MŰEGYETEM 1782

Graphicity of the union of matroids

PhD Dissertation

Written by: Csongor Gy. Csehi

Supervisor: Professor András Recski

2017.

Contents

Chapter 1

Introduction

1.1 Structure of the dissertation

There is a conjecture that if the union (also called sum) of graphic matroids is not graphic, then it is nonbinary [35]. Some special cases have been proved only, for example if several copies of the same graphic matroid are given.

In the first chapter we collect all the definitions and the most important earlier results on related topics. In Chapter 2 we study new results related to the conjecture. These results (Sections 2.1–2.6 and Chapter 5) give some necessary and some sufficient conditions and leave the original conjecture open for very special matroids only. We present some consequences of our results in electric network theory (Section 2.7). Some new interesting matroid classes are defined in Chapter 3. Lastly, we present an additional byproduct of these investigations (Chapter 4).

1.2 Acknowledgement

I am grateful to my supervisor András Recski, who introduced me to the research topic of matroid theory. He is supporting my research with his indispensable pieces of idea and advice for almost seven years now.

The highly valuable remarks and suggestions of the referees of our papers are much appreciated.

Research was partially supported by grant No. OTKA 108947 of the Hungarian Scientific Research Fund.

1.3 Definitions

We mostly apply the notations of Oxley [31]. Many previous result in our research topic are summarized in the books [36] and [45].

A *matroid* $M(E, I)$ is a pair (E, I), where E is the *ground set* and I is a family of subsets of E with the following properties:

1. $\emptyset \in I$
2. For all $X \subseteq E$ and $Y \subseteq X$, if $X \in I$, then $Y \in I$
3. For all $X \in I$ and $Y \in I$, if $|X| > |Y|$, then there exists an $x \in X - Y$ such that $Y \cup \{x\} \in I$

The second property is the *hereditary property* and the third one is the *augmentation property*. The elements of I are the *independent* sets in M. A subset of E is *dependent* if it is not independent.

A maximal independent set of a matroid is a *basis*. A minimal dependent set of a matroid is a *circuit* (or cycle or polygon; sometimes cycle stands for the union of some circuits, but in this dissertation we use them as synonymes). A minimal set of the matroid which contains at least one element from each basis is a *cut set* (or cocircuit, cocycle or bond). A one element circuit is a *loop* and a one element cut set is a *coloop* (or bridge, but there are other structures which are called bridges so we will use only coloop). Some noncoloop elements of a matroid are *serial* if they belong to exactly the same circuits. Two elements x and y are *parallel* if $\{x, y\}$ is a circuit. A *series (or parallel) class* is a maximal set of pairwise serial (or parallel) elements.

Given a basis B each element x in $E - B$ determines a unique circuit contained in $B \cup \{x\}$. These circuits form the *set of fundamental circuits with respect to* B. Similarly each element y in B determines a unique cocircuit contained in $(E - B) \cup \{y\}$. These cocircuits form the *set of fundamental cocircuits with respect to* B.

The *rank* of a set X is the size of its largest independent subset (it is denoted by $r(X)$ or $r_M(X)$). Let $L(M)$ and $NL(M)$ denote the set of loops and nonloops, respectively, in the matroid M. This means that $L(M) \bigcap NL(M) = \emptyset$, $L(M) \bigcup NL(M) = E$, $r(L(M)) = 0$ and $r(NL(M)) = r(E)$.

Let $\sigma(X)$ denote the *closure* of a set $X \subseteq E$ in M, that is, $\sigma(X) = \{e | r(X \cup \{e\}) = r(X)\}$. A set $X \subseteq E$ is *closed* if $\sigma(X) = X$. The closed sets are also called *flats*. In particular, $L(M) = \sigma(\emptyset)$ is the smallest and E is the largest flat.

The *uniform* matroid $U_{k,n}$ denotes the matroid on n elements, where a subset is independent if and only if it has at most k elements. The matroid $U_{n,n}$ is the *free matroid*.

The set of columns of a matrix forms a *vector matroid*, where the independent sets are the linearly independent vector sets. We say that a matroid is *representable* over a field F (F-representable) if there exists a matrix over that field for which the matroid is isomorphic to the vector matroid of that matrix. A matroid is *linear* if there exists a field F for which the matroid is F-representable. A matroid is *binary* if it is $GF(2)$-representable. A matroid is *regular* if it is representable over every field.

The *dual* of a matroid M is the matroid where the bases are the complements of the bases of M. It is denoted as M^*.

A matroid is *graphic* if there exists an undirected graph for which the ground set of the matroid is the set of edges and the independent sets are the forests (circuit free subgraphs) of the graph. The graph is called the *graphic representation* of the matroid (and the matroid is called the *circuit matroid* (or polygon matroid) of the graph). A matroid is *cographic* if its dual is graphic. A cographic matroid can be obtained from the graphic representation G of its dual matroid, the ground set is the set of edges and the independent sets are the cut free subgraphs of G (this matroid is the *bond matroid* of G).

The *deletion* of $X \subset E$ from a matroid M gives a new matroid on the ground set $E - X$ where a subset is independent if and only if it is independent in M. It is denoted as $M \setminus X$. The *contraction* of $X \subset E$ in a matroid M gives a new matroid on the ground set $E - X$ where the rank of a set Y is $r_M(Y \cup X) - r_M(X)$. This new matroid is denoted as M/X. A matroid M' is a *submatroid* of M if it can be obtained from M by a deletion. A matroid M' is a *minor* of M if it can be obtained from M by a deletion and a contraction. A *series-contraction* in a matroid is a contraction of an element which is serial with an other element of the matroid. A matroid M' is a *series-minor* of M if it can be obtained from M by a deletion and some series-contractions. The inverse operation of the series-contraction is the *series extension*, where we add a new element to the matroid which will be serial with one element and contracting this new element we get the original matroid. The *parallel extension* can be defined similarly, where we add a new element which will be parallel to an other one, and deleting this new element we get the original matroid. A *parallel-deletion* in a matroid is a deletion of an element which is parallel with an other element of the matroid. A matroid M' is a *parallel-minor* of M if it can be obtained from M by a contraction and some parallel-deletions.

The *direct sum* of $M_1(E_1, I_1)$ and $M_2(E_2, I_2)$ is a matroid whose ground set is the disjoint union of E_1 and E_2 and the independent sets are the disjoint unions of a set from

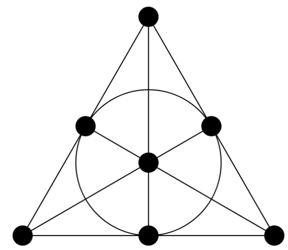

Figure 1.1: The Fano matroid.

I_1 and a set from I_2. It is denoted as $M_1 \oplus M_2$.

The *union* (also called sum) of $M_1(E, I_1)$ and $M_2(E, I_2)$ is a matroid on the ground set E whose independent sets are the unions of a set from I_1 and a set from I_2. We shall refer to M_1 and M_2 as *addends*. It is denoted as $M_1 \vee M_2$.

The *intersection* of $M_1(E, I_1)$ and $M_2(E, I_2)$ is a subset of the power set of E, whose elements are the sets that are in both I_1 and I_2. Unfortunately this structure is not always a matroid.

A matroid is *connected* (or *nonseparable*) if it does not arise as the direct sum of two smaller matroids. A maximal connected submatroid is a *component* (or *elementary separator*). Equivalently, a component is a minimal nonempty set S with the property $r(S) + r(E \setminus S) = r(E)$. If M is not connected and X is the ground set of a connected component of M, then $M/X = M \setminus X$.

Let C_1 be a circuit of a matroid M. $\emptyset \neq A \subseteq C_1$ is an *arc* of C_1 if there exists a circuit C_2 such that $A \cup C_2$ contains at least two circuits, and A is minimal with respect to this property. The set $\{A_1, A_2, A_3\}$ of distinct arcs of a circuit is *incompatible* if $A_1 \cap A_2 \cap A_3 \neq \emptyset$ and no one of A_1, A_2 or A_3 is contained in the union of the other two.

The Fano matroid is a matroid on 7 elements, where each two element subset is independent, each four element subset is dependent and exactly those three element subsets are dependent which form a line or the indicated circle in Figure 1.1. This matroid is denoted as F_7.

The R_{10} matroid is the binary matroid with binary representation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

A graph consisting of three internally disjoint paths (each of length at least one) between two points is called a θ graph. The complete graph with 5 vertices (it has one edge between every pair of vertices) is denoted by K_5. The complete bipartite graph with 3 vertices in each color-class (it has one edge between two vertices if and only if they are in different color-classes) is denoted by $K_{3,3}$. $K^i_{3,3}$ is the graph which can be obtained from $K_{3,3}$ by adding i edges among the 3 nodes of one color-class.

The graph W_n consisting of a circuit C_n on n vertices and a node to which there goes an edge from each vertices of the circuit is called a *wheel*. The cicuit matroid of the wheel graph is also called a wheel. The *whirl* W^n is a matroid on the same ground set as W_n, where the circuits are the circuits of W_n but C_n, and the sets $\{e\} \cup C_n$ for each element $e \notin C_n$.

A graph G is called *k-connected* (or k-vertex-connected) if it has at least $k+1$ vertices and for any two vertices there exist k internally vertex disjoint paths between them. A *j-separation* of a matroid is a partition $\{E_1, E_2\}$ of E such that $r(E_1)+r(E_2) \leq r(E)+j-1$ and $r(E_1), r(E_2) \geq j$. A matroid M is called *k-connected* if for all j $(1 \leq j < k)$, M has no j-separation. The definition for matroids generalizes the original one, so a graph with at least $k+1$ vertices is k-connected if and only if its circuit matroid is k-connected [14]. The previous definition of the connected matroid (using the direct sum) is equivalent to the 2-connectivity of the matroid.

A cocircuit C^* is called *nonseparating* if $M \setminus C^*$ is connected. For a connected graphic matroid, a nonseparating cocircuit corresponds to the star of a vertex whose deletion from the associated graph keeps it 2-connected.

1.4 History

Graphic matroids form one of the most significant classes in matroid theory. When introducing matroids, Whitney concentrated on relations to graphs. The definition of some basic operations like deletion, contraction and direct sum were straightforward generalizations of the respective concepts in graph theory. Most matroid classes, for example those of binary, regular or graphic matroids, are closed with respect to these operations. This is not the case for the union. The union of two graphic matroids can be nongraphic.

The fundamental results of [29] and [17] characterize those graphic matroids whose union is the free matroid (the cycle matroid of a tree). The first paper studying whether the union of graphic matroids is graphic (that is, the cycle matroid of any graph) was probably that of Lovász and Recski [25], they examined the case if several copies of the same graphic matroid are given.

Another possible approach is to fix a graph G_0 and characterize those graphs G where the union of their cycle matroids $M(G_0) \vee M(G)$ is graphic. (Observe that we may clearly disregard the cases if G_0 consists of loops only, or if it contains coloops.) As a byproduct of some studies on the application of matroids in electric network analysis, this characterization has been performed for the case if G_0 consists of loops and a single circuit of length two only, see the first graph of Figure 1.2. (In view of the above observation this is the simplest nontrivial choice of G_0.)

Theorem 1.4.1 *[34] Let A and B be the cycle matroids of the graphs shown in Figure 1.2 on ground sets $E_A = \{1, 2, ..., n\}$ and $E_B = \{1, 2, i, j, k\}$, respectively. Let M be an arbitrary graphic matroid on E_A.*

Then the union $A \vee M$ is graphic if and only if B is not a minor of M with any triplet i, j, k.

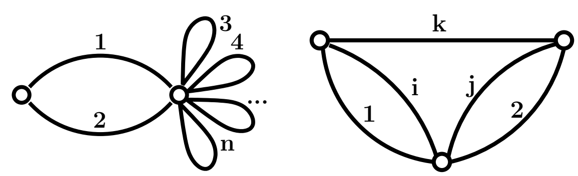

Figure 1.2: A graphic representation of A (left) and B (right)

Recski [35] conjectured some thirty years ago the following.

Conjecture 1.4.2 *If the union of two graphic matroids is not graphic, then it is nonbinary.*

This is known to be true if the two graphic matroids are identical or if one of them is A as given in Theorem 1.4.1. These results follow in a straightforward way from [25] and from [34], respectively.

The main purpose of the dissertation is to extend the result of Theorem 1.4.1. First we extend the result if G_0 either consists of loops and two points joined by n parallel edges ($n \geq 2$, see Section 2.3), or if it consists of loops and a single circuit of length n ($n \geq 2$, see Section 2.2). We prove equivalent conditions for the graphicity of $M(G_0) \vee M(G)$, if G_0 is one of these two matroids. Our results will then imply that the above conjecture is true if one of these two types of graphs plays the role of G_0.

1.5 Related results

In this section we collect the results in different areas of matroid theory which are most strongly related to our questions.

Tutte gave a characterization of binary [46] and graphic matroids [47]:

Theorem 1.5.1 *A matroid is binary if and only if it has no $U_{2,4}$ minor.*

Theorem 1.5.2 *A matroid is graphic if and only if it has no $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$ minor.*

A useful (and possibly the shortest) proof of the previous theorem can be found in [53].

According to these results Conjecture 1.4.2 can be reformulated as follows: the union of two graphic matroids is graphic or has a $U_{2,4}$ minor. Observe that it is a consequence of these theorems that every minor of a graphic (resp. binary) matroid is also graphic (resp. binary).

Bixby gave a stronger version of a trivial consequence of this theorem [2]

Theorem 1.5.3 *A binary matroid is graphic if and only if it has no F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$ parallel-minor.*

Bixby also gave a condition for the series minors, but we present this in an equivalent form (such as Wagner in [54]). This result is about those non graphic cases which are minimal with respect to deletion.

Theorem 1.5.4 *Let M be a non graphic binary matroid. If for every element $e \in E$ the matroid $M \setminus e$ is graphic, then M is a series extension of F_7, F_7^*, R_{10}, $M^*(K_5)$, or $M^*(K_{3,3}^i)$, for some $0 \geq i \geq 3$.*

Wagner also gave a different characterization of those binary matroids that are graphic. It is unique from the point of view that it is naturally expressed in terms of the circuits of a matroid.

Theorem 1.5.5 *A binary matroid is graphic if and only if it does not contain an incompatible set of arcs.*

The Fournier triplets [19] are the main concepts of many algorithms in the topic.

Theorem 1.5.6 *A matroid is graphic if and only if, for every three distinct cocircuits having a common element, one of them separates the other two.*

Nash-Williams gave a good formula for the rank function of the homomorph image of a matroid [30].

Theorem 1.5.7 *The structure $\phi(M(E,I)) = M(\phi(E), \{\phi(X) : X \in I\})$ is a matroid. Its rank function is $r_\phi(Y) = min\{r(\phi^{-1}(Z)) + |Y - Z| : Z \subseteq Y\}$*

As a consequence of this theorem, we get a formula for the rank function of the union:

Theorem 1.5.8 *The rank function of the union of matroids M_i for $i = 1, ..., n$ is $R(Y) = min\{\Sigma_{i=1}^{n} r_i(Z) + |Y - Z| : Z \subseteq Y\}$*

There are some useful properties of binary matroids which can be useful for our aims [55], [32] [49].

Theorem 1.5.9 *A matroid is binary if and only if the cardinality of the intersection of any circuit and cut set is even (this number can not be one in any matroid).*

Theorem 1.5.10 *A matroid is binary if and only if the symmetric difference of every two circuits is a disjoint union of circuits.*

Theorem 1.5.11 *A matroid is binary if and only if the symmetric difference of every two distinct circuits contains a circuit.*

The regular matroids form a famous matroid class which is between the binary and graphic matroid classes. Every graphic and cographic matroid is regular, and every regular matroid is binary. The following results are from Tutte [46] [49] [52]

Theorem 1.5.12 *A matroid is regular if and only if it has no $U_{2,4}$, F_7 or F_7^* minor.*

Theorem 1.5.13 *The followings are equivalent for a matroid M:*

- *M is regular*
- *M has a real representation matrix that is totally unimodular.*
- *M can be represented over $GF(2)$ by such a matrix which becomes totally unimodular by properly assigning ± 1 values to the original 1's. (In fact, any representation of M over $GF(2)$ has then this property.)*
- *M is representable over $GF(2)$ and $GF(3)$.*

- *M is representable over $GF(3)$, and every representation matrix over $GF(3)$ is totally unimodular when considered as a real matrix.*

Bixby [4] and Seymour [38] proved the following.

Theorem 1.5.14 *A matroid is representable over $GF(3)$ if and only if it it has no $U_{2,5}$, $U_{3,5}$, F_7 or F_7^* minor.*

Just like for the binary and graphic case, every minor of a regular matroid is regular. The dual of a regular matroid is also regular.

Theorem 1.5.15 *[39] A 3-connected regular matroid is graphic (resp. cographic) if and only if it has no R_{10} (resp. R_{12}) minor.*

The problem of deciding if a binary matroid is graphic is often called the *graph-realization* problem. We mention that this name is often used for two other problems as well. In matroid theory one can ask that, if a binary matroid is represented by a matrix N, does there exist a tree T so that the columns of N are the incidence vectors of the paths in T [7]. In classical graph theory one can ask the existence of a simple graph with a given degree seqence.

The graph-realization is also related to the *consecutive-ones* problem and its generalization, the *arborescence-realization* problem [43]. The consecutive ones problem is that of determining whether the rows of a given $\{0,1\}$-matrix can be permuted such that the 1's in each column are consecutive. An *arborescence* is a digraph T, the underlying graph of which is a tree that has exactly one vertex of indegree zero. The arborescence-realization problem is that for a given $\{0,1\}$-matrix A determine whether there exists an arborescence T such that the arcs of T are indexed on the rows of A and the columns of A are the incidence vectors of the arc sets of dipaths of T (the construction of T, if it exists, is also a part of the problem). Note that in this paragraph *arc* means directed edge, unlike everywhere else in the dissertation.

The graph-realization problem plays an important role in converting linear programs into network problems [6].

Tutte gave a polynomial time algorithm for the graph-realization problem [48]. It is based on chain groups and Y-components. We will show some similar results later using Y-components. Later on he refined his algorithm [49] using new results such as:

Theorem 1.5.16 *A nonseparable binary matroid M, with $r(M) > 2$, has an F_7 minor or a separating cocircuit.*

Tutte's algorithm can also be applied to recognize 3-connected matroids [3]. For these algorithms it is important to find separating cocircuits efficiently. Cunningham gave a method [15] for finding a separating cocircuit or an F_7 minor. These algorithms are also useful for recognizing cographic matroids, such as in [15].

Later many other, faster, simpler algorithms have been proposed for the graph-realization problem, see [20] (it builds the graph representation from fundamental circuits, based on PQ-graphs) and [26] (based on Fournier triplets and avoidance graphs). However, in our studies we could not use these methods or the underlying concepts efficiently, because even if we suppose that the union is binary, we do not know the binary matrix representation.

Seymour gave a polynomial time algorithm for the general version of the problem, that is deciding if an arbitrary matroid is graphic [40] (based on the stars of nodes). The following theorem is the tool for this algorithm:

Theorem 1.5.17 *Suppose that M is a matroid, G is a graph and $E(M) = E(G)$. Then $M = M(G)$ if and only if $r(M) \leq r(M(G))$ and the star of every node in G is a union of cocircuits in M.*

To upgrade Seymour's algorithm Truemper gave a similar theorem [44].

