PHD THESIS – BOOKLET

ISOPTIC CURVES AND SURFACES

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BUTE
2017

0.1 Introduction

The topic of my thesis concerns isoptic curves and surfaces in Euclidean and non-Euclidean geometries. This is the problem of generalizing the inscribed angle theorem. The question is quite simple: "What is the locus of points where a given object subtends a given angle?" It is well known that the locus of points in the plane from a segment subtends a given angle α $(0 < \alpha < \pi)$ is the union of two arcs except for the endpoints with the segment as common chord. If this α is equal to $\frac{\pi}{2}$ then we get the Thales circle. It is also known, that any point on the main circle of an ellipse has this property, that the tangent lines drawn from this point to the ellipse form a right angle. For the parabola, only the points on its directrix satisfy this property.

Although the name "isoptic curve" was suggested by Taylor in 1884 ([62]), reference to former results can be found in [63]. In the obscure history of isoptic curves, we can find the names of la Hire (cycloids 1704) and Chasles (conics and epitrochoids 1837) among the contributors of the subject. A very interesting table of isoptic and orthoptic curves is introduced in [63], unfortunately without any exact reference of its source. However, recent works are available on the topic, which shows its timeliness. In [6] and [7], the Euclidean isoptic curves of closed strictly convex curves are studied using their support function. Papers [34, 68, 69] deal with Euclidean curves having a circle or an ellipse for an isoptic curve. Further curves appearing as isoptic curves are well studied in Euclidean plane geometry $\mathbf{E}^2$, see e.g. [36, 67]. Isoptic curves of conic sections have been studied in [25] and [55]. There are results for Bezier curves by Kunkli et al. as well, see [32]. Many papers focus on the properties of isoptics, e.g. [37, 38, 39], and the references therein. There are some generalizations of the isoptics as well *e.g.* equioptic curves in [52] by Odehnal or secantopics in [49, 57] by Skrzypiec.

I encountered this question along with my supervisor during the third semester of my BSc studies within the framework of Independent Research Task I. The simplicity of the proposition encouraged me to continue this research alongside my studies even after the semester ended. On the recommendation of my supervisor, I began to study the topic in non-Euclidean geometries, especially in the Bolyai-Lobachevsky hyperbolic geometry. I did not find any existing result in this direction, despite the fact that the examination of problems in non-Euclidean geometries is crucial to understand and describe both our small-scale and astronomical world. Simultaneously, efforts were made to expand the examination to higher dimensional spaces. Of course it is possible to examine the question in non-Euclidean spaces and we obtained some results in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ geometry, but further reaching results of this research is beyond this thesis.

This research topic was interesting to me due to the many applications. In Euclidean space $\mathbf{E}^3$ our investigation concerns sight. Therefore the results can be used in any area which is concerned with the quality of visibility. Isoptic curves are related to the discipline of shape recognition from X-ray images called geometric tomography (see [21]). Furthermore, recent results in the topic were motivated by the packing problem (see [64] Remark 4.12). It seems natural that every result in the Euclidean geometry can be applied to computer graphics. Perhaps one day isoptic curves will be implemented in CAD systems as well.

In the first chapter we examine isoptic curves on the Euclidean plane. First we give an overview of some preliminary results concerning strictly convex closed and differentiable curves. The isoptic curves of conic sections have been well known for a long time [5], but we give a new approach using constructional procedures for their study. Next investigating isoptic curves of polygons, we find an algorithm to determine the isoptic curve for any finite point set. Finally, we examine the inverse question investigated by Kurusa in [34], does the isoptic of a convex body determines the convex body itself?

In the second part of the thesis, we consider the Euclidean space $\mathbf{E}^3$. There are several possibilities of how to generalize the definition of isoptic curves to that of the isoptic surface, here we will choose the notion of solid angle. The disadvantage of this approach is that restricting the isoptic surface to any plane will not necessarily result in the isoptic curve of the restricted curve. However, some results are connected to with the detector placement problem, and the radiation level problem in nuclear physics ([24]). We will close the chapter with the discussion of isoptic surfaces of some Platonic and Archimedean solids.

In the third chapter we again return to the plane, but this time the geometry is no longer Euclidean. We will consider the elliptic and the extended hyperbolic plane as embedded into three dimensional projective space $\mathcal{P}^3$. This will be useful for computations as real coordinates can be assigned to each point and straight line. The main subject of our examination will be the conic sections which are quite interesting both in the elliptic and hyperbolic plane. The literature on the classification of hyperbolic conic section is huge ([18, 19, 41]). To determine isoptic curves it will be inevitable to clarify the definition of the generalized angle and distance. Naturally, we give the equation of the isoptic curve to every type of conic and visualize them as well.

In the last chapter, we get acquainted with one of the eight Thurston geometries, that of $\widetilde{\mathbf{SL_2R}}$ [62]. We introduce the geodesic and translation curves that are different in this geometry, and we give some interesting results concerning the sum of interior angles in triangles. The isoptic surfaces in this geometry is a brand new direction in our research. However, even the isoptic surface of a line segment is quite interesting due to the twisted nature of this geometry.

The *Wolfram Mathematica* software package was utilized to create all the figures and perform all the necessary computations presented in this thesis.

Chapter 1

Isoptic curves on the Euclidean plane

1.1 Isoptics of closed strictly convex curves

Definition 1.1.1 ([63]) *The locus of the intersection of tangents to a curve (or curves) meeting at a constant angle α $(0 < \alpha < \pi)$ is the $\alpha-$isoptic of the given curve (or curves). The isoptic curve with right angle called orthoptic curve.*

In order to conduct further investigations on isoptics we need to summarize some preliminary results on the support function.

Definition 1.1.2 *Let $\mathcal{C}$ be a closed, strictly convex curve which surrounds the origin. Let $p(t)$ where $t \in [0, 2\pi[$ be the distance from 0 to the support line of $\mathcal{C}$ being perpendicular to the vector e^{it}. The function p is called a support function of $\mathcal{C}$.*

It is well-known [2] that the support function of a planar, closed, strictly convex curve $\mathcal{C}$ is differentiable. For now we would like to express the isoptic of $\mathcal{C}$ using the support function.

