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Graphicity of the union of matroids

Summary of PhD Dissertation

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Contents

Abstract

There is a conjecture that if the union (also called sum) of graphic matroids is not graphic then it is nonbinary [23]. We obtained new results related to this question. These results give necessary conditions and sufficient conditions and leave the original conjecture open for very special matroids only. We also define and study new interesting matroid classes. We present some consequences of our results in electric network theory, and an additional byproduct of our investigations for the bases of the union of matroids.

1 Introduction

1.1 Definitions

We mostly apply the notations of Oxley [19].

A *matroid* $M(E, I)$ is a pair (E, I), where E is the *ground set* and I is a family of subsets of E with the following properties:

1. $\emptyset \in I$
2. For all $X \subseteq E$ and $Y \subseteq X$, if $X \in I$, then $Y \in I$
3. For all $X \in I$ and $Y \in I$, if $|X| > |Y|$, then there exists an $x \in X - Y$ such that $Y \cup \{x\} \in I$

The elements of I are the *independent* sets in M. A maximal independent set of a matroid is a *basis*. A minimal dependent set of a matroid is a *circuit*. A minimal set of the matroid which contains at least one element from each basis is a *cut set*. A one element circuit is a *loop*, and a one element cut set is a *coloop*. Some non-coloop elements of a matroid are *serial* if they belong to exactly the same circuits. Two elements x and y are *parallel* if $\{x, y\}$ is a circuit. Let $L(M)$ and $NL(M)$ denote the set of loops and non-loops, respectively, in the matroid M.

The *rank* of a set X is the size of its largest independent subset (it is denoted by $r(X)$). Let $\sigma(X)$ denote the *closure* of a set $X \subseteq E$ in M, that is, $\sigma(X) = \{e | r(X \cup \{e\}) = r(X)\}$. A set $X \subseteq E$ is *closed* (or is a flat) if $\sigma(X) = X$.

The *uniform* matroid $U_{k,n}$ denotes the matroid on n elements, where a subset is independent if and only if it has at most k elements. The matroid $U_{n,n}$ is the *free matroid*.

The set of columns of a matrix forms a *vector matroid*, where the independent sets are the linearly independent vector sets. We say that a matroid is *representable* over a field F if there exists a matrix over that field for which the matroid is isomorphic to the vector matroid of that matrix. A matroid is *linear* if there exists a field F for which the matroid is F-representable. A matroid is *binary* if it is $GF(2)$-representable. A matroid is *regular* if it is representable over every field.

The *dual* of a matroid M is the matroid where the bases are the complements of the bases of M. It is denoted as M^*.

A matroid is *graphic* if there exists an undirected graph for which the ground set of the matroid is the set of edges and the independent sets are the forests (circuit free subgraphs) of the graph. A matroid is *cographic* if its dual is graphic.

The *deletion* of $X \subset E$ from a matroid M gives a new matroid on the ground set $E-X$ where a subset is independent if and only if it is independent in M. It is denoted as $M \setminus X$. The *contraction* of $X \subset E$ in a matroid M gives a new matroid on the ground set $E - X$ where the rank of a set Y is $r_M(Y \cup X) - r_M(X)$. This new matroid is denoted as M/X. A matroid M' is a *submatroid* of M if it can be obtained from M by a deletion. A matroid M' is a *minor* of M if it can be obtained from M by a deletion and a contraction.

The *direct sum* of $M_1(E_1, I_1)$ and $M_2(E_2, I_2)$ is a matroid whose ground set is the disjoint union of E_1 and E_2 and the independent sets are the disjoint unions of a set from I_1 and a set from I_2. It is denoted as $M_1 \oplus M_2$.

The *union* (also called sum) of $M_1(E, I_1)$ and $M_2(E, I_2)$ is a matroid on the ground set E whose independent sets are the unions of a set from I_1 and a set from I_2. We shall refer to M_1 and M_2 as *addends*. It is denoted as $M_1 \vee M_2$.

A matroid is *connected* if it does not arise as the direct sum of two smaller matroids. A maximal connected submatroid is a *component*. A *j-separation* of a matroid is a partition $\{E_1, E_2\}$ of E such that $r(E_1)+r(E_2) \leq r(E)+j-1$ and $r(E_1), r(E_2) \geq j$. A matroid M is called *k-connected* if for all j $(1 \leq j < k)$, M has no j-separation. A cocircuit C^* is called *non-separating* if $M \setminus C^*$ is connected.