Theorem 1.5.18 *$M = M(G)$ if the followings hold:*

- *M and $M(G)$ are connected*
- *There exists a common base B of M and $M(G)$, and the set of fundamental cocircuits with respect to B are the same in M and $M(G)$*
- *The star of every node of G contains a cocircuit of M*

Bixby proved the following equivalent condition for graphicness [5] (earlier Tutte gave a similar condition in [51]). In order to formulate it we have to introduce some concepts. A *bridge* of a cocircuit Y is a component of $M \setminus Y$. The *Y-components* are the matroids $M/(E - (B \cup Y))$ where B is a bridge of Y. The *bridge graph* of a cocircuit Y has one node for each bridge, and there is an edge between two such bridges A_1 and A_2 if there does not exist A_i-segments S_i (for $i = 1, 2$) so that $S_1 \cup S_2 = Y$. Here an *X-segment* is a parallel class of $= (M \setminus B)/(E - (B \cup Y))$.

Theorem 1.5.19 *Let M be a matroid and let Y be a cocircuit of M. Then M is graphic if and only if the Y-components of M are graphic and the bridge graph of Y is bipartite.*

There are some important results for the case where the input is a 3-connected binary matroid. Lemos showed that if a 3-connected binary matroid has rank $r(M) \geq 1$, then it is graphic if and only if each element avoids exactly $r(M) - 1$ nonseparating cocircuits of M [24] (or equivalently M contains exactly $r(M) + 1$ nonseparating cocircuits). The antecedent of this result is the following [3]:

Theorem 1.5.20 *If M is a 3-connected binary matroid having at least four elements, then each element belongs to at least two nonseparating cocircuits, and M is graphic if and only if each element is contained in exactly two nonseparating cocircuits.*

There is a result [1] about the minors of the 3-connected nongraphic matroids:

Theorem 1.5.21 *Let $\{e, f, g\}$ be a circuit of a 3-connected matroid M. If M is non-graphic, then it has a minor using e, f, g isomorphic to $U_{2,4}$, F_7 or $M^*(K_{3,3})$.*

To use the previous results for our cases we might need some results on 3-connected matroids.

Theorem 1.5.22 *[50] Let M be a 3-connected matroid in which for each element e $M \setminus e$ and M/e are not 3-connected. Then $r(M) \geq 3$ and M is a wheel or a whirl.*

Theorem 1.5.23 *[42] Let M be a 3-connected matroid. If for all 3-connected matroids N which has $|E(M)| + 1$ elements and M as a minor it is true that any pair of elements is contained in a minor isomorphic to M, then this property is true for all 3-connected matroids N having a minor isomorphic to M.*

Theorem 1.5.24 *[41] If a 3-connected matroid M is nonbinary, then every pair of elements are in a $U_{2,4}$ minor.*

Theorem 1.5.25 *[27] A matroid is 3-connected if and only if every 4-element subset is contained in a minor isomorphic to a wheel of rank 3 or 4, a whirl of rank 2, 3, or 4, or the the relaxation of a rank-3 whirl.*

1.6 Dead end approach

One can think that the conjecture can be solved by checking if the matroids F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$ can be obtained as the union of two graphic matroids or not. This property could be checked easily, for example by an exhausting search. It is true that all submatroids of a union $M = M_1 \vee M_2$ can be obtained by the union of a submatroid of M_1 and a submatroid of M_2. However, it is not true that if a matroid M can be obtained as the union of two graphic matroids M_1 and M_2, then its minors can be obtained as the union of a minor of M_1 and a minor of M_2 (formally as $(M_1 \setminus X_1/Y_1) \vee (M_2 \setminus X_2/Y_2)$ where $X_1 \cup Y_1 = X_2 \cup Y_2$). A trivial counterexample is when both matroid have two parallel elements and a loop, but the loops are different. In this case the union is $U_{2,3}$ and the $U_{1,2}$ minor, which contains both of the original loops, can not be obtained as the union of minors of the addends. This example shows that the above mentioned property can not be true even for series or parallel-minors of the union. However, according to Theorem 1.5.4 it would be enough for us to prove that for all series class S of the union $M = M_1 \vee M_2$, there exist an element $x \in S$ so that M/x can be obtained by the union of a minor of M_1 and a minor of M_2. Now consider the case $M = A \vee B$ from Theorem 1.4.1. In this case 1 and 2 are serial in the union. Moreover $M/1 = U_{2,4}$. However, $U_{2,4}$ can not be obtained as the union of a minor of M_1 and a minor of M_2. This means that it is a counterexample for the above mentioned property.

To finish the explanation of the topic in the previous paragraph, we show that there are infinitely many binary but nongraphic matroids that are minimal with respect to deletion. We know that the matroids F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$ are binary but nongraphic. According to Theorem 1.5.2 their series extensions are not graphic either. It is easy to see that the series extension M' of a binary matroid M along e is binary (we will denote the new element by e'). We can obtain a binary matrix representation of this matroid by adding a new row and a column to the binary representation of M, where the new column has 1 in the new row and 0s elsewhere, and there is an other 1 in the new row in the column of e (the other elements of the new row are 0s). That way each base of the obtained matroid contains at least one element from $\{e, e'\}$ and a base B can not contain both e and e' if and only if $B \setminus \{e, e'\}$ spans e in M. This means that this new matrix is in fact the binary representation of M'. This completes the proof that there are infinitely many binary but nongraphic matroids that are minimal with respect to deletion. This together with the results of the previous paragraph shows that the approach to show that the forbidden minors can not be obtained as the union of two graphic matroids, is a dead end (despite that the number of forbidden minors/parallel-minors/series-minors are finite).

The only positive result of the previous concept is that Conjecture 1.4.2 can be reformulated equivalently as follows:

Conjecture 1.6.1 *None of the series extensions of the matroids F_7, F_7^*, R_{10}, $M^*(K_5)$, and $M^*(K_{3,3}^i)$, for some $0 \geq i \geq 3$, can be obtained as the union of two graphic matroids.*

Chapter 2

Graphicity of the union of matroids

2.1 Reduction steps

The results studied in Sections 2.1 through 2.3 have been published in [10].

Observe that the first graph of Figure 1.2, representing A, has only two nonloop edges (1 and 2), while the second graph, representing B, has the property that the complement of the set $\{1, 2\}$ of nonloop edges of A contains both a circuit and a spanning tree. This property will turn out to be crucial if we consider a larger set of nonloop edges which are either all parallel or all serial, see Remark 2.3.5.

In this section we formulate some reduction steps for arbitrary graphic matroids M_1 and M_2 on the same ground set. We will use these reductions to give equivalent conditions for the graphicity of the union in some special cases. Then as a corollary, we can prove the conjecture in these two special cases: If the nonloop edges of a graph are either all parallel or all serial, then the union of its cycle matroid with any graphic matroid is either graphic or contains a $U_{2,4}$ minor, hence it is nonbinary (by Theorem 1.5.1).

Throughout M_1 and M_2 will be graphic matroids on the same ground set E. We shall refer to them as addends. It is well known that if a matroid is graphic, then so are all of its submatroids and minors. Hence if a matroid has a nongraphic minor, then the matroid is not graphic.

The following lemmata contain the main opportunities to simplify our addend matroids. Throughout, $M \setminus X$ and M/X will denote deletion and contraction, respectively, of the set X in a matroid M, while $X - Y$ will denote the difference of the sets X and Y. We shall write $Y \cup x$, $Y - x$, $M \setminus x$ and M/x instead of $Y \cup \{x\}$, $Y - \{x\}$, $M \setminus \{x\}$ and $M/\{x\}$, respectively.

Lemma 2.1.1 *Let X and Y denote the set of coloops in M_1 and in M_2, respectively. The union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic.*

PROOF: If an element of a matroid M is a coloop, then it will be a coloop in the union of M with any other matroid. Therefore if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic, then we can extend its representing graph with coloops for $X \cup Y$ and this way we get a graphic representation of $M_1 \vee M_2$.

On the other hand if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is nongraphic, then $M_1 \vee M_2$ can't be graphic because it has a nongraphic submatroid. $\square$

Lemma 2.1.2 *If the ground set of a connected component X of the matroid M_1 is a subset of $L(M_2)$, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is graphic.*

PROOF: It is easy to see that in this case the matroid which is the direct sum of $(M_1 \setminus X) \vee (M_2 \setminus X)$ and $M_1 \setminus (E - X)$ is isomorphic to $M_1 \vee M_2$. The direct sum of graphic matroids is also graphic, hence $M_1 \vee M_2$ is graphic.

On the other hand if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is not graphic, then $M_1 \vee M_2$ can't be graphic because it has a nongraphic submatroid. $\square$

Recall that the cycle matroid of a loopless graph with no isolated vertices is connected if and only if the graph is 2-vertex-connected.

Lemma 2.1.3 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ in which $X \subset L(M_2)$ determines a connected subgraph and $E - X$ has exactly two common vertices with X (call them a and b). Then the union $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic, where M_1' is the cycle matroid of $G' := G(V, (E - X) \cup \{(a, b)\})$ and $M_2' := (M_2 \setminus X) \cup loop(a, b)$ (Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').*

PROOF: If $M_1' \vee M_2'$ is graphic, then delete the edge (a, b) from the graph of the union and then put the original subgraph of X (from G) in the place of this deleted edge (put the original a and b to the endpoints of (a, b) in the union) and we get a graphic representation of $M_1 \vee M_2$.

On the other hand if $M_1' \vee M_2'$ is nongraphic, then we show that this union arises as a minor of $M_1 \vee M_2$ hence this latter cannot be graphic either. There has to be a path between a and b in X in G; let α denote one of its edges. There is a subset C of X so that $\{\alpha\}$ will be a basis in the contraction $[M_1 \setminus (E - X)]/C$.

$$((M_1 \vee M_2)/C) \setminus (X - (C \cup \{\alpha\})) = (M_1/C \setminus [X - (C \cup \{\alpha\})]) \vee (M_2 \setminus [X - \{\alpha\}]) = M_1' \vee M_2'$$

□

After these preliminaries we can define the reduction that will be the most important concept to reduce the infinite number of cases for Theorems 2.2.1 and 2.3.1.

Definition 2.1.4 *We say that a pair M_1, M_2 is reduced if none of the above lemmata can help us to decrease the number of elements.*

Theorem 2.1.5 *Suppose that M_1 and M_2 are graphic matroids and the application of Lemmata 2.1.1, 2.1.2, 2.1.3 to M_1 and M_2 leads to a reduced pair of matroids M_1', M_2'. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

Proposition 2.1.6 *Assume that M_1 and M_2 are given by their graphs G_1 and G_2, respectively. Then we can perform the reduction of these matroids in polynomial time.*

Proof: The number of elements decreases with every step of reduction so we have to see that each step can be performed in polynomial time and that we can check in polynomial time whether we can apply a reduction step.

If we found a coloop or a component which is a subset of $L(M_{3-i})$ in order to apply Lemma 2.1.1 or Lemma 2.1.2, respectively, then we can delete them quickly and it is also easy to replace a subset by an edge as in Lemma 2.1.3 (once we have found the subset).

We can find the 2-vertex-connected components of a graph in polynomial time. This way we can easily identify all the coloops. Moreover we can determine the number of the elements from $NL(M_i)$ in any set and delete those components in M_{3-i} which have none.

In order to apply Lemma 2.1.3 we have to recognize these sets X effectively in spite of the fact that the same matroid may have many graphic representations. For this purpose we define a relation on the edge set of a coloopless graph G so that e and f are in relation if and only if either $e = f$ or $\{e, f\}$ is a cut set in G. This is an equivalence relation and using the operation "twisting" (see [31], Section 5.3) one can change G to G' so that edges in each equivalence class form paths in G' and $M(G) = M(G')$. This can be performed in polynomial time (for each equivalence class contract all but one of the edges and replace the remaining edge by a path formed by all the edges in this class). Finally pick all pairs of points in both graphs and decide whether they separate their component into two parts so that one of them is a subset of the set of loops in the other graph. □

Now we are ready to give this polynomial algorithm which reduces a pair of given graphic matroids. We formulate the algorithm for coloopless matroids only, in order to keep its later application simpler.

Algorithm 2.1.7 *INPUT: Two coloopless graphic matroids M_1 and M_2 on the same ground set given by their graphs G_1 and G_2, respectively.*

OUTPUT: The reduced pair M_1', M_2'.

1. *If G_i has a 2-vertex-connected component which does not have edges from $NL(M_{3-i})$, then delete this component from G_i and the corresponding loops from G_{3-i}.*

2. *Change G_i if necessary, to a new one where the equivalence classes (as described in the proof of Proposition 2.1.6) are paths.*

3. *If $X \subseteq E$ determines a connected subgraph of G_i which does not have edges from $NL(M_{3-i})$ and the subgraph has exactly two common vertices a and b with $E - X$ in G_i, then delete from G_{3-i} all the loops of X except a single (arbitrary) one denoted by x and replace G_i by $(G_i \setminus X) \cup edge(a, b)$ (where $edge(a, b)$ will play the role of x).*

4. *If during the last step the matroids are changed, then go to Step 2 otherwise let M_1', M_2' denote the reduced pair.*

We close this section with one more reduction related statement which can be used for results similar as Proposition 2.3.6.

Lemma 2.1.8 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ and E_0 is the edge set of a 2-connected component X of G which has only one edge x from $NL(M_2)$. Then the union $M_1 \vee M_2$ is graphic if and only if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is graphic.*

Proof: If $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is graphic, then we can obtain the graph of $M_1 \vee M_2$ by replacing edge x with the subgraph X in the following way:

Let a and b denote the end vertices of x in X. Cut vertex a into two vertices a_1 and a_2 in X. Among the edges incident to a in X, join x to a_1 (b remains the other endpoint of x) and all the others to a_2, let X' denote the resulting graph. Now replace x in the graph of $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ with the graph X' along the vertices a_1 and a_2.

On the other hand if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is nongraphic, then since this union arises as a minor of $M_1 \vee M_2$, this latter cannot be graphic either.

$$(M_1 \vee M_2)/(E_0 - \{x\}) = ((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$$

□

2.2 The case when all the nonloop edges of G_0 are serial

From now on we study the union $M_1 \vee M_2$ where $M_1 = M(G_0)$ is the matroid which consists of a circuit of length n and k loops and $M_2 = M(G)$ is an arbitrary graphic matroid given by a representation G. We shall write $[n]$ for the set of the edges of the circuit in G_0.

Theorem 2.2.1 *Let $M_1' = M(G_0')$ and $M_2' = M(G')$ be coloopless graphic matroids after all the possible reductions of Algorithm 2.1.7. If G_0' consists of loops and a single circuit of length n ($n \geq 2$), then $M_1' \vee M_2'$ is graphic if and only if either $NL(M_1')$ contains a cut set in G' or $M_2' \setminus NL(M_1')$ is the free matroid.*

Proof: Observe that $NL(M_1')$ is the set of edges of the only nonloop circuit of G_0'. For the if part of the proof the following two propositions solve the two possible cases separately.

It is easy to see that if we have elements from $NL(M_1')$ which are serial in M_2', then the union will be graphic (because these elements can destroy the circuit of G_0', so $[n]$ will consists of coloops in the union). In fact, the slightly more general statement of the following proposition is also true.

Proposition 2.2.2 *If $[n]$ contains a cut set $[c]$ in M_2', then the union $M_1' \vee M_2'$ will be the cycle matroid of the graph obtained from G' by replacing the edges of $[n]$ with coloops.*

Proof: For every set $X \subset L(M_1')$ we have to prove that $[n] \cup X$ is independent in the union if and only if X is independent in M_2'.

If X is not independent in M_2', then it will not be independent in the union either because every element of X is in $L(M_1')$. It means $[n] \cup X$ is not independent either in the union.

On the other hand if X is independent in M_2', then it is a part of a base. A cut set intersects all the bases so $[c]$ has an element a such that $X \cup \{a\}$ is independent in M_2'. $[n]$ is a circuit in M_1', so every proper subset of it is independent thus $[n] - \{a\}$ is independent. Now $[n] \cup X$ has a partition so that $[n] - \{a\}$ is independent in M_1' and $X \cup \{a\}$ is independent in M_2' so it is independent in the union $M_1' \vee M_2'$. □

Notice that this proposition is more general than the if part of Theorem 2.2.1, since the matroids need not to be reduced.

If a set contains no cut set in a matroid that means it is spanned by its complement. This means $[n]$ is spanned in M_2' by elements from $L(M_1')$.

Proposition 2.2.3 *If $M_2' \setminus [n]$ is the free matroid and $[n]$ does not contain a cut set in M_2', then the union $M_1' \vee M_2'$ is graphic, namely it is a circuit formed by all the elements.*

Proof: Now $L(M_1')$, that is $E-[n]$, is a spanning forest of G' according to the assumption.

$L(M_1') \cup [n]$ is not independent in the union because we can pick only $n-1$ independent elements in M_1' and for every element i of $[n]$ the set $L(M_1') \cup \{i\}$ is not independent in M_2' (because $L(M_1')$ spans every element of $[n]$). On the other hand every proper subset of $L(M_1') \cup [n]$ is independent in the union $M_1' \vee M_2'$. In order to prove this we give a suitable partition for all cases when we delete only one element α:

- If $\alpha \in [n]$, then $[n] - \{\alpha\}$ is independent in M_1' and $L(M_1')$ is independent in M_2'.
- If $\alpha \in L(M_1')$, then let F' denote the spanning tree of G' in $L(M_1')$ which contains α and let F_1' and F_2' be the two parts of F' (in the two sides of α). There exists a subset of $NL(M_1')$ which is a path in G' between F_1' and F_2', since otherwise α would be a coloop in G' which contradicts Step 1. There exists such a path with exactly one edge e, because if the lengths of all these paths are at least two, then there will be a point which is not covered by F' in G' so there will be a cut set in $[n]$. In that case $L(M_1') \cup \{e\} - \{\alpha\}$ is independent in M_2' and $[n] - \{e\}$ is independent in M_1'.

□

This means that we proved the if part of Theorem 2.2.1 because if G' contains more edges than a spanning forest and $[n]$, then there must be a circuit in it which is a subset of $L(M_1')$.

For the only if part suppose that $M_2' \setminus [n]$ is not the free matroid and $[n]$ does not contain a cut set in M_2'. M_2' contains a circuit I in $E - [n]$ by the assumption, and its length is at least three due to the reduction (length 1 or 2 would contradict to Lemmata 2.1.1 and 2.1.3, respectively).

In the proof we shall give a nongraphic minor of the union, to do this the following "nonequivalent reduction" will be our tool.

Lemma 2.2.4 *If we contract an element $e \in L(M_i)$ in an addend M_{3-i} and delete the corresponding loop in the other one (M_i), then the union of the new matroids will be a minor of that of the originals.*

PROOF: Contract e in the union. Then the independent sets will be those which can be partitioned so that the first part is independent in M_i and the second is independent with e in M_{3-i} because e is a loop in M_i. This description is exactly the union of the above described matroids. $\square$

Thus we can reduce our study to the 2-connected cases only, as follows. We can contract (like in the previous lemma) a spanning tree of elements from $L(M_1')$ in all 2-connected components but the one containing I. We can apply now the steps of the reduction, so we get a reduced connected matroid from it which contains the circuit $I \subset L(M_1')$ and the new M_1' will still consist of a circuit and loops.

We can clearly assume that there is only one circuit from $L(M_1')$ because if there were more we could delete one element from both matroids so that there remain at least one circuit from $L(M_1')$ in M_2' (but less than before) and $[n]$ still does not contain a cut set. After that we can reduce the matroids and we still have all the necessary conditions.