Theorem 1.1.3 ([66]) *Given a planar, closed, strictly convex curve $\mathcal{C}$ in polar coordinates with the radius z a function of angle t, where $t \in [0, 2\pi)$. Then the following equation holds*

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}.$$

The corollary of this theorem is that we may use this parametrization to determine the isoptic curve of $\mathcal{C}$. The angle of $p(t)$ and $p(t + \pi - \alpha)$ is α.

Theorem 1.1.4 ([6]) *Let $\mathcal{C}$ be a plane, closed, strictly convex curve and suppose that the origin is in the interior of $\mathcal{C}$. Let $p(t)$, $t \in [0, 2\pi]$ be the support function of $\mathcal{C}$. Then the α-isoptic curve of $\mathcal{C}$ has the form*

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t)\cot(\pi - \alpha) + \frac{1}{\sin(\pi - \alpha)}p(t + \pi - \alpha)\right) ie^{it}.$$

1.1.1 Example

A typical example may be an ellipse with semi-axes a and b such that they coincide with x and y coordinate axes respectively *i.e.* we have a central ellipse. In this case we have a well-known simple parametrization: $(a\cos(t), b\sin(t))$, $t \in [0, 2\pi)$. Now we calculate the support function of the ellipse and use Theorem 1.1.4 to determine the isoptic curve.

$$p(t) = \sqrt{a^2\cos^2(t) + b^2\sin^2(t)} \tag{1.1}$$

Finally we can apply Theorem 1.1.4 with some simplification to obtain:

$$\begin{gathered} z_\alpha(t) = \sqrt{a^2\cos^2(t) + b^2\sin^2(t)}e^{it} + \\ + \left(\cot(\alpha)\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} + \frac{1}{\sin(\alpha)}\sqrt{a^2\cos^2(t-\alpha) + b^2\sin^2(t-\alpha)}\right) ie^{it} \end{gathered} \tag{1.2}$$

1.2 Isoptic curves of Euclidean conic sections

In this section, we determine the isoptic curves of Euclidean conic sections. There are a lot of possibilities to give the equations of the isoptics (see *e.g* [36]). However we give here a new approach using the *construction method of the tangent lines from an outer point* (see *e.g* in [58]). This procedure requires less computation.

1.2.1 Ellipse

Theorem 1.2.1 *Given a canonical ellipse with semi-axis a and b in a coordinate system on the Euclidean plan $\mathbf{E}^2$. The α-isoptic curve $(0 < \alpha < \pi)$ of the considered ellipse has the equation*

$$\cos\alpha = \frac{x^2 + y^2 - a^2 - b^2}{\sqrt{\left(-a^2 + b^2 + x^2\right)^2 + 2y^2\left(a^2 - b^2 + x^2\right) + y^4}}.$$

1.2.2 Hyperbola

Theorem 1.2.2 *Given a canonical hyperbola with semi-axis a and b in a coordinate system on the Euclidean plane $\mathbf{E}^2$. The α and $\pi - \alpha$-isoptic curves $(0 < \alpha < \pi)$ of the hyperbola considered has the equation*

$$\cos^2\alpha = \frac{(-a^2 + b^2 + x^2 + y^2)^2}{\left(a^2 + b^2 - x^2\right)^2 + 2y^2\left(a^2 + b^2 + x^2\right) + y^4}.$$

Remark 1.2.3 *In some papers e.g [52] the equations of isoptic curves of the ellipse and hyperbola arise on the square to get not only the α-isoptic but the $\pi - \alpha$-isoptic as well. For the ellipse, this is not essential. In the case of a hyperbola, the two asymptotes split the space into four domains, two of them contain a hyperbola branch (focal domains), the other two are empty. Let K be an outer point of the hyperbola. If K is in a focal domain, then the tangent lines are tangent to the same branch of the hyperbola, else they touch both branches. In these cases, the isoptic angles are complementary to each other, i.e. they sum up to π. Therefore, we take the square of the equation and thus obtain both types of isoptic curves.*

Remark 1.2.4 *It is easy to see that the origin is a "saddle point" with respect to the angle of the tangent lines. From any points inside focal domains, the angle of the tangents are greater, and from the other domains they are less then from the origin. Therefore, the isoptic curves do not exists in the interval*

$$\left(\arccos\left(\frac{b^2 - a^2}{b^2 + a^2}\right), \arccos\left(\frac{a^2 - b^2}{a^2 + b^2}\right)\right)$$

if the condition $b > a$ holds.

1.2.3 Parabola

Theorem 1.2.5 *Given an axial parabola with its foci $F(0, p)$ in a coordinate system on the Euclidean plan $\mathbf{E}^2$. The α-isoptic curves $(0 < \alpha < \pi)$ of the considered parabola has the equation*

$$\cos\alpha = -\frac{y}{\sqrt{(p-y)^2 + x^2}}.$$

1.3 Isoptic curves to finite point sets

We have to clarify, what the isoptic curve of a point set is.

Definition 1.3.1 *A given finite point set $\mathcal{P}$ is seen under α $(0 < \alpha < \pi)$ from P if the smallest angle with vertex P which contains all points in $\mathcal{P}$ is α.*

Definition 1.3.2 *The isoptic curve of an arbitrarily given finite point set $\mathcal{P}$ is the locus of points P where $\mathcal{P}$ is seen under a given fixed angle α $(0 < \alpha < \pi)$. The isoptic curve with a right angle is called orthoptic curve.*

Now, we can easily see from the definition that:

Corollary 1.3.3 *To determine the isoptic curve of a finite point set it is sufficient to determine the isoptic curve of its convex hull.*

We may suppose that the points are in general position, *i.e* no three points are collinear. There are numerous algorithms to determine the convex hull of finitely many points (see [1]), therefore we may assume that we have n points (X_i where $i \in 1, \dots, n$) in clockwise order. Let $\mathcal{C}$ be a closed curve, obtained by drawing every $\overline{X_i X_{i+1}}$ segment (if $i = n$, then $X_{i+1} = X_1$). The resulting curve will be a closed simple n-gon, and the isoptic curve of the given finite point set $\mathcal{P}$ will be the isoptic curve of $\mathcal{C}$.

