1. The Legendre polynomials can be computed recursively as follows,

\[ l_0(x) = 1, \quad l_1(x) = x, \quad l_i(x) = \frac{2i - 1}{i} x l_{i-1}(x) - \frac{i - 1}{i} l_{i-2}(x), \] (4.53)

and its integrals

\[ L_0(x) = -1, \quad L_1(x) = x, \quad \text{after computing all it should be replaced by } L_1(x) = x + 1, \quad L_i(x) = \frac{2i - 3}{i} x L_{i-1}(x) - \frac{i - 1}{i} L_{i-2}(x), \] (4.54)

The basis functions are expressed in terms of the volume coordinates \( N_1, N_2 \) and \( N_3 \) for the 2D triangular element (Fig. 2.2).

The shape functions can be obtained for a general triangle by scaling the Legendre polynomials. Therefore the scaled Legendre polynomials are introduced

\[ l_n^S(\xi, \eta) = \eta^n l_n \left( \frac{\xi}{\eta} \right), \] (4.55)

and scaled integrated Legendre polynomials

\[ L_n^S(\xi, \eta) = \eta^n L_n \left( \frac{\xi}{\eta} \right), \] (4.56)
on triangular domain with vertices in $(-1, 1)$, $(1, 1)$ and $(0, 0)$, $\xi \in (0, 1]$ and $\eta \in [-t, t]$. We can observe that $L_n^S$ vanishes on the two edges $\xi = -\eta$ and $\xi = \eta$ and corresponds on the third edge $\eta = 1$.

### 4.3.1 Higher order scalar shape function

The hierarchical triangular nodal element of order $p$ can be defined by using scaled Legendre polynomials:

- for vertex-based functions: $\phi^V_i = N_i$, $i = 1, 2, 3$;
- for edge-based functions for the edge $E$ with vertices 1 and 2 ($0 \leq i \leq p - 2$): $\phi^E_i = L_i^S(N_1 - N_2, N_1 + N_2)$;
- interior functions for $i, j \geq 0$ and $i + j \leq p - 3$: $\phi^I_{ij} = L_i^S(N_1 - N_2, N_1 + N_2)N_j^S(N_1 - N_2, N_1 + N_2)$.

The following is an example how the triangular basis function can be derived from the basis of the quadrilateral element in Fig. 4.7.

*Fig. 4.7. Reference element for deriving of triangular shape functions from quadrilateral element*

constructed as

\[
\begin{align*}
N_i^{E1} &= \left(\frac{1 - y}{2}\right)^i L_i(x), \\
N_i^{E2} &= \left(\frac{1 + x}{2}\right)^i L_i(y), \\
N_i^{E3} &= \left(\frac{1 - x}{2}\right)^i L_i(y), \\
N_i^{E4} &= \left(\frac{1 + y}{2}\right)^i L_i(x),
\end{align*}
\]
The basis functions of the triangular element can be obtained by using the transformation

\[ x :\rightarrow \frac{2x}{1-y}, \quad y :\rightarrow y. \] (4.58)

The edge basis functions of a triangular element are

\[
N_e^1_i = \left( \frac{1 - y}{2} \right)^i L_i \left( \frac{2x}{1 - y} \right), \\
N_e^2_i = \left( \frac{1 - y + 2x}{2 - 2y} \right)^i L_i(y), \\
N_e^3_i = \left( \frac{1 - y - 2x}{2 - 2y} \right)^i L_i(y). \]

The summary of the triangular basis functions are:

- in vertices: \( N^V_1 = \frac{1 - 2x - y}{4}, N^V_2 = \frac{1 + 2x - y}{4} \) and \( N^V_3 = \frac{y + 1}{2} \). The number of vertex functions is three;
- on edges: \( N_e^1_i = \left( \frac{1 - y}{2} \right)^i L_i \left( \frac{2x}{1 - y} \right), N_e^2_i = \left( \frac{1 - y + 2x}{2 - 2y} \right)^i L_i(y) \) and \( N_e^3_i = \left( \frac{1 - y - 2x}{2 - 2y} \right)^i L_i(y) \), where \( i = 2, 3, \ldots, p \). These basis functions are required if \( p \geq 2 \). Number of edge functions is \( 3(p - 1) \);
- within the element (interior functions): \( N^I_{ij} = \left( \frac{1 - y}{2} \right)^i L_i \left( \frac{2x}{1 - y} \right) L_j(y) = N_e^1_i L_j(y). \) These interior basis functions are required if \( p \geq 3 \). The number of interior functions is \( \frac{(p - 1)(p - 2)}{2} \).

I have started with using hierarchical higher order finite elements for approximating scalar nodal functions. These functions must be continuous over element boundaries. This means that the point values in element vertices, and the polynomials along edges must be the same. This is obtained by defining basis functions associated with vertices, edges, and the interior of elements. The vertex basis functions are the standard linear nodal functions being one in one vertex, and vanishing in the other ones. The edge based functions have to be non-zero on one edge, and must vanish at the other two edges. The scaled integrated Legendre polynomials fulfill this property. In 2D, the interior basis functions must vanish on the boundary of the triangle. They are defined as the tensor product of 1D basis functions. One factor has to vanish on two edges, the second one has to eliminate the third edge.
4.3.2 Higher order vector shape function

The basic idea is to include the gradients of the scalar basis functions into the set of basis functions. If the scalar function is continuous, then the gradient has continuous tangential components. First, we take the lowest order Nedelec elements, which have one basis function for each edge. The additionally needed edge-based basis functions on the edge can be taken as the gradients of the scalar edge-based functions. Since the edge-based functions are gradients, the curl of them vanishes. The interior basis functions for the triangle need vanishing tangential components. This is obtained by taking the gradients of the scalar functions. But now, the gradient fields are not enough to span all vector valued functions. Recall that the scalar basis is built as tensor product $u_i v_j$. The evaluation of the gradient gives $\nabla (u_i v_j) = (\nabla u_i)v_j + u_i \nabla v_j$. Note that not only the sum, but both individual terms have vanishing tangential boundary values. Thus, also different linear combinations like $(\nabla u_i)v_j - u_i \nabla v_j$ can be taken as basis functions. By counting the dimension one finds that still $p-1$ functions are missing. These can be chosen linearly independent as the product of the edge-element function, and a polynomial $v_j$. By means of the scaled Legendre polynomials, the 2D construction is easily extended to 3D elements, which is detailed in section 4.4 on page 61.

Hierarchical triangular vector (curl) element of order $p$ using scaled Legendre polynomials is as follows [53]

- edge-based shape functions for an edge $e$ bounded by vertices 1 and 2;
  - edge-element shape function: $\phi^e = N_2 \nabla N_1 - N_1 \nabla N_2$;
  - higher order edge-based functions: $\phi_i^e = \nabla L_{i+2}^S(N_1 - N_2, N_1 + N_2)$ for $2 \leq i \leq p+1$;

- Element-based functions:
  let $u_i = L_{i+2}^S(N_2 - N_1, N_1 + N_2)$ and $v_j = N_3 I_j(N_3 - N_1 - N_2)$;
  - Type 1: gradient fields $\phi_{ij}^1 = \nabla (u_i v_j)$ for $i, j > 0$ and $i + j \leq p - 2$;
  - Type 2: $\phi_{ij}^2 = (\nabla u_i)v_j - u_i \nabla v_j$ for $i, j > 0$ and $i + j \leq p - 2$;
  - Type 3: $\phi_{ij}^3 = (\nabla N_1 N_2 - N_1 \nabla N_2)v_j$ for $0 \leq j \leq p - 2$.

4.4 Higher order tetrahedral elements

The higher order triangular elements are described in section 4.3 on page 56. Those triangular Schöberl-type elements are generalized to 3D tetrahedral elements.

Electromagnetic problems naturally contain the continuity of the tangential components across sub-domain interfaces. This property has led to the construction of finite elements with tangential continuity. The most prominent one is the edge element, which is the lowest order member of the first family of Nedelec elements. Up from the lowest order element all of them contain edge-based degrees of freedom. Higher order ones also contain unknowns in the element faces, and in the interior of the elements. To match the functions over the element-interfaces, the orientation of the edges and faces has to be taken into account. While on edges this is just changing the sign, the orientation of triangular faces is more involved.

The complete sequence property is essential for the convergence of the finite element approximation, in particular for eigenvalue problems. Also the kernel-preserving multigrid precon-
4.4.1 Higher order scalar shape function

The hierarchical tetrahedral nodal (scalar) element of order \( p \) can be defined by using scaled Legendre polynomials:

- vertex based functions: \( \phi_i^V = N_i \), for \( i = 1, 2, 3, 4 \);
- edge based functions for one of the six edges bounded by vertices 1 and 2: \( \phi_i^E = L_i^S(N_1 - N_2, N_1 + N_2) \), where \( 2 \leq i \leq p - 2 \);
- face based functions for one of the four faces defined by points 1, 2 and 3: \( \phi_{ij}^F = L_{ij}^S(N_1 - N_2, N_1 + N_2)N_3l_{ij}^S(N_3 - N_1 - N_2, N_1 + N_2 + N_3) \), where \( i, j \geq 0 \) and \( i + j \leq p - 3 \);
- interior based functions for \( 0 \leq i + j + k \leq p - 4 \): \( \phi_{ijk}^F = L_{ijk}^S(N_1 - N_2, N_1 + N_2)N_3l_{ijk}^S(N_3 - N_1 - N_2, N_1 + N_2 + N_3)N_4l_{ijk}(N_4 - N_1 - N_2 - N_3) \).

4.4.2 Higher order vector shape function

The hierarchical tetrahedral vector (curl) element of order \( p \) can be defined by using scaled Legendre polynomials:

- edge based functions for one of the six edges bounded by vertices 1 and 2:
  - edge element shape function: \( \phi_i^E = N_2\nabla N_1 - N_1\nabla N_2 \);
  - higher order edge based functions for \( 2 \leq i \leq p + 1 \): \( \phi_{i-1}^E = \nabla N_i^S(N_1 - N_2, N_1 + N_2) \);
- face based functions for one of the four faces defined by points 1, 2 and 3:
  \( u_i = L_{i+2}^S(N_1 - N_2, N_1 + N_2) \),
  \( v_j = N_3l_j^S(N_3 - N_1 - N_2, N_3 + N_1 + N_2) \),
  - Type 1: gradient fields for \( i, j \geq 0 \) and \( i + j \leq p - 2 \),
    \( \phi_{ij}^F = \nabla(u_i v_j) \);
  - Type 2: for \( i, j \geq 0 \) and \( i + j \leq p - 2 \),
    \( \phi_{ij}^F = (\nabla u_i)v_j - u_i\nabla v_j \);
  - Type 3: for \( 0 \leq i \leq p - 2 \),
    \( \phi_{ij}^F = (\nabla N_1 N_2 - N_1\nabla N_2)v_j \);
- element based functions:
  \( u_i = L_{i+2}^S(N_1 - N_2, N_1 + N_2) \),
  \( v_j = N_3l_j^S(N_3 - N_1 - N_2, N_3 + N_1 + N_2) \),
  \( w_k = N_4l_k^S(N_4 - N_1 - N_2 - N_3) \),
  - Type 1: gradient fields for \( i, j, k \geq 0 \) and \( i + j + k \leq p - 3 \),
    \( \phi_{ijk}^F = \nabla(u_i v_j w_k) \);
  - Type 2: for \( i, j, k \geq 0 \) and \( i + j + k \leq p - 3 \),
    \( \phi_{ijk}^F = (\nabla u_i)v_j w_k - u_i\nabla v_j w_k + u_i v_j\nabla w_k \),
    \( \phi_{ijk}^F = (\nabla u_i)v_j w_k - u_i\nabla v_j w_k - u_i v_j\nabla w_k \);
  - Type 3: for \( j, k \geq 0 \), \( j + k \leq p - 3 \),
    \( \phi_{jk}^F = (\nabla N_1 N_2 - N_1\nabla N_2)v_j w_k \).
4.5 Thesis 2