An element from $NL(M_1')$ will be called an *essential diagonal* of I if it connects two distinct vertices of I in G'.

Proposition 2.2.5 *If we contract all elements from $L(M_1')$ except the edges of I in M_2', then there will be at least two different essential diagonals of I.*

PROOF: Indirectly suppose that I has at most two vertices incident to edges from $NL(M_1')$ after we contracted all other elements of $L(M_1')$ in M_2'. It means that all paths between the vertices of I and the endpoints of the edges from $NL(M_1')$ go through these points. This is a contradicton because $I \subseteq L(M_1')$, and I is a connected subset which has at most two common vertices with $E - I$ so I either disappears or must be a simple edge in the reduced matroid (see Lemma 2.1.2 and Lemma 2.1.3). $\square$

Now use Lemma 2.2.4 to contract M'_2 in two steps. First contract the subset as in Proposition 2.2.5. We get a circuit which is a subset of $L(M'_1)$, with at least two different essential diagonals. Then contract all but three suitable edges of I such that we get a circuit of length three with at least two different essential diagonals. (See the second graph of Figure 2.1 below)

Now we have a minor M''_2 of M'_2 and the corresponding submatroid M''_1 of M'_1 (which is M'_1 without the loops which we contracted in M'_2) so that if the union $M''_1 \vee M''_2$ is nongraphic, then the original union is not graphic either.

We have to apply Lemma 2.2.4 again to the loops from $NL(M''_1)$ in M''_2 to contract the corresponding elements in M'_1 and after that we can delete all the remaining elements which are loops in both matroids. Let M'''_2 and M'''_1 denote the new matroids.

Proposition 2.2.6 *$M'''_1 \vee M'''_2$ is not graphic, morever, not binary.*

Proof: For now M'''_1 consists of a circuit of length h (let $[h]$ denote the set of its elements) and three loops m, n and o while M'''_2 has a circuit formed by m, n and o and $[h]$ can be partitioned to sets of elements M, N and O such that all elements of the sets are parallel to the corresponding element from $L(M'''_1)$ (see Figure 2.1).

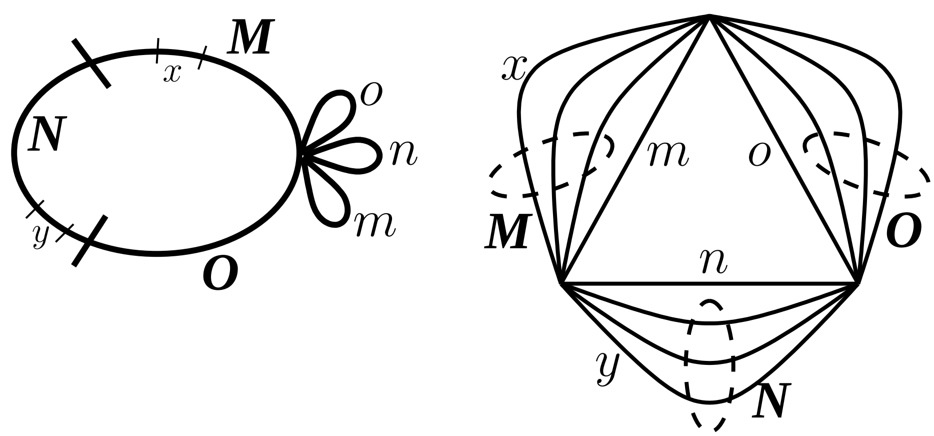

Figure 2.1: Graphic representation of M'''_1 and M'''_2

According to Proposition 2.2.5 at least two of the three sets M, N and O are nonempty. Then suppose that $x \in M$ and $y \in N$ are two elements from $NL(M'''_1)$. We show that $(M'''_1 \vee M'''_2) \,/\, ([h] - \{x\})$ is $U_{2,4}$. The rank of M'''_1 is $h-1$ and the rank of M'''_2 is 2 so in order to obtain a basis of the union we can choose $h-1$ elements of the first matroid and 2 elements of the second one. It is easy to check that $\{x, m\}$, $\{x, n\}$, $\{x, o\}$, $\{m, n\}$,

$\{m, o\}$ and $\{n, o\}$ are independent in $(M_1''' \vee M_2''') / ([h] - \{x\})$: for the first pick $[h] - \{y\}$ from M_1''' and y and m from M_2''' and for the last five simply pick $[h] - \{x\}$ from M_1''' and the others from M_2'''. This means it is really a $U_{2,4}$. $\square$

This proposition completes the proof of the only if part of Theorem 2.2.1 because we gave a nongraphic minor of the union. $\square$

Using the condition of Theorem 2.2.1 we can achieve an algorithm for this cases:

Proposition 2.2.7 *If G_0 consists of loops and a single circuit of length n ($n \geq 2$) and $M(G)$ is an arbitrary graphic matroid on the same ground set, then the graphicity of their union can be decided with the following polynomial time algorithm.*

Algorithm 2.2.8 *INPUT: Two matroids M_1 and M_2 on the same ground set given by the graphs G_0 and G, respectively, where G_0 consists of two parts: a circuit (with edge set $[n]$) and loops.*

OUTPUT: Decision whether the union $M_1 \vee M_2$ is graphic.

1. *If $[n]$ has an element which is a coloop in G, then the union is graphic. If the complement of $[n]$ has elements which are coloops in G, then delete these elements from both G and G_0.*

2. *Run Algorithm 2.1.7 to the pair M_1, M_2, that gives us the reduced pair M_1', M_2'.*

3. *If $[n]$ contains a cut set in M_2' or if $M_2' \setminus [n]$ is the free matroid, then the union is graphic otherwise it is not (neither binary).*

Step 1 uses the statement of Lemma 2.1.1 and the special structure of G_0. Algorithm 2.1.7 preserves the graphicity or nongraphicity of the union according to Lemma 2.1.2 and Lemma 2.1.3. The correctness of the algorithm will, then follow from Theorem 2.2.1.

In view of Proposition 2.1.6 this algorithm is polynomial – in Step 3 we only have to check whether the deletion of the edges of $[n]$ disconnects G' or leads to a circuit-free subgraph.

2.3 The case when all the nonloop edges of G_0 are parallel

From now on we study the union $M_1 \vee M_2$ where M_1 is the cycle matroid of G_0 which consists of loops and two points joined by n parallel edges and $M_2 = M(G)$ is an arbitrary graphic matroid. We shall write $[n]$ for the set of the parallel elements in M_1. In this section we shall prove a neccessary and sufficient condition for M_2, like in the previous section, for the graphicity of the union $M_1 \vee M_2$.

Theorem 2.3.1 *Let $M_1' = M(G_0')$ and $M_2' = M(G')$ be coloopless graphic matroids after all the possible reductions of Algorithm 2.1.7. If G_0' consists of loops and two points joined by n parallel edges, then $M_1' \vee M_2'$ is graphic if and only if no 2-connected component of G' has two nonserial edges a and b from $NL(M_1')$ so that $M_2' \setminus \{a, b\}$ is not the free matroid.*

Proof: Observe that $NL(M_1')$ is the set of serial edges of G_0'. For the **only if part** assume that there exist two such edges a and b. The assumption implies that:

- There exists a circuit C_1 in G' containing a and b (since they are in the same component)
- There exists a circuit C_2 in G' containing a but not b (since they are not serial)
- There exists a circuit C_3 in G' containing none of them (since the remaining is not the free matroid)

Observe that since a and b are not coloops in G', the condition that a and b are not serial is equivalent to that $\{a, b\}$ does not contain a cutset (which more strongly resembles Theorem 2.2.1).

The existence of C_1 and C_2 must generate a subgraph of G' as shown in the first graph of Figure 2.2. Here there are three internally disjoint paths between the vertices V_1 and

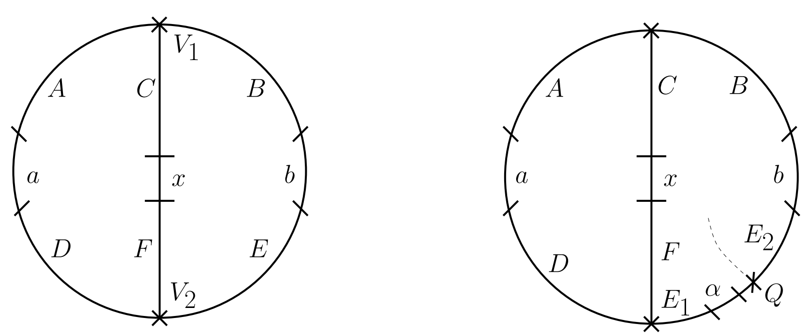

Figure 2.2: The main structure

V_2 (let θ denote this structure). Each path must have length at least one, hence there exists at least one edge x in the "central" path. However any of the paths indicated by A, B, C, D, E and F may be of length zero.

We have to consider two different cases.

Case 1. Let C_3 be in the same component of G' as a and b are. That is, θ is a proper subgraph of the component of G' under consideration. The following lemma handles this case. We speak about paths in a matroid in the sense of equivalence classes, see the proof of Proposition 2.1.6.

Lemma 2.3.2 *If a 2-connected component of the graph G' of the reduced matroid M_2' contains θ as a proper subgraph, then the union $M_1' \vee M_2'$ has a $U_{2,4}$ minor.*

PROOF: First delete all the other 2-connected components of G' from both matroids. If there are more than a θ structure and an additional path remaining, then we can delete at least one path such that the result reduces to a matroid which still has at least four paths (but less than before). That way we can suppose that there is exactly one path in addition to θ (throughout we refer to this path as the *additional path*) in the reduced matroid M_2'. The three paths of the θ structure are shown in the first graph of Figure 2.2, let P and Q denote the endpoints of the additional path.

The following case study examines all the possible cases. Recall that if there are two paths (equivalence classes) between two points, then at least one of the paths must contain an element from $NL(M_1')$, since we consider a pair of reduced matroids.

1. If both P and Q are in $C \cup F$ (the points V_1, V_2 are also permitted), then there must be at least one element c from $NL(M_1')$ in the additional path or in the one which forms a circuit with it, let y denote an arbitrary element from the other one. If we contract all elements but a, b, c and y in the union we get a $U_{2,4}$ (since any pair of elements will be independent and any triple will be dependent).

2. Otherwise without loss of generality we may suppose that Q is in E and it is separated from V_2 by at least one edge α, see the second graph of Figure 2.2. The other endpoint P can be in A, B, C, D, E_1, E_2 and F (see the second graph of Figure 2.2). Let p denote an edge in the additional path.

   (a) If P is in A, B or C: if we contract all elements but b, x, p and α in the union, then we get a $U_{2,4}$.

   (b) If $P = V_2$ or $P \in E_1$: there must be an edge c from $NL(M_1')$ in one of the two paths between P and Q (according to the reduction). If we contract all elements except b, c, x and the one of α or p which is not serial with c in M_2' in the union, then we get a $U_{2,4}$.

(c) If $P \in E_2$: as before there must be an edge c from $NL(M_1')$ in one of the two paths between P and Q. If p is in the same path as c, then let the role of p be played by an arbitrary element from the part of the other path which forms a circuit with the additional path. If we contract all elements but b, c, x and p in the union, then we get a $U_{2,4}$.

(d) If $P \in D$ (but not V_2): there must be an edge α_2 in D between V_2 and P. If we contract all elements but b, α, α_2 and p in the union, then we get a $U_{2,4}$.

(e) If $P \in F$ (but not V_2): there must be an edge β in F between V_2 and P. If we contract all elements but b, α, β and p in the union, then we get a $U_{2,4}$.

□

This case study is mainly the same as in [5: Figure 18.11], where Lemma 2.1.3 has implicitly been used as well.

Case 2 (in the proof of the only if part of Theorem 2.3.1). C_3 is in another component of G, but according to Lemma 2.1.1, that component must contain at least one element c from $NL(M_1')$. Let C_3' be a circuit of G' containing c.

We shall find a $U_{2,4}$ minor in $M_1' \vee M_2'$. If we contract all elements of $C_1 \cup C_2 \cup C_3'$ but $\{a, b, c, x\}$ and delete all the other elements in the union, then we get a $U_{2,4}$ in $M_1' \vee M_2'$.

For the **if part** of Theorem 2.3.1 suppose that there are no such edges a and b. Then we distinguish two cases again.

Either there are two nonserial edges a, b from $NL(M_1')$ in G' but $G' \setminus \{a, b\}$ is the free matroid. Then we have the situation as in the first graph of Figure 2.2 and so the union is graphic (a large circuit formed by all the elements which were not coloops in G').

In the other case all components of G' are circuits, consisting of all but one elements from $NL(M_1')$ (see Figure 2.3). Let $X \subseteq E$ be an arbitrary subset and let $k(X)$ denote the number of circuits of G' which are fully contained in X. If $k(X) \geq 2$, then X is dependent in the union, because we can only choose one element from M_1' and $j-1$ elements of a circuit of length j from M_2'. If $k(X) = 1$, then it is independent in the union, because we can choose one element from M_1' (one from the circuit of G' which is in X) and all the others from M_2'. If $k(X) = 0$, then X is independent in M_2' so it is independent in $M_1' \vee M_2'$ too. This means that a set is independent in the union if and only if it does not contain all the edges of more than one circuit of G'. Then the union is graphic, the circuits of G' will become parallel paths (see an example in Figure 2.3). □

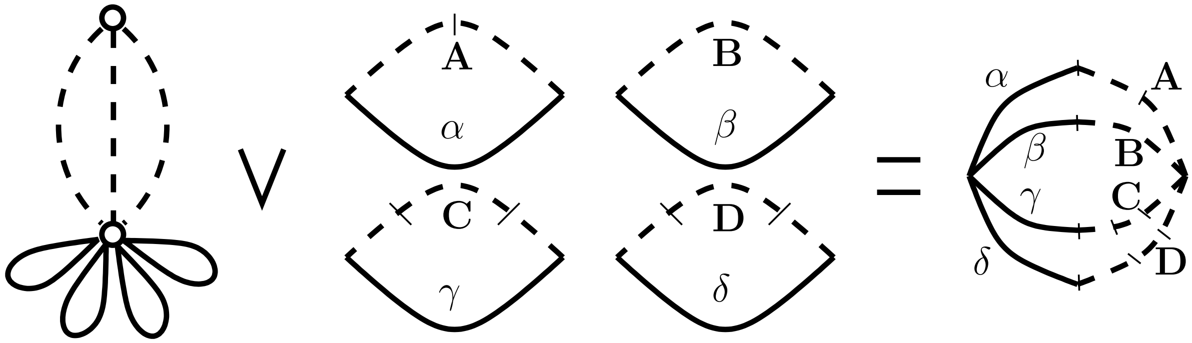

Figure 2.3: A schematic representation of the second case of the proof of the if part

Using the condition of Theorem 2.3.1 we can achieve an algorithm for this cases:

Proposition 2.3.3 *If G_0 consists of loops and two points joined by n parallel edges and $M(G_1)$ is an arbitrary graphic matroid on the same ground set, then the graphicity of their union can be decided with the following polynomial time algorithm.*

Algorithm 2.3.4 *INPUT: Two matroid M_1 and M_2 on the same ground set given by the graphs G_0 and G, respectively, where M_1 consists of two parts: a set $[n]$ of parallel elements and loops.*

OUTPUT: Decision whether the union $M_1 \vee M_2$ is graphic.

1. *Delete all the coloops of G (if any) from both G_0 and G.*
2. *Run Algorithm 2.1.7 to the pair M_1, M_2, that gives us the reduced pair M_1', M_2'.*
3. *If there exist two elements a and b of $[n]$ so that $M_2' \setminus \{a, b\}$ is not the free matroid and a and b are in the same component but not serial in M_2', then the union is not graphic (neither binary) otherwise it is graphic.*

Step 1 uses the statement of Lemma 2.1.1 and the special structure of G_0. Algorithm 2.1.7 preserves the graphicity or nongraphicity of the union according to Lemma 2.1.2 and Lemma 2.1.3. The correctness of the algorithm will then follow from Theorem 2.3.1.

In view of Proposition 2.1.6 this algorithm is polynomial – in Step 3 there are $\binom{n}{2}$ possible choices of a and b, and in each case a spanning forest has to be constructed only.

Remark 2.3.5 *Observe that the final conditions in Algorithms 2.2.8 and 2.3.4 can be formulated in a uniform way as well:* $M_1 \vee M_2$ *is graphic if and only if for every circuit* C *of length* ≥ 2 *in* M_1' *either* $M_2' \setminus C$ *is the free matroid or* C *contains a cut set in* M_2'*.*

One can see that the lemmata in Section 2.1 can be applied to any pair of graphic matroids. In addition we only use the structure of the reduced matroids in Theorem 2.2.1 and 2.3.1. Hence we have a slightly more general result:

Proposition 2.3.6 *Using the previous algorithms, one can decide in polynomial time whether the union of* M_1 *and* M_2 *is graphic, if using all the possible reductions described by Lemmata 2.1.1-2.1.2-2.1.3-2.1.8 leads to a case where* M_1' *or* M_2' *consists of loops and either a single circuit, or some parallel elements, and the other matroid is graphic.*

Proof: We only have to verify that we can reduce the matroids in polynomial time until one of them consists of loops and a single circuit, or loops and parallel elements (as in Algorithms 2.2.8 or 2.3.4). This is true since every step of the reduction (including the steps described by Lemma 2.1.8) can be performed in polynomial time and decreases the number of elements. The last part of the proof follows from Propositions 2.2.7 and 2.3.3. □

In the last two subsections we have given necessary and sufficient conditions for the problem (when is the union of two graphic matroids graphic) in two special cases. These conditions can be checked in polynomial time. Our results also prove the conjecture (if it is not graphic, then it is nonbinary) in these two cases.

The generalization of the statement in Remark 2.3.5 for arbitrary matroids M_1, M_2 is not true. However, in the next sections we will give necessary conditions and sufficient conditions which are of similar character.

2.4 Further reduction steps

The results studied in Sections 2.4 through 2.6 and in Chapter 3 have been published in [11].

We can formalize some lemmata similar to the ones in Subsection 2.1. The main observation is that we can reduce the addends if there are some elements which are in special relation in both matroids, namely if two elements are parallel, serial, one of them is a loop, or they don't have a common circuit.

Lemma 2.4.1 *If two parallel elements x and y of M_1 are serial in M_2, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus x) \vee (M_2/x)$ is graphic.*

Proof: Let N denote the union $(M_1 \setminus x) \vee (M_2/x)$. One can easily see that $M_1 \vee M_2$ can be obtained from N by a serial extension $\{x, y\}$ of the element y. Since serial extension cannot change graphicity or nongraphicity, this proves the assertion. □

Lemma 2.4.2 *If two serial elements x and y of M_1 are serial in M_2 as well, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus \{x, y\}) \vee (M_2 \setminus \{x, y\})$ is graphic.*

Proof: x and y will be coloops in the union, so they don't influence the graphicity of the union. □

Observe that the case if two serial elements of M_1 are loops in M_2 has been covered by Lemma 2.1.3.