Despite our first approach being useful to visualize the isoptic curve, it will not appropriate for giving a formula or a parametrization of the curve, however the idea is very simple and useful. A tangent line of a polygon may be of two types, it is either an edge line or it passes through exactly one vertex. When both support lines pass through a vertex then we can assign a diagonal. This segment does not change until at least one of the tangents changes its point of tangency. Therefore the isoptic curve of the polygon will be the union of finitely many circular arcs.

Lemma 1.3.4 (CsG) *Given an angle α $(0 < \alpha < \pi)$ and an n-gon with $X_1, X_2, \dots, X_n$ vertices. We consider all segments formed by $\overline{X_i X_j}$. Let Ω be the union of the discs each one of which corresponds to a circular arc where a segment $\overline{X_i X_j}$ subtends α. Then the isoptic curve of the n-gon will be $\partial\Omega$.*

Remark 1.3.5 *The statement that '...the isoptic curve of a convex polygon is the union of circular arcs' can be found in [34] without any proof, only the inscribed angle theorem is mentioned as justification.*

Now, our goal is to give an implicit equation for $\partial\Omega$. It is supposed that the polygon is closed, convex and it has n vertices $X_1, X_2, \dots X_n$ in clockwise order.

If we want to determine the subtended angle of a line segment from a planar point P then we project it to a unit circle around P and compute its arc length. Notice that if our n-gon is convex then its projection will cover the circular arc exactly twice. Therefore if a point P is on the isoptic curve of the polygon, then the sum of all $X_i P X_{i+1}\angle$ ($i = 1 \dots n$ and $X_{n+1} = X_1$) have to be 2α.

Theorem 1.3.6 (CsG) *Given a convex n-gon with the vertices $X_1, X_2, \dots X_n$ $(X_i = (x_i, y_i)^T)$ in clockwise order and α, $(0 < \alpha < \pi)$. Then the α-isoptic curve of the n-gon considered has the equation:*

$$2\alpha = \sum_{i=1}^{n} \arccos\left(\frac{\left\langle \overrightarrow{PX}_i, \overrightarrow{PX}_{i+1}\right\rangle}{\sqrt{\left\langle \overrightarrow{PX}_i, \overrightarrow{PX}_i\right\rangle \left\langle \overrightarrow{PX}_{i+1}, \overrightarrow{PX}_{i+1}\right\rangle}}\right) \tag{1.3}$$

where $X_{n+1} = X_1$ and $P = (x, y)^T$.

Chapter 2

Isoptic surfaces in the Euclidean space

2.1 Isoptic hypersurfaces in $\mathbf{E}^n$

First, we generalize the notion of angle to Euclidean space $\mathbf{E}^3$. From this statement it is possible to generalize it to Euclidean spaces of any dimension n ($n \geq 3$). The notion of *solid angle* is well known and widely studied in the literature (see [22]).

Definition 2.1.1 *The solid angle $\Omega_{S(P)}$ subtended by a surface S in $\mathbf{E}^3$ is defined as the surface area of the projection of S onto the unit sphere around P.*

In the International System of Units (SI), a solid angle is expressed in a dimensionless unit called a *steradian* (symbol: sr)[1], e.g. the solid angle subtended by the whole Euclidean space $\mathbf{E}^3$ is equal to 4π steradians. Moreover, this notion has several important applications in physics, (in particular in astrophysics, radiometry or photometry) (see [4]) computational geometry (see [29]) and we can easily generalize it to any dimension. Some results for higher dimensional solid angles can be found in [53]. The Definition 1.1.1 can also be generalized:

Definition 2.1.2 ([11]) *The isoptic hypersurface $\mathcal{H}_{\mathcal{D}}^{\alpha}$ in $\mathbf{E}^n$ ($n \geq 3$) of an arbitrary $2 \leq d \leq n$ dimensional compact set $\mathcal{D}$ is the locus of points P where the $d-$dimensional measure of the projection of $\mathcal{D}$ onto the unit $(n-1)$-sphere around P is a given fixed value α $(0 < \alpha < \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})})$.*

Remark 2.1.3 *Another approach can be find in [50], with numerous possibilities for application, see e.g. [33] Section 3.*

2.1.1 Isoptic hypersurface of $(n-1)-$dimensional compact hypersurfaces in $\mathbf{E}^n$

Now, we consider a compact $(n-1)$ dimensional hypersurface $\mathcal{D}$ ($n \geq 3$) lying in a hyperplane of $\mathbf{E}^n$, given by a usual parametric system. Applying an appropriate transformation, we may assume parametrization of $\mathcal{D}$ has the form:

$$\widetilde{\phi}(x_1, x_2, \ldots, x_{n-1}) = \begin{pmatrix} \widetilde{f}_1(x_1, x_2, \ldots, x_{n-1}) \\ \widetilde{f}_2(x_1, x_2, \ldots, x_{n-1}) \\ \vdots \\ \widetilde{f}_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\ 0 \end{pmatrix}, \tag{2.1}$$

where $x_i \in [a_i, b_i]$,$(a_i, b_i \in \mathbb{R})$, $(i = 1, \ldots, n-1)$. Take a point $P(x_1^0, x_2^0, \ldots, x_n^0) = P(\mathbf{x}^0)$ assuming $x_n^0 > 0$. Projecting $\mathcal{D}$ onto the unit sphere with center P, we have the following parametrization:

$$\phi(x_1, x_2, \ldots, x_{n-1}) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_{n-1}) \\ f_2(x_1, x_2, \ldots, x_{n-1}) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_{n-1}) \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}, \tag{2.2}$$