I have implemented the hysteresis simulation to the higher order vector shape functions of FEM for solving magnetostatic and eddy current problems and to handle the nonlinear iteration and I have worked out an efficient speed up technique of the fixed point method by combining the relaxation method and the local permeability update.

a) I have extended the field equations for nonlinear magnetostatic and eddy current problems considering the gauge fixing with the Lagrange multiplier method. I have proposed formulations by assuming general hysteretic relationship between the $H$ and $B$ to offer for researchers the ability to use any type of hysteresis model and numerical method.

b) To handle the nonlinear iteration I have worked out the relaxation method with the permeability updates to speed up the fixed-point iteration process.

c) I have combined the solution of the field equations with higher order Schöberl type edge elements for nonlinear case. The introduced element type allows to choose finite elements with arbitrary polynomial degree utilizing the local complete sequence property.
Chapter 5

Applications

I have proposed a method in the previous section to solve electromagnetic problems taking the hysteresis effect into account. Here I prove that the proposed method is efficient in the field analysis. I have worked out three examples; one for the solution of one dimensional diffusion equation for ferromagnetic half-space, another one for the solution of a two dimensional (axisymmetric) eddy current problem for hysteresis measurement simulation and the third one for the solution of a three dimensional magnetostatic problem for the benchmark problem of COMPUMAG.

5.1 1D example – Ferromagnetic half-space

The sketch of the half-space problem can be seen in Fig. 5.1. The half-space, where \( z > 0 \) is conductive ferromagnetic material. A sinusoidal magnetic field is prescribed at \( z = 0 \) as a Dirichlet boundary condition. Indeed, eddy currents are induced in the electrically conductive material, decreasing with \( z \). The problem can be described by a 1D model, because the field quantities are not changing with \( x \) and \( y \). A finite element model of this half-space is very simple, which enables to set up the field computation procedure with hysteresis [54–60].

Three problems are solved in this example going from the simplest linear case to the computation including the JAM.

\[
\begin{align*}
\sigma &= 0 \\
\mu_r &= 1 \\
H_D(z = 0, t) &= H_{\text{amp}} \sin(\omega t) \\
\text{Dirichlet condition}
\end{align*}
\]

Fig. 5.1. The ferromagnetic half-space. The \( z > 0 \) half-space is the ferromagnetic and electrically conductive material, the \( z < 0 \) can be considered as air or vacuum, where the sinusoidal magnetic field is coming from. It is prescribed as a boundary condition at \( z = 0 \).
5.1.1 Solution of the linear problem

The following transient problem is solved

\[
\frac{\partial}{\partial z} \left( \frac{1}{\sigma} \frac{\partial H}{\partial z} \right) + \mu \frac{\partial H}{\partial t} = 0.
\]  

(5.1)

The weak form of the equation above can be obtained by integrating over the problem domain

\[
-\int_{\Omega} 1 \frac{\partial N_j}{\partial z} \frac{\partial H}{\partial z} d\Omega - \int_{\Omega} \mu N_j \frac{\partial H}{\partial t} d\Omega = 0,
\]  

(5.2)

\footnote{The analytical solution of (5.1) \( H_D(z = 0, t) = H_{\text{amp}} \sin(\omega t) \) in time harmonic case is \( H(z, t) = H_{\text{amp}} e^{-z/\delta} \sin(\omega t - z/\delta), \)
where \( \delta = \sqrt{\frac{1}{\mu \sigma \omega}} \) is the penetration depth.}

Fig. 5.2. Analytical and numerical solutions of the linear half-space problem in the first period of the equation multiplied by the nodal basis function of FEM. After applying Green’s theorem, the weak equation has the form

\[
-\int_{\Omega} 1 \frac{\partial N_j}{\partial z} \frac{\partial H}{\partial z} d\Omega - \int_{\Omega} \mu N_j \frac{\partial H}{\partial t} d\Omega = 0,
\]  

(5.2)
where the $N_j$ are the nodal basis functions of the FEM and $j = 1 \ldots N$, where $N$ is the nodal degrees of freedom (DOF) on the finite element mesh. In this case the finite element mesh is on the positive $z$ axis.

The linear problem is solved by the following parameters

- frequency: $f = 50$ Hz;
- electrical conductivity: $\sigma = 1.12 \cdot 10^7$ S/m;
- relative permeability: $\mu_r = 4000$;
- amplitude of the Dirichlet condition at $z = 0$: $H_{\text{amp}} = 5000$ A/m;
- number of samples per period: $N = 100$;
- number of finite elements: 1000.

The first period of the analytical and the numerical solutions are plotted in Fig. 5.2. It can be seen that the transient numerical solution (blue dashed line) is different from the steady state analytical solution (red continuous line). The numerical solution is approaching the analytical one e.g. in the fifth period the two solutions are almost the same in Fig. 5.3.

**Fig. 5.3.** Analytical and numerical solutions of the linear half-space problem in the fifth period
5.1.2 Solution of the nonlinear problem with Langevin characteristics

The half-space problem is solved in this section in case of non-linear magnetic material. The material characteristics are described by the Langevin function shown in Fig. 5.4.

\[ B(H) = B_s \left[ \coth \left( \frac{H}{a} \right) - \frac{a}{H} \right], \quad (5.3) \]

where \( B_s \) is the induction at saturation and \( a \) is a form parameter, which ensures the desired slope at the origin. The differential relative permeability is the derivative of the equation above with respect to \( H \),

\[ \mu(H) = \frac{dB}{dH} = B_s \left[ \frac{1}{a} - \frac{1}{a} \coth^2 \left( \frac{H}{a} \right) + \frac{a}{H^2} \right]. \quad (5.4) \]

The slope of the Langevin characteristics at the origin is \( \mu = 4000\mu_0 \) (\( B_s = 1 \) T, \( a = 66.3 \) A/m), because the relative permeability should be the same as in the linear case. A kind of verification can be carried out, where the characteristics can be considered as linear for small signal excitations.

![Fig. 5.4. The material nonlinearity follows the Langevin curve](image)

**Formalism**

The procedure written in section 4.2.3 is applied for this 1D problem. The above described material non-linearity is taken into account by the fixed point method, whereas a constant linearization point is assumed (\( \mu_{FP} \)) and the error of the linearization is corrected by a residual
vector $R$. So the constitutive relationship for the magnetic flux density can be written as

$$B = \mathcal{H} \{H\} = \mu_{FP} H + R,$$

(5.5)

where the operator $\mathcal{H} \{}$ represents the material characteristics. Substituting (5.5) into the differential equation

$$\frac{\partial}{\partial z} \left( \frac{1}{\sigma} \frac{\partial H}{\partial z} \right) = -\frac{\partial B}{\partial t},$$

(5.6)

the linearized fixed point equation is obtained

$$\frac{\partial}{\partial z} \left( \frac{1}{\sigma} \frac{\partial H}{\partial z} \right) = -\mu_{FP} \frac{\partial H}{\partial t} + \frac{\partial R}{\partial t} \rightarrow H = \phi_2(R),$$

(5.7)

where $R$ is the residual. The weak form of the non-linear half space problem can be formulated as

$$- \int_{\Omega} \frac{\partial H}{\partial z} \frac{\partial N_j}{\partial z} d\Omega - \int_{\Omega} \left( N_j \mu_{FP} \frac{\partial H}{\partial t} + \frac{\partial R}{\partial t} \right) d\Omega = 0,$$

(5.8)

where $N_j$ are nodal shape functions of FEM chosen as the test functions of $H$. After solving (5.7) for $H$, the new residual can be computed as

$$R = \mathcal{H} \{H\} - \mu_{FP} H \rightarrow R = \phi_1(H),$$

(5.9)

where $H$ is the value of the magnetic field intensity in the current iteration step.

**Solution with fixed point iteration**

In this section, the half-space problem is solved by the traditional FP iteration method without any speed up technique. The block scheme of the FP procedure can be seen in Fig. 5.5, where two FP algorithms are shown instead of one as in Fig. 4.4. The reason is that in the further computations two FP steps will be needed (relaxation) and the running times will be able to be compared easily if I use two FP algorithms here too, but I have to emphasize that one FP step would be enough here.

Beside the FP algorithm, $\phi_2$ represents the solution of the differential equation,

$$- \frac{\partial}{\partial z} \left( \frac{1}{\sigma} \frac{\partial H}{\partial z} \right) = \mu_{FP} \left( \frac{H - H_{old}}{\Delta t} \right) + \frac{R - R_{old}}{\Delta t} = 0,$$

(5.10)

where $H_{old}$ and $R_{old}$ represent the magnetic field and the residual values at the previous time step respectively. After solving the differential equation (5.10) based on the previous value of $R$, the new value of the residual can be computed by $\phi_1$, which is as follows

$$R = \mathcal{H} \{H\} - \mu_{FP} H,$$

(5.11)
**Table 5.1.** Number of required FP iteration steps with different $\mu_{\text{FP}}$ starting value

<table>
<thead>
<tr>
<th>FP</th>
<th>Rel.</th>
<th>$\mu_{\text{FP}}$</th>
<th>$H_{\text{amp}}$</th>
<th>Number of iteration steps (for two periods, 80 time steps)</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>–</td>
<td>2000</td>
<td>100</td>
<td>–</td>
<td>divergent</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>2000</td>
<td>1000</td>
<td>–</td>
<td>divergent</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>2000</td>
<td>10000</td>
<td>–</td>
<td>not convergent after 37125 steps</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>4000</td>
<td>100</td>
<td>474</td>
<td>avg. steps: 5.925</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>4000</td>
<td>1000</td>
<td>5194</td>
<td>avg. steps: 64.925</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>4000</td>
<td>10000</td>
<td>73176</td>
<td>avg. steps: 914.7</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>6000</td>
<td>100</td>
<td>1294</td>
<td>avg. steps: 16.175</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>6000</td>
<td>1000</td>
<td>7836</td>
<td>avg. steps: 97.95</td>
</tr>
<tr>
<td>+</td>
<td>–</td>
<td>6000</td>
<td>10000</td>
<td>109559</td>
<td>avg. steps: 1369.4875</td>
</tr>
</tbody>
</table>

where the nonlinear relationship between the $B$ and $H$ is denoted by the operator $H \{ \}$. 

