# Non-perturbative Methods in Quantum Field Theories

PhD Thesis

**Péter Mati**

*Supervisor:* Prof. Antal Jakovác

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
Department of Theoretical Physics
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# 1 Introduction

In the subatomic world the physical objects behave in such ways that seem completely bizarre from the human perspective, and at some point we even lose our intuition based on everyday classical physics. The mathematical model which was manifested under the name of Quantum Field Theory (QFT), has proved to be the most successful strategy in the description of elementary particle interactions. Nowadays, it is regarded as a fundamental part of the contemporary theoretical physics, however, in most textbooks the emphasis is on the effectiveness of the theory, which at present essentially means perturbative QFT. Undoubtedly, the perturbative description of Quantum Electrodynamics (QED) and of the Standard Model (SM) of electroweak interactions had an enormous success in giving theoretical predictions to experimental results with an extraordinary accuracy. To mention one example: the precision that is achieved by the prediction in QED of the magnetic moment of the electron is one part in $10^{10}$ [1]. However, one must not consider Perturbation Theory (PT) as the fundamental definition of QFT, rather it must be viewed as a systematic technique to approximate the full theory, taking into account the errors in a well-controlled way. On the other hand, there are serious mathematical problems: not only that the perturbation series may not converge for most of the theories, but there are QFTs for which they are not even asymptotic series and we know that they do not converge. Therefore, a non-perturbative description seems to be indispensable to capture the true qualitative nature of such situations, for instance in the case of strongly interacting physical systems.

The Bloch-Nordsieck (BN) model (1937) [2] was build in order to get rid of the infrared singularities by which QED is being plagued. The infrared catastrophe occurs as the consequence of the presence of the massless gauge fields (photons). The most simple illustration of the problem can be already observed in a semi-classical level: to a finite amount of radiated electromagnetic energy an infinite number of soft photons can be associated. On the level of the PT in QED this phenomenon can be observed as a very bad behaviour of the perturbation series: every correction term diverges logarithmically, so there is no hope to extract physically sensible results from the series. However, in the case of the BN model there exists a non-perturbative way to compute the full fermion propagator (cf. [2, 3]), hence the model can be considered as non-perturbatively solvable. Thus, it provides a perfect opportunity to test other non-perturbative techniques and compare the results to the exact one.

To obtain information on a strongly correlated statistical system we can have another non-perturbative approach. In the framework of the Kadanoff-Wilson's idea on renormalisation group (RG) [14] the differential RG transformations are realised through a blocking construction, consisting in the successive elimination of the rapid degrees of freedom. Consequently, the effective theory defined by the effective Hamiltonian contains quantum fluctuations whose frequencies are smaller than the momentum cut-off. We can have the similar line of thought but from the opposite direction: we can introduce one-parameter family of models via an effective average action on a given scale $k$, where all the rapid fluctuations are already taken into account above $k$ [4]. By performing the limit $k \to 0$ one will arrive to the full quantum effective action. The $\mathrm{O}(N)$ is one of the most studied quantum field theory, which, apparently, falls into the the same universality class of statistical physical systems as the Ising model for $N = 1$, the XY model for $N = 2$, the Heisenberg model $N = 3$ and it is used as the toy model for the Higgs sector in the Standard model or to describe meson phenomenology for $N = 4$. The limit $N \to \infty$ describes the spherical model [5]. Thus, the O($N$) model represents an excellent playground to study its phase structure by Functional Renormalisation Group (FRG), and the results show good agreement with those achieved by using other non-perturbative methods, e.g. lattice simulations.

## 2 Objectives

One of the main subject of my doctoral research was to study the BN model using the non-perturbative functional technique called the Two-Particle Irreducible (2PI) formalism [6]. The 2PI functional technique was developed in many-body theory for non-perturbative studies. The 2PI effective action in general is needed to be truncated to make real computations possible. My goal was to improve the 2PI resummation technique in a way that it incorporates vertex corrections, too. If the theory can be solved in the framework of the non-perturbative functional technique it could make it possible to extend the method to other theories.

Another aim in the Bloch-Nordsieck model was to extend to finite temperature the procedure developed previously at zero temperature. My goal was to find a solution for the finite temperature Dyson-Schwinger equation. Finding the finite temperature spectral function for the fermion could give predictions regarding the behaviour of the full QED, too.

The 2PI technique was developed to describe collective phenomena. As a consequence, the 2PI description of the Bloch-Nordsieck model at zero temperature did not give the right result despite giving an infrared safe solution. My goal was to study the finite temperature Bloch-Nordsieck in the framework of the 2PI formalism without vertex resummation. At finite temperature the 2PI formalism expected to provide better description of the theory.The result could be compared to the exact solution at finite temperature and the 2PI technique could be validated.