[1] https://en.wikipedia.org/wiki/Solid_angle

where if $i \neq n$

$$f_i(\mathbf{x}) = \frac{\widetilde{f}_i(x_1,\dots,x_{n-1}) - x_1^0}{\sqrt{(\widetilde{f}_1(x_1,\dots,x_{n-1}) - x_1^0)^2 + \dots + (\widetilde{f}_{n-1}(x_1,\dots,x_{n-1}) - x_{n-1}^0)^2 + (x_n^0)^2}}$$

otherwise

$$f_n(\mathbf{x}) = \frac{-x_n^0}{\sqrt{(\widetilde{f}_1(x_1,\dots,x_{n-1}) - x_1^0)^2 + \dots + (\widetilde{f}_{n-1}(x_1,\dots,x_{n-1}) - x_{n-1}^0)^2 + (x_n^0)^2}}$$

It is well known, that the surface area can be calculated using the formula bellow:

$$S(x_1^0, x_2^0, \dots, x_n^0) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\dots\int_{a_{n-1}}^{b_{n-1}} \sqrt{\det G} \;\; \mathrm{d}x_{n-1}\,\mathrm{d}x_{n-2}\dots\mathrm{d}x_1 \tag{2.3}$$

where

$$G = J^T J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_{n-1}} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \frac{\partial f_{n-1}}{\partial x_2} & \dots & \frac{\partial f_{n-1}}{\partial x_{n-1}} \end{pmatrix}^T \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_{n-1}} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \frac{\partial f_{n-1}}{\partial x_2} & \dots & \frac{\partial f_{n-1}}{\partial x_{n-1}} \end{pmatrix}.$$

The isoptic hypersurface $\mathcal{H}_{\mathcal{D}}^{\alpha}$ by Definition 2.1.2 is the following:

$$\mathcal{H}_{\mathcal{D}}^{\alpha} = \left\{\mathbf{x}^0 \in \mathbf{E}^n | \alpha = S(x_1^0, x_2^0, \dots, x_n^0)\right\}, \text{ where } \alpha \in \left(0\,,\, \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\right). \tag{2.4}$$

In the general case, the isoptic hypersurfaces can only be determined through numerical computations.

2.1.2 Isoptic surface of rectangles

Using basic analysis we can propose the following theorem given by Gotoh and Yagi in [24] and Csima in [11] independently:

Theorem 2.1.4 ([11, 24]) *Given a rectangle $\mathcal{D} \subset \mathbf{E}^2$ lying in the $[x, y]$ plane in a given Cartesian coordinate system. Moreover, we can assume that it is centered at the origin with edges $(2a, 2b)$. Then the isoptic surface for a given angle α $(0 < \alpha < 2\pi)$ is determined by the following equation:*

$$\alpha = \arctan\left(\frac{(a-x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) + \arctan\left(\frac{(a+x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) +$$
$$+ \arctan\left(\frac{(a-x)(b+y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right) + \arctan\left(\frac{(a+x)(b+y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}}\right).$$

Remark 2.1.5 *This topic has numerous application possibilities, for example designing stadiums, theaters or cinemas. It can be interesting, to have a stadium, with the property that from every seat on the grandstand, the field can be seen under the same angle.*

Designing a lecture hall, it is important, that the screen or the blackboard is clearly visible from every seat. In this case, an isoptic lecture hall is not feasible, but the hall can be optimized.

2.1.3 Isoptic surface of ellipsoids

We use a standard analytic approach to determine the isoptic surface of an ellipsoid, but only a numerical solution can be derived due to its computational complexity. It is well known that the ellipsoid has the following parametrization:

$$\widetilde{\phi}(u, v) = \begin{pmatrix} a\cos u \cos v \\ b \cos u \sin v \\ c \sin u \end{pmatrix}, \tag{2.5}$$

where $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $v \in [0, 2\pi]$ and $a, b, c \in \mathbb{R}^+$. Projecting this onto the unit sphere around $P(x_0, y_0, z_0)$ we obtain:

$$\phi(u,v) = \begin{pmatrix} \frac{a\cos u\cos v - x_0}{\sqrt{(a\cos u\cos v - x_0)^2 + (b\cos u\sin v - y_0)^2 + (c\sin u - z_0)^2}} \\ \frac{b\cos u\sin v - y_0}{\sqrt{(a\cos u\cos v - x_0)^2 + (b\cos u\sin v - y_0)^2 + (c\sin u - z_0)^2}} \\ \frac{c\sin u - z_0}{\sqrt{(a\cos u\cos v - x_0)^2 + (b\cos u\sin v - y_0)^2 + (c\sin u - z_0)^2}} \end{pmatrix}. \tag{2.6}$$

Now, we can determine the isoptic surface of the ellipsoid:

$$2\alpha = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\phi_u(u,v) \times \phi_v(u,v)| \, du \, dv. \tag{2.7}$$

This integral can be computed only numerically.

2.2 Isoptic surfaces of polyhedra

Now we discuss the algorithm developed to determine the isoptic surface of a given polyhedron.

1. We assume that an arbitrary polyhedron $\mathcal{P}$ is given by the usual data structure. This consists of the list of facets $\mathcal{F}_\mathcal{P}$ with the set of vertices V_i in clockwise order. Each facet can be embedded into a plane.

   It is well known, that if $\mathbf{a} \in \mathbf{R}^3 \setminus \{\mathbf{0}\}$ and $b \in \mathbf{R}$ then $\{\mathbf{x} \in \mathbf{R}^3 | \mathbf{a}^T\mathbf{x} = b\}$ is a plane and $\{\mathbf{x} \in \mathbf{R}^3 | \mathbf{a}^T\mathbf{x} \leq b\}$ define a half space. Every polyhedron is the intersection of finitely many half spaces. Therefore an arbitrary polyhedron can also be given by a system of inequalities $A\mathbf{x} \leq \mathbf{b}$ where $A \in \mathbf{R}^{m\times 3}$ $(4 \leq m \in \mathbb{N})$, $\mathbf{x} \in \mathbf{R}^3$ and $\mathbf{b} \in \mathbf{R}^m$.

2. For an arbitrary point $P \in \mathbf{E}^3$ with coordinates $\mathbf{p}$ we have to decide, which facets of $\mathcal{P}$ 'can be seen' from it. Let us denote the i^{th} facet of $\mathcal{P}$ by $\mathcal{F}^i_\mathcal{P}$ $(i = 1, \dots, m)$ and by $\mathbf{a}^i$ the vector defined by the i^{th} row of the matrix A which characterize the facet $\mathcal{F}^i_\mathcal{P}$.