Lemma 2.4.3 *Suppose that x and y are serial elements in M_1 and they are not contained in any common circuit of M_2. Assume x is not a loop of M_2. Let $M_1' = M_1/x$ and relabel y to z. Let M_2' be obtained from M_2 as shown in Figure 2.4. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

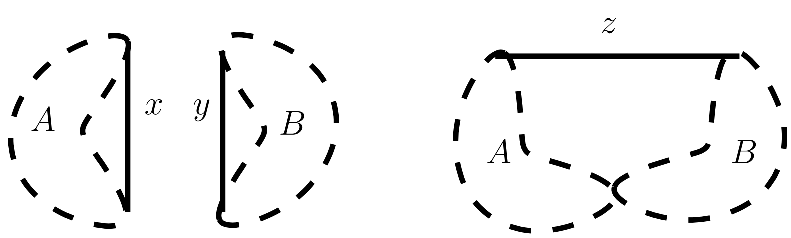

Figure 2.4: The structure of M_2 and M_2'

Proof: If there exists a circuit C_x in the union $M_1 \vee M_2$ which contains x but not y, then $C_x - x$ is independent in the union, that is, $C_x - x = I_1 \cup I_2$ where I_j is independent in M_j. This is a contradiction since $I_1 \cup x$ will also be independent in M_1 (since x and y were serial), hence C_x would be independent in the union.

So x and y are both coloops in the union, or they are serial. In both cases the graphicity of the union is equivalent to the graphicity of $M_1' \vee M_2'$. (Moreover $M_1 \vee M_2$ can easily be constructed from $M_1' \vee M_2'$.) □

The last two reductions below are different from the previous ones because they state that if some conditions don't hold, then the union is nonbinary, but if the conditions hold, then we can make an equivalent reduction. This means that these lemmata can not help us to give a necessary and sufficient condition to the graphicity of the union of graphic matroids, but can help when thinking about a possible minimal counterexample of Conjecture 1.4.2.

Definition 2.4.4 *We will call a partition S_1, S_2 of a set S a* **good partition** *if S_i is independent in M_i for $i = 1, 2$.*

Observe that exactly the independent sets of the union $M_1 \vee M_2$ have good partitions.

Lemma 2.4.5 *Let two parallel elements x and y of M_1 be parallel in M_2 too. If x and y are coloops or serial in the union, then there exists a subscript $k \in \{1, 2\}$ so that $M_1 \vee M_2$ is graphic if and only if $(M_k/x) \vee (M_{3-k} \setminus x)$ is graphic. If they are neither serial nor coloops, then the union is nonbinary.*

PROOF: **Case 1:** If x and y are neither serial nor coloops, then there exists a circuit C_x which contains x but not y. Then $(C_x - x) \cup y$ will be a circuit too (because of the symmetric role of x and y in both M_1 and M_2). In a binary matroid there must be a circuit in the symmetric difference of two other ones, but $\{x, y\}$ can not be dependent in the union so it means that the union is not binary.

Case 2: On the other hand if x and y are both coloops in the union, then $(M_1 \vee M_2) \setminus x$ equals the modified union $(M_k/x) \vee (M_{3-k} \setminus x)$ for both values $k = 1, 2$.

Case 3: Finally suppose that x and y are serial in the union. We claim that there exists a $k \in \{1, 2\}$ so that every independent set S of the union with $x, y \notin S$ has such a good partition $S = S_1 \cup S_2$ where $S_k \cup x$ is also independent in M_k, leading to a good partition of $S \cup x$.

Indirectly suppose that there exist two independent sets P, Q in $M_1 \vee M_2$ so that $x, y \notin P, Q$, $P_1 \cup x$ is dependent in M_1 for every good partition $P = P_1 \cup P_2$ (let us call this *Property 1*) and $Q_2 \cup x$ is dependent in M_2 for every good partition $Q = Q_1 \cup Q_2$ (let us call this *Property 2*). Observe that an independent set of $M_1 \vee M_2$ avoiding x and y cannot have both Property 1 and Property 2 because x and y are serial in $M_1 \vee M_2$.

Choose P and Q such that $|P \cap Q|$ is maximum. If $P \cup b$, for $b \in Q - P$, is an independent set of $M_1 \vee M_2$, then $P \cup b$ has Property 1; a contradiction to the choice of P. Hence P spans Q in $M_1 \vee M_2$. If $(P - a) \cup b$ is independent in $M_1 \vee M_2$, for $b \in Q - P$

and $a \in P - Q$, then, by the choice of P, $(P - a) \cup b$ does not have Property 1. Therefore $(P - a) \cup b$ has Property 2.

Claim 2.4.6 *Q is dependent in the union.*

PROOF: P does not have Property 2, hence there is a good partition $P = P_1 \cup P_2$ so that $P_2 \cup x$ is independent in M_2. Let b denote a fixed element of $Q - P$. Construct $Z_1 \subseteq P_1$ and $Z_2 \subseteq P_2$ in the following way. Put an element e into Z_i if there exists an ordered sequence of elements $\langle a_0, a_1, ..., a_k \rangle$ so that $a_0 = b$, $a_k = e$ and the following property is true:

For all $j \leq k$: $(P_1 - A(i,j,k)) \cup B(i,j,k)$ is independent in M_1, and $(P_2 - B(i,j,k)) \cup A(i,j,k) \cup x$ is independent in M_2. Here $A(i,j,k)$ and $B(i,j,k)$ denote the elements of the sequence with the same parity as $i + j + k$ preceeding a_j, and with different parity, respectively. Formally, $A(i,j,k) = \cup_{t=0}^{t \leq \frac{j-1}{2}} a_{2t+1}$ if $i + j + k$ is odd, $A(i,j,k) = \cup_{t=0}^{t \leq \frac{j}{2}} a_{2t}$ if $i + j + k$ is even and $B(i,j,k) = \cup_{t=0}^{t \leq j} a_t - A(i,j,k)$.

We will call such an element sequence an *alternating sequence*. Observe that the modifed version of P_1 or P_2 along an alternating sequence (by the proper A and B) have the same closure as the original ones in M_1 or M_2, respectively.

Claim 2.4.6 follows from the fact that $Z = Z_1 \cup Z_2$ is a subset of Q but $Z \cup b$ is dependent in the union.

In order to prove $Z \subseteq Q$ suppose the contrary. Let the last element a_k of an alternating sequence $\langle a_0, ..., a_k \rangle$ be not in Q but suppose that $a_i \in Q$ for all $i < k$. Then $P' = (P - a_k) \cup b$ were independent in the union but would not have Property 2 contradicting to the definition of P.

We claim that $Z \cup b$ is dependent in the union because $r_1(Z_1) = r_1(Z \cup b)$ and $r_2(Z_2) = r_2(Z \cup b)$. It is easy to see that the edges of the unique path between the end vertices of b from both P_1 (in a graph of M_1) and P_2 (in a graph of M_2) are elements of Z. Observe that exactly these elements will have a corresponding alternating sequence with $k = 1$. This means that b is spanned by Z_1 in M_1 and by Z_2 in M_2. This argument remains true even if b happens to be a loop in M_i for $i = 1$ or 2 since then, b is obviously spanned by any subset Z_i.

Indirectly suppose that there exists an element e in Z_i which is not spanned by Z_{3-i} in M_{3-i}. Consider the alternating sequence $\langle a_0, ..., a_k = e \rangle$. We can suppose that a_i $(i \leq k)$ is in Q. We know that P_{3-i} spans e in M_{3-i} (otherwise $P \cup b$ would be independent in the union). This means that the modified version of P_{3-i} along any alternating sequence will also span e in M_{3-i}.

If an edge f is in the unique path between the two end vertices of e in $(P_{3-i} - B(0,0,k)) \cup A(0,0,k)$ in a graph of M_{3-i}, then either f has an alternating sequence $\langle a_0, ..., a_k, f\rangle$ or $f \in A(k)$. The first case means that we have to put f into Z_{3-i}. The second one means that f is one of the elements from the alternating sequence of e with same parity of subscript as k, which means that f is spanned by Z_{3-i} in M_{3-i} (because e is the first in that sequence which is not spanned). These together give that there is a path between the end vertices of e in M_{3-i} which consists of elements of Z_{3-i} and elements which are spanned by Z_{3-i}, so e is spanned by Z_{3-i} in M_{3-i} what is a contradiction.

This proves Claim 2.4.6. □

Our indirect assumption contained that Q is independent, so we get a contradiction. This means that there exists a $k \in \{1,2\}$ so that every independent set of the union without x and y has a good partition A_1, A_2 so that $A_k \cup x$ is independent in M_k. That way if $k = 1$, then $(M_1/x) \vee (M_2 \setminus x) = (M_1 \vee M_2)/x$ or if $k = 2$, then $(M_1 \setminus x) \vee (M_2/x) = (M_1 \vee M_2)/x$. □

Lemma 2.4.7 *Let x and y be two parallel elements of M_1 and suppose that x is a loop, but y is not a loop in M_2. Let x and y be coloops or serial in the union. Then the union is graphic if and only if $(M_1/x) \vee (M_2 \setminus x)$ is graphic. On the other hand, if they are neither serial nor coloops, then the union is not binary.*

Recall that the case if both x and y are loops in M_2 has been covered by Lemma 2.1.3.

PROOF: If there exists a circuit C in the union so that $x \in C$ but $y \notin C$, then $(C - x) \cup y$ is also a circuit. This is because for every $\alpha \in C - x$ we know that $(C - \alpha) \cup y$ is independent, since if something in the union is independent with x, it means that we chose x from M_1 and in M_1 the role of x and y are exactly the same (they are parallel). In a binary matroid there must be a circuit in the symmetric difference of two other ones, but $\{x, y\}$ can not be dependent in the union because x, y is a good partition, so it means that the union is not binary.

On the other hand if x and y are both coloops in the union, then $(M_1 \vee M_2) \setminus x = (M_1/x) \vee (M_2 \setminus x)$.

Finally suppose that x and y are serial in the union, this means that x is a coloop in $(M_1 \vee M_2) \setminus y$. Then every independent set S in the union has a good partition S_1, S_2 so that $S_1 \cup x$ is independent in M_1. That way $(M_1/x) \vee (M_2 \setminus x) = (M_1 \vee M_2)/x$. □

2.5 Sufficient condition

Theorems 2.2.1 and 2.3.1 can be reformulated as follows:

Theorem 2.5.1 *Suppose that G_1 consists of loops and a single circuit of length n ($n \geq 2$) and $M(G_2)$ is an arbitrary graphic matroid on the same ground set. The union $M_1 \vee M_2$ is graphic if and only if for the reduced pair M_1', M_2' every non loop circuit C of M_1' contains a cut set in M_2' or $M_2' \setminus C$ is the free matroid.*

Theorem 2.5.2 *Suppose that G_1 consists of loops and two points joined by n ($n \geq 2$) parallel edges and $M(G_2)$ is an arbitrary graphic matroid on the same ground set. The union $M_1 \vee M_2$ is graphic if and only if for the reduced pair M_1', M_2' every non loop circuit C of M_1' contains a cut set in M_2' or $M_2' \setminus C$ is the free matroid or the elements of C are not in the same 2-connected component of G'.*

Now let both matroids be arbitrary. The following theorems in this section will show that these conditions can be formalized together to a sufficient but no longer necessary condition for the graphicity of the union.

Theorem 2.5.3 *Let M_1, M_2 be two matroids defined on the same ground set E. Then $M_1 \vee M_2$ is graphic if for every circuit C in M_1 either $r_2(E-C) < r_2(E)$ or $r_2(E-C) = |E-C|$ holds.*

Proof: We shall apply the following observation:

Proposition 2.5.4 *If there exists an element $\alpha \in E$ so that $E - \{\alpha\}$ is independent in a matroid M, then M is graphic.*

Proof: If E is independent as well, then M is the free matroid which is the cycle matroid of a tree. Otherwise E contains a unique circuit C hence M is the cycle matroid of a graph composed of a circuit (formed by the elements of C) and some coloops (corresponding to the elements of $E - C$). □

Lemma 2.5.5 *If M_1 has a circuit C such that the set $E - C$ is independent in M_2, then $M_1 \vee M_2$ is graphic.*

Proof: For every element α of C the set $C - \{\alpha\}$ is independent in M_1 and $E - C$ is independent in M_2. This means $E - \{\alpha\}$ is independent in the union, hence $M_1 \vee M_2$ is graphic by Proposition 2.5.4. □

We consider the cases according to the circuits of M_1:

1. If there exists a circuit C of M_1 so that $r_2(E-C) = |E-C|$, then $M_1 \vee M_2$ is graphic by Lemma 2.5.5.

2. Let $C_1, C_2, ..., C_k$ be the circuits of M_1. The only remaining case is that $r_2(E-C_i) < r_2(E)$ holds for every i. This means that every basis of M_2 intersects every circuit C_i. Let $X \subseteq E$ be a basis of M_2, then $E-X$ must be independent in M_1 (since it can not contain a circuit). This means that $X \cup (E-X) = E$ is independent in the union $M_1 \vee M_2$ so the union is the free matroid.

In summary, the union contains at most one circuit. $\square$

$U_{0,2} \vee U_{0,2}$ is the simplest example to show that this condition is not necessary.

If the requirements of Theorem 2.5.3 are prescribed for circuits of length at least two only, then a slightly weaker condition will still suffice.

Theorem 2.5.6 *Assume that M_2 is graphic. Then $M_1 \vee M_2$ is graphic if for every circuit C of length at least two in M_1 either $r_2(E-C) < r_2(E)$ or $r_2(E-C) = |E-C|$.*

Proof: We follow the same line of thought as in Theorem 2.5.3.

1. If there exists a circuit C of M_1 so that $r_2(E-C) = |E-C|$, then $M_1 \vee M_2$ is graphic by Lemma 2.5.5.

2. Suppose now that $r_2(E-C) < |E-C|$ for every circuit C, $|C| > 1$ of M_1 and let γ be a noncoloop element. Let $C_1, C_2, ..., C_k$ be the circuits of M_1 containing γ (we may suppose that $k > 0$). Now $r_2(E-C_i) < r_2(E)$ holds for every i, hence every basis of M_2 intersects every circuit C_i. Let $X \subseteq E - \{\gamma\}$ be an independent set in $M_1 \vee M_2$ and let X_1, X_2 be a good partition of X. If $X_1 \cup \{\gamma\}$ is dependent in M_1 it must contain a unique circuit C_1. X_2 is independent in M_2 so it is a subset of a basis B. Then $B \cap C_1$ is not empty, let a denote one of its elements. $X_1 \cup \{\gamma\} - \{a\}$ is independent in M_1 since C_1 is the only circuit in $X_1 \cup \{\gamma\}$ and $a \in C_1$. $X_2 \cup \{a\}$ is independent in M_2 (it is a subset of B). This means $(X_1 \cup \{\gamma\} - \{a\}) \cup (X_2 \cup \{a\}) = X \cup \{\gamma\}$ is independent in the union $M_1 \vee M_2$. So every γ element of this type will be coloop in the union.

We have to study the loops of M_1.

We may suppose that there is no circuit C of M_1 with $r_2(E-C) = |E-C|$ (see Case 1). Observe that the elements which are only contained by circuits as in the second case can not ruin the graphicity of the union, since they will be coloops. This way we can simply delete all elements like that from both matroids and the union will be graphic if and only if the union of the original matroids is graphic. The initial condition changes to the requirement that there can be only loops in M_1'. This means that the union $M_1' \vee M_2' = M_2'$ namely $M_1 \vee M_2 = M_2' \oplus \{\text{coloops}\}$. $\square$

Now $(U_{1,2} \oplus U_{0,1}) \vee U_{0,3}$ is the simplest example to show that this condition is still not necessary.

In order to obtain further, gradually weaker conditions which will still suffice, first we may form a symmetric version of Theorem 2.5.6, that is, the union is graphic if the circuits of one of the matroids satisfy the rank requirements in the other matroid. However, $(U_{1,2} \oplus U_{0,2}) \vee (U_{0,2} \oplus U_{1,2})$ is the simplest example to show that this condition is still not necessary (the loops of the first matroid are the parallel elements in the second matroid).

Next it is enough to require this property to a reduced pair of matroids only. However $(U_{1,3} \oplus U_{0,3}) \vee (U_{0,2} \oplus U_{0,2} \oplus U_{0,2})$, where every component of the second matroid has exactly one loop from the first matroid shows that even this condition is not necessary.

It is easy to see that Lemma 2.4.3 eliminates this case because there exist serial elements in M_2 so that one is a loop in M_1. The following example shows that even with all these extensions, and with the help of Lemmata 2.4.1, 2.4.2 and 2.4.3 the property is not necessary for the graphicity of the union.

Example 2.5.7 *Let M_1 be the matroid which is the direct sum of three parallel elements a, b, c and $M(K_4)$ where $1, 2, 3$ are three edges incident to a common vertex with other endpoints P, Q, R, respectively, $f_1 = (P, Q)$, $f_2 = (Q, R)$ and $f_3 = (R, P)$.*

Let M_2 be the matroid which is the direct sum of a three long circuit $1, 2, 3$, two parallel elements f_1, f_2, three parallel elements a, b, f_3 and a loop c.

The union will have a circuit a, b, c and coloops hence graphic. However a, b is a length two circuit in M_1 so that $M_2 \setminus \{a, b\}$ contains a spanning tree and a circuit too. Nevertheless M_1, M_2 is a reduced pair. This means this is a counterexample for the necessity of the property.

In fact in Theorem 2.5.2 where one of the matroids consists of parallel elements and loops, we stated this property in a slightly different way, there the circuit C for which $M_2 \setminus C$ consists spanning tree and circuit too were in one component of M_2. Observe that a and b

are in the same component of M_2 in Example 2.5.7 hence that remains a counterexample for the necessity even if we add this condition.

However if we use Lemmata 2.4.5 and 2.4.7, then this example can be reduced too. In fact either of the two will do, because a and b are parallel in both matroids (Lemma 2.4.5), on the other hand c is a loop in M_2 and c is parallel to a in M_1 (Lemma 2.4.7). Recall that Lemmata 2.4.5 and 2.4.7 are not about equivalent reduction (just in the case where the union is binary), so we can no longer speak about necessity of the extended version of the conditions.

2.6 Necessary condition

In this section we show a necessary condition for the binarity of the union of two graphic matroids. This condition is formalized in a similar way to the sufficient condition in the previous section. Unfortunately they are not exactly the same so there remains a gap which consists of those cases where there might exist a counterexample for Conjecture 1.4.2 (a pair of graphic matroids which have a nongraphic but binary union). This is the main motivation of the lemmata in Section 2.1 and 2.4. They imply that a possible minimal counterexample must have some special properties.

Theorem 2.6.1 *Let M_1 and M_2 be graphic matroids. If all of the following conditions hold, then the union $M_1 \vee M_2$ is not binary.*

1. *There exist X_i dependent sets in M_i for $i \in \{1, 2\}$*
2. *$X_1 \cap X_2 = \emptyset$*
3. *There exist a circuit C_i of M_i in X_i so that $|C_i| \geq 2$ for $i \in \{1, 2\}$*
4. *$r_i(X_i) = r_i(X_1 \cup X_2)$ for $i \in \{1, 2\}$*
5. *There are two distinct elements $a, b \in C_1 \cup C_2$ such that for $i \in \{1, 2\}$:*
    - *if $a \in C_i$ and $b \in C_{3-i}$, then a and b are in the same component in both matroids*
    - *if $a, b \in C_i$, then there exists $X'_{3-i} \subset X_{3-i}$ so that if we contract X'_{3-i} in M_{3-i}, then a and b are diagonals of C_{3-i} connecting distinct pairs of vertices*

Condition 1 is obviously implied by Condition 3, it is mentioned separately for the simplification of the discussion below. Observe that the two minimal examples of graphic pairs which have nonbinary union motivate the last condition.