1.2 History

Graphic matroids form one of the most significant classes in matroid theory. When introducing matroids, Whitney concentrated on relations to graphs. The definition of some basic operations like deletion, contraction and direct sum were straightforward generalizations of the respective concepts in graph theory. Most matroid classes, for example those of binary, regular or graphic matroids, are closed with respect to these operations. This is not the case for the union. The union of two graphic matroids can be non-graphic.

The fundamental results of [17] and [7] characterize those graphic matroids whose union is the free matroid (the cycle matroid of a tree). The first paper studying whether the union of graphic matroids is graphic (that is, the cycle matroid of any graph) was probably that of Lovász and Recski [14], they examined the case if several copies of the same graphic matroid are given.

Another possible approach is to fix a graph G_0 and characterize those graphs G where the union of their cycle matroids $M(G_0) \vee M(G)$ is graphic. (Observe that we may clearly disregard the cases if G_0 consists of loops only, or if it contains coloops.) As a byproduct of some studies on the application of matroids in electric network analysis, this characterization has been performed for the case if G_0 consists of loops and a single circuit of length two only, see the first graph of Figure 1. (In view of the above observation this is the simplest nontrivial choice of G_0.)

Theorem 1 *[22] Let A and B be the cycle matroids of the graphs shown in Figure 1 on ground sets $E_A = \{1, 2, ..., n\}$ and $E_B = \{1, 2, i, j, k\}$, respectively. Let M be an arbitrary graphic matroid on E_A.*
Then the union $A \vee M$ is graphic if and only if B is not a minor of M with any triplet i, j, k.

Recski [23] conjectured some thirty years ago the following.

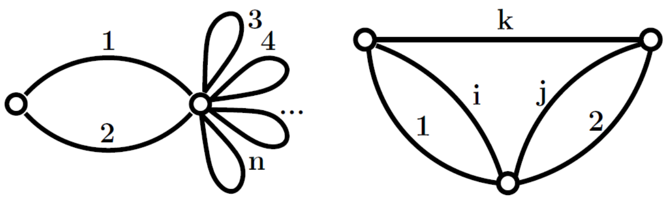

Figure 1: A graphic representation of A (left) and B (right)

Conjecture 2 *If the union of two graphic matroids is not graphic then it is nonbinary.*

This is known to be true if the two graphic matroids are identical or if one of them is A as given in Theorem 1 – these results follow in a straightforward way from [14] and from [22], respectively. In this section we extend the result of Theorem 1.

1.3 Related Results

Tutte gave a characterization of binary [27] and graphic matroids [28]:

Theorem 3 *A matroid is binary if and only if it has no $U_{2,4}$ minor.*

Theorem 4 *A matroid is graphic if and only if it has no $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$ minor.*

According to these results Conjecture 2 can be reformulated as follows: the union of two graphic matroids is graphic or has a $U_{2,4}$ minor. Observe that it is a consequence of these theorems that every minor of a graphic (resp. binary) matroid is also graphic (resp. binary).

For those non graphic cases which are minimal with respect to deletion Wagner gave a characterization in [31].

Theorem 5 *Let M be a non graphic binary matroid. If for every element $e \in E$ the matroid $M \setminus e$ is graphic, then M is a series extension of F_7, F_7^*, R_{10}, $M^*(K_5)$, or $M^*(K_{3,3}^i)$, for some $0 \geq i \geq 3$.*

The Fournier triplets [9] are the main concepts of many algorithms in the topic.

Theorem 6 *A matroid is graphic if and only if, for every three distinct cocircuits having a common element, one of them separates the other two.*

There are some useful properties of binary matroids which can be useful for our aims [32], [20] [30].

Theorem 7 *A matroid is binary if and only if the cardinality of the intersection of any circuit and cut set is even (this number can not be one in any matroid).*

Theorem 8 *A matroid is binary if and only if the symmetric difference of every two circuits is a disjoint union of circuits.*

Theorem 9 *A matroid is binary if and only if the symmetric difference of every two distinct circuits contains a circuit.*

The problem of deciding if a binary matroid is graphic is often called the *graph-realization* problem. Tutte gave a polynomial time algorithm for the graph-realization problem [29]. Later on he refined his algorithm [30] using new results such as:

Theorem 10 *A nonseparable binary matroid M, with $r(M) > 2$, has an F_7 minor or a separating cocircuit.*

For these algorithms it is important to find separating cocircuits efficiently. Cunningham gave a method [6] for finding a separating cocircuit or an F_7 minor.

Later many other, faster, simpler algorithms have been proposed for the graph-realization problem, see [10] (it builds the graph representation from fundamental circuits, based on PQ-graphs) and [15] (based on Fournier triplets and avoidance graphs). However, in our studies we could not use these methods or the underlying concepts efficiently, because even if we suppose that the union is binary, we do not know the binary matrix representation.