   Since the polyhedron $\mathcal{P}$ is given by the system of inequalities $A\mathbf{x} \leq \mathbf{b}$, where each inequality $\mathbf{a}^i\mathbf{x} \leq b_i$ $(i \in \{1, 2 \dots, m\})$ is assigned to a certain facet, therefore the facet $\mathcal{F}^i_\mathcal{P}$ is visible from P if an only if the inequality $\mathbf{a}^i\mathbf{p} > b_i$ holds.

   Now, we define the characteristic function $\mathbb{I}^i_\mathcal{P}(\mathbf{x})$ for each facet $\mathcal{F}^i_\mathcal{P}$:

   $$\mathbb{I}^i_\mathcal{P}(\mathbf{x}) = \begin{cases} 1 & \mathbf{a}^i\mathbf{x} > b_i \\ 0 & \mathbf{a}^i\mathbf{x} \leq b_i. \end{cases}$$

3. Using the Definition 2.1.1, let $\Omega_i(P) := \Omega_{\mathcal{F}^i_\mathcal{P}}(P)$ be the solid angle of the facet $\mathcal{F}^i_\mathcal{P}$ regarding the point P.

   To determine $\Omega_i(P)$, we use the methods of spherical geometry. Let us suppose that $\mathcal{F}^i_\mathcal{P}$ contains n_i vertices, $V_{i_j}(\mathbf{x}_{i_j})$, $(j = 1, \dots, n_i)$ where the vertices are given in clockwise order. Projecting these vertices onto the unit sphere centered at P we get a spherical n_i-gon,whose area can be calculated by the usual formula

   $$\Omega_i(P) = \Theta - (n_i - 2)\pi.$$

   Here Θ is the sum of the angles τ_j of the spherical projection of the polygon $\mathcal{F}^i_\mathcal{P}$ where the angles are measured in radians.

4. To obtain angles τ_j, we need to determine the angles between the two planes containing the neighboring edges $\overline{PV}_{i_{j-1}}$, $\overline{PV}_{i_j}$ and $\overline{PV}_{i_j}$, $\overline{PV}_{i_{j+1}}$. Thus for $j = 1, 2, \dots, n$ $(i_0 := i_{n_i}$ and $i_{n_i+1} := i_1)$, we have:

   $$\tau_j = \pi - \arccos\left(\frac{\left\langle \overrightarrow{PV}_{i_{j-1}} \times \overrightarrow{PV}_{i_j}, \overrightarrow{PV}_{i_j} \times \overrightarrow{PV}_{i_{j+1}} \right\rangle}{\left|\overrightarrow{PV}_{i_{j-1}} \times \overrightarrow{PV}_{i_j}\right| \left|\overrightarrow{PV}_{i_j} \times \overrightarrow{PV}_{i_{j+1}}\right|}\right).$$

Finally, we get the solid angle function $\Omega_i(\mathbf{x})$ of the facet $\mathcal{F}_{\mathcal{P}}^i$ for any $\mathbf{x} \in \mathbf{R}^3$:

$$\Omega_i(\mathbf{x}) = 2\pi - \sum_{j=1}^{n_i} \arccos\left(\frac{\left\langle \overrightarrow{XV}_{i_{j-1}} \times \overrightarrow{XV}_{i_j}, \overrightarrow{XV}_{i_j} \times \overrightarrow{XV}_{i_{j+1}} \right\rangle}{\left| \overrightarrow{XV}_{i_{j-1}} \times \overrightarrow{XV}_{i_j} \right| \left| \overrightarrow{XV}_{i_j} \times \overrightarrow{XV}_{i_{j+1}} \right|} \right).$$

We can summarize our results in the following

Theorem 2.2.1 ([13]) *Consider a solid angle α ($0 < \alpha < 2\pi$) and a convex polyhedron $\mathcal{P}$ given by its data structure and its set of inequalities. Then the isoptic surface of $\mathcal{P}$ can be determined by the equation*

$$\alpha = \sum_{i=1}^{m} \mathbb{I}_{\mathcal{P}}^i(\mathbf{x}) \Omega_i(\mathbf{x}).$$

Remark 2.2.2

1. *The algorithm can easily be extended to non-closed directed surfaces e.g. for subdivision surfaces.*
2. *If we have a **convex** polyhedron, then projecting its whole surface to the unit sphere, we obtain a double coverage (double solid angle) of the given polyhedron, therefore the algorithm can be changed i.e. it is not necessary to determine the visible facets. In this case the isoptic surfaces are determined by the following **implicit** equation:*

$$\alpha = \frac{1}{2} \sum_{i=1}^{m} \Omega_i(\mathbf{x}).$$

3. *Despite the equation being obtainable in $\mathcal{O}(e)$ steps, where e is the number of edges, the rendering of these figures by Wolfram Mathematica takes 20–40 minutes. The implicit equation of the isoptic surface is so complicated that it seems difficult to draw further consequences from it.*

Chapter 3

Isoptic curves of conics in non-Euclidean constant curvature geometries

3.1 Isoptic curve of the line segment on the hyperbolic and elliptic plane

Theorem 3.1.1 (CsG) *Let us suppose that a line segment is given in the $\mathbf{H}^2$ hyperbolic or $\mathcal{E}^2$ elliptic plane by its endpoints $A = (a : 0 : 1)$ and $B = (-a : 0 : 1)$ where $a \in]0, 1]$. Then for a given $\alpha (0 < \alpha < \pi)$, the α-isoptic curve of AB in the hyperbolic and elliptic plane has an equation of the form:*

$$\cos(\alpha) = \frac{\epsilon(\epsilon - \frac{1}{a^2} + \frac{a^2-x^2}{y^2a^2})}{\sqrt{(\epsilon + \frac{1}{a^2} + (\frac{a-x}{ya})^2)(\epsilon + \frac{1}{a^2} + (\frac{a+x}{ya})^2)}}, \tag{3.1}$$

where $\epsilon = \pm 1$ if G is either the elliptic or the hyperbolic plane.