![Flowchart](image)

**Fig. 5.5.** The Fixed Point Algorithm

The expression of the error is the following

$$\text{error} = ||R - R_{\text{prev}}||_\nu,$$

(5.12)
where $\nu$ is the magnetic susceptibility at the current iteration point,

$$\nu(H) = \left(\frac{d\mathcal{H}}{dH}\right)^{-1}. \quad \text{(5.13)}$$

The weighting by $\nu(H)$ ensures that the error is computed justly even at the positions, where the slope of the characteristics is much lower.

The parameters of the computation are

- frequency: $f = 50 \text{ Hz}$;
- electrical conductivity: $\sigma = 1.12 \cdot 10^7 \text{ S/m}$;
- material characteristics is given by the Langevin function $B = \coth(H/66.3) - 66.3/H$;
- amplitude of the Dirichlet condition at $z = 0$: $H_{\text{amp}} = 1000 \text{ A/m}$;
- number of samples per period: $N = 40$ and two periods are considered (80 time steps);
- tolerance of the FP iteration $\varepsilon = 10^{-8}$.

The FP method is convergent if $\phi_1$ is a contraction ($\mu_{\text{FP}} > \mu_{\text{max}}/2$). As it can be seen in Table 5.1 if $\mu_{\text{FP}} = 2000\mu_0$, which is lower than the $\mu_{\text{max}}/2$, then the method will be divergent.

The total number of required FP iteration steps are summarized in Table 5.1. (In fact, this value should be divided by two, because there are two FP algorithm in one iteration step.) Only FP algorithm is used to obtain the solutions without relaxation. Three different so
called optimal relative permeability values are used as a starting value. As it has already been mentioned the case \( \mu_{FP} = 2000 \) is unstable. Only a few iteration steps are required to achieve the prescribed tolerance (in this example it is \( \varepsilon < 10^{-8} \)) by choosing \( \mu_{FP} = 4000 \), since the slope of characteristics at the origin is 4000.9. If the optimal permeability is exactly the same as the slope of the material characteristics at the origin and the amplitude of the excitation is sufficiently small for holding on the linear part, then no FP iterations are required to achieve the desired tolerance. The problem can be considered as linear, where the optimal permeability is 4000 and the amplitude of the magnetic field intensity is 100 A/m like in Table 5.1. This is the explanation of the very small number of iterations. If the amplitude of the excitation increases (\( H_{amp}=1000 \text{ A/m},10000 \text{ A/m} \)), then the characteristics can not be considered as linear any more, because the saturation part is reached. Therefore more iteration steps are required as it can be seen in Fig. 5.6. It is straightforward that less iterations are required when the excitation is crossing zero (Fig. 5.6). The solutions are plotted in Fig. 5.7 and more plots can be found in appendix A on page 1.

**Solution with fixed point iteration + relaxation**

The fixed-point iteration with relaxation has been discussed in section 4.2.5. The block scheme of the fixed point iteration with relaxation can be seen in Fig. 4.5. Using this method, same results have been obtained like in the previous section, where only fixed point method were used without any speed-up procedure. The only difference can be found in the convergence speed, which can be seen after comparing the Figs. 5.6 and 5.8. The FP method with relaxation is about two times faster than without relaxation (Tables 5.1, 5.2).

![Fig. 5.8. Number of iteration steps required at specified time steps in the case of fixed-point method with relaxation](image)

**Solution with fixed point iteration + relaxation and \( \mu \) update**

The main purpose of this investigation is to speed up the original fixed point iteration. One possible way is the relaxation, but if the Tables 5.1 and 5.2 are considered, then it can be noticed that the number of required iteration steps depends on the optimal permeability \( \mu_{FP} \) as well. In this case, if the \( \mu_{FP} = 4000 \mu_0 \), which is very close to the initial slope of the nonlinear curve, then the solution requires less iteration steps. It is straightforward that the chosen permeability \( \mu_{FP} \) should be chosen as close as possible to the current slope of the nonlinear curve. Of course the slope is different at each time instance. Therefore the permeability values must be updated.
Table 5.2. Number of required FP+Rel. iteration steps with different $\mu_{FP}$ starting value

<table>
<thead>
<tr>
<th>FP</th>
<th>Rel.</th>
<th>$\mu_{FP}$</th>
<th>$H_{amp}$</th>
<th>Number of iteration steps (for two periods, 80 time steps)</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>2000</td>
<td>100</td>
<td>1248</td>
<td>avg. steps: 15.6</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>2000</td>
<td>1000</td>
<td>2761</td>
<td>avg. steps: 34.5125</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>2000</td>
<td>10000</td>
<td>34639</td>
<td>avg. steps: 432.9875</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>4000</td>
<td>100</td>
<td>397</td>
<td>avg. steps: 4.9025</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>4000</td>
<td>1000</td>
<td>2771</td>
<td>avg. steps: 34.6375</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>4000</td>
<td>10000</td>
<td>33587</td>
<td>avg. steps: 419.8375</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>6000</td>
<td>100</td>
<td>894</td>
<td>avg. steps: 11.175</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>6000</td>
<td>1000</td>
<td>4124</td>
<td>avg. steps: 51.55</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>6000</td>
<td>10000</td>
<td>49 812</td>
<td>avg. steps: 622.65</td>
</tr>
</tbody>
</table>

Fig. 5.9. Number of iteration steps required at specified time steps in the case of fixed-point method with relaxation and $\mu$ update

in each time iteration step. It is important to emphasize that the permeability is the originally chosen value at FP iterations, it is modified only in the time iteration steps.

The choice of the optimal permeability is not arbitrary, because the condition of contraction must be always fulfilled. This is the reason why the optimal permeability can not be the exact slope of the nonlinear material characteristics. It this case the fixed-point permeability must be always greater than the half of the maximal permeability, $\mu_{FP} > \mu_{max}/2$. Therefore the fixed-point permeability is always chosen as the previous slope of the characteristics, if it is greater than $\mu_{max}/2$. Otherwise, the fixed-point permeability is chosen as $\mu_{max}/2$. Furthermore, a more sophisticated procedure results if the optimal permeability varies by the location.

The flow chart of the procedure can be seen in Fig. 4.6. A new block has been added, where the optimal permeability is set according to the rules described above. I also plot (Fig. 5.9) the number of required FP iteration steps as a function of the time to be able to compare with the previous ones (Figs. 5.6, 5.8). The required iteration steps are summarized in Table 5.3. If the amplitude of the magnetic field is low $H_{amp} = 100$ A/m and the $\mu_{FP}$ is very close to the real slope of the material characteristics (in this case it is 4000), then the path of the nonlinear iteration remains on the linear part of the Langevin function. This is the reason that in the first rows of Tables 5.2 and 5.3 almost the same values (397 and 395 steps) can be found since no better optimal permeability value can be found than 4000.

The conclusion is that a much faster method has been obtained by using the relaxation and the $\mu$ update technique and the solution remained the same as in the case of the original fixed...
Table 5.3. Number of required FP+Rel. iteration steps with $\mu$ update

<table>
<thead>
<tr>
<th>FP</th>
<th>Rel.</th>
<th>$\mu_{FP}$</th>
<th>$H_{amp}$</th>
<th>Number of iteration steps (for two periods, 80 time steps)</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>update</td>
<td>100</td>
<td>395</td>
<td>avg. steps: 4.9375</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>update</td>
<td>1000</td>
<td>1696</td>
<td>avg. steps: 21.2</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>update</td>
<td>10000</td>
<td>10765</td>
<td>avg. steps: 134.5625</td>
</tr>
</tbody>
</table>

point method. This method is used in the further computations.

5.1.3 Solution of the nonlinear problem with Jiles-Atherton model of hysteresis

In this section, I have solved the half-space problem taking the hysteresis into account [61, 62]. It is assumed that the material of the half-space is ferromagnetic and electrically conductive. The hysteretic property of the half-space material is described by the rate independent Jiles-Atherton model (JAM) of hysteresis. The problem is discretized by the finite element method (FEM) and the nonlinear equations are solved by the fixed point method with relaxation and $\mu$ update as it is detailed in the previous sections.

It is important to emphasize that the Langevin function, which is applied to describe the material nonlinearity in the previous section, is the anhysteretic curve of the JAM. By the way, the choice of the anhysteretic curve is almost arbitrary, not only the Langevin function is allowed e.g. the Fröhlich function is also often used in JAM. In this investigation I use the JAM described in section 3.1 on page 20.

\[ H [\text{A/m}] \]

\[ B [\text{T}] \]

Fig. 5.10. The Langevin function used in the previous section and the JAM. The JAM parameters are $M_s = 8.1 \cdot 10^5$ A/m, $\alpha = 10^{-3}$, $a = 300$ A/m, $c = 0.5$, $k = 800$ A/m

The parameters of the JAM have been determined to be similar to the Langevin function (Fig. 5.10). The JAM parameters are $M_s = 8.1 \cdot 10^5$ A/m, $\alpha = 10^{-3}$, $a = 300$ A/m, $c = 0.5$, $k = 800$ A/m. The number of required iteration steps are plotted as a function of the time and the number of required iteration steps are shown by different amplitudes in Fig. 5.11 and Table 5.4 respectively. The tolerance of the nonlinear iteration is $\varepsilon = 10^{-8}$. To achieve this tolerance
Fig. 5.11. Number of iteration steps required at specified time steps in the case of JAM

the nonlinear iteration follows the block scheme drawn in Fig. 4.6. The solution at $t = 35$ ms can be seen in Fig. 5.12 and more results are presented in appendix B.

Table 5.4. Number of required iteration steps in the case of JAM

<table>
<thead>
<tr>
<th>FP</th>
<th>Rel.</th>
<th>$\mu_{FP}$</th>
<th>$H_{amp}$</th>
<th>Number of iteration steps (for two periods, 80 time steps)</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>update</td>
<td>100</td>
<td>463</td>
<td>avg. steps: 5.7875</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>update</td>
<td>1000</td>
<td>936</td>
<td>avg. steps: 11.7</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>update</td>
<td>10000</td>
<td>2990</td>
<td>avg. steps: 37.375</td>
</tr>
</tbody>
</table>

It means that same procedure has been used for solving the problem with hysteresis as in the case of the Langevin function. The operator of the material nonlinearity must be uniformly monotonous and Lipschitzian. The running time is almost the same in both (Langevin and JAM) cases. One iteration step takes about 0.5 sec. on my portable computer with PentiumM 2 GHz processor and 2 GB of RAM. This means that the total process takes about 25 minutes if we consider the longest case (last row) in Table 5.4. Comparing Tables 5.4 and 5.1 it can be

Fig. 5.12. Solutions with JAM, $H_{amp} = 100$ A/m, 1000 A/m, 10000 A/m at $t=35$ ms

concluded that the speed up of FP algorithm has been successfully achieved by the combination of relaxation and $\mu$ update, since the average FP iteration steps reduced from 419.8 to 37.4.
5.2 2D example – Hysteresis measurement simulation

The simulation of the ferromagnetic hysteresis measurement has been performed in this section [63–67]. The detailed description of the hysteresis measurement can be found in section 3.4 on page 29. A ring shaped specimen made from structural steel is excited by a current fed coil (Fig. 5.13). The induced voltage is recorded during the measurement and after that the magnetic flux density is computed from it.