Seymour gave a polynomial time algorithm for the general version of the problem, that is deciding if an arbitrary matroid is graphic [25] (based on the stars of nodes). The following theorem is the tool for this algorithm:

Theorem 11 *$M = M(G)$ if and only if $r(M) \leq r(M(G))$ and the star of every node in G is a union of cocircuits in M.*

To upgrade Seymour's algorithm Truemper gave a similar theorem [26].

Theorem 12 *$M = M(G)$ if the followings hold:*

- *M and $M(G)$ are connected*
- *There exists a common base B of M and $M(G)$, and the set of fundamental cocircuits with respect to B are the same in M and $M(G)$*
- *The star of every node of G contains a cocircuit of M*

2 Reduction steps

We gave some reduction methods, which can decrease the number of elements in the matroids (see Lemmata 13, 14, 15, 16, published in [2] and Lemmata 17, 18 and 19, published in [3]). In all cases we suppose that the two addends are graphic.

Lemma 13 *Let X and Y denote the set of coloops in M_1 and in M_2, respectively. The union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic.*

Lemma 14 *If the ground set of a connected component X of the matroid M_1 is a subset of $L(M_2)$, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is graphic.*

Lemma 15 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ in which $X \subset L(M_2)$ determines a connected subgraph and $E - X$ has exactly two common vertices with X (call them a and b). Then the union $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic, where M_1' is the cycle matroid of $G' := G(V, (E - X) \cup \{(a, b)\})$ and $M_2' := (M_2 \setminus X) \cup loop(a, b)$ (Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').*

Lemma 16 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ and E_0 is the edge set of a 2-connected component X of G which has only one edge x from $NL(M_2)$. Then the union $M_1 \vee M_2$ is graphic if and only if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is graphic.*

Lemma 17 *If two parallel elements x and y of M_1 are serial in M_2, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus x) \vee (M_2/x)$ is graphic.*

Lemma 18 *If two serial elements x and y of M_1 are serial in M_2 as well, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus \{x, y\}) \vee (M_2 \setminus \{x, y\})$ is graphic.*

Observe that the case if two serial edges of M_1 are loops in M_2 has been covered by Lemma 15.

Lemma 19 *Suppose that x and y are serial elements in $M(G_1)$ and they are not contained in any common circuit of $M(G_2)$.Let a and b be the two endvertices of x and c and d be the two endvertices of y in G_2. Assume x is not a loop of G_2 ($a \neq b$). Let $G'_1 = G_1/x$ and relabel y to z. Let G'_2 be obtained from G_2 by deleting x and y, identifying vertices b and c and adding a new edge z between a and d. Then $M(G_1) \vee M(G_2)$ is graphic if and only if $M(G'_1) \vee M(G'_2)$ is graphic.*

These results together gives the following:

Theorem 20 *Suppose that M_1 and M_2 are graphic matroids and the application of Lemmata 13, 14, 15, 16, 17, 18 and 19 to M_1 and M_2 leads to a reduced pair of matroids M'_1, M'_2. Then $M_1 \vee M_2$ is graphic if and only if $M'_1 \vee M'_2$ is graphic.*

The last two reductions below (published in [3]) are equivalent for the graphicity of the union in some cases depending also on the union, not just on the addends. This means that these lemmata can not help us to give a necessary and sufficient condition to the graphicity of the union of graphic matroids, but can help when thinking about a possible minimal counterexample of Conjecture 2.

Lemma 21 *Let two parallel edges x and y of M_1 be parallel in M_2 too. If x and y are coloops or serial in the union then there exists a subscript $k \in \{1, 2\}$ so that $M_1 \vee M_2$ is graphic if and only if $(M_k/x) \vee (M_{3-k} \setminus x)$ is graphic. If they are neither serial nor coloops then the union is nonbinary.*

Lemma 22 *Let x and y be two parallel edges of M_1 and suppose that x is a loop, but y is not a loop in M_2. If x and y are coloops or serial in the union, then the union is graphic if and only if $(M_1/x) \vee (M_2 \setminus x)$ is graphic. If they are neither serial nor coloops then the union is not binary.*

3 The special cases

We gave equivalent condition for the graphicity of the union for two cases, using the reductions. In both cases the first matroid is special, it has n parallel or serial edges and some loops. The other matroid is an arbitrary graphic matroid. (those results have been published in [2]).