The other main topic of my doctoral research was the application of the FRG to the O($N$) model. According to the Mermin-Wagner (MW) theorem a continuous symmetry cannot be spontaneously broken in two dimensions: $D_L = 2$. My goal was to study this statement in the framework of the FRG in the Local Potential Approximation (LPA) and give a proof for it. Showing that the MW theorem holds when using FRG in LPA is essential because so far this has only been conjectured.

The Mermin-Wagner theorem is violated in the framework of the FRG using LPA when the effective potential is approximated as a Taylor series and theories in $D \geq 4$ does not seem to be trivial as they should. My goal was to develop a method which allow me to show that even when the potential is approximated with a polynomial resulting from a Taylor expansion the MW theorem gets restored and that the systems in $D \geq 4$ shows the triviality property. The method consists in taking the truncation of the Taylor series to infinity.

# 3 New scientific results

The scientific results of my doctoral research are summarised as follows.

### 3.1

It turned out that in the BN model the Ward-Takahashi identities make a a one-to-one correspondence between the fermion propagator and the vertex function. As a consequence, it is possible to extend the 2PI self-energy equation with the vertex function by expressing it with the fermion propagator. The result is an integro-differential equation equivalent to the Dyson-Schwinger (DS) equation. This equation is linear in the fermion propagator, and taking good care of renormalisation, it is possible to solve analytically, providing the exact solution of the BN model. This is a new way of obtaining the exact solution in the Bloch-Nordsieck model. And, while the original solution method is very hard to generalise to other theories, the generalisation of the Ward identity improved 2PI equations, could be easier. The results are published in [I].

### 3.2

After generalising the Ward identities to finite temperature the a DS equation similar to the $T = 0$ case can be obtained in the BN model. The exact fermionic spectral function of the BN model is derived at finite temperature. Analytic results are presented for some special parameters, namely when we perform the computation in the rest frame of the fermion, for other values we have numerical results. The spectral function is finite and normalisable for any nonzero temperature values. The real time dependence of the retarded Green's function is power-like for small times and exhibits exponential damping for large times. Treating the temperature as an infrared regulator, a safe interpretation of the zero temperature result is also given. The results are published in [II].

### 3.3

The BN spectral function was numerically determined at finite temperature in the framework of the 2PI approximation. The finding is that the results of the 2PI computation nicely agree with the exact one, provided that a matching of the coupling constant is performed. The mapping between the two parameters results in the finite temperature running of the 2PI coupling constant. This result may apply to the finite temperature behaviour of the coupling constant in QED, too. The results are published in [III].

### 3.4

The occurrence of spontaneous symmetry breaking (SSB) was studied for O($N$) models using FRG. It is shown that even the Local Potential Approximation (LPA), when treated exactly, is sufficient to give qualitatively correct results for systems with continuous symmetry, in agreement with the Mermin-Wagner theorem. This was shown analytically both for $N < \infty$ and $N \to \infty$. The results are published in [IV].

### 3.5

We discussed the derivation of the so-called Vanishing Beta Function curves, which can be used to explore the fixed point structure of the theory under consideration, for arbitrary field components $N$ and dimensions $D$. The technique is based on the most popular approximation scheme, namely, the polynomial expansion of the effective potential in the LPA. In this case, as an artefact of the approximation, spurious fixed points show up. Using statistical arguments, we can extract the physical fixed points of the theory in accordance with earlier results regarding $D < 4$. For $D \geq 4$ triviality of the O($N$) model was shown which is a new result using FRG in LPA. For the large-$N$ O($N$) model in dimensions $4 < D < 6$, we found a new interacting fixed point, that defines a metastable critical potential. The existence of this fixed point is a subject of current studies in connection with the asymptotically safe scenario of the model [7, 8]. The results are published in [V].

**The cited articles from my thesis points which contain my scientific results:**

I A. Jakovac, P. Mati, *R*esummations in the Bloch-Nordsieck model, Phys. Rev. D **85**, e-Print: arXiv:1112.3476 [hep-ph].

II A. Jakovac, P. Mati, *S*pectral function of the Bloch-Nordsieck model at finite temperature, Phys. Rev. D **87**, 125007, e-Print: arXiv:1301.1803 [hep-th].

III A. Jakovac, P. Mati, *V*alidating the 2PI resummation: The Bloch-Nordsieck example, Phys. Rev. D **90**, 045038, e-Print: arXiv:1405.6576 [hep-th].

IV N. Defenu, P. Mati, I. G. Marian, I. Nandori, A. Trombettoni, *Truncation Effects in the Functional Renormalisation Group Study of Spontaneous Symmetry Breaking*, [arXiv:1410.7024 [hep-th]],
The article is under publication at JHEP.

V P. Mati, *The Vanishing Beta Function curves from the Functional Renormalisation Group*, e-Print: arXiv:1501.00211 [hep-th],
To be published in PRD (currently under consideration, only minor revisions were required by the referee).

**Other cited articles:**

# References

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