Proof: There are two cases. At first we study if $a, b \in C_1$, and then, if $a \in C_1$ and $b \in C_2$. These two cases cover all the possibilities by the symmetries of the conditions.

Suppose that $a, b \in C_1$. We can extend X_2' to X_2'' from C_2 so that if we contract X_2'' in M_2, then there remains only three elements α, β, γ from C_2 in M_2' and a is parallel to α, b is parallel to β. Also we can contract a proper subset $P \subset X_1$ in M_1 so that a and b will be parallel in M_1' and $r_1'(X_1 - P) = 1$.

Now examine the union contracted to $X_2'' \cup P \cup a$. We show that the elements b, α, β and γ form a $U_{2,4}$. Consider the rank of the set $X_1 \cup X_2$ in the union: $r_{union}(X_1 \cup X_2) \leq r_1(X_1 \cup X_2) + r_2(X_1 \cup X_2) = r_1(X_1) + r_2(X_2)$. We know that $r_2(X_2'') = r_2(X_2) - 2$ and $r_1(P \cup a) = r_1(X_1)$, this shows that $r'_{union}(\{b, \alpha, \beta, \gamma\}) \leq 2$. For $x_1 \in \{a, b\}$ and $A \in \{\{a, \beta\}, \{\alpha, \beta\}, \{a, \gamma\}, \{\alpha, \gamma\}, \{b, \alpha\}, \{\beta, \gamma\}\}$, $(x_1 \cup A) - a$ will be independent if $(x_1 \cup A)$ contains a, as the partition $(P \cup x_1) \cup (X_2'' \cup A)$ shows, where the first subset is independent in M_1 and the second is independent in M_2. Note that x_1 and A can be chosen so that $(x_1 \cup A) - a$ is equal to any 2-subset of $\{b, \alpha, \beta, \gamma\}$. Hence we have a $U_{2,4}$ minor in the union, thus it is not binary.

For the other case suppose that $a \in C_1$, $b \in C_2$. According to the fifth condition of the theorem there exist $X_1' \subset X_1$ and $X_2' \subset X_2$ so that if we contract X_1' in M_1, then C_1 is contracted to a and an other element c, and b is parallel to them and if we contract X_2' in M_2, then C_2 is contracted to b and an other element d, and a is parallel to them. According to the second condition of the theorem c and d are different elements. Choose X_1' and X_2' to be maximal. Thus $r(M_1/X_1') = r(M_2/X_2') = 1$.

Now examine the union contracted to $X_1' \cup X_2'$. We show that the elements a, b, c and d form a $U_{2,4}$. Again $r_{union}(X_1 \cup X_2) \leq r_1(X_1 \cup X_2) + r_2(X_1 \cup X_2) = r_1(X_1) + r_2(X_2)$. Now $r_1(X_1') = r_1(X_1) - 1$ and $r_2(X_2') = r_2(X_2) - 1$ so $r'_{union}(\{a, b, c, d\}) \leq 2$. For $x_1 \in \{a, b, c\}$ and $x_2 \in \{a, b, d\}$, $x_1 \neq x_2$, $\{x_1, x_2\}$ will be independent, as the partition $(X_1' \cup x_1) \cup (X_2' \cup x_2)$ shows, where the first subset is independent in M_1 and the second is independent in M_2. Note that x_1 and x_2 can be chosen so that $\{x_1, x_2\}$ is equal to any 2-subset of $\{a, b, c, d\}$. Hence we have a $U_{2,4}$ minor in the union, thus it is not binary. □

While it is not quite apparent at first, the conditions of Theorem 2.6.1 are similar to those of the symmetric version of Theorem 2.5.6. If the sufficient condition is not met, then circuits C_1, C_2 of length at least two must exist in the respective matroids, satisfying $r_{3-i}(E - C_i) = r_{3-i}(E)$ and $r_{3-i}(E - C_i) < r_{3-i}(E - C_i)$ for $i = 1, 2$. Only the fifth condition seems to be different but it just denies the degenerate cases (which did not come up in the earlier cases either).

As we already mentioned the minimal counterexample must be unreducible if exists.

Remark 2.6.2 *As we have a necessary and a sufficient condition for the graphicity of the union, we could use them to prove Theorems 2.2.1 and 2.3.1. However, the necessary part of those deductions would not be easy (the sufficiency is almost trivial). For the case of Theorem 2.2.1 we would have to show that the 5/2 condition of Theorem 2.6.1 holds, and this would be mainly the same as a part of the proof of Theorem 2.2.1. For the case of Theorem 2.3.1 we would have to show that the 5/1 condition of Theorem 2.6.1 holds (if $n \geq 3$), and this would be mainly the same as a part of the proof of Theorem 2.3.1. On the other hand, the conditions of Theorems 2.2.1 and 2.3.1 are the antecedents of the sufficient and the necessary conditions.*

2.7 Applications of the results of Sections 2.2 and 2.3 for Linear Active Networks

The results studied in this chapter have been published in [13].

In this chapter the results of Sections 2.2 and 2.3 are formalized from the point of view of linear active networks.

Consider a linear network composed of 2-terminal devices. Its interconnection structure is described by a graph G. The voltages or the currents of a subset of devices can independently be prescribed if and only if the subset of the corresponding edges in the graph G is circuit-free or cut set free, respectively. This classical result of Kirchhoff can be generalized for networks containing multiterminal devices as well: the independence structure can be described by the circuits and cut sets of a matroid M. However, this matroid will not always be graphic. It is useful in the application if this matroid is graphic, because the representing graph gives a good visualization. On the other hand, the graphicity of this matroid means that there exists a linear network with only 2-terminal devices which behaves as the original one (with the multiports).

The independence properties of a network, obtained by the interconnection of linear multiports, can be obtained as the union of two matroids, where the first one is the matroid of the graph of the interconnection structure and the second one describes the algebraic relations among the port voltages and currents.

Applying our results of Theorems 2.2.1 and 2.3.1 we give a physical characterization of subclasses of those active networks where M happens to be graphic.

2.7.1 Introduction

Here we briefly introduce the theory of linear active networks, but for a comprehensive overview the reader should see the book [36]. Electric network analysis was the first real application of graph theory, almost 170 years ago. The laws of Kirchhoff [23] related the voltages and the currents of the devices to the circuits and cut sets, respectively, of the graph of the interconnection.

These classical results can be applied if the network consists of 2-terminal devices only. If the multiterminal devices are modelled by controlled sources, then the interconnection can still be described by a graph but, due to the controls among the edges, the independence structure of the network will not always remain graphic. Since the network is linear, it can be described by a matrix and we can describe the combinatorial properties of the independence structure of the columns of this matrix but the resulting matroids will rarely be graphic.

The matroid operation union turned out to be the appropriate tool to describe the effect of control, as found independently by [33], [21] and [28]. However, the subset of graphic matroids is not closed with respect to union, in fact, the union of two graphic matroids is often outside the more general subset of binary matroids.

The fundamental results of [29] and [17] characterize those graphic matroids whose union is the free matroid (the cycle matroid of a tree). If the union of several copies of the same graphic matroid is considered, then one can decide if this union is graphic [25] but the question is still open for general addends. As discussed in Chapter 2, a possible approach is to fix a graph G_0 or its cycle matroid $M_0 = M(G_0)$ and study those graphs G where the union of $M(G)$ and M_0 is graphic. If M_0 consists of loops only or it contains coloops, then the problem is trivial hence the first interesting question was if G_0 consists of a circuit of length two (two parallel edges) and any number of loops. In the language of electric network analysis this corresponds to the linear active networks composed of 2-terminal passive devices plus a single current controlled current source. This case has been solved in [34] – mathematically it was a Kuratowski-type characterization of G which had a physical interpretation as the lack of feedback, see Theorems 2.7.1 through 2.7.4

below.

We have generalized the results of [34] for the case if G_0 consists of either n serial or n parallel edges in addition to the loops, see [9] for $n = 3$ and Sections 2.2 and 2.3 for any n. In the present chapter we study the interpretation of the structure of G_0 in terms of controlled sources, and formulate the mathematical meaning of these recent results in the language of electric network analysis.

2.7.2 Former Results

Suppose that a network is composed of 2-terminal devices and current controlled current sources (CCCS). The graph of the network is defined in the usual way (each CCCS corresponds to a pair of edges), and we assign orientation to each edge arbitrarily. There are several equations among the currents of the devices, some of them are the Kirchhoff Current Laws, describing the topology of the network, some others describe the controls. In what follows, we shall refer to these sets of equations as the graphic and the algebraic sets of equations, respectively. For example, the graph of the network on the left hand side of Figure 2.5 is shown in the right hand side, the set of the graphic equations consists of

$$i_1 + i_2 + i_3 = 0$$
$$i_3 - i_4 - i_5 = 0$$

(and any linear combinations of them), while there is a single algebraic equation $i_5 = c \cdot i_2$.

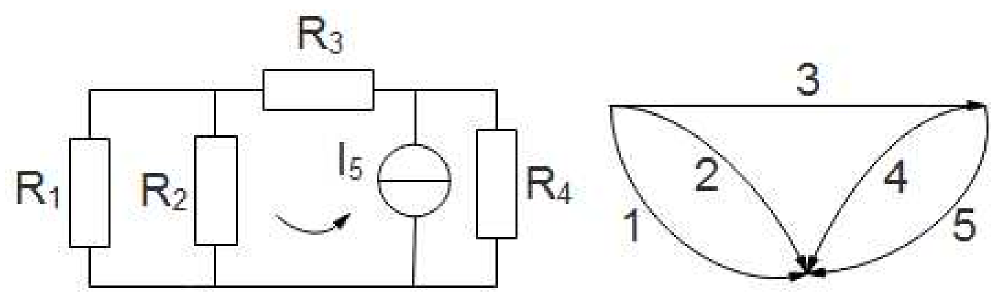

Figure 2.5: A network and the corresponding graph

Hence there are three linear equations referring to the five currents and these equations can be summarized by the coefficient matrix

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & c & 0 & 0 & -1 \end{pmatrix}$$

In contrast, the network of Figure 2.6 has a different kind of control, namely $i_5 = c \cdot i_3$, hence our matrix will be

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & c & 0 & -1 \end{pmatrix}$$

If the five column vectors of the matrices are considered, then in case of M_1 *any three*

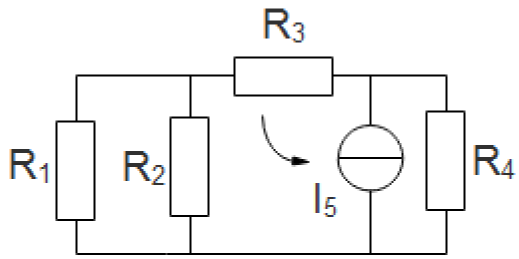

Figure 2.6: The network of Figure 2.5 with a different type of control

out of the five vectors except $\{1, 3, 4\}$ *are linearly independent while in case of* M_2 *a set of three vectors are linearly independent if and only if the set does not contain both the first and the second vectors.* (In the first case we suppose $c \neq 0$, $c \neq -1$ and in the second case we suppose that $c \neq 0$, $c \neq 1$, see Remark 2.7.9 below)

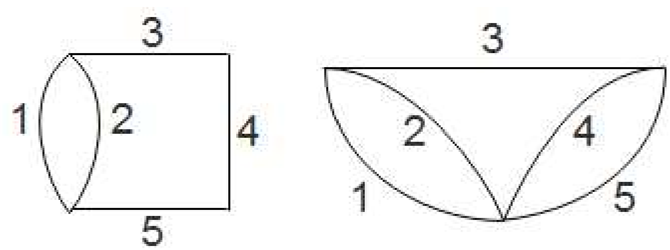

Figure 2.7: A graph representing M_2 and the graph of the graphic submatrices of both M_1 and M_2

In the second case the italicized sentence can be rephrased as follows: *a set of three vectors are linearly independent if and only if the corresponding three edges form a spanning tree in the graph on the left hand side of Figure 2.7.* On the other hand, no such reformulation is possible in the first case – no one can draw a graph with four vertices and five edges so that $\{1, 3, 4\}$ is a circuit and any other set of three edges forms a spanning tree.

Using the terminology of matroid theory we may conclude that the matroid describing the second network is graphic while that for the first one is nongraphic.

In both cases the first two rows of the matrices refer to the graphic set of equations and the last row refers to the algebraic one. This partition of the rows leads to a graphic and to an algebraic submatrix. One can easily see that a subset of columns of both graphic submatrices is linearly independent if and only if the corresponding edges form a circuit-free subgraph of the graph on the right hand side of Figure 2.7.

Theorem 2.7.1 *[34] Let G_0 consist of a circuit of length two (two parallel edges a, b) and any number of loops, let $M_0 = M(G_0)$. Let G be an arbitrary graph on the same edge-set. Then the union of $M(G)$ and M_0 is graphic if and only if G does not contain any subgraph isomorphic to the graph of Figure 2.8 or to its subdivision, with a and b in the indicated positions.*

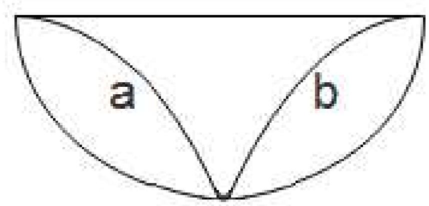

Figure 2.8: The graph whose existence means the presence of feedback

If a network is composed from 2-terminal devices and of a single CCCS (whose edges will play the role of a and b), then the existence of the subgraph of Figure 2.8 or its subdivision (with a and b in the requested positions) means the presence of a feedback F, no matter what kind of subnetworks N_1, N_2 are interconnected, see Figure 2.9. Hence the above theorem can be reformulated as follows:

Theorem 2.7.2 *[34] Suppose that a network is composed of 2-terminal devices and of a single current controlled current source. The independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

The graph G in Theorem 2.7.1 was arbitrary. In network theory applications we may always suppose that the underlying graph of the electric network is connected, in fact, even 2-connected if there is no control in the network. Moreover, if a subgraph is connected along two points to the rest of the graph and none of the edges of this subgraph is a controlling or a controlled element, then the whole subgraph can be replaced by a single edge. Using these replacements if applicable, we obtain the reduced graph of the network. For a more formal description of this matroid theoretical reduction see Section 2.1.

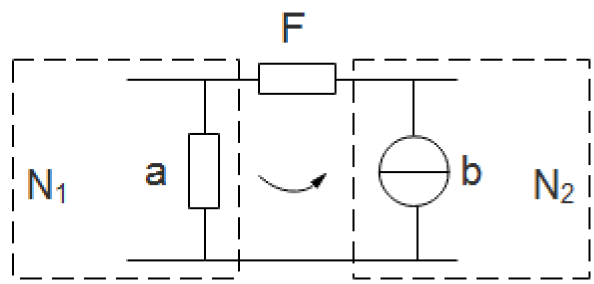

Figure 2.9: Feedback in a general network

In view of this, feedback is formally defined as the presence of at least one circuit in the complement of $\{a, b\}$ in the reduced network graph. Then one can reformulate Thorem 2.7.1 as follows:

Theorem 2.7.3 *Suppose that the reduced graph of the network is* 2*-connected and* a, b *are two nonserial edges. Then there is a subgraph isomorphic to Figure 2.8 or its subdivision, with* a *and* b *in the specified positions, if and only if the complement of* $\{a, b\}$ *in the reduced network graph contains at least one circuit.*

In the next subsection we shall refer to the negative of this reformulation:

Theorem 2.7.4 *Suppose that a network is composed of 2-terminal devices and of a single current controlled current source involving the nonserial edges* a *and* b*. Then the independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

Here feedback is a circuit in the complement of $\{a, b\}$ in the reduced network graph. We may suppose without loss of generality that the reduced graph of the network is 2-connected. The condition that a and b are nonserial can be supposed if we speak about a control between them.

In what follows we shall generalize Theorems 2.7.2 and 2.7.4 for more general types of control. Recall that in case of a CCCS the current of a single source is controlled by the current of a single resistor. We have found analogous results if only one of these restrictions remains.

2.7.3 New Results

Several Controlled Sources and a Single Controlling Element

Suppose that the current of a single resistor R_0 controls several current sources $I_1, I_2, ..., I_k$ as described by the respective equations $i_j = c_j \cdot i_0$ for every $j = 1, 2, ..., k$. We may suppose that the set $[n]$ of the corresponding edges $e_0, e_1, e_2..., e_k$ does not contain any cut-set in the graph of the network, since otherwise there were an additional equation $\Sigma i_j = 0$ among some of these currents, which, together with the control equations $i_j = c_j \cdot i_0$, would lead to a singular network.

Since there are k controls in the network, the above definition of the feedback is modified as the presence of at least one circuit in the complement of the set $[n]$ in the reduced network graph.

Theorem 2.7.5 *Suppose that a network is composed of 2-terminal devices and the current of a resistor R_0 controls several current sources $I_1, I_2, ..., I_k$ as described by the respective equations $i_j = c_j \cdot i_0$ for every $j = 1, 2, ..., k$ (where the control constants $c_1, c_2..., c_k$ are generic parameters). Then the independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

Here feedback is a circuit in the complement of $[n]$ in the reduced network graph. We call some numbers generic if they are algebraically independent over the field of the rational numbers. We may suppose without loss of generality that $[n]$ is cut set free. The condition that a and b are nonserial can be supposed if we speak about a control between them.

PROOF: The system of equations $i_j = c_j \cdot i_0$ for every $j = 1, 2, ..., k$ leads to an algebraic submatrix representing a matroid M_1 which consists of loops and a single circuit of length $k + 1$. Let M_2 denote the matroid, represented by the graph of the interconnetion. Theorem 2.2.1 states that the union of the reduced matroids M_1' and M_2' is graphic if and only if either $[n]$ contains a cut-set or M_2' $[n]$ is the free matroid. Since the former case is excluded, the reduced network graph without the edges in $[n]$ must be circuit-free. □

Several Controlling Elements and a Single Controlled Source

Suppose that a single current source i_0 is controlled by the current of several resistors $R_1, R_2, ..., R_k$ as described by the equation $i_0 = \Sigma c_j \cdot i_j$ where the summation is for every $j = 1, 2, ..., k$. We may suppose without loss of generality that the network graph is either 2-connected or the set $[n]$ of the corresponding edges $e_0, e_1, e_2..., e_k$ has at least one edge from each 2-connected component.

Since there is a single control involving $k + 1$ elements in the network, the above definition of the feedback is modified as the presence of at least one circuit in the complement of any two-element subset of the set $[n]$ in the reduced network graph.

Theorem 2.7.6 *Suppose that a single current source i_0 is controlled by the current of several resistors $R_1, R_2, ..., R_k$ as described by the equation $i_0 = \Sigma c_j \cdot i_j$ where the summation is for every $j = 1, 2, ..., k$. Like in Theorem 2.7.5, suppose that the control constants $c_1, c_2..., c_k$ are generic parameters. Then the independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

Here feedback is a circuit in the complement of the edge set $\{a, b\}$ for two nonserial edges a, b of $[n]$ in the same 2-connected component of the reduced network graph. We call some numbers generic if they are algebraically independent over the field of the rational numbers. We may suppose without loss of generality that the network graph is either 2-connected or the set $[n]$ of the corresponding edges $e_0, e_1, e_2..., e_k$ has at least one edge from each 2-connected component. The condition that a and b are nonserial can be supposed if we speak about a control between them.