Theorem 23 *Let $M'_1 = M(G'_0)$ and $M'_2 = M(G')$ be graphic matroids after all the possible reductions using Lemmata 13, 14 and 15. If G'_0 consists of loops and a single circuit of length n ($n \geq 2$), then $M'_1 \vee M'_2$ is graphic if and only if either $NL(M'_1)$ contains a cut set in G' or $M'_2 \setminus NL(M'_1)$ is the free matroid.*

Theorem 24 *Let $M'_1 = M(G'_0)$ and $M'_2 = M(G')$ be graphic matroids after all the possible reductions using Lemmata 13, 14 and 15. If G'_0 consists of loops and two points joined by n parallel edges, then $M'_1 \vee M'_2$ is graphic if and only if no 2-connected component of G' has two non-serial edges a and b from $NL(M'_1)$ so that $M'_2 \setminus \{a, b\}$ is not the free matroid.*

The proof of these results also showed that these cases can not give a counterexample for the conjecture.

We present applications of these two theorems for linear active networks in Section 8.

4 Sufficient conditions

It turned out that the common generalization of Theorems 23 and 24 gives a general sufficient condition for the graphicity of the union (published in [3]).

Theorem 25 *Let M_1, M_2 be two matroids defined on the same ground set E. Then $M_1 \vee M_2$ is graphic if for every circuit C in M_1 either $r_2(E-C) < r_2(E)$ or $r_2(E-C) = |E-C|$ holds.*

We showed that this condition remains sufficient if the length of the circuit is at least 2, but for this case we have to suppose that M_2 is graphic (published in [3]).

Theorem 26 *Assume that M_2 is graphic. Then $M_1 \vee M_2$ is graphic if for every circuit C of length at least two in M_1 either $r_2(E-C) < r_2(E)$ or $r_2(E-C) = |E-C|$.*

5 Necessary condition

We proved that adding some additional conditions to the sufficient ones we get a necessary condition for the binarity of the union of arbitrary graphic matroids (published in [3]).

Theorem 27 *Let M_1 and M_2 be graphic matroids. If all of the following conditions hold, then the union $M_1 \vee M_2$ is not binary.*

1. *There exist X_i dependent sets in M_i for $i \in \{1,2\}$*
2. *$X_1 \cap X_2 = \emptyset$*
3. *There exist a circuit C_i of M_i in X_i so that $|C_i| \geq 2$ for $i \in \{1,2\}$*
4. *$r_i(X_i) = r_i(X_1 \cup X_2)$ for $i \in \{1,2\}$*
5. *There are two distinct elements $a, b \in C_1 \cup C_2$ such that for $i \in \{1,2\}$:*
   - *if $a \in C_i$ and $b \in C_{3-i}$, then a and b are in the same component in both matroids*

- *if $a, b \in C_i$, then there exists $X'_{3-i} \subset X_{3-i}$ so that if we contract X'_{3-i} in M_{3-i}, then a and b are diagonals of C_{3-i} connecting distinct pairs of vertices*

6 Further sufficient conditions

We improved the sufficient condition for the graphicity of the union to a form resembling to the necessary condition (unpublished).

Theorem 28 *Suppose that M_1 and M_2 are two matroids on the same ground set and at least one of them is graphic. If there are no such sets C_1 and C_2 so that C_i is a circuit of M_i, $C_1 \cap C_2 = \emptyset$ and C_i does not contain a cut set in M_{3-i}, then $M_1 \vee M_2$ is graphic.*

We further improved the sufficient condition to bring it closer to the necessary condition (unpublished).

Theorem 29 *Suppose that M_1 and M_2 are graphic matroids and all the elements of the union are in the same component (the union is connected). If there are no such sets X_1 and X_2 so that X_i is dependent in M_i for $i \in \{1, 2\}$, $X_1 \cap X_2 = \emptyset$ and $r_i(X_i) = r_i(X_i \cup X_{3-i})$ for $i \in \{1, 2\}$, then $M_1 \vee M_2$ is graphic.*

7 New questions

In order to put Conjecture 2 into a more general framework, we formally define eight matroid classes as follows. The first column of the table represents the name of the sets. 'nb' stands for nonbinary and 'gr' stands for graphic

A	those gr matroids which give a	gr or nb union with	any gr matroid.
B	those gr matroids which give a	gr union with	any gr matroid.
C	those gr matroids which give a	gr or nb union with	any matroid.
D	those gr matroids which give a	gr union with	any matroid.
E	those matroids which give a	gr or nb union with	any gr matroid.
F	those matroids which give a	gr union with	any gr matroid.
G	those matroids which give a	gr or nb union with	any matroid.
H	those matroids which give a	gr union with	any matroid.