![Fig. 5.13. The drawing and the photo about the specimen](image)

5.2.1 Model

I have determined the electromagnetic field in the examined specimen, which is used in my measurements. The description of the magnetic field in the ring shaped specimen can be done in cylindrical coordinate system \((r, z, \varphi)\), where the axis \(z\) and the axis of the ring are identical. Since the arrangement is axisymmetric, the same magnetic field distribution can be obtained for all value of \(\varphi\). As a result, a two dimensional model can be introduced for the variables \(r\) and \(z\). The 2D model of the ferromagnetic ring measurement in cylindrical coordinate system can be seen in Fig. 5.14. No air layer is needed around the ring since the Ampere’s law is fulfilled by prescribing magnetic field on the surface of the ring by (5.14). The situation is the same as in the real case.

![Fig. 5.14. The geometry model of the ferromagnetic ring simulation](image)

The magnetic field generated by the excitation coil is prescribed as

\[
H_s(r,t) = \frac{N_e \cdot i(t)}{2\pi r}, \tag{5.14}
\]
on the perimeter of the ring, where $N_e$ is the number of turns of the excitation coil, $i(t)$ is the current flowing through the coil wires and $r$ is the radius variable.

### 5.2.2 Mesh parameters

A triangular mesh is applied to solve the finite element problem. The mesh can be seen in the Fig. 5.15. The edges of the triangular elements within the ring are no greater than 1 mm. The number of triangular elements is 698. The mesh is only applied within the ring since there is no magnetic field outside the ring. Quadratic Schöberl-type edge elements are applied for solving

![Fig. 5.15. The mesh of the 2D problem](image)

the problem as it is described in section 4.3 on page 56.

### 5.2.3 Results

The rate-independent JAM is introduced for modeling the material characteristics with the

![Fig. 5.16. The measured and simulated hysteresis curves at 0.2 Hz](image)
Fig. 5.17. The measured and simulated hysteresis curves are plotted at 1 Hz. From left to right the figures show simulation results with the rate-independent and the rate-dependent JAM parameters $M_s = 1.22 \cdot 10^6$, $\alpha = 4.7 \cdot 10^{-4}$, $a = 326.01$ A/m, $c = 0.00565$, $k = 468.985$ A/m. The model parameters are chosen to obtain same hysteresis curve as it was measured in section 3.4. The “measured” magnetic field intensity can be determined exactly the same way as in the experiments:

$$H(t) = \frac{N_e i(t)}{l},$$

(5.15)

where $N_e = 134$ is the number of turns of the excitation coil and $l$ is the average magnetic length in this case $l = 50\pi$ mm. So there is no field computation needed to determine $H(t)$ since the coil is current-fed as in the experiment.

The corresponding magnetic flux density can be determined from the magnetic vector potential $B = \text{curl} \ A$, which has the form in cylindrical coordinates

$$B = B_\phi \hat{e}_\phi = \begin{vmatrix} e_r / r & e_\phi & e_z / r \\ \partial / \partial r & 0 & \partial / \partial z \\ A_r & 0 & A_z \end{vmatrix} = \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) e_\phi.$$  

(5.16)
Fig. 5.19. Solutions with the rate-independent JAM at $f = 0.2$ Hz. The $\varphi$ component of magnetic flux density $B_\varphi$ is plotted by surface and the magnetic vector potential $A$ by arrows.

The average of magnetic flux density is computed with respect to the cross-section area of the ring

$$B(t) = \frac{1}{S} \int B_\varphi(t) dS,$$

where $S$ is the cross-section area of the ring. The measured and the simulated hysteresis curves
Fig. 5.20. Solutions with the rate-dependent JAM at $f = 5$ Hz. The $\phi$ component of magnetic flux density $B_\phi$ is plotted by surface and the magnetic vector potential $A$ by arrows are plotted in Fig. 5.16. The frequency of the excitation current is 0.2 Hz, where the dynamic effects can be neglected. Therefore the simulation with rate-independent JAM provides a good agreement with the measured curve. Distribution of magnetic field in the cross-section of the ring can be seen in Fig. 5.19.

The measured and simulated hysteresis curves are plotted in Fig. 5.17 at 1 Hz. The measured curves are the same in all three figures (blue line). The simulation is done by rate independent JAM with the parameters $M_0 = 1.22 \cdot 10^6$ A/m, $\alpha = 4.7 \cdot 10^{-4}$, $a = 326.01$ A/m,
The simulation is done by the rate dependent JAM with the parameters $M_s = 1.22 \cdot 10^6$ A/m, $\alpha = 4.7 \cdot 10^{-4}$, $a = 326.01$ A/m, $c = 0.00565$, $M_s = 1.22 \cdot 10^6$ A/m, $\alpha = 4.7 \cdot 10^{-4}$, $a = 326.01$ A/m, $c = 0.00565$, $k = 468.985$ A/m, $k_{EC} = 1.8 \cdot 10^{-6}$, $k_A = 0.01$ at right of Fig. 5.17. The simulated and measured curves are good agreement.

The same investigations have been carried out at 5 Hz. The results can be seen in Fig. 5.18. In the figure at the left the simulation is performed by applying the rate independent JAM for modeling the material. The figure at right shows the simulation (red line) result with rate dependent JAM with same parameters above. Distribution of magnetic field in the cross-section of the ring can be seen in Fig. 5.20.
5.3 3D example – Solution of the problem T.E.A.M 13

The developed method works very well for large problems as well, as it can be seen in the follows. I have proved the efficiency of the developed nonlinear finite element method by solving a large 3D magnetostatic problem proposed by the COMPUMAG Society, which can be considered as a benchmark problem to compare different kind of numerical methods. The original problem can be found in Appendix E on page 15 or [1].

5.3.1 Introduction

The magnetic field distribution of a coil surrounded by steel channels and plates is determined (see Fig. 5.21). The coil is excited by DC current (magnetostatic case). The magnetostatic field is calculated at various positions. The experimental setup contains search coils to measure the magnetic flux density allowed to compare with the simulation results [1].

Fig. 5.21. Sketch of the problem. The coil is surrounded by iron plates

The thickness of plates and channels is very small in comparison with the other sizes of the geometry. Therefore some difficulties may occur in the meshing process. It is not easy to generate a suitable mesh with tetrahedral elements since the finite element mesh must be finer in the region of the iron plates despite in the region of air it can be coarser. It is not easy to make connection between the various density meshes.
By means of very fine meshing the degrees of freedoms (DOFs) become very large \((n \cdot 100000)\). Therefore direct solvers cannot be employed to solve the algebraic equations. The current of the coil is prescribed directly as a current density and not the Biot-Savart law is applied to take into account the excitation current. In this case it is not enough to solve the system of equations for the components of the magnetic vector potential since the gauge fixing must be ensured by introducing Lagrange multipliers as well. The algebraic equations can be solved by the general mean residual (GMRES) iterative method. For large indefinite problems like this with zeros in the diagonal of the system matrix the geometric multigrid [Appendix F on page 30] preconditioner with Vanka pre-/postsmoother can successfully be applied [68] [Appendix G on page 32].

The setup of the problem will be done on the linear case. The following parameters will be optimized: the mesh size, the element order, the radius of enclosing sphere and the number of multigrid levels. They are determined by computing the inductance of the linear problem assuming \(\mu_r = 2300\). The inductance is proportional to the stored magnetic energy in the arrangement. Therefore it is suitable to indicate the accuracy of the solution [35, 45].

The formalism with respect to the magnetic vector potential for solving this magnetostatic problem has already been discussed in section 4.1.1 on page 42.

## 5.3.2 Setup of the linear problem

Fine tuning of the linear problem is important since it is solved several times during the nonlinear iteration. Any minor improvement in the running time or accuracy could become essential in the nonlinear problem.

Purpose of the linear problem setup to determine the following parameters:

- radius of enclosing sphere;
- choosing a suitable solver;
- mesh density;
- element order;
- geometrical symmetry.

The radius of the enclosing sphere is sufficient since the magnetic field disappears at the infinity. However it is impossible to mesh that. Instead of that the arrangement is closed by a sphere where the magnetic field can be considered as zero. The enclosure could be for instance a cube but the sparsity pattern of a sphere is much better. The values in the Table 5.5 below have been computed by first order edge elements with 2 level multigrid method where size of the coarser mesh is approximately quarter of the original one. The absolute values of the magnetic flux densities \(B_1, B_8\) and \(B_{25}\) have been calculated by

\[
B = \frac{1}{S} \int_S \text{curl} \mathbf{A} \, dS = \frac{1}{S} \oint \mathbf{A} \cdot d\mathbf{l},
\]

where \(S\) is the cross-section area of the plates. The arrangement is symmetrical to the plane \(z = 0\), where homogeneous Neumann condition is prescribed \(\mathbf{n} \times \mathbf{H} = 0\) in Fig. 5.22.
The Neumann condition is fulfilled automatically on the symmetry plane. More accurate Neumann condition can be achieved by finer mesh around the Neumann plane. The differences between the values of $B_1$ and $B_{25}$ in Tables 5.5 and 5.6 can be avoided by mesh refinement around the symmetry plane. The second row in Table 5.6 is computed by second order edge elements. The $r = 0.5$ m outer sphere radius is used in the further computations.

### 5.3.3 Nonlinearity

In magnetostatic case the first magnetization curve can describe the nonlinear behavior of the ferromagnetic material. It is given in the problem specification in Appendix E on page 19. Since the output quantity of the $A$ formulation is the magnetic flux density $B$. Therefore the inverse characteristics must be used, which is detailed in section 3.3 on page 28.

The measured magnetization curve is approximated by the Jiles-Atherton model with the parameters $M_s = 1.4 \cdot 10^6$ A/m, $\alpha = 7 \cdot 10^{-4}$, $a = 200$ A/m, $c=0.1$, $k=175$ A/m (see Fig. 5.23).

### 5.3.4 Results

The problem TEAM 13 is solved in two steps. In the first step first order tetrahedral elements are used on a coarse mesh to determine a suitable fixed point reluctivity $\nu_{FP}$. In the second step

<table>
<thead>
<tr>
<th>$r$ [m]</th>
<th>$B_1$ [T]</th>
<th>$B_8$ [T]</th>
<th>$B_{25}$ [T]</th>
<th>$L$ [mH]</th>
<th>DOF</th>
<th>NoE</th>
<th>$t$ [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.66</td>
<td>0.60</td>
<td>1.35</td>
<td>368.69</td>
<td>120868</td>
<td>89481</td>
<td>301</td>
</tr>
<tr>
<td>0.3</td>
<td>1.68</td>
<td>0.60</td>
<td>1.3</td>
<td>373.78</td>
<td>127848</td>
<td>94668</td>
<td>316</td>
</tr>
<tr>
<td>0.5</td>
<td>1.70</td>
<td>0.60</td>
<td>1.27</td>
<td>374.49</td>
<td>131213</td>
<td>97196</td>
<td>393</td>
</tr>
<tr>
<td>1</td>
<td>1.71</td>
<td>0.61</td>
<td>1.27</td>
<td>375.59</td>
<td>128126</td>
<td>94891</td>
<td>489</td>
</tr>
<tr>
<td>1.5</td>
<td>1.71</td>
<td>0.61</td>
<td>1.27</td>
<td>375.57</td>
<td>129314</td>
<td>95757</td>
<td>585</td>
</tr>
<tr>
<td>2</td>
<td>1.71</td>
<td>0.62</td>
<td>1.28</td>
<td>373.2</td>
<td>130943</td>
<td>96996</td>
<td>582</td>
</tr>
</tbody>
</table>
Fig. 5.23. The experimental first magnetization curve is approximated by the Jiles-Atherton model with the parameters $M_s = 1.4 \cdot 10^6$ A/m, $\alpha = 7 \cdot 10^{-4}$, $a = 200$ A/m, $c=0.1$, $k=175$ A/m and the problem is solved by second order edge elements on a finer mesh with the above determined fixed point reluctivity. The used finite element mesh can be seen in Fig. 5.24.