Remark 3.1.2

1. *Choosing $\alpha = \pi/2$ results in the orthoptic curve:*

$$\frac{x^2}{a^2} + \frac{y^2}{\frac{a^2}{1-\epsilon a^2}} = 1. \tag{3.2}$$

   This is an ellipse (without endpoints of the given segment) in the Euclidean sense, and it can be called the Thales curve.

   In the hyperbolic plane, if we increase the parameter a, then the Thales curve tends to a hypercycle (or an equidistant curve). That means the hypercycle is a special type of orthoptic curves with equation

$$x^2 + 2y^2 = 1.$$

2. *In the hyperbolic plane, if $a \to 1$, then equation* (3.1) *converges to the following equation*

$$x^2 + \left(\frac{y}{\cos\left(\frac{\alpha}{2}\right)}\right)^2 = 1.$$

3. *In the elliptic plane, ignoring the condition $a \in]0, 1]$ if $a \to \infty$, the right side of equation* (3.1) *converges to* 1*, therefore the full line in elliptic geometry has no isoptic curve at all.*

3.2 The general method

The procedure for the segment can be used to develop a more general method to determine the isoptic curves of a given hyperbolic or elliptic object. However we describe the method for conics. Let a hyperbolic or elliptic conic section C and one of its points P be given. First, we determine the equation of the tangent line in P. After that, a system of equations for the coordinates of the tangent point from an exterior point K can be given. This point has to satisfy the equation of the given curve and the tangent lines to this point have to contain K. This system can be solved for every $K = (x^0 : y^0 : 1)$ outer point with respect to the parameters x^0, y^0. The equation of the tangent lines from K can be determined by solving a system of equations for its coordinates. Finally, we have to fix the angle of the straight lines and we get the equation of the isoptic curve. But first of all, we need the conic equations.

3.3 Elliptic conics and their isoptics

We remark that in the elliptic plane the maximum distance between two points is less than $\frac{\pi}{2}$; therefore, in some cases the given curve cannot be seen under an arbitrarily small angle.

3.3.1 Equation of the elliptic ellipse and hyperbola

Definition 3.3.1 *The elliptic ellipse is the locus of all points of the elliptic plane whose distances to two fixed points sum up to the same constant $2a$.*

Definition 3.3.2 *The elliptic hyperbola is the locus of points where the absolute value of the difference of the distances from the two foci is a constant $2a$.*

We get the following equation:

$$\left(\frac{x}{\tan(a)}\right)^2 + \frac{y^2}{\frac{1}{(1+f^2)\cos^2(a)} - 1} = 1. \tag{3.3}$$

If the distance between the two foci is less than $2a$, it is an ellipse; if it is greater, then it is a hyperbola since cosine is monotonically decreasing on $[0, \pi]$ and we have

$$\begin{aligned} d(F_1, F_2) <> 2a \quad &\Leftrightarrow \quad \cos(2a) <> \cos(d(F_1, F_2)) = \frac{\langle \mathbf{f}_1, \mathbf{f}_2 \rangle}{\sqrt{\langle \mathbf{f}_1, \mathbf{f}_1 \rangle \langle \mathbf{f}_2, \mathbf{f}_2 \rangle}} = \frac{1 - f^2}{1 + f^2} \\ &\Leftrightarrow \quad 2\cos^2(a) - 1 <> \frac{2}{1+f^2} - 1 \Leftrightarrow 1 <> \frac{1}{\cos^2(a)(1+f^2)} \\ &\Leftrightarrow \quad 0 <> \frac{1}{\cos^2(a)(1+f^2)} - 1. \end{aligned}$$

Therefore, the elliptic ellipse and hyperbola are also an ellipse and a hyperbola in the model.

3.3.2 Equation of the elliptic parabola

Definition 3.3.3 *An elliptic parabola is the set of points ($X = (x : y : 1) \in \mathcal{E}^2$) in the elliptic plane that are equidistant to a proper point (the focus F) and a proper line (the directrix e) ($s = d(X; F) = d(X; e)$).*

We obtain the equation of the elliptic parabola:

$$-x^2 + \frac{(1+py)^2}{1+p^2} = 1. \tag{3.4}$$

3.3.3 Isoptic curves of elliptic ellipse and hyperbola

Now, we will use the method described in Section 3.2 to determine the isoptic curves to the elliptic ellipses and hyperbolas.

Theorem 3.3.4 ([12]) *Let an elliptic ellipse or hyperbola be centered at the origin of the projective model given by its semimajor axis a ($a \in (0, \frac{\pi}{2})$ and its foci $F_1 = (f : 0 : 1)$, $F_2 = (-f : 0 : 1)$, $(0 < f < 1)$ such that $2a > d(F_1, F_2)$ or $2a < d(F_1, F_2)$ holds. The α-isoptic and $(\pi - \alpha)$-isoptic curves $(0 < \alpha < \pi)$ of the considered ellipse or hyperbola in the elliptic plane have the equation*

$$\cos^2(\alpha) = \frac{\left((1+f^2)\cos(2a)\left(x^2+y^2+1\right) + f^2x^2 - 1\right)^2}{2\left(1+f^2\right)y^2\left(f^2+x^2\right) + \left(f^2-x^2\right)^2 + \left(1+f^2\right)^2 y^4}. \tag{3.5}$$

Remark 3.3.5

1. *The orthoptic curve of the elliptic ellipse and hyperbola is an ellipse with the following equation:*

$$\left(1+f^2\right)\cos(2a)\left(x^2+y^2+1\right)+f^2x^2=1.$$

2. *The isoptic curve of the elliptic hyperbola exists if the following formula holds true*

$$\left(\cos\alpha \le \max\left(\frac{1-(1+f^2)\cos(2a)}{f^2}, f^2+(1+f^2)\cos(2a)\right)\right)$$
$$\wedge\left(a\ge\frac{\pi}{6}\vee\left(f\le\sqrt{\frac{1}{\cos(2a)}-1}\right)\vee(\alpha\notin I)\right),$$

where

$$I=\left(\arccos\left(\frac{(1+f^2)\cos(2a)-1}{f^2}\right),\arccos\left(\frac{1-(1+f^2)\cos(2a)}{f^2}\right)\right).$$