PROOF: The equation $i_0 = \Sigma c_j \cdot i_j$ leads to an algebraic submatrix representing a matroid M_1 which consists of loops and $k+1$ parallel edges. Theorem 2.3.1 states that the union of the reduced matroids M_1' and M_2' is graphic if and only if no 2-connected component of the reduced network graph G has two nonserial edges a, b so that $G \setminus \{a, b\}$ contains a circuit. This is clearly equivalent to the condition of Theorem 2.7.6. $\square$

2.7.4 Examples and a Remark

Example 2.7.7 *Consider the network of Figure 2.10 where $i_0 = c_1 \cdot i_1 + c_2 \cdot i_2$. The graph of the network is given in Figure 2.11. The coefficient matrix for the system of equations for the currents of the elements will be*

$$\begin{pmatrix} -1 & c_1 & c_2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The matroid represented by the columns of this matrix has six elements and rank four. This matroid is nongraphic – if we contract elements 4 and 5, then the resulting minor is the rank 2 uniform matroid on the set $\{0, 1, 2, 3\}$ which is known not to be binary, let alone graphic. Based on Theorem 2.7.6 one could reach the same conclusion: The elements 0 and 1 are nonserial edges in the same 2-connected component of the graph of Figure 2.11, still the complement of the set $\{0, 1\}$ contains a circuit, namely $\{2, 5\}$.

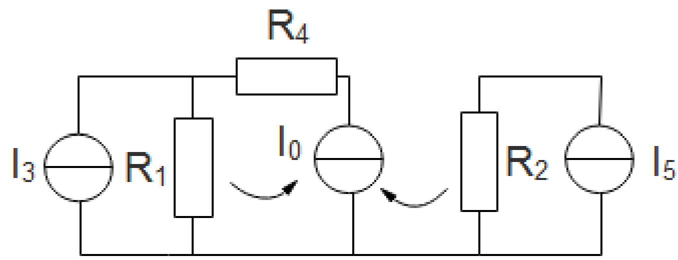

Figure 2.10: The network of Example 2.7.7

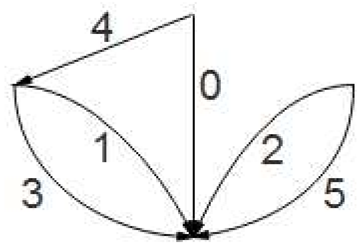

Figure 2.11: The graph of the network of Example 2.7.7

Example 2.7.8 *The network of Figure 2.12 illustrates Theorem 2.7.5. Let the controls be* $i_1 = c_1 \cdot i_0$ *and* $i_2 = c_2 \cdot i_0$*. The graph of the network is given in Figure 2.13 and the coefficient matrix for the system of equations for the currents of the elements will be*

$$\begin{pmatrix} c_1 & -1 & 0 & 0 & 0 & 0 & 0 \\ c_2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

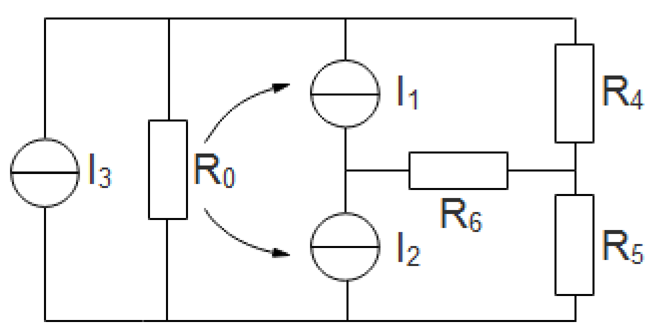

Figure 2.12: The network of Example 2.7.8

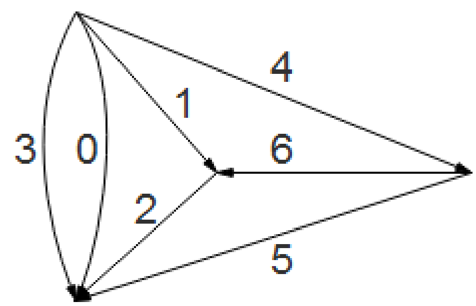

Figure 2.13: The graph of the network of Example 2.7.8

The corresponding matroid has seven elements and rank five. One can see that it is nongraphic – if we contract elements $0, 2$ and 6, the resulting minor is the rank 2 uniform matroid on the set of the remaining elements. Based on Theorem 2.7.5 one could reach the same conclusion: If we delete the edges of the set $[n] = \{0, 1, 2\}$ from the graph of Figure 2.13, the remaining graph contains a circuit, namely $\{3, 4, 5\}$.

Remark 2.7.9 *Results applying matroid union for engineering applications frequently require a genericity-type condition like the one we had in Theorems 2.7.5 and 2.7.6 concerning the control constants $c_1, c_2 ..., c_k$. The basic reason of this has been discovered by Edmonds [18] during his study about the relation between rank and term rank of the matrices. If such an assumption is missing, the statement might be wrong.*

For example, suppose that $c_1 = 1$ in Example 2.7.7. Then the set $\{0, 1, 4\}$ will become a circuit and the matroid will be graphic (a circuit formed by $\{0, 1, 4\}$ and another formed by $\{1, 2, 3, 5\}$, sharing a common edge). Physically, it corresponds to a singular network: The relation $c_1 = -1$ leads to a control equation $i_0 = -i_1 + c_2 \cdot i_2$; hence the Kirchhoff equation $i_3 = -(i_1 + i_0)$ would lead to a relation $i_3 = c_2 \cdot i_5$ between two independent current sources.

Chapter 3

New questions

In order to put Conjecture 1.4.2 into a more general framework, we formally define eight matroid classes as follows.

The first column of the table represents the name of the sets. 'gr' stands for graphic.

A	those gr matroids which give a	gr or nonbinary union with	any gr matroid.
B	those gr matroids which give a	gr union with	any gr matroid.
C	those gr matroids which give a	gr or nonbinary union with	any matroid.
D	those gr matroids which give a	gr union with	any matroid.
E	those matroids which give a	gr or nonbinary union with	any gr matroid.
F	those matroids which give a	gr union with	any gr matroid.
G	those matroids which give a	gr or nonbinary union with	any matroid.
H	those matroids which give a	gr union with	any matroid.

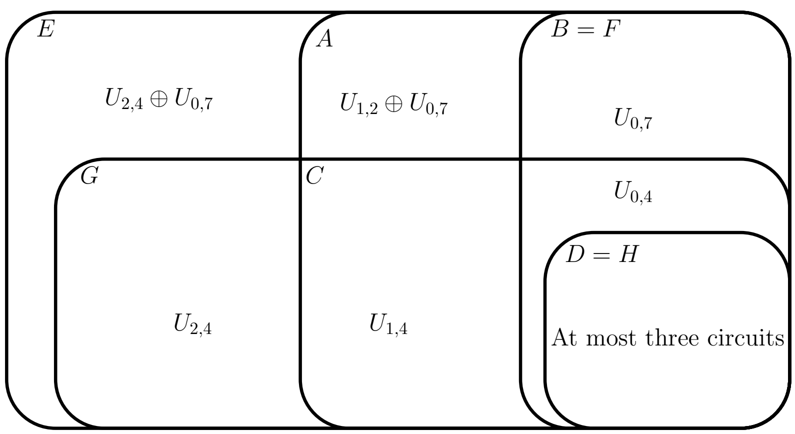

Figure 3.1: Examples for each nonempty subset

Observe that Conjecture 1.4.2 states that A is the set of all graphic matroids.

Most of the relationships between the sets are trivial ($D \subseteq C \subseteq A$, $D \subseteq B \subseteq A$,

$H \subseteq G \subseteq E$, $H \subseteq F \subseteq E$, $A \subseteq E$, $B \subseteq F$, $C \subseteq G$, $D \subseteq H$) see Figure 3.1. For $D = H$ recall that the union of M and $U_{0,k}$ is M so if the union is graphic, then M is also graphic. Since $U_{0,k}$ is graphic $F = B$ follows similarly. $(A \cap G) - C$ is empty because if a matroid is in G but not in C, then it is not graphic.

Theorem 3.0.10 *A matroid is in D if and only if it contains at most three circuits.*

PROOF: The condition that a matroid contains at most three circuits holds if and only if it contains only a θ graph in addition to coloops or it is the direct sum of at most three circuits and some coloops. Let M denote a matroid in D. Let M_2 denote the other matroid in the union. According to the reduction we can suppose that neither M nor M_2 contains any coloop.

For the if part of the proof we consider two cases.

Case 1. Let M be the cycle matroid of a θ graph. Let P, Q, R denote the three paths of the θ graph and $\alpha_i, \beta_i, \gamma_i$ be their elements, respectively.

- If $r(M_2) = 0$, then the union is isomorphic to M.

- If $r(M_2) = 1$, then we have the case as in Theorem 2.5.2, let $[n]$ denote the parallel elements in M_2. This means that the union is graphic if no two elements $a, b \in [n]$ exist such that $M \setminus \{a, b\}$ contains both a spanning tree and a circuit. If we pick two elements from the same path in the θ graph, then no spanning tree remains, while if we pick two from different paths, then no circuit remains. Thus the union is graphic.

- If $r(M_2) \geq 2$ and there is a basis B in M_2 so that it contains at least one element from two different paths of the θ graph, then the union is the free matroid since we can pick B from M_2 and $E - B$ from M_1.

- Finally let $r(M_2) \geq 2$ and suppose that there is no basis in M_2 containing at least one element from two different paths of the θ graph. We claim that in this case all the elements in two paths of the θ graph of M are loops in M_2. This will suffice since then, the union $M \vee M_2$ is graphic, namely a circuit of these two paths of the θ graph with coloops for the third path. Indirectly suppose that $\alpha_1 \in P$ and $\beta_1 \in Q$ are nonloop elements of M_2. Let B_1 be a basis of M_2 containing α_1 and, by the assumption, all of its further elements are in P. Since both B_1 and $\{\beta_1\}$ are independent and $|B_1| > 1$, there must exist an element $\alpha \in B_1$ so that $\{\alpha, \beta_1\}$ is also independent, hence it can be extended to a basis of M_2, a contradiction.

Case 2. Let M be the direct sum of at most three circuits. Suppose that there are exactly three circuits in M. Let C_1, C_2, C_3 denote the three circuits of M and a_i, b_i, c_i be their elements, respectively.

Fortunately the cases where $r(M_2) \leq 1$ are the same as before. The only difference is that we have to pay attention to the fact that the two elements a, b which are parallel in M_2 so that $M \setminus \{a, b\}$ contains a spanning tree and a circuit, must be in the same component of M.

Again, we examine the cases according to the bases ($r(M_2) \geq 2$).

- If there is a basis B in M_2 containing at least one element from every circuit of M, then B is independent in M_2 and $E - B$ is independent in M so the union is the free matroid.

- If there is no basis in M_2 containing at least one element from two distinct circuits of M, then, just like in the last subcase of Case 1, only one of the three circuits of M has nonloop elements in M_2 so the union $M \vee M_2$ is graphic (two circuits remain the same as in M, while the elements of the third become coloops).

- If there is no basis in M_2 containing at least one element from every circuit of M, but there exist bases B_1, B_2, B_3 so that $B_1 \cap C_1 \neq \emptyset; B_1 \cap C_2 \neq \emptyset; B_2 \cap C_2 \neq \emptyset; B_2 \cap C_3 \neq \emptyset; B_3 \cap C_1 \neq \emptyset; B_3 \cap C_3 \neq \emptyset$, then the union $M \vee M_2$ will be graphic, namely a single circuit.

- If there is no basis in M_2 containing at least one element from every circuit of M, there exist bases B_1, B_2 so that $B_1 \cap C_i \neq \emptyset; B_1 \cap C_j \neq \emptyset; B_2 \cap C_j \neq \emptyset; B_2 \cap C_k \neq \emptyset$ ($\{i, j, k\} = \{1, 2, 3\}$) but does not exist a B_3 such that $B_3 \cap C_i \neq \emptyset; B_3 \cap C_k \neq \emptyset$, then the union $M \vee M_2$ will be a circuit of the elements of C_i and C_k and coloops for the elements of C_j, hence graphic.

- If there is no basis in M_2 containing at least one element from every circuit of M, but there exists a basis B_1 so that $B_1 \cap C_i \neq \emptyset; B_1 \cap C_j \neq \emptyset$ and does not exists a basis B_2 such that $B_2 \cap (C_i \cup C_j) \neq \emptyset; B_2 \cap C_k \neq \emptyset$ ($\{i, j, k\} = \{1, 2, 3\}$), then all the elements of C_k are loops in M_2. This means that the union $M \vee M_2$ will be a circuit of the elements of C_k and coloops for the elements of C_i and C_j, hence graphic.

On the other hand we shall show that if M contains more than three circuits, then its union with an appropriately chosen matroid will contain a $U_{2,4}$ minor.

Lemma 3.0.11 *If a graphic matroid contains at least four circuits, then it contains at least one of the following three minors: $U_{1,4}$, $U_{0,4}$, $U_{1,3} \oplus U_{0,1}$.*

PROOF: We have already seen that a matroid M containing at least three circuits either contains three pairwise disjoint circuits or a θ-graph. In the former case the extension of three disjoint circuits with a fourth one either leads to a minor $U_{0,4}$ (if the fourth circuit is disjoint to the previous ones) or to $U_{1,3} \oplus U_{0,1}$ (if the fourth circuit intersects at least one of the old ones).

On the other hand, if M contains a θ-graph, then the fourth circuit may be disjoint to it, leading to $U_{1,3} \oplus U_{0,1}$ or contributes to the θ-graph and we obtain a $U_{1,4}$ as a minor. □

For all the three cases we construct M_2 so that $M \vee M_2$ contains $U_{2,4}$, hence not graphic. For the set X of those elements which are not in the minor (M_{minor}) we can simply make loops in M_2, leading to $(M \vee M_2)/X = M_{minor} \vee (M_2 \setminus X)$.

Case 1: If M has a $U_{1,4}$ minor, then let $M_2 \setminus X = U_{1,4}$. Then $(M \vee M_2)/X = U_{1,4} \vee U_{1,4} = U_{2,4}$, hence not graphic.

Case 2: If M has a $U_{0,4}$ minor, then let $M_2 \setminus X = U_{2,4}$. Then $(M \vee M_2)/X = U_{0,4} \vee U_{2,4} = U_{2,4}$, hence not graphic.

Case 3: If M has a $U_{1,3} \oplus U_{0,1}$ minor, then let $M_2 \setminus X = U_{1,4}$. Then $(M \vee M_2)/X = (U_{1,3} \oplus U_{0,1}) \vee U_{1,4} = U_{2,4}$, hence not graphic.

□

All the containments as indicated in Figure 3.1 are proper, as shown by the examples. The position of these examples are straightforward for all but one case, see Theorem 3.0.12 below.

Theorem 3.0.12 *The set $E - (G \cup A)$ is not empty, it contains the matroid $K = U_{2,4} \oplus U_{0,7}$.*

PROOF: K is not graphic because it has a $U_{2,4}$ minor, hence it is not in A. $(U_{4,4} \oplus F_7) \vee K$ is not graphic but binary, hence K is not in G. To show that K is in E let M be an arbitrary graphic matroid. We have to prove that $M \vee K$ is either graphic or not binary. Now we apply the possible reduction steps to obtain a reduced pair M', K' on the common underlying set E', as in Theorem 2.1.5.

Proposition 3.0.13 *If $NL(K')$ does not contain a cut set in M', then the union $M' \vee K'$ is not binary.*

PROOF: If $NL(K')$ does not contain a cut set, then $L(K')$ contains a basis of M', so $(M' \vee K')/L(K') = U_{2,4}$ $\square$

Proposition 3.0.14 *If $L(K')$ is independent in M' but $NL(K')$ contains a cut set X in M', then the union $M \vee K$ is graphic.*

PROOF: Since X is a cut set there exists $x \in X$ so that $L(K') \cup \{x\}$ is independent in M'. Every two element subset of $NL(K')$ is independent in K' so there exist two distinct elements y, z in $NL(K') - \{x\}$ such that $L(K') \cup \{x, y, z\}$ is independent in the union. Now the statement follows from Proposition 2.5.4. $\square$

Proposition 3.0.15 *If $r_{M'}(E') - r_{M'}(E' - NL(K')) \geq 2$, then $M \vee K$ is graphic.*

PROOF: The above condition means that every basis of M' contains at least two elements of $NL(K')$. In that case $M' \vee K' = (M' \setminus NL(K')) \oplus U_{4,4}$ and that is graphic, since M' is graphic. This means that the union $M \vee K$ is also graphic . $\square$

Proposition 3.0.16 *If $r_{M'}(E') - r_{M'}(E' - NL(K')) = 1$ and $L(K')$ is not independent in M', then the union $M \vee K$ is not binary.*

PROOF: Since M', K' is a reduced pair of matroids, a circuit C of M' in $L(K')$ must be spanned by the set $NL(K')$ and the length of C is at least three. Let X denote the cut set of M' in $NL(K')$ and x denote an element of it. Let $1, 2, 3$ be three different elements of C. Let Z denote the set $C \cup x - \{1, 2, 3\}$ extended by all the elements z of $L(K') - \{1, 2, 3\}$ for which it is true that $r_{M'}(Z \cup z) > r_{M'}(Z \cup \{1, 2, 3\})$ (in this way Z is independent in M'). Consider the matroid M'/Z, it is a graphic matroid which can be represented by a graph of three vertices, where $\{1, 2, 3\}$ forms a circuit, and there are parallel edges to at least two of 1,2 and 3. Suppose that a is parallel to 1, b is parallel to 2, and c is the third element of $NL(K')$ which remained. With this notation we show that $M' \vee K'/(\{b, c, x\} \cup Z) = U_{2,4}$. The rank of this matroid is trivially not larger than two. We have to show that every pair of elements is independent. $\{a, 1\}$ will be independent, as the partition $(Z \cup \{b, 1\}) \cup \{a, c\}$ shows, where the first subset is independent in M' and the second subset is independent in K'. For any other 2-subset P of $\{a, 1, 2, 3\}$, P will be independent, as the partition $(Z \cup P) \cup \{b, c\}$ shows, where the first subset is independent in M' and the second is independent in K'. We found a $U_{2,4}$ minor in a minor of $M \vee K$, hence it is not binary. □

The above propositions cover all the possible cases for M' and in every case the union is graphic or not binary, so K is in E. □

Chapter 4

Partitioning the bases of the union of matroids

The results studied in this chapter have been published in [12].

In this chapter we present further useful theorems on the bases of the union of matroids. These theorems are anticipated to help solving the remaining cases of Conjecture 1.4.2.

Let $B = \cup_{i=1}^{n} B_i$ be a partition of basis B in the union of n matroids into independent sets B_i of M_i. We prove that every other basis B' has such a partition where B_i and B'_i span the same set in M_i for $i = 1, 2, \ldots, n$.

4.1 Introduction

Throughout this chapter let E denote the common underlying set of every matroid and let $r_1, r_2, \ldots, r_n$ denote the rank functions of the matroids $M_1, M_2, \ldots, M_n$, respectively. M will denote the union $\vee_{i=1}^{n} M_i$ of these matroids, and R will denote the rank function of M. A subset $X \subseteq E$ is independent in M if and only if it arises as $X = \bigcup_{i=1}^{n} X_i$ with X_i independent in M_i for each i. Recall that $R(X) = min\{\Sigma_{i=1}^{n} r_i(Y) + |X - Y| : Y \subseteq X\}$ (Theorem 1.5.8).