![Finite Element Mesh](image)

**Fig. 5.24.** The finite element mesh

<table>
<thead>
<tr>
<th>$r$ [m]</th>
<th>$B_1$ [T]</th>
<th>$B_8$ [T]</th>
<th>$B_{25}$ [T]</th>
<th>$L$ [mH]</th>
<th>DOF</th>
<th>NoE</th>
<th>$t$ [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.57</td>
<td>0.64</td>
<td>1.18</td>
<td>381.95</td>
<td>68527</td>
<td></td>
<td>156</td>
</tr>
<tr>
<td>0.5</td>
<td>1.53</td>
<td>0.71</td>
<td>1.11</td>
<td>404.76</td>
<td>388899</td>
<td></td>
<td>1197</td>
</tr>
</tbody>
</table>

**Table 5.6.** Results based on the half geometry with Neumann condition
5.3.5 Solution with first order elements

The purpose of this investigation is to determine the best fixed point reluctivity based on a relatively coarse finite element mesh and with first order triangular elements. The benefit of this simplified problem is that a suitable reluctivity can be found in a short time and the further more accurate computations can be performed based on the fixed point reluctivity determined here.

![Figure 5.25](image) The measured [1] and simulated results with first order elements on a coarse finite element mesh

The problem has been solved by supposing different fixed-point reluctivities and the results have been summarized in Table 5.7.

![Table 5.7](image) Chosen relative reluctivity versus required iteration steps

<table>
<thead>
<tr>
<th>$\nu_{FP,r}$</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/100</td>
<td>93</td>
</tr>
<tr>
<td>1/500</td>
<td>57</td>
</tr>
<tr>
<td>1/1000</td>
<td>56</td>
</tr>
<tr>
<td>1/2000</td>
<td>divergent</td>
</tr>
</tbody>
</table>

The computational results obtained by a direct SPOOLES LU solver [69] with respect to the finite element mesh with 6608 elements and 9298 DOFs can be observed in Fig. 5.25. I have compared the simulation results with the measured data [1]. Indeed the results are not in so good agreement with the measured data, but they are in the same magnitude with the measurements and they are quantitatively correct. My purpose is to improve the accuracy of the method by introducing second order edge elements, which is done in the next section.

5.3.6 Solution with second order elements

The finite element mesh used in previous section has been a little bit improved. Now it consists of 13 706 tetrahedral elements. The total DOFs are 116 168 by means of the second order elements in Fig. 5.26.

The direct LU solver is not capable to solve such a huge problems like this. Therefore the GMRES\(^2\) method has been applied to solve the linearized system. The two levels geometric

\(^2\)General Mean Residual
**Fig. 5.26.** Magnetic field lines

![Magnetic field lines](image)

**Fig. 5.27.** The measured [1] and simulated results with second order elements

![Simulated vs Measured Results](image)

**Fig. 5.28.** Relative error of the measured and the simulated data

![Relative error](image)

multigrid method is applied as preconditioner. The lower level of the multigrid method is the first order approach of the problem, which can be solved by LU coarse solver since the DOFs
are 27 544 in this case. Moreover the top level of the multigrid method is the second order approach, which is determined by GMRES with the trial value extracted form the first order solution. In this way the convergence of the GMRES is very good only a few steps are required to achieve the solution. Furthermore it is important to note that the Vanka algorithm has been used as a pre-/postsmoother in the multigrid method [68].

Simulation results with higher order edge elements can be observed in Fig. 5.27. I have got better agreement between the simulation data and the measured ones. The average magnetic flux is along the vertical part of the "plate U" can be seen at left side. The magnetic flux along horizontal part of "plate U" can be observed in the middle and the flux in the middle "plate I" are plotted at right.

Important differences can be observed at two locations. One is where the "plate U" is connecting to the air domain $x = 0$ in the middle figure. The other one is where the "plate I" connects to the Neumann plane $z = 0$ in the figure at right.

The Table 5.8 below summarizes the computational details based on the table suggested on page 24.

**Table 5.8.** Description of computer program

<table>
<thead>
<tr>
<th>No.</th>
<th>Item</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Code name</td>
<td>TEAM13</td>
</tr>
<tr>
<td>2</td>
<td>Formulation</td>
<td>FEM (Finite Element method)</td>
</tr>
<tr>
<td>3</td>
<td>Governing equations</td>
<td>$(4.1)$</td>
</tr>
<tr>
<td>4</td>
<td>Solution variables</td>
<td>magnetic vector potential $\mathbf{A}$, Lagrange multipliers $\lambda$</td>
</tr>
<tr>
<td>5</td>
<td>Gauge condition</td>
<td>imposed by Lagrange multiplier method</td>
</tr>
<tr>
<td>6</td>
<td>Fraction of geometry</td>
<td>The half of the original geometry is considered by applying homogeneous Neumann condition on symmetry plane</td>
</tr>
<tr>
<td>7</td>
<td>Technique for nonlinear problem</td>
<td>Fixed-point method</td>
</tr>
<tr>
<td></td>
<td>Convergence criterion</td>
<td>$\varepsilon &lt; 10^{-8}$</td>
</tr>
<tr>
<td>8</td>
<td>Approximation method of $B$–$H$ curve</td>
<td>JAM</td>
</tr>
<tr>
<td>9</td>
<td>Technique for open boundary problem</td>
<td>Truncation</td>
</tr>
<tr>
<td>10</td>
<td>Calculation method of magnetic field produced by exciting current</td>
<td>Taking into account exciting current in governing equations directly</td>
</tr>
<tr>
<td>11</td>
<td>Property of coefficient matrix of linear equations</td>
<td>sparse, asymmetric</td>
</tr>
</tbody>
</table>
Table 5.8. Description of computer program

<table>
<thead>
<tr>
<th>No.</th>
<th>Item</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Solution method for linear equations</td>
<td>Linear system solver: GMRES; Pre-conditioner: Geometric multigrid with 2 levels (the first and second order elements); Presmoother: Vanka; Postsmoother: Vanka; Coarse solver: LU</td>
</tr>
<tr>
<td></td>
<td>Convergence criterion for iteration method</td>
<td>Relative tolerance of GMRES: $10^{-6}$</td>
</tr>
<tr>
<td>13</td>
<td>Element type</td>
<td>Tetrahedron</td>
</tr>
<tr>
<td>14</td>
<td>Number of elements</td>
<td>13,706</td>
</tr>
<tr>
<td>15</td>
<td>Number of nodes</td>
<td>2,497</td>
</tr>
<tr>
<td>16</td>
<td>Number of unknowns (DOFs)</td>
<td>116,168</td>
</tr>
<tr>
<td>17</td>
<td>Computer name</td>
<td>HP Compaq nx8220 portable computer</td>
</tr>
<tr>
<td></td>
<td>speed</td>
<td>MIPS: 7,396; MFlops: 3,071 FPU/3916 SSE2</td>
</tr>
<tr>
<td></td>
<td>main memory</td>
<td>2 GB</td>
</tr>
<tr>
<td></td>
<td>used memory</td>
<td>1.25 GB including the operation system</td>
</tr>
<tr>
<td></td>
<td>precision data (bits)</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>CPU time (sec)</td>
<td>total 2052 solving linear equations 36</td>
</tr>
</tbody>
</table>

5.3.7 Comparison

Same problem has been solved in the paper [70, 71], where a hybrid method described to solve magnetostatic problems, which combines Finite Formulation of Electromagnetic Fields (FFEF) with integral boundary conditions based on Green function applied to magnetization. The magnetostatic problem is formulated in terms of line integrals of magnetic vector potential while integral boundary conditions are obtained by magnetic scalar potential. The two potentials are coupled by the Ampere structural constraint. The method has been applied both to linear problems with magnetization sources inside the domain, and to nonlinear ones, handled by means of the Fixed Point technique which introduces fictitious magnetization sources.

I am grateful to Prof. Repetto for providing simulation data (literature data) [71] for this problem allowing the comparison between the method I used and his hybrid formulation method. The magnetic flux density in the middle of central column (Plate I) according to the literature data (at left in green) and the error between the literature and measured data (at
right) are plotted in Fig. 5.29. These can be compared by the results in Figs. 5.27 and 5.28. The simulation data were computed by 100 nonlinear iterations with the $L_2$ norm of $B$ error.

![Plate I](image1)

![Plate I](image2)

**Fig. 5.29.** Comparison with the literature

It is very hard to compare the literature values with my computation results as it can be seen in the Table 5.9, because I used finer finite element mesh and the obtained results are indeed more accurate. It is important to emphasize that less number of nonlinear iterations were required in my computation, but the CPU time is not known from the literature. Therefore the comparison of the computation time is not possible. The order of tetrahedral finite elements is also unknown from the literature.

<table>
<thead>
<tr>
<th></th>
<th>my value</th>
<th>literature values [71]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of finite elements</td>
<td>13 706</td>
<td>2688</td>
</tr>
<tr>
<td>Number of nonlinear iterations</td>
<td>57</td>
<td>100</td>
</tr>
<tr>
<td>Relative error</td>
<td>$-5%\ldots7%$</td>
<td>$-10%\ldots-70%$</td>
</tr>
<tr>
<td>Convergence criterion</td>
<td>$10^{-8}$ for $A$</td>
<td>0.45 · $10^{-5}$ for $B$</td>
</tr>
</tbody>
</table>

*Table 5.9. Comparison between literature values and my results*
5.4 Thesis 3

*I have confirmed the nonlinear field computational method with higher order finite elements by examples regarding magnetostatics, eddy current problems and diffusion equation. I have worked out three examples; one for the solution of one dimensional diffusion equation for ferromagnetic half-space, another one for the solution of a two dimensional (axisymmetric) eddy current problem for hysteresis measurement simulation and the third one for the solution of a three dimensional magnetostatic problem for the benchmark problem of COMPUMAG.*

a) I have illustrated the nonlinear simulation method on the one dimensional half-space problem. I have applied the nonlinear iteration method developed in the previous chapter for solving the half-space problem with the Langevin characteristics and with the Jiles-Atheron model, as well. The problem has been solved by the original fixed-point method and the fixed-point method with relaxation (R), which is different from the Hantila’s over-relaxation method [52] since $0 < \omega < 1$ is allowed and finally this FP+R method is improved by the local update of the permeability obtaining significant speed up.

b) The simulation of the ferromagnetic hysteresis measurement known from section 3.4 has been worked out by using a two dimensional axisymmetric model. The simulations have been carried out at different frequencies using both the rate-independent and the rate-dependent JAM.

c) I have developed the three dimensional nonlinear simulation technique for the T.E.A.M 13 test problem of COMPUMAG. I have used edge finite element method and $A$-vector potential formalism in combination with Lagrange multipliers $\lambda$. I have validated the simulation of the three dimensional nonlinear test problem by the available measurements. I have combined the higher order edge finite element method with the fixed-point method for solving this three dimensional magnetostatic problem. I have used geometric multigrid method for preconditioning the system by utilizing the hierarchical property of the introduced higher order edge elements. The coarse grid is represented by the DOFs of first order elements though the finer grid is defined by the DOFs of second order elements.
Chapter 6

Summary of Theses

Thesis 1

*I have extended the JAM of hysteresis to avoid the non-physical solutions and to represent the rate-dependent variation of the response with the input for the direct H-B and for the inverse B-H characteristics as well.*

a) I have developed an engineering approach based on an energy balance equation for obtaining the differential equation of the rate-independent Jiles-Atherton model. I have introduced a parameter $\delta_M$ to avoid the non-physical solution (negative slope) of the model differential equation. I have validated the developed rate-independent model by experiments.

b) I have extended the energy balance equation of the rate-independent Jiles-Atherton model by two additional energy (eddy current and anomalous loss) terms to take into account the time variation of excitation magnetic field. I have validated the developed rate-dependent model by experiments.

c) I have built a magnetic measurement system for measuring magnetic hysteresis in LabVIEW environment. I have proposed measurement method for measuring hysteresis loops, minor loops, first order reversal curves and I have worked out an under-relaxation method to obtain sinusoidal waveform in $B$.

d) I have worked out a procedure for parameter identification of both the rate-dependent and rate independent Jiles-Atherton model of hysteresis based on the nonlinear least square method.