Observe that Conjecture 2 states that A is the set of all graphic matroids. Most of the relationships between the sets are trivial ($D \subseteq C \subseteq A$, $D \subseteq B \subseteq$

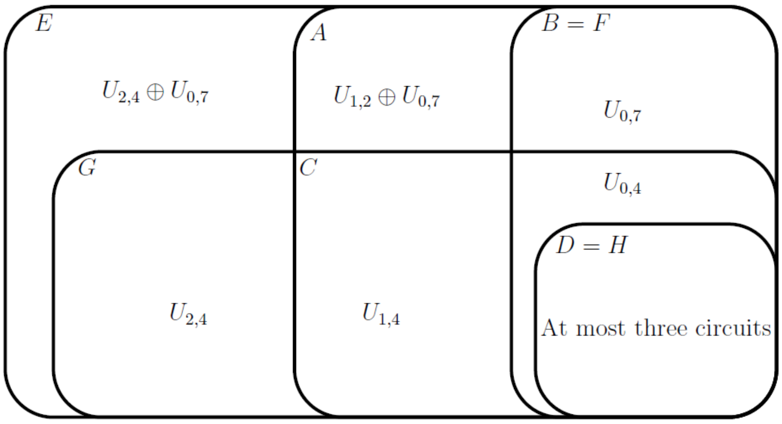

Figure 2: Examples for each nonempty subset

A, $H \subseteq G \subseteq E$, $H \subseteq F \subseteq E$, $A \subseteq E$, $B \subseteq F$, $C \subseteq G$, $D \subseteq H$) see Figure 2. For $D = H$ recall that the union of M and $U_{0,k}$ is M so if the union is graphic then M is also graphic. Since $U_{0,k}$ is graphic $F = B$ follows similarly. $(A \cap G) - C$ is empty because if a matroid is in G but not in C then it is not graphic.

We gave an equivalent condition for a matroid to be in D (published in [3]).

Theorem 30 *A matroid is in D if and only if it contains at most three circuits.*

All the containments as indicated in Figure 2 are proper, as shown by the examples. The position of these examples are straightforward for all but one case, see the following result (published in [3]).

Theorem 31 *The set $E - (G \cup A)$ is not empty, it contains the matroid $K = U_{2,4} \oplus U_{0,7}$.*

8 The independence structure of linear active networks

In this section the results of the previous section are reformulated for linear active networks.

Consider a linear network composed of 2-terminal devices. Its interconnection structure is described by a graph G. The voltages or the currents of a subset of devices can independently be prescribed if and only if the subset of the corresponding edges in the graph G is circuit-free or cut set free, respectively. This classical result of Kirchhoff [13] can be generalized for networks containing multiterminal devices as well: the independence structure can be described by the circuits and cut sets of a matroid M. However, these matroids will not always be graphic.

Here we present applications of Theorems 23 and 24 for linear active networks (published in [5]).

Theorem 32 *Suppose that a network is composed of 2-terminal devices and the current of a resistor R_0 controls several current sources $I_1, I_2, ..., I_k$ as described by the respective equations $i_j = c_j \cdot i_0$ for every $j = 1, 2, ..., k$ (where the control constants $c_1, c_2 ..., c_k$ are generic parameters). Then the independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

Theorem 33 *Suppose that a single current source i_0 is controlled by the current of several resistors $R_1, R_2, ..., R_k$ as described by the equation $i_0 = \Sigma c_j \cdot i_j$ where the summation is for every $j = 1, 2, ..., k$. Like in Theorem 32, suppose that the control constants $c_1, c_2 ..., c_k$ are generic parameters. Then the independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

9 Partitioning the bases of the union of matroids

We proved some interesting facts for the bases of the union (published in [4]).

Theorem 34 *Let $M_1, M_2, \ldots, M_n$ be matroids and let M be their union. Let B be a basis of M with a good partition $B_1, B_2, \ldots, B_n$. For any basis B' of M there is a good partition $\cup_{i=1}^n B'_i$ so that $\sigma_i(B_i) = \sigma_i(B'_i)$ for $i = 1, 2, \ldots, n$.*

Theorem 35 *Let B be an arbitrary basis of the union $M = \vee_{i=1}^{n} M_i$, with an arbitrary good partition $\cup_{i=1}^{n} B_i$. Let M_i' be obtained from M_i by replacing all the elements of $E - \sigma_i(B_i)$ by loops. Then $\vee_{i=1}^{n} M_i' = M$.*

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