3.3.4 Isoptic curves of elliptic parabola

Theorem 3.3.6 ([12]) *Let an elliptic parabola be given in the projective model by its focus $F=(0:p:1)$ and its directrix e which coincides with an x-axis. The α-isoptic curve of this parabola $(0<\alpha<\pi)$ in the elliptic plane has the equation*

$$\cos(\alpha)=\frac{y(py+1)}{\sqrt{(x^2+1)((p^2(x^2+1)-2py+y^2+x^2)}}. \tag{3.6}$$

Remark 3.3.7 *The orthoptic curve of the elliptic parabola contains two straight lines $y=0$ and $y=-\frac{1}{p}$.*

Remark 3.3.8 *The figures of the isoptic curves also confirm the fact that in elliptic geometry there is only* ***one class of conic sections****. However, in the affine model of the projective plane used, these can be considered separately.*

3.4 Hyperbolic conic sections and their isoptics

3.4.1 Generalized angle of straight lines

Having regard to the fact that the majority of the generalized conic sections have ideal and outer tangents as well, it is inevitable to introduce the generalized concept of the hyperbolic angle. In the extended hyperbolic plane there are three classes of lines by the number of common points with the absolute conic AC:

1. The straight line $u=\mathbb{R}\boldsymbol{u}$ is *proper* if $\mathrm{card}(u\cap AC)=2 \Leftrightarrow \langle\boldsymbol{u},\ \boldsymbol{u}\rangle>0$.
2. The straight line $u=\mathbb{R}\boldsymbol{u}$ is *non-proper* if $\mathrm{card}(u\cap AC)<2$.
   (a) If $\mathrm{card}(u\cap AC)=1 \Leftrightarrow \langle\boldsymbol{u},\ \boldsymbol{u}\rangle=0$ then $u=\mathbb{R}\boldsymbol{u}$ is called *boundary* straight line.
   (b) If $\mathrm{card}(u\cap AC)=0 \Leftrightarrow \langle\boldsymbol{u},\ \boldsymbol{u}\rangle<0$ then $u=\mathbb{R}\boldsymbol{u}$ is called *outer* straight line.

We define the generalized angle between straight lines using the results of the papers [3], [26] and [65] in the projective model.

Definition 3.4.1 *1. Suppose that $u=\mathbb{R}\boldsymbol{u}$ and $v=\mathbb{R}\boldsymbol{v}$ are both proper lines.*

(a) If $\langle\boldsymbol{u},\boldsymbol{u}\rangle\langle\boldsymbol{v},\boldsymbol{v}\rangle-\langle\boldsymbol{u},\boldsymbol{v}\rangle^2>0$ then they intersect in a proper point and their angle $\alpha(\boldsymbol{u},\boldsymbol{v})$ can be measured by

$$\cos\alpha=\frac{\pm\langle\boldsymbol{u},\boldsymbol{v}\rangle}{\sqrt{\langle\boldsymbol{u},\boldsymbol{u}\rangle\langle\boldsymbol{v},\boldsymbol{v}\rangle}}. \tag{3.7}$$

(b) *If* $\langle \boldsymbol{u}, \boldsymbol{u}\rangle\langle \boldsymbol{v}, \boldsymbol{v}\rangle - \langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 < 0$ *then they intersect in a non-proper point and their angle is the length of their normal transverse and it can be calculated using the formula below:*

$$\cosh \alpha = \frac{\pm\langle \mathbf{u}, \mathbf{v}\rangle}{\sqrt{\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle}}. \tag{3.8}$$

(c) *If* $\langle \boldsymbol{u}, \boldsymbol{u}\rangle\langle \boldsymbol{v}, \boldsymbol{v}\rangle - \langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 = 0$ *then they intersect in a boundary point and their angle is* 0.

2. *Suppose that* $u = \mathbb{R}\boldsymbol{u}$ *and* $v = \mathbb{R}\boldsymbol{v}$ *are both outer lines of* $\mathbf{H}^2$. *The angle of these lines will be the distance of their poles using the formula (3.8).*

3. *Suppose that* $u = \mathbb{R}\boldsymbol{u}$ *is a proper and* $v = \mathbb{R}\boldsymbol{v}$ *is an outer line. Their angle is defined as the distance of the pole of the outer line to the real line and can be computed by*

$$\sinh \alpha = \frac{\pm\langle \mathbf{u}, \mathbf{v}\rangle}{\sqrt{-\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle}}. \tag{3.9}$$

4. *Suppose that at least one of the straight lines* $u = \mathbb{R}\boldsymbol{u}$ *and* $v = \mathbb{R}\boldsymbol{v}$ *is boundary line of* $\mathbf{H}^2$. *If the other line fits the boundary point, the angle cannot be defined, otherwise it is infinite.*

Remark 3.4.2 *In the previous definition we fixed that except case 1. (a) we use real distance type values instead of complex angles which arise in other cases. The* $\pm$ *on the right sides are justifiable because we consider complementary angles i.e.* α *and* $\pi - \alpha$ *together.*

3.4.2 Classification of generalized conic sections on the hyperbolic plane in dual pairs

The literature of the hyperbolic conic section classification is very wide and it goes back until 1902 when it was first given by Liebmann (see [35]). We also note that there is a detailed theory of conic sections in the work of both Cooligde and Kagan(see [8] and [30]). There are numerous other works in this topic (see [17] and [54]), but in this section we will summarize and extend the results of K. Fladt (see [18] and [19]) about the generalized conic sections on the extended hyperbolic plane.