Thesis 2

*I have implemented the hysteresis simulation to the higher order vector shape functions of FEM for solving magnetostatic and eddy current problems and to handle the nonlinear iteration and I have worked out an efficient speed up technique of the fixed point method by combining the*
relaxation method and the local permeability update.

a) I have extended the field equations for nonlinear magnetostatic and eddy current problems considering the gauge fixing with the Lagrange multiplier method. I have proposed formulations by assuming general hysteretic relationship between the $H$ and $B$ to offer for researchers the ability to use any type of hysteresis model and numerical method.

b) To handle the nonlinear iteration I have worked out the relaxation method with the permeability updates to speed up the fixed-point iteration process.

c) I have combined the solution of the field equations with higher order Schöberl type edge elements for nonlinear case. The introduced element type allows to choose finite elements with arbitrary polynomial degree utilizing the local complete sequence property.

 Thesis 3

I have confirmed the nonlinear field computational method with higher order finite elements by examples regarding magnetostatics, eddy current problems and diffusion equation. I have worked out three examples; one for the solution of one dimensional diffusion equation for ferromagnetic half-space, another one for the solution of a two dimensional (axisymmetric) eddy current problem for hysteresis measurement simulation and the third one for the solution of a three dimensional magnetostatic problem for the benchmark problem of COMPUMAG.

a) I have illustrated the nonlinear simulation method on the one dimensional half-space problem. I have applied the nonlinear iteration method developed in the previous chapter for solving the half-space problem with the Langevin characteristics and with the Jiles-Atheron model, as well. The problem has been solved by the original fixed-point method and the fixed-point method with relaxation (R), which is different from the Hantila’s over-relaxation method [52] since $0 < \omega < 1$ is allowed and finally this FP+R method is improved by the local update of the permeability obtaining significant speed up.

b) The simulation of the ferromagnetic hysteresis measurement known from section 3.4 has been worked out by using a two dimensional axisymmetric model. The simulations have been carried out at different frequencies using both the rate-independent and the rate-dependent JAM.

c) I have developed the three dimensional nonlinear simulation technique for the T.E.A.M 13 test problem of COMPUMAG. I have used edge finite element method and $A$ vector potential formalism in combination with Lagrange multipliers $\lambda$. I have validated the simulation of the three dimensional nonlinear test problem by the available measurements. I have combined the higher order edge finite element method with the fixed-point method for solving this three dimensional magnetostatic problem. I have used geometric multigrid method for preconditioning the system by utilizing the hierarchical property of the introduced higher order edge elements. The coarse grid is represented by the DOFs of first order elements though the finer grid is defined by the DOFs of second order elements.
Chapter 7

Conclusions and future work

The purpose of this work is to find an efficient way to compute magnetic fields in the presence of highly nonlinear even hysteretic materials. Both magnetostatic and eddy current fields were investigated.

Further investigations could be done by choosing hysteresis models other than the Jiles-Atherton. It would be very interesting to apply the Preisach model of hysteresis.

A different approach could be developed to take into account the ferromagnetic hysteresis, because these hysteresis models (Preisach, Jiles-Atherton etc.) were developed not for finite element computations. We have a finite element mesh for discretizing the electromagnetic field equations but same mesh could be applied for solving the equation(s) of the hysteresis model. The currently used hysteresis models do not utilize that the region is discretized by the finite element method. Both the electromagnetic field equations and the differential equations of the material could be solved simultaneously on the same finite element mesh as a coupled problem. In this way a more sophisticated approach could be achieved. The geometrical shape, the temperature of the material, the interaction between the magnetic domains and the effect of the mechanical stress could be considered by this coupled problem approach.

An interactive user friendly CAD software package should be developed for solving electromagnetic field equations in the presence of nonlinear magnetic material, which would be useful in designing electrical machines or power transformers.

In my further investigations I would like to deal with the chaotic behavior of ferromagnetic hysteresis.
Appendix A

Results of the half-space problem with Langevin function

Fig. A.1. Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at t=0.5 ms

Fig. A.2. Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at t=5 ms
Fig. A.3. Solutions with Langevin characteristics, $H_{\text{amp}} = 100 \, \text{A/m}, 1000 \, \text{A/m}, 10000 \, \text{A/m}$ at $t=10 \, \text{ms}$

Fig. A.4. Solutions with Langevin characteristics, $H_{\text{amp}} = 100 \, \text{A/m}, 1000 \, \text{A/m}, 10000 \, \text{A/m}$ at $t=15 \, \text{ms}$

Fig. A.5. Solutions with Langevin characteristics, $H_{\text{amp}} = 100 \, \text{A/m}, 1000 \, \text{A/m}, 10000 \, \text{A/m}$ at $t=20 \, \text{ms}$
Fig. A.6. Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=25$ ms

Fig. A.7. Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=30$ ms

Fig. A.8. Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=35$ ms
Fig. A.9. Solutions with Langevin characteristics, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=40$ ms.
Appendix B

Results of the half-space problem with modified JAM

Fig. B.1. Solutions with JAM, $H_{\text{amp}} = 100 \text{ A/m}, 1000 \text{ A/m}, 10000 \text{ A/m}$ at $t=0.5\text{ ms}$

Fig. B.2. Solutions with JAM, $H_{\text{amp}} = 100 \text{ A/m}, 1000 \text{ A/m}, 10000 \text{ A/m}$ at $t=5\text{ ms}$
Fig. B.3. Solutions with JAM, $H_{amp} = 100$ A/m, 1000 A/m, 10000 A/m at $t=10$ ms

Fig. B.4. Solutions with JAM, $H_{amp} = 100$ A/m, 1000 A/m, 10000 A/m at $t=15$ ms

Fig. B.5. Solutions with JAM, $H_{amp} = 100$ A/m, 1000 A/m, 10000 A/m at $t=20$ ms
Fig. B.6. Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=25$ ms

Fig. B.7. Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=30$ ms

Fig. B.8. Solutions with JAM, $H_{\text{amp}} = 100$ A/m, 1000 A/m, 10000 A/m at $t=35$ ms
Fig. B.9. Solutions with JAM, $H_{amp} = 100$ A/m, 1000 A/m, 10000 A/m at $t=40$ ms
Appendix C

Hysteresis measurement simulation at 0.2 Hz

Simulation results of the hysteresis measurement simulation is presented, which are detailed in section 5.2. The magnetic flux density is plotted in Tesla. The horizontal and vertical cylindrical-coordinate axes are $r$ [m] and $z$ [m] respectively.
Appendix D

Hysteresis measurement simulation at 5 Hz

Simulation results of the hysteresis measurement simulation are presented, which are detailed in section 5.2. The magnetic flux density is plotted in Tesla. The horizontal and vertical cylindrical-coordinate axes are $r$ [m] and $z$ [m] respectively.
Appendix E

Problem T.E.A.M. 13

Team Problem 13

3-D Non-Linear Magnetostatic Model

1. General Description

The model is shown in Fig.1. An exciting coil is set between two steel channels, and a steel plate is inserted between the channels. The coil is excited by dc current. The ampere turns are 1000 and 3000 AT which is sufficient to saturate the steel. The problem is to calculate magnetic fields at various positions.

2. Analyzed Region and Boundary Conditions

If the symmetrical and periodic boundary conditions can be used, the 1/4 region shown in Fig.2(a) is enough to be analyzed. The analysis of 1/2 region shown in Fig.2(b) using only symmetrical boundary condition is also acceptable.

3. Mesh Description

The mesh is not specified.

4. Nonlinearity

The B-H curve of the steel shown in Fig.3 is to be used. The typical values of B(T) and H(A/m) are also shown in Fig.3. The curve for high flux densities (B>1.8T) should be approximated by Eq.(1):

\[
B = \mu_0 H + (aH^2 + bH + c) \quad (1.8 \leq B \leq 2.22T)
\]

\[
B = \mu_0 H + M_s \quad (B \geq 2.22T)
\]

where \(\mu_0\) is the permeability of free space. The constants a, b and c are -2.381x10^{-10}, 2.327x10^{-5}, and 1.590 respectively. Ms is the saturation magnetization (2.16T) of the steel. Equation (1) shows that the steel part is assumed to be completely saturated when B is higher than 2.22T.

5. Quantities and Distributions to be Calculated

5a. Points where flux densities are compared

To compare results, please complete Tables 1, 2 and 3. Fig.4 shows the specified positions for average flux density in the steel and flux density in the air. Fig.5 shows the recommended points to be compared. The points 1 to 4 are for comparison between various numerical methods of analysis. The points...
where large errors may occur, such as due to large flux density changes, are chosen. The points 5 to 8 show the recommended points to be compared with experiment. Around these points, flux densities can be measured accurately because the gradient of flux density is not so high.

5b. Distributions of flux density vectors
Distributions of flux density vectors on the x-y plane at z=63.2mm, and on the y-z plane at x=0mm are to be presented.

6. Description of Computer Program
To compare formulations, variables, etc., please complete Table 4. The used memory in the item No.17 in Table 4 is defined as the sum of dimensions declared in the program.