An element of the underlying set E of a matroid is a *loop* if it is dependent as a single element subset, and it is a *coloop* if it is contained in every base. We shall need the following observation ([25], independently rediscovered in [22]):

Proposition 4.1.1 *If M has no coloops, then $R(E) = \sum_{i=1}^{n} r_i(E)$.*

The *weak map* relation is defined as follows: the matroid B is *freer* than A (denoted by $A \preceq B$) if every independent set of A is independent in B as well. Clearly $M_j \preceq \vee_{i=1}^{n} M_i$

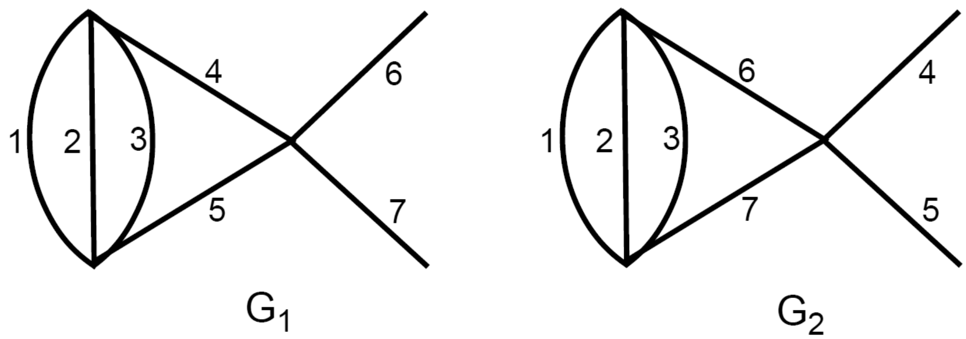

Figure 4.1: The graphs of Example 4.2.1

for every $j = 1, 2, \ldots, n$ and $A \preceq B$ implies $A \vee C \preceq B \vee C$ for every C.

Let $\sigma_i(X)$ denote the closure of a set $X \subseteq E$ in M_i, and let $\sigma(X)$ denote the closure of X in M. We shall need the following easy property of the closure function:

Proposition 4.1.2 *Let $S_1, S_2 \subseteq E$ be independent subsets with $\sigma(S_1) = \sigma(S_2) = S$. Let, furthermore, $S_0 \subseteq E$ so that $S \cap S_0 = \emptyset$ and $S_1 \cup S_0$ is independent. Then $S_2 \cup S_0$ is also independent.*

PROOF: Observe that $|S_1| = |S_2|$ since both are independent and span the same subset S. Indirectly suppose that $r(S_2 \cup S_0) < |S_2| + |S_0| = |S_1| + |S_0| = |S_1 \cup S_0|$. Since $S_1 \cup S_0$ is independent, there exists an element $x \in S_1 - S_2$ so that $r(S_2 \cup S_0 \cup \{x\}) > r(S_2 \cup S_0)$. However, $x \in S_1 \subseteq S = \sigma(S_2)$ implies that $r(S_2 \cup \{x\}) = r(S_2)$, a contradiction. □

4.2 Partitioning the bases

Let B be a basis of M. The partition $B_1, B_2, \ldots, B_n$ of B is a *good partition* if B_i is independent in M_i for $i = 1, 2, \ldots, n$.

Let $F_i = \sigma_i(B_i)$ for every i. This collection of flats $F_1, F_2, \ldots, F_n$ depends on the actual good partition of B, as illustrated by the following example.

Example 4.2.1 *If M_1 and M_2 are the cycle matroids of the graphs G_1 and G_2 of Figure 4.1, respectively, then M will be the cycle matroid of the graph of Figure 4.2. The basis $B = \{1, 2, 4, 5, 6, 7\}$ of M has 18 good partitions, see the second and third columns of Table 4.1, where each row represents two good partitions (put $a, b \in \{1, 2\}$, $a \neq b$ in both way). These good partitions lead to 9 different collections of flats, see the fourth and fifth columns of Table 4.1.*

Table 4.1: Good partitions and the corresponding flats of Example 4.2.1

	B_1	B_2	F_1	F_2
1	$\{a,4,6,7\}$	$\{b,5\}$	E	$\{1,2,3,5\}$
2	$\{a,5,6,7\}$	$\{b,4\}$	E	$\{1,2,3,4\}$
3	$\{a,4,6\}$	$\{b,5,7\}$	$E-\{7\}$	$E-\{4\}$
4	$\{a,4,7\}$	$\{b,5,6\}$	$E-\{6\}$	$E-\{4\}$
5	$\{a,5,6\}$	$\{b,4,7\}$	$E-\{7\}$	$E-\{5\}$
6	$\{a,5,7\}$	$\{b,4,6\}$	$E-\{6\}$	$E-\{5\}$
7	$\{a,6,7\}$	$\{b,4,5\}$	$E-\{4,5\}$	$E-\{6,7\}$
8	$\{a,6\}$	$\{b,4,5,7\}$	$\{1,2,3,6\}$	E
9	$\{a,7\}$	$\{b,4,5,6\}$	$\{1,2,3,7\}$	E

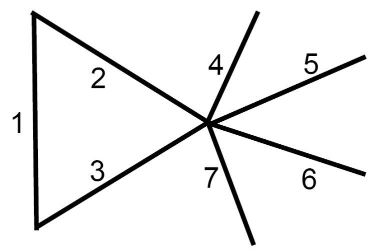

Figure 4.2: The union of the matroids of Example 4.2.1

Surprisingly if we consider any other basis of the union, the list of the possible collections of flats will always be the same.

Theorem 4.2.2 *Let $M_1, M_2, \ldots, M_n$ be matroids and let M be their union. Let B be a basis of M with a good partition $B_1, B_2, \ldots, B_n$. For any basis B' of M there is a good partition $\cup_{i=1}^n B_i'$ so that $\sigma_i(B_i) = \sigma_i(B_i')$ for $i = 1, 2, \ldots, n$.*

PROOF: Suppose that B' is a basis of the union with a good partition $X_1, X_2, \ldots, X_n$.

Let A denote the set of the noncoloop elements of the union. B' is independent in the union so $|B' \cap A| = R(B' \cap A)$. Clearly $R(B' \cap A) = R(A)$ since B' is a basis in the union, and $\sigma(A) = A$. According to Proposition 4.1.1 $\Sigma_{i=1}^n r_i(A) = R(A)$. Now $r_i(A) \geq r_i(X_i \cap A)$ since $X_i \cap A \subseteq A$, and $r_i(X_i \cap A) = |X_i \cap A|$ since X_i is independent in M_i. These together give the following:

$$|B' \cap A| = R(B' \cap A) = R(A) = \Sigma_{i=1}^n r_i(A) \geq \Sigma_{i=1}^n r_i(X_i \cap A) = |B' \cap A|$$

Since the two sides are equal, the inequality must be satisfied as equality, so $r_i(A) = r_i(X_i \cap A)$. This means that every good partition $X_1, X_2, \ldots, X_n$ of a basis B' of the union will satisfy $\sigma_i(A \cap X_i) = A$, that is, $X_i \cap A$ spans A in M_i for $i = 1, 2, \ldots, n$.

These results are true for B, too, so $B_i \cap A$ spans A in M_i for $i = 1, 2, \ldots, n$. All the coloops of M are in $B \cap B'$, this way we can get a good partition of B', namely $B'_i = (X_i \cap A) \cup (B_i \setminus A)$ according to Proposition 4.1.2. This partition satisfies the requirements of Theorem 4.2.2. □

Richard A. Brualdi [8] observed that the following well-known property of transversal matroids is a special case of Theorem 4.2.2.

Corollary 4.2.3 *Suppose that $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ is a family of subsets of a set E with the maximum cardinality of a partial transversal equal to k. If there exists a transversal in the set $A_{i_1}, \ldots, A_{i_k}$, then this set has all partial transversals of size k of $\mathcal{A}$ among its transversals. This means that the transversal matroid of $(A_{i_1}, \ldots, A_{i_k})$ is identical with the transversal matroid of $\mathcal{A}$.*

This property is mainly the same as the Mendelson-Dulmage theorem [16].

Theorem 4.2.4 *Let $G(A, B, E)$ denote a bipartite graph with color classes A and B, and edge set E. If there exists a matching T_1 covering $A' \subseteq A$ and a matching T_2 covering $B' \subseteq B$, then there exists a matching covering both T_1 and T_2.*

4.3 Weak maps with the same union

Theorem 4.3.1 *Let B be an arbitrary basis of the union $M = \vee_{i=1}^n M_i$, with an arbitrary good partition $\cup_{i=1}^n B_i$. Let M'_i be obtained from M_i by replacing all the elements of $E - \sigma_i(B_i)$ by loops. Then $\vee_{i=1}^n M'_i = M$.*

Proof: Let $F_i = \sigma_i(B_i)$ for every i. According to the construction M'_i has ground set E and $X \subseteq E$ is independent in M'_i if and only if $X \subseteq F_i$ and X is independent in M_i.

Clearly $M'_i \preceq M_i$, and therefore $\vee_{i=1}^n M'_i \preceq \vee_{i=1}^n M_i = M$.

On the other hand we have to prove that any independent set X of M is independent in $\vee_{i=1}^n M'_i$ as well.

Let B' be a basis of M, containing X. By Theorem 4.2.2, there exists a good partition $\cup_{i=1}^n B'_i$ of B' so that $\sigma_i(B'_i) = F_i$ for every i. Since B'_i is independent in M'_i, so is $B'_i \cap X$. Hence $X = \cup_{i=1}^n (B'_i \cap X)$ is independent in $\vee_{i=1}^n M'_i$, as requested. □

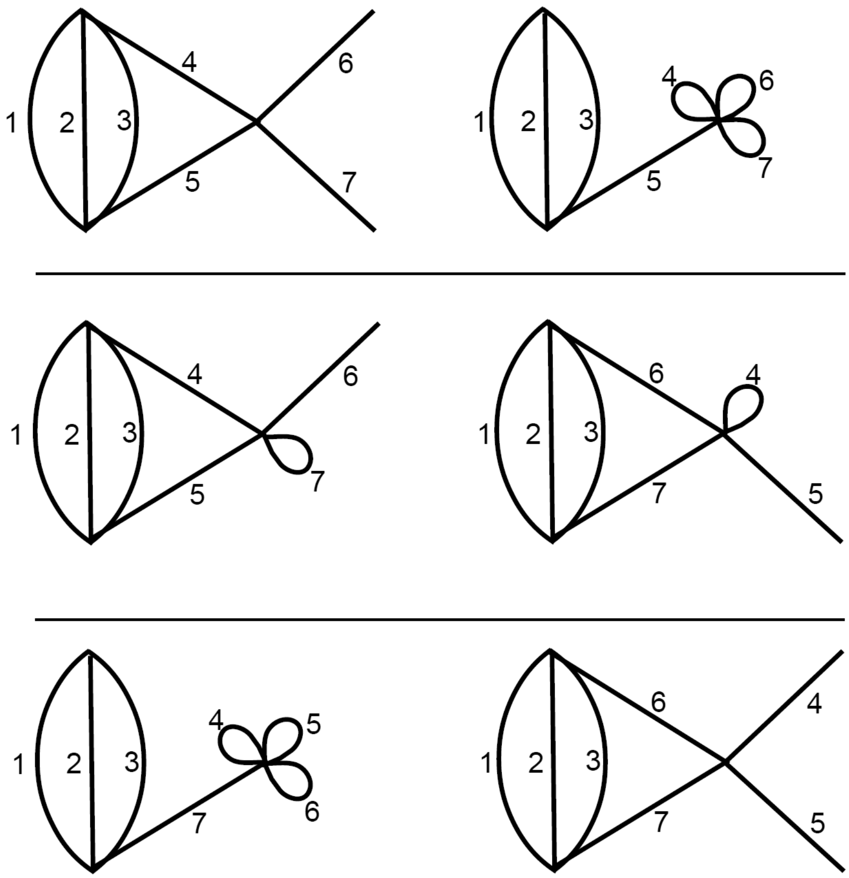

Figure 4.3: The graphs representing the restricted matroids of Example 4.3.2

Theorem 4.3.1 is not just a consequence of Theorem 4.2.2 but they are equivalent. Example 4.3.2 illustrates Theorem 4.3.1.

Example 4.3.2 *Let M_1 and M_2 be the cycle matroids of the graphs G_1 and G_2 of Figure 4.1, as in Example 4.2.1. Consider the pair of flats $E, \{1, 2, 3, 5\}$ as in the first row of Table 4.1. The corresponding restricted matroids M_1', M_2' are represented by the graphs of the first row of Figure 4.3. One can easily see that $M_1' \vee M_2'$ is still the cycle matroid of the graph of Figure 4.2. Similarly, the pairs of flats, given by rows 3 and 9 of Table 4.1 lead to the second and third rows of Figure 4.3, respectively.*

As Csaba Király noticed, that a generalization of Theorem 4.3.1 can be proved using the rank function formula given in Theorem 1.5.7. It is the following:

Theorem 4.3.3 *Suppose that B is a base of $\phi(M)$ and $\phi(F) = B$. Then $\phi(M \setminus (E - \sigma(F))) = \phi(M)$.*

Chapter 5

The unknown cases of Conjecture 1.4.2

In this chapter we present some further unpublished results on Conjecture 1.4.2. These results extend our knowledge on the remaining cases of Conjecture 1.4.2, which are not covered by the conditions of Chapter 2.

5.1 Reducing the gap

In the proof of Theorem 2.5.6 we proved more than the statement of the theorem. The theorem said that every circuit must fit one of the two conditions. However we proved that if there exists a circuit C in M_1 so that $r_2(E - C) = |E - C|$, then the union is graphic. On the other hand if $r_2(E - C) < r_2(E)$ for every circuit C of M_1, then the union is graphic. This approach helps us to a better sufficient condition.

In fact if we think about the symmetrized version of Theorem 2.5.6 we can get a more general result. The following theorem gives a sufficient condition for the graphicity of the union in a form resembling to Theorem 2.6.1.

Theorem 5.1.1 *Suppose that M_1 and M_2 are two matroids on the same ground set and at least one of them is graphic. If there are no such sets C_1 and C_2 so that C_i is a circuit of M_i, $C_1 \cap C_2 = \emptyset$ and C_i does not contain a cut set in M_{3-i}, then $M_1 \vee M_2$ is graphic.*

Proof: Without loss of generality suppose that M_2 is graphic.

In the case if there is no circuit in M_1 which does not contain a cutset in M_2 the union is graphic according to Theorem 2.5.6. Let α denote one element of C_1, where C_1 is a circuit in M_1 to which $r_2(E - C_1) = r_2(E)$. Let B denote a basis in M_1 so that $(C_1 - \alpha) \in B$. Since every circuit of M_2 which is disjoint to C_1 must contain a cut set in

M_1 (according to the assumption), the set $E - B - \alpha$ is independent in M_2. This means that $(E - B - \alpha) \cup B = E - \alpha$ is independent in the union. Then the union must be graphic according to Proposition 2.5.4. $\square$

This proof helps us to speak about disjoint circuit pairs, but there remains the property in Theorem 2.6.1 that those circuits are spanned by disjoint sets in the other matroid. The following theorem shows that it is a straightforward generalization and the extended condition remains sufficient to the graphicity of the union.

We have to speak about a connected union. We can suppose that the minimal counterexample for the condition is connected.

Theorem 5.1.2 *Suppose that M_1 and M_2 are graphic matroids and all the elements of the union are in the same component (the union is connected). If there are no such sets X_1 and X_2 so that X_i is dependent in M_i for $i \in \{1, 2\}$, $X_1 \cap X_2 = \emptyset$ and $r_i(X_i) = r_i(X_i \cup X_{3-i})$ for $i \in \{1, 2\}$, then $M_1 \vee M_2$ is graphic.*

PROOF: If there exists a spanning pair $A_1 \cap A_2 = \emptyset$ so that A_i spans M_i, then the union is the free matroid (which is a contradiction) or has only one circuit (with all the elements because of the connectivity of the union). This is because in that case there can be only one element which is not in $A_1 \cup A_2$ according to the assumption (also see Proposition 2.5.4).

We will show that if there does not exist a spanning pair, then the union can not be connected.

Proving Theorem 5.1.2 led us to the questions studied in Chapter 4. The following lemma is a consequence of Theorem 4.2.2.

Lemma 5.1.3 *Suppose that B is a basis in the union of two arbitrary graphic matroids M_1 and M_2. If B_1, B_2 is a good partition of B, then each basis of the union have a good partition which spans the same components in M_i as B_i.*

Here we present two different proofs. The first one is similar to that of Theorem 4.2.2. However, the proof is much easier than the that of Theorem 4.2.2 since the lemma is only a special case of that theorem. On the other hand the second proof is completely different. It uses alternating paths, which can be a useful tool for proving related statements. The main difference between the two proofs is that the second proof is not using Proposition 4.1.1.

PROOF: Suppose that X is a basis of the union with a good partition X_1, X_2.

Suppose that A is the set of non coloop elements in the union. $r_1(A) + r_2(A) = r(A)$ by Proposition 4.1.1. This means that every good partition of a basis X of the union will span A by $A \cap X_i$ in M_i.

This means that $X_i \cap A$ spans A in M_i for both i (what is the same as $B_i \cap A$). All the coloops are in $B \cap X$, this way we can get a good partition of X, namely $(X_1 \cap A) \cup (B_1 \setminus A), (X_2 \cap A) \cup (B_2 \setminus A)$. This partition completes the proof of Lemma 5.1.3. $\square$

Proof: Let $p(i)$ denote the parity of i so that $p(i)$ is 1 if i is odd and 2 otherwise. Observe that $3 - p(i) = p(i + 1)$.

Indirectly suppose that X is a basis in the union, but has no good partition with the required property. Let X_1, X_2 be a good partition of X. Suppose that X_1 has an element a_1 which is not spanned by B_1 in M_1. We shall prove that we can obtain a good partition of X which spans more common elements with B_1, B_2. To show this we shall show a sequence of elements which can be switched between X_1 and X_2. To find such a sequence start with a_1 and pick the next element a_{i+1} so that $a_{i+1} \in X_{3-p(i)}$ and a_{i+1} is in the only circuit C_{i+1} of $X_{3-p(i)} \setminus [(\bigcup_k a_k) \cap X_{3-p(i)}] \cup [(\bigcup_k a_k) \setminus X_{3-p(i)}]$ in $M_{3-p(i)}$ (note: $a_i \in C_{i+1}$).

Observe that the element of such a sequence must be in B. This is because if a_i is not in B, then $B_1 \cup (\bigcup_k a_{2k+1}) \setminus (\bigcup_k a_{2k})$ and $B_2 \cup (\bigcup_k a_{2k}) \setminus (\bigcup_k a_{2k+1})$ will be a good partition of a set which has more element than B (and this is a contradiction since B is a basis in the union).

If there is no such circuit C_{i+1}, then the sequence $a_1 ... a_i$ is proper for the switch. This is because this means that a_i is independent to $X_{3-p(i)} \setminus [(\bigcup_k a_k) \cap X_{3-p(i)}] \cup [(\bigcup_k a_k) \setminus X_{3-p(i)}]$ in $M_{3-p(i)}$.

Otherwise pick a_{i+1} so that it is not in $B_{3-p(i)}$. We can do this because $a_i \in B_{3-p(i)}$ and $B_{3-p(i)}$ is independent in $M_{3-p(i)}$, so can not contain a circuit (so C_{i+1} must have an element which is not in $B_{3-p(i)}$).