7. References

Fig. 1. 3-D nonlinear magnetostatic model
Fig. 2. Analyzed region

(a) 1/4 region  (b) 1/2 region
Fig. 3. B-H curve of steel
Fig. 4. Specified positions for flux density (see Tables 1 and 2)
Fig. 5. Recommended points to be compared (see Table 1, 2, and 3)

Comparison between various numerical methods:
* the points where the flux densities change suddenly \( \ldots \ldots \cdot 1, 2 \)
* the point where the permeability changes suddenly \( \ldots \ldots \cdot 3 \)
* the point where the error due to the cancellation may be large \( \ldots \ldots \cdot 4 \)

Comparison between calculation and experiment:
* the average flux densities \( \ldots \ldots \cdot 6, 7, 7 \) (No.1, 12 and 23)
* the point where the flux density is high and it does not change suddenly \( \ldots \ldots \cdot 8 \) (No.27)
Table 1 Average flux density $|\mathbf{B}|$ (T) in the steel (see Fig.4)

<table>
<thead>
<tr>
<th>No.</th>
<th>x mm</th>
<th>y mm</th>
<th>z mm</th>
<th>1000</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>$\leq 25.0$</td>
<td>0.0</td>
<td>10.0</td>
<td>20.0</td>
</tr>
<tr>
<td>2</td>
<td>10.0</td>
<td>$\leq z\leq 25.0$</td>
<td>30.0</td>
<td>40.0</td>
<td>50.0</td>
</tr>
<tr>
<td>3</td>
<td>20.0</td>
<td>$\leq z\leq 25.0$</td>
<td>60.0</td>
<td>70.0</td>
<td>80.0</td>
</tr>
<tr>
<td>4</td>
<td>30.0</td>
<td>$\leq z\leq 25.0$</td>
<td>90.0</td>
<td>100.0</td>
<td>110.0</td>
</tr>
<tr>
<td>5</td>
<td>40.0</td>
<td>$\leq z\leq 25.0$</td>
<td>120.0</td>
<td>130.0</td>
<td>140.0</td>
</tr>
<tr>
<td>6</td>
<td>50.0</td>
<td>$\leq z\leq 25.0$</td>
<td>150.0</td>
<td>160.0</td>
<td>170.0</td>
</tr>
<tr>
<td>7</td>
<td>60.0</td>
<td>$\leq z\leq 25.0$</td>
<td>180.0</td>
<td>190.0</td>
<td>200.0</td>
</tr>
<tr>
<td>8</td>
<td>70.0</td>
<td>$\leq z\leq 25.0$</td>
<td>210.0</td>
<td>220.0</td>
<td>230.0</td>
</tr>
<tr>
<td>9</td>
<td>80.0</td>
<td>$\leq z\leq 25.0$</td>
<td>240.0</td>
<td>250.0</td>
<td>260.0</td>
</tr>
<tr>
<td>10</td>
<td>90.0</td>
<td>$\leq z\leq 25.0$</td>
<td>270.0</td>
<td>280.0</td>
<td>290.0</td>
</tr>
<tr>
<td>11</td>
<td>100.0</td>
<td>$\leq z\leq 25.0$</td>
<td>300.0</td>
<td>310.0</td>
<td>320.0</td>
</tr>
<tr>
<td>12</td>
<td>110.0</td>
<td>$\leq z\leq 25.0$</td>
<td>330.0</td>
<td>340.0</td>
<td>350.0</td>
</tr>
<tr>
<td>13</td>
<td>120.0</td>
<td>$\leq z\leq 25.0$</td>
<td>360.0</td>
<td>370.0</td>
<td>380.0</td>
</tr>
<tr>
<td>14</td>
<td>130.0</td>
<td>$\leq z\leq 25.0$</td>
<td>390.0</td>
<td>400.0</td>
<td>410.0</td>
</tr>
<tr>
<td>15</td>
<td>140.0</td>
<td>$\leq z\leq 25.0$</td>
<td>420.0</td>
<td>430.0</td>
<td>440.0</td>
</tr>
<tr>
<td>16</td>
<td>150.0</td>
<td>$\leq z\leq 25.0$</td>
<td>450.0</td>
<td>460.0</td>
<td>470.0</td>
</tr>
<tr>
<td>17</td>
<td>160.0</td>
<td>$\leq z\leq 25.0$</td>
<td>480.0</td>
<td>490.0</td>
<td>500.0</td>
</tr>
<tr>
<td>18</td>
<td>170.0</td>
<td>$\leq z\leq 25.0$</td>
<td>510.0</td>
<td>520.0</td>
<td>530.0</td>
</tr>
<tr>
<td>19</td>
<td>180.0</td>
<td>$\leq z\leq 25.0$</td>
<td>540.0</td>
<td>550.0</td>
<td>560.0</td>
</tr>
<tr>
<td>20</td>
<td>190.0</td>
<td>$\leq z\leq 25.0$</td>
<td>570.0</td>
<td>580.0</td>
<td>590.0</td>
</tr>
<tr>
<td>21</td>
<td>200.0</td>
<td>$\leq z\leq 25.0$</td>
<td>600.0</td>
<td>610.0</td>
<td>620.0</td>
</tr>
<tr>
<td>22</td>
<td>210.0</td>
<td>$\leq z\leq 25.0$</td>
<td>630.0</td>
<td>640.0</td>
<td>650.0</td>
</tr>
<tr>
<td>23</td>
<td>220.0</td>
<td>$\leq z\leq 25.0$</td>
<td>660.0</td>
<td>670.0</td>
<td>680.0</td>
</tr>
<tr>
<td>24</td>
<td>230.0</td>
<td>$\leq z\leq 25.0$</td>
<td>690.0</td>
<td>700.0</td>
<td>710.0</td>
</tr>
<tr>
<td>25</td>
<td>240.0</td>
<td>$\leq z\leq 25.0$</td>
<td>720.0</td>
<td>730.0</td>
<td>740.0</td>
</tr>
</tbody>
</table>
Table 2 Flux density $|B| (T)$ (see Fig.4)

<table>
<thead>
<tr>
<th>No.</th>
<th>coordinates (mm)</th>
<th>ampere turn (AT)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>26</td>
<td>10.0</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>20.0</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>30.0</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>40.0</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>50.0</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>60.0</td>
<td>20.0</td>
</tr>
<tr>
<td>32</td>
<td>70.0</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>80.0</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>90.0</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>100.0</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>110.0</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 Flux density $|B| (T)$ (see Fig.5)

<table>
<thead>
<tr>
<th>No.</th>
<th>coordinates (mm)</th>
<th>ampere turn (AT)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>①</td>
<td>2.2</td>
<td>15.1</td>
</tr>
<tr>
<td>②</td>
<td>2.0</td>
<td>14.9</td>
</tr>
<tr>
<td>③</td>
<td>1.5</td>
<td>0.0</td>
</tr>
<tr>
<td>④</td>
<td>1.5</td>
<td>0.0</td>
</tr>
</tbody>
</table>

The points ⑤ to ⑧ are for comparison between various numerical methods of analysis. The points where large errors may occur, such as due to large flux density changes, are chosen.

The points ⑤ to ⑧ show the recommended po
Table 4 Description of computer program

<table>
<thead>
<tr>
<th>No.</th>
<th>Item</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Code name</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Formulation</td>
<td>1. FEM (Finite Element Method)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. BEM (Boundary Element Method)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3. IEM (Integral Equation Method)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. FDM (Finite Difference Method)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5. combination (               +               )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6. others (                          )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Please write references in item No.18)</td>
</tr>
<tr>
<td>3</td>
<td>Governing equations</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Solution variables</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Gauge condition</td>
<td>1. not imposed</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. imposed</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a) impose the condition on governing equations directly</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) penalty function method</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c) Lagrange multiplier method</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(d) others (                          )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Please write references in item No.18)</td>
</tr>
<tr>
<td>6</td>
<td>Fraction of geometry</td>
<td>1. 1/4 [1]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. 1/2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. Modified Newton-Raphson-method</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3. Incremental method</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. SOR[4]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5. others (                          )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Please write references in item No.18)</td>
</tr>
<tr>
<td></td>
<td>Convergence criterion</td>
<td></td>
</tr>
<tr>
<td>No.</td>
<td>Item</td>
<td>Specification</td>
</tr>
<tr>
<td>-----</td>
<td>------</td>
<td>---------------</td>
</tr>
</tbody>
</table>
| 8   | Approximation method of B-H curve | 1. spline  
3. straight lines  
4. others (please write references in item No.18) |
| 9   | Technique for open boundary problem [6] | 1. truncation  
2. mapping  
3. ballooning  
4. Zienkiewicz's infinite element  
5. Tong’s infinite element[7]  
6. BEM or IEM  
7. others (please write references in item No.18) |
| 10  | Calculation method of magnetic field produced by exciting current | 1. Biot-Savart law (analytical)  
2. Biot-Savart law (numerical)  
3. taking into account exciting current in governing equations directly |
| 11  | Property of coefficient matrix of linear equations | 1. symmetric  
(la) sparse  
(lb) full  
2. asymmetric  
(2a) sparse  
(2b) full  
3. combination |
| 12  | Solution method for linear equations | 1. ICCG  
2. ILUBCG  
3. ILUCGS[7]  
4. SOR  
5. LDL^T  
6. LU  
7. Gauss elimination method  
8. others (please write references in item No.18) |
<p>|     | Convergence criterion for iteration method |</p>
<table>
<thead>
<tr>
<th>No.</th>
<th>Item</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>Element type</td>
<td>1. tetrahedron</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. triangular prism</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3. hexahedron</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. triangle</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5. rectangle</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6. others (please write references in item No.18)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1. nodal element (nodes)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. edge element (edges)</td>
</tr>
<tr>
<td>14</td>
<td>Number of elements</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Number of nodes</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Number of unknowns</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Computer name</td>
<td>speed (MIPS), (MFLOPS).</td>
</tr>
<tr>
<td></td>
<td></td>
<td>main memory (MB)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>used memory (MB)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>precision of data (bits)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CPU time (sec) total solving linear equations</td>
</tr>
<tr>
<td>18</td>
<td>References on Nos.1 to 13, etc.</td>
<td></td>
</tr>
</tbody>
</table>
## Average flux density |B| in steel plate (1000AT, measured)

| No. | x       | y       | z       | average flux density |B|(T) | G=0.52(mm) | G=0.47(mm) |
|-----|---------|---------|---------|--------------------|-----|------------|------------|
| 1   |         |         | 0.0     | 1.333              | 1.354 |
| 2   | 1.6     | 0.0     | 0.0     | 1.329              | 1.339 |
| 3   |         | 10.0    | 0.0     | 1.286              | 1.304 |
| 4   | 0.0     | 20.0    | 0.0     | 1.252              | 1.245 |
| 5   | 25.0    | 30.0    | 0.0     | 1.129              | 1.138 |
| 6   | 60.0    | 40.0    | 0.0     | 0.985              | 0.982 |
| 7   | 60.0    | 50.0    | 0.0     | 0.985              | 0.982 |
| 8   | 60.0    | 60.0    | 0.0     | 0.655              | 0.674 |
| 9   | 2.1     | 10.0    | 0.0     | 0.259              | 0.263 |
| 10  | 20.0    | 15.0    | 60.0    | 0.453              | 0.451 |
| 11  | 30.0    | 60.0    | 60.0    | 0.554              | 0.563 |
| 12  | 40.0    | 60.0    | 60.0    | 0.637              | 0.641 |
| 13  | 50.0    | 60.0    | 60.0    | 0.698              | 0.706 |
| 14  | 60.0    | 60.0    | 60.0    | 0.755              | 0.763 |
| 15  | 80.0    | 60.0    | 60.0    | 0.809              | 0.819 |
| 16  | 100.0   | 60.0    | 60.0    | 0.901              | 0.907 |
| 17  | 110.0   | 60.0    | 60.0    | 0.954              | 0.958 |
| 18  | 122.1   | 60.0    | 60.0    | 0.956              | 0.968 |
| 19  | 122.1   | 15.0    | 60.0    | 0.960              | 0.971 |
| 20  | 122.1   | 30.0    | 60.0    | 0.965              | 0.973 |
| 21  | 122.1   | 40.0    | 60.0    | 0.970              | 0.982 |
| 22  | 122.1   | 50.0    | 60.0    | 0.974              | 0.985 |
| 23  | 122.1   | 60.0    | 60.0    | 0.981              | 0.991 |
| 24  | 122.1   | 70.0    | 60.0    | 0.984              | 0.995 |
| 25  | 122.1   | 80.0    | 60.0    | 0.985              | 0.995 |
The curve for high flux densities ($B > 1.8T$) is approximated as follows:

$$\mu_0 H + (aH^2 + bH + c) \quad (1.8T \leq B \leq 2.22T)$$

$$\mu_0 H + M_s \quad (B \geq 2.22T)$$

$\mu_0$ : permeability of free space

$a$ : $-2.822 \times 10^{-10}$

$b$ : $2.529 \times 10^{-5}$

$c$ : $1.591$

$M_s$ : saturation magnetization ($2.16T$)

B-H curve of steel.
<table>
<thead>
<tr>
<th>No.</th>
<th>B (T)</th>
<th>H (A / m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.025</td>
<td>45</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>75</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>120</td>
</tr>
<tr>
<td>5</td>
<td>0.20</td>
<td>173</td>
</tr>
<tr>
<td>6</td>
<td>0.30</td>
<td>201</td>
</tr>
<tr>
<td>7</td>
<td>0.40</td>
<td>222</td>
</tr>
<tr>
<td>8</td>
<td>0.50</td>
<td>240</td>
</tr>
<tr>
<td>9</td>
<td>0.60</td>
<td>250</td>
</tr>
<tr>
<td>10</td>
<td>0.70</td>
<td>265</td>
</tr>
<tr>
<td>11</td>
<td>0.80</td>
<td>280</td>
</tr>
<tr>
<td>12</td>
<td>0.90</td>
<td>300</td>
</tr>
<tr>
<td>13</td>
<td>1.00</td>
<td>330</td>
</tr>
<tr>
<td>14</td>
<td>1.10</td>
<td>365</td>
</tr>
<tr>
<td>15</td>
<td>1.20</td>
<td>415</td>
</tr>
<tr>
<td>16</td>
<td>1.30</td>
<td>500</td>
</tr>
<tr>
<td>17</td>
<td>1.40</td>
<td>640</td>
</tr>
<tr>
<td>18</td>
<td>1.50</td>
<td>890</td>
</tr>
<tr>
<td>19</td>
<td>1.55</td>
<td>1150</td>
</tr>
<tr>
<td>20</td>
<td>1.60</td>
<td>1940</td>
</tr>
<tr>
<td>21</td>
<td>1.65</td>
<td>3100</td>
</tr>
<tr>
<td>22</td>
<td>1.70</td>
<td>4370</td>
</tr>
<tr>
<td>23</td>
<td>1.75</td>
<td>6347</td>
</tr>
<tr>
<td>24</td>
<td>1.80</td>
<td>8655</td>
</tr>
</tbody>
</table>
Appendix F

The geometric multigrid method

The geometric multigrid solver or preconditioner is a fast and memory-efficient iterative method for elliptic and parabolic models. It performs one or several cycles of the geometric multigrid method. The classical multigrid algorithm uses one or several auxiliary meshes that are coarser than the original (fine) mesh. The idea is to perform just a fraction of the computations on the fine mesh. Instead, it performs computations on the coarser meshes when possible, which leads to fewer operations. The size of the extra memory used for the coarser meshes and associated matrices is comparable to the size of the original data. This leads to an iterative algorithm that is both fast and memory efficient.

A hierarchy of multigrid levels can be used where each corresponds to a mesh and a choice of shape functions. Thus, in addition to coarsening the mesh it is possible to construct a new “coarser” level by lowering the order of the shape functions. The number of degrees of freedom decreases when you go to a coarser multigrid level. There is also an option to generate the finer meshes from the coarsest mesh by successive mesh refinements, which leads to aligned (nested) meshes.

To describe the multigrid algorithm, assume that you have $N + 1$ multigrid levels numbered from 0 to $N$, where 0 is the finest level (the level for which you seek the solution). To solve the linear system $A_0 x = b$ (corresponding to level 0), the algorithm must reform the system matrices $A_1, \ldots, A_N$ for the coarse multigrid levels. It must also compute the prolongation matrices $P_i$ that map a solution $x$ vector on level $i$ to the corresponding solution vector $P_i x$ on the next finer level $i - 1$.

The prolongation matrices are constructed using plain interpolation from one multigrid level to the other. The system matrices for the coarse levels can be constructed in two ways:

- By assembling $A_i$ on the mesh of level $i$.
- By projection from the finer level: $A_i = P_i^T A_{i-1} P_i$ where $T$ denotes a matrix transposition. This is also called the Galerkin method. It typically leads to more nonzero elements in the system matrix $A_i$, but the convergence can be faster than in the previous method.

The following algorithm describes one multigrid cycle:

1. The input to the algorithm is some initial guess $x_0$ to the solution of the system $A_0 x = b$.
2. Starting with $x_0$, apply a few iterations of a presmoother to the linear system $A_0 x = b$. 

yielding a more accurate iterate \( x_{0s} \). Typically the presmoother is some simple iterative algorithm such as SOR, but you also chose an arbitrary iterative solver.

3. Compute the residual \( r_0 = b - A_0 x_{0s} \). The presmoother “smooths” the residual so the oscillations in \( r \) have such a long wavelength that they are well resolved on the next coarser level (1). Therefore, project the residual onto level 1 by applying the transpose of the prolongation: \( r_1 = P^T_1 r_0 \).

4. If \( N = 1 \) the coarse solver can be used to solve the system \( A_1 x_1 = r_1 \). The coarse solver is typically a direct solver. The number of degrees of freedom on level 1 is less than for level 0, which means that solving \( A_1 x_1 = r_1 \) is less expensive. If instead \( N > 1 \), solve the system \( A_1 x_1 = r_1 \) (approximately) by recursively applying one cycle of the multigrid algorithm for levels 1, 2, \ldots, \( N \). In both cases the obtained solution \( x_1 \) is called the coarse grid correction.

5. Map the coarse grid correction to level 0 using the prolongation matrix: \( x_{0c} = x_{0s} + P_1 x_1 \).

6. Starting with \( x_{0c} \), a few iterations of a postsmoother must be applied to the linear system \( A_0 x = b \), yielding a more accurate iterate \( x_{0mg} \). The iterate \( x_{0mg} \) is the output of the multigrid cycle.

The cycle just described is called the V-cycle. The recursive call in Step 4 (when \( N > 1 \)) is also a V-cycle. For the W-cycle and the F-cycle, the Steps 1-6 above are the same but with the twist that the recursive call in Step 4 is substituted with two multigrid calls for the coarser levels. For the W-cycle these two calls are recursive calls, they are W-cycle calls. For the F-cycle the first call is a W-cycle and the second a V-cycle.

For only two multigrid levels (\( N = 1 \)) these cycles are the same, because the algorithm uses the coarse solver in Step 4. Also note that the amount of work on the finest level is the same for the different cycles. Normally the V-cycle is sufficient, but the W-cycle and the F-cycle can be useful for more difficult problems.

When using multigrid as a preconditioner on a right-hand side \( b \), the preconditioned solution \( M^{-1} b \) is obtained by applying a fixed number of multigrid cycles starting with the initial guess 0. When using multigrid as a solver, the multigrid cycle repeats until it reaches convergence.

When using multigrid as a preconditioner for the conjugate gradients method for a symmetric matrix \( A \), the preconditioning matrix \( M \) should also be symmetric. This requirement is fulfilled if the matrices \( M \) associated with the presmoother and the postsmoother are transposes of each other. For instance, this is the case if the presmoother is SOR and the postsmoother is SORU, and if the same number of smoothing steps is used.
Appendix G

The Vanka Algorithm

This preconditioner/smoother is intended for, but not restricted to, problems involving the Navier-Stokes equations. Formally it applies to saddle-point problems. A saddle-point problem is a problem where the (equilibrium) solution is neither a maximum nor a minimum. The corresponding linear system matrix is indefinite, and often it has zeros on the diagonal. This is the case for the Navier-Stokes equations but also for problems formulated with weak constraints.

The algorithm is a local smoother/preconditioner of Vanka type. It is based on the ideas in [68]. It is possible to describe it as a block SOR method, where the local coupling of the degrees of freedom (DOFs) determines the blocks. The important idea in this algorithm is to use the Lagrange multiplier variable (or set of variables) to form the blocks. For illustration purposes, consider the Navier-Stokes equations. For these equations the pressure variable plays the role of Lagrange multiplier. The linearized equations on discrete form has the following structure:

\[ A \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} S & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} \] (G.1)

where \( U \) and \( P \) are the velocity and pressure degrees of freedom, respectively. The algorithm loops over the Lagrange multiplier variable DOFs, here the pressure DOFs \( P_j \), and finds the direct connectivity to this DOF. To do so, the algorithm locates the nonzero entries in the matrix column corresponding to \( P_j \). The row indices of the nonzero entries defines the DOFs \( U_k \), and the software forms a local block matrix based on this connectivity:

\[ A_j = \begin{bmatrix} S_j & D_j^T \\ D_j & 0 \end{bmatrix} \] (G.2)

One Vanka update loops over all \( P_j \) and updates

\[ \begin{bmatrix} U_j \\ P_j \end{bmatrix} \leftarrow \begin{bmatrix} U_j \\ P_j \end{bmatrix} + \omega A_j^{-1} \left( \begin{bmatrix} F \\ G \end{bmatrix} - A \begin{bmatrix} U \\ P \end{bmatrix} \right)_j \] (G.3)

where the \((.)_j\) denotes the restriction of a vector to the rows corresponding to the block \( j \). \( \omega \) is a relaxation parameter. The algorithm does not form the inverses of the block matrices.
explicitly. Instead, it computes the Vanka update either with a direct solver subroutine call or by a GMRES iterative method subroutine call. The GMRES method is the restarted GMRES without preconditioning. The algorithm relies on that it is possible to invert the submatrices $A_j$. If it is not possible, the algorithm fails. Note that a zero on the diagonal of $A$ or $A_j$ is not necessarily a problem for this updating strategy.

In general, the Vanka update does not necessarily update all DOFs. This is, for example, the case for problems with weak constraints, where only a small subset of the problem’s DOFs are directly coupled to the Lagrange multipliers for the constraints. Another example is the Navier-Stokes equations (or similar types of equations) coupled to other equations, but where the coupling is not directly through the pressure variable. The Vanka algorithm automatically detects DOFs that are not updated by the above Vanka updating procedure and performs, for each Vanka update, a number of SSOR sweeps for these DOFs. This part of the algorithm is denoted the SSOR update. The SSOR update only works for a submatrix that has a nonzero diagonal. Just as the SOR and Jacobi preconditioner algorithms, this fails if it finds zeros on the diagonal for the DOFs in the SSOR update.
References