Every such sequence must have a last element a_n because the elements are all different. To show this suppose indirectly that $a_i = a_m$ but $i \neq m$. If $p(i) \neq p(m)$, then $a_i \in B_{3-p(i)}$ and $a_m \in B_{p(i)}$ which is a contradiction since B_1 and B_2 are disjoint. Now suppose that $p(i) = p(m)$. Then $a_m \in X_{p(m)} \setminus [(\bigcup_k a_k) \cap X_{p(m)}] \cup [(\bigcup_k a_k) \setminus X_{p(m)}]$ which is a contradiction since $a_i \in X_{p(m)}$.

These together mean that we must have such a sequence, so X has a good partition which spans more common elements with B_1, B_2. This is a contradiction since in this way we can reach a good partition of X which spans exactly the same subsets as B_i in M_i for both $i = 1, 2$. $\square$

Now suppose that there does not exist a spanning pair $A_1 \cap A_2 = \emptyset$ so that A_i spans M_i. Let B denote a basis of the union and B_1, B_2 is a good partition. Without loss of generality suppose that B_1 does not span M_1, let C denote $E - \sigma_{M_1}(B_1)$ (as a consequence $r_1(C) \geq 1$). This means that $C \subseteq B$ because otherwise we could get a larger basis of the union (if $f \in C \setminus B$, then $B_1 \cup f, B_2$ is a good partition of $B \cup f$). According to the previous lemma each basis can be partitioned so that C remains a cut set. This means that all elements of C are in all the bases of the union so they are coloops. This would be a contradiction so $C = \emptyset$ and this completes the proof of Theorem 5.1.2. $\square$

Theorem 5.1.2 essentially states that the necessary condition of Theorem 2.6.1 contains such a subset of conditions which is sufficient (1, 2 and 4). This subset of conditions gives a generalization of Theorem 2.5.6.

Chapter 6

Summary

First we studied the graphicity of the union of graphic matroids. The main motivation was Conjecture 1.4.2:

Conjecture 6.0.4 *If the union of two graphic matroids is not graphic, then it is nonbinary.*

We gave some reduction methods, which can decrease the number of elements in the matroids (see Lemmata 2.1.1, 2.1.2, 2.1.3, 2.1.8, published in [10] and Lemmata 2.4.1, 2.4.2 and 2.4.3, published in [11]). In all cases we suppose that the two addends are graphic.

Lemma 6.0.5 *Let X and Y denote the set of coloops in M_1 and in M_2, respectively. The union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic.*

Lemma 6.0.6 *If the ground set of a connected component X of the matroid M_1 is a subset of $L(M_2)$, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is graphic.*

Lemma 6.0.7 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ in which $X \subset L(M_2)$ determines a connected subgraph and $E - X$ has exactly two common vertices with X (call them a and b). Then the union $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic, where M_1' is the cycle matroid of $G' := G(V, (E - X) \cup \{(a, b)\})$ and $M_2' := (M_2 \setminus X) \cup loop(a, b)$ (Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').*

Lemma 6.0.8 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ and E_0 is the edge set of a 2-connected component X of G which has only one edge x from $NL(M_2)$. Then* *the union $M_1 \vee M_2$ is graphic if and only if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is graphic.*

Lemma 6.0.9 *If two parallel elements x and y of M_1 are serial in M_2, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus x) \vee (M_2/x)$ is graphic.*

Lemma 6.0.10 *If two serial elements x and y of M_1 are serial in M_2 as well, then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus \{x, y\}) \vee (M_2 \setminus \{x, y\})$ is graphic.*

Lemma 6.0.11 *Suppose that x and y are serial elements in $M(G_1)$ and they are not contained in any common circuit of $M(G_2)$.Let a and b be the two endvertices of x and c and d be the two endvertices of y in G_2. Assume x is not a loop of G_2 ($a \neq b$). Let $G_1' = G_1/x$ and relabel y to z. Let G_2' be obtained from G_2 by deleting x and y, identifying vertices b and c and adding new edge z between a and d. Then $M(G_1) \vee M(G_2)$ is graphic if and only if $M(G_1') \vee M(G_2')$ is graphic.*

These results together gives the following (it is Theorem 2.1.5 extended with all the reduction steps):

Theorem 6.0.12 *Suppose that M_1 and M_2 are graphic matroids and the application of Lemmata 6.0.5, 6.0.6, 6.0.7, 6.0.8, 6.0.9, 6.0.10 and 6.0.11 to M_1 and M_2 leads to a reduced pair of matroids M_1', M_2'. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

We have also given some reduction steps which are equivalent for the graphicity of the union in some cases depending also on the union, not just on the addends (see Lemmata 2.4.5 and 2.4.7).

We gave equivalent condition for the graphicity of the union for two cases, using the reductions. In both cases the first matroid is special, it has n parallel or serial edges and some loops. The other matroid is an arbitrary graphic matroid (see Theorems 2.2.1 and 2.3.1, published in [10]). These results are the following:

Theorem 6.0.13 *Let $M_1' = M(G_0')$ and $M_2' = M(G')$ be coloopless graphic matroids after all the possible reductions using Lemmata 6.0.5, 6.0.6 and 6.0.7. If G_0' consists of loops and a single circuit of length n ($n \geq 2$), then $M_1' \vee M_2'$ is graphic if and only if either $NL(M_1')$ contains a cut set in G' or $M_2' \setminus NL(M_1')$ is the free matroid.*

Theorem 6.0.14 *Let $M_1' = M(G_0')$ and $M_2' = M(G')$ be coloopless graphic matroids after all the possible reductions using Lemmata 6.0.5, 6.0.6 and 6.0.7. If G_0' consists of loops and two points joined by n parallel edges, then $M_1' \vee M_2'$ is graphic if and only if no 2-connected component of G' has two nonserial edges a and b from $NL(M_1')$ so that $M_2' \setminus \{a, b\}$ is not the free matroid.*

The proof of these results also showed that this cases can not give a counterexample for the conjecture.

We presented applications of these two theorems for linear active networks (see Theorems 2.7.5 and 2.7.6, published in [13]).

Theorem 6.0.15 *Suppose that a network is composed of 2-terminal devices and the current of a resistor R_0 controls several current sources $I_1, I_2, ..., I_k$ as described by the respective equations $i_j = c_j \cdot i_0$ for every $j = 1, 2, ..., k$ (where the control constants $c_1, c_2 ..., c_k$ are generic parameters). Then the independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

Theorem 6.0.16 *Suppose that a single current source i_0 is controlled by the current of several resistors $R_1, R_2, ..., R_k$ as described by the equation $i_0 = \Sigma c_j \cdot i_j$ where the summation is for every $j = 1, 2, ..., k$. Like in Theorem 6.0.15, suppose that the control constants $c_1, c_2 ..., c_k$ are generic parameters. Then the independence structure describing the currents of the devices is graphic if and only if there is no feedback in the network.*

It turned out that the common generalization of Theorems 6.0.13 and 6.0.14 gives a general sufficient condition for the graphicity of the union (see Theorem 2.5.3, published in [11]).

Theorem 6.0.17 *Let M_1, M_2 be two matroids defined on the same ground set E. Then $M_1 \vee M_2$ is graphic if for every circuit C in M_1 either $r_2(E-C) < r_2(E)$ or $r_2(E-C) = |E-C|$ holds.*

We showed that this condition remains sufficient if the length of the circuit is at least 2, but for this case we have to suppose that M_2 is graphic (see Theorem 2.5.6, published in [11]).

Theorem 6.0.18 *Assume that M_2 is graphic. Then $M_1 \vee M_2$ is graphic if for every circuit C of length at least two in M_1 either $r_2(E-C) < r_2(E)$ or $r_2(E-C) = |E-C|$.*

We proved that adding some additional conditions to the sufficient ones we get a necessary condition for the graphicity of the union of arbitrary graphic matroids (see Theorem 2.6.1, published in [11]).

Theorem 6.0.19 *Let M_1 and M_2 be graphic matroids. If all of the following conditions hold, then the union $M_1 \vee M_2$ is not binary.*

1. *There exist X_i dependent sets in M_i for $i \in \{1, 2\}$*

2. $X_1 \cap X_2 = \emptyset$

3. *There exist a circuit C_i of M_i in X_i so that $|C_i| \geq 2$ for $i \in \{1, 2\}$*

4. $r_i(X_i) = r_i(X_1 \cup X_2)$ *for* $i \in \{1, 2\}$

5. *There are two distinct elements $a, b \in C_1 \cup C_2$ such that for $i \in \{1, 2\}$:*

   - *if $a \in C_i$ and $b \in C_{3-i}$, then a and b are in the same component in both matroids*
   - *if $a, b \in C_i$, then there exists $X'_{3-i} \subset X_{3-i}$ so that if we contract X'_{3-i} in M_{3-i}, then a and b are diagonals of C_{3-i} connecting distinct pairs of vertices*

We improved the sufficient condition for the graphicity of the union to a form resembling to the necessary condition (see Theorem 5.1.1, unpublished).

Theorem 6.0.20 *Suppose that M_1 and M_2 are two matroids on the same ground set and at least one of them is graphic. If there are no such sets C_1 and C_2 so that C_i is a circuit of M_i, $C_1 \cap C_2 = \emptyset$ and C_i does not contain a cut set in M_{3-i}, then $M_1 \vee M_2$ is graphic.*

We further improved the sufficient condition to bring it closer to the necessary condition (see Theorem 5.1.2, unpublished).

Theorem 6.0.21 *Suppose that M_1 and M_2 are graphic matroids and all the elements of the union are in the same component (the union is connected). If there are no such sets X_1 and X_2 so that X_i is dependent in M_i for $i \in \{1, 2\}$, $X_1 \cap X_2 = \emptyset$ and $r_i(X_i) = r_i(X_i \cup X_{3-i})$ for $i \in \{1, 2\}$, then $M_1 \vee M_2$ is graphic.*

In order to put the conjecture into a more general framework, we defined eight matroid classes of interest. We visualized the relation among these sets and gave examples for all nonempty subsets. We got two interesting results for these sets (see Theorems 3.0.10 and 3.0.12, published in [11]).

Theorem 6.0.22 *A matroid gives a graphic union with any matroid if and only if it contains at most three circuits.*

Theorem 6.0.23 *There exists such a nongraphic matroid which gives a graphic or non-binary union with any graphic matroid, but there exists a nongraphic matroid with which the union is nongraphic and binary. For example $U_{2,4} \oplus U_{0,7}$ is such a matroid.*

Lastly we proved some interesting facts for the bases of the union (see Theorems 4.2.2 and 4.3.1, published in [12]).

Theorem 6.0.24 *Let $M_1, M_2, \ldots, M_n$ be matroids and let M be their union. Let B be a basis of M with a good partition $B_1, B_2, \ldots, B_n$. For any basis B' of M there is a good partition $\cup_{i=1}^{n} B_i'$ so that $\sigma_i(B_i) = \sigma_i(B_i')$ for $i = 1, 2, \ldots, n$.*

Theorem 6.0.25 *Let B be an arbitrary basis of the union $M = \vee_{i=1}^{n} M_i$, with an arbitrary good partition $\cup_{i=1}^{n} B_i$. Let M_i' be obtained from M_i by replacing all the elements of $E - \sigma_i(B_i)$ by loops. Then $\vee_{i=1}^{n} M_i' = M$.*

Bibliography

[1] T. Asano, T. Nishizeki and P.D. Seymour, A Note on Nongraphic Matroids, *Journal of Combinatorial Theory B* **37** (1984) 290–293.

[2] R.E. Bixby, Kuratowski's and Wagner's theorems for matroids, *Journal of Combinatorial Theory B* **22** (1977) 31–53.

[3] R.E. Bixby, W.H. Cunningham, Matroids, graphs, and 3-connectivity, *in: J.A. Bondy, U.S.R. Murty (Eds.), Graph Theory and Related Topics, Academic Press, New York* (1979) 91–103.

[4] R.E. Bixby, On Reids characterization of the ternary matroids, *Journal of Combinatorial Theory B* **26** (1979) 174–204.

[5] R.E. Bixby, Bridges in general matroids, *Bell Telephone Laboratories Technical Memorandum* (1979)

[6] R.E. Bixby, W.H. Cunningham, Converting linear programs to network problems, *Mathematics of Operations Research* **5** (1980) 321–357.

[7] R.E. Bixby and D.K. Wagner, An Almost Linear-Time Algorithm for Graph Realization, *Mathematics of Operations Research* **13** (1988) 99–123.

[8] R. A. Brualdi, private communication, (February 2017)

[9] Cs. Gy. Csehi, Matroidelméleti vizsgálatok és alkalmazásaik, *Master's Thesis, Budapest University of Technology and Economics,* (2012)

[10] Cs.Gy. Csehi and A. Recski, The graphicity of the union of graphic matroids, *European Journal of Combinatorics* **50** (2015) 38–47. doi: 10.1016/j.ejc.2015.03.022

[11] Cs.Gy. Csehi and A. Recski, Matroid union – Graphic? Binary? Neither?, *Discrete Applied Mathematics* **209** (2016) 75–83. doi: 10.1016/j.dam.2015.10.037

[12] Cs.Gy. Csehi and A. Recski, Partitioning the bases of the union of matroids, *Discrete Mathematics* **340/4** (2017) 691-694. doi: 10.1016/j.disc.2016.12.011

[13] Cs.Gy. Csehi and A. Recski, On the Graphicity of the Independence Structure of Linear Active Networks, *Periodica Polytechnica Electrical Engineering* **61/2** (2017) 193-197. doi: https://doi.org/10.3311/PPee.9982

[14] W.H. Cunningham, On matroid connectivity, *Journal of Combinatorial Theory B* **30** (1981) 94–99.

[15] W.H. Cunningham, Separating Cocircuits in Binary Matroids, *Linear Algebra and its Applications* **43** (1982) 69–86.

[16] A.L. Dulmage, N.S. Mendelsohn, Coverings of bipartite graphs, *Canadian Journal of Mathematics* **10.4** (1958) 516–534.

[17] J. Edmonds, Minimum partition of a matroid into independent subsets, *J. Res. Nat. Bur. Stand* **69B** (1965) 67–72.

[18] J. Edmonds, Systems of distinct representatives and linear algebra, *J. Res. Nat. Bur. Standards 71B* **71B** (1967) 241–245.

[19] J. C. Fournier, Une relation de sparation entre cocircuits d'un matroide, *Journal of Combinatorial Theory B* **16** (1974) 181–190.

[20] S. Fujishige, An efficient PQ-graph algorithm for solving the graph-realization problem, *Journal of Computer and Systems Sciences* **21** (1980) 63–68.

[21] M. Iri and N. Tomizawa, A unifying approach to fundamental problems in network theory by means of matroids, *Trans. Inst. Electron. & Commun. Eng. Jpn.* **57A, 8** (1975) 35–41.

[22] A. K. Kelmans, M. V. Lomonosov, and V. P. Polesskii, On minimum coverings in matroids *Problemy Peredachi Informatsii* **12/3** (1976) 94–107.

[23] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, *Ann. Phys. Chem.* **72** (1847) 497–508.

[24] M. Lemos, A characterization of graphic matroids using non-separating cocircuits, *Advances in Applied Mathematics* **42** (2009) 75–81.

[25] L. Lovász and A. Recski, On the sum of matroids, *Acta Math. Acad. Sci. Hungar.* **24** (1973) 329–333.

[26] J. Mighton, A new characterization of graphic matroids, *Journal of Combinatorial Theory B* **98** (2008) 1253–1258.

[27] T. Moss, A minor-based characterization of matroid 3-connectivity, *Advances in Applied Mathematics* **50** (2013) 132–141.

[28] J. Narayanan, Theory of Matroids and Network Analysis, *Ph.D. Thesis in Elect. Engin., Indian Institute of Tech., Bombay, India* (1974)

[29] C. St. J. A. Nash-Williams, Decomposition of finite graphs into forests, *Journal of the London Mathematical Society* **39 (1)** (1964) 12–12.

[30] C. St. J. A. Nash-Williams, On applications of matroids to graph theory, *Theory of Graphs Intern. Symposium* (1967) 263–265.

[31] J. Oxley, *Matroid Theory,* Second Edition, Oxford University Press (2011)

[32] R. Rado, Note on independence functions, *Proceedings of the London Mathematical Society* **7** (1957) 300–320.

[33] A. Recski, On partitional matroids with applications, *Colloq. Math. Sot. J. Bolyai* **10** (1973) 1169–1179.

[34] A. Recski, On the sum of matroids II, *Proc. 5th British Combinatorial Conf. Aberdeen* (1975) 515–520.

[35] A. Recski, Some open problems of matroid theory, suggested by its applications, *Coll. Math. Soc. J. Bolyai* **40** (1982) 311–325.

[36] A. Recski, Matroid Theory and its Applications in Electric Network Theory and in Statics, *Springer, Berlin* (1989)

[37] A. Recski, Matroids – the Engineers' Revenge, W J Cook, L Lovász, J Vygen (eds.) *Research trends in combinatorial optimization,* Springer (2008) 387–398.

[38] P.D. Seymour, Matroid representation over GF(3), *Journal of Combinatorial Theory B* **26** (1979) 159–173.

[39] P.D. Seymour, Decomposition of regular matroids, *Journal of Combinatorial Theory B* **28** (1980) 305–359.

[40] P.D. Seymour, Recognizing Graphic Matroids, *Combinatorica 1* (1981) 75–78.

[41] P.D. Seymour, On minors of non-binary matroids, *Combinatorica 1* **4** (1981) 387–394.

[42] P.D. Seymour, Minors of 3-Connected Matroids, *European Journal of Combinatorics* **6** (1985) 375–382.

[43] R.P. Swaminathan and D.K. Wagner, The arborescence-realization problem, *Discrete Applied Mathematics* **59** (1995) 267–283.

[44] K. Truemper, On the efficiency of representability tests for matroids, *European Journal of Combinatorics* **3** (1982) 275–291.

[45] K. Truemper, Matroid Decomposition, *Elsevier Science* (2014)

[46] W.T. Tutte, A homotopy theorem for matroids I, II, *Transactions of the American Mathematical Society* **88** (1958) 144–174.

[47] W.T. Tutte, Matroids and graphs, *Trans. Am. Math. Soc.* **90** (1959) 527–552.

[48] W.T. Tutte, An Algorithm for Determining Whether a Given Binary Matroid is Graphic, *Proceedings of the American Mathematical Society* **11** (1960) 905–917.

[49] W.T. Tutte, Lectures on matroids, *Journal of Research of the National Bureau of Standards B* **69** (1965) 1–47.

[50] W.T. Tutte, Connectivity in matroids, *Can. J. Math.* **18** (1966) 1301–1324.

[51] W.T. Tutte, On even matroids, *I. Res. Nat. Bur. Standards Sect. B* **71** (1967) 213–214.

[52] W.T. Tutte, *Introduction to the Theory of Matroids,* American Elsevier, New York (1971)

[53] D.K. Wagner, On theorems of Whitney and Tutte, *Discrete Mathematics* **57** (1985) 147–154.

[54] D.K. WAGNER, A circuit characterization of graphic matroids, *Journal of Combinatorial Theory B* **118** (2016) 284–290.

[55] H. WHITNEY, On the abstract properties of linear dependence, *American Journal of Mathematics* **57** (1935) 509–533.