1. **Theorem 3.4.3 ([14])** *If the conic section has the normalform* $ax^2 + by^2 = 1$ *then we get the following types of central conic sections:*

   (a) Absolute conic: $a = b = 1$
   (b) i. Circle: $1 < a = b$
   ii. Circle enclosing the absolute: $a = b < 1$
   (c) i. Hypercycle: $1 = a < b$
   ii. Hypercycle enclosing the absolute: $0 < a < 1 = b$
   (d) Hypercycle excluding the absolute: $a < 0 < 1 = b$
   (e) Concave hyperbola: $0 < a < 1 < b$
   (f) i. Convex hyperbola: $a < 0 < 1 < b$
   ii. Hyperbola excluding the absolute: $a < 0 < b < 1$
   (g) i. Ellipse: $1 < a < b$
   ii. Ellipse enclosing the absolute: $0 < a < b < 1$
   (h) empty: $a \leq b \leq 0$

   where either the conic and its dual pair lies in the same class or (i) and (ii) are dual pairs with $a' = \frac{1}{a}$ *and* $b' = \frac{1}{b}$.

2. **Theorem 3.4.4 ([14])** *The parabolas have the normalform* $ax^2 + (b + 1)y^2 - 2y = b - 1$ *and the following cases arise:*

(a) i. Horocycle: $0 < a = b$

ii. Horocycle enclosing the absolute: $a = b < 0$

(b) i. Elliptic parabola: $0 < b < a$

ii. Parabola enclosing the absolute: $b < a < 0$

(c) i. Two sided parabola: $a < b < 0$

ii. Concave hyperbolic parabola: $0 < a < b$

(d) i. Convex hyperbolic parabola: $a < 0 < b$

ii. Parabola excluding the absolute: $b < 0 < a$

where all (i) and (ii) are dual pairs with parameters $a' = -\frac{b^2}{a}$ *and* $b' = -b$.

3. **Theorem 3.4.5 ([14])** *The so-called **semi-hyperbola** has the normalform* $ax^2 + 2by^2 - 2y = 0$ *where* $|b| < 1$ *and its dual pair is projectively equivalent with another semi-hyperbola with* $a' = \frac{1}{a}$ *and* $b' = -b$.

4. **Theorem 3.4.6 ([14])** *If the conic has the normalform* $(1-x^2-y^2)+2ay(x+1) = 0$ *where* $a > 0$ *then it is called **osculating parabola**. Its dual is also an osculating parabola by a convenient reflection.*

3.4.3 Isoptic curves of generalized hyperbolic conic sections

3.4.3.1 Isoptic curves of central conics

Theorem 3.4.7 ([14]) *Let a central conic section be given by its equation* $ax^2 + by^2 = 1$ *(see Theorem 3.4.3). Then the compound* α*-isoptic curve* $(0 < \alpha < \pi)$ *of the considered conic has the equation*

$$\frac{\left(a\left((b+1)x^2-1\right)+(a+1)by^2-b\right)^2}{\left|(a-1)^2b^2y^4+2(a-1)b\left(b+a\left((b-1)x^2-1\right)\right)y^2+\left(a(b-1)x^2+a-b\right)^2\right|} =$$

$$= \begin{cases} \cosh^2(\alpha), & aby^2\left(ax^2+by^2-1\right) \geq 0 \wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right) > 0 \wedge \\ & x^2+y^2 > 1 \\ & \\ \cos^2(\alpha), & aby^2\left(ax^2+by^2-1\right) \geq 0 \wedge \\ & x^2+y^2 < 1 \\ & \\ \sinh^2(\alpha), & aby^2\left(ax^2+by^2-1\right) \geq 0 \wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right) < 0, \end{cases}$$

wherein $u_{1,2}$ *and* $v_{1,2}$ *are*

$$\begin{array}{ll} u_1 = -\frac{ax+\sqrt{aby^2(ax^2+by^2-1)}}{ax^2+by^2} & v_1 = \frac{-ax+\sqrt{aby^2(ax^2+by^2-1)}}{ax^2+by^2} \\ u_2 = \frac{-by^2+x\sqrt{aby^2(ax^2+by^2-1)}}{ax^2y+by^3} & v_2 = -\frac{by^2+x\sqrt{aby^2(ax^2+by^2-1)}}{ax^2y+by^3}. \end{array} \tag{3.10}$$

3.4.3.2 Isoptic curves of parabolas

Theorem 3.4.8 ([14]) *Let a parabola be given by its equation* $ax^2+(b+1)y^2-2y = b-1$ *(see Theorem 3.4.4). Then the compound* α*-isoptic curve* $(0 < \alpha < \pi)$ *of the considered conic has the equation*

$$\left(a\left(b\left(2x^2+y^2-1\right)+(y-1)^2\right)+b^2\left(y^2-1\right)\right)^2\Big|(y-1)^2\Big((y+1)^2b^4-$$

$$-2a\left(2x^2+y^2+b(y+1)^2-1\right)b^2+a^2\left((y-1)^2+b^2(y+1)^2+2b\left(2x^2+y^2-1\right)\right)\Big)\Big|^{-1} =$$

$$= \begin{cases} \cosh^2(\alpha), & ab^2x^2\left(ax^2+b\left(y^2-1\right)+(y-1)^2\right) \geq 0 \wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right) > 0 \wedge \\ & x^2+y^2>1 \\ \\ \cos^2(\alpha), & ab^2x^2\left(ax^2+b\left(y^2-1\right)+(y-1)^2\right) \geq 0 \wedge \\ & x^2+y^2<1 \\ \\ \sinh^2(\alpha), & ab^2x^2\left(ax^2+b\left(y^2-1\right)+(y-1)^2\right) \geq 0 \wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right) < 0, \end{cases}$$

wherein $u_{1,2}$ *and* $v_{1,2}$ *are*

$$\begin{aligned} u_1 &= -\tfrac{ax^2(b+y-1)+y\sqrt{ab^2x^2(ax^2+b(y^2-1)+(y-1)^2)}}{a(b-1)x^3+b^2xy^2} \\ u_2 &= \tfrac{ax^2-b^2y-\sqrt{ab^2x^2(ax^2+b(y^2-1)+(y-1)^2)}}{a(b-1)x^2+b^2y^2} \\ \\ v_1 &= \tfrac{-ax^2(b+y-1)+y\sqrt{ab^2x^2(ax^2+b(y^2-1)+(y-1)^2)}}{a(b-1)x^3+b^2xy^2} \\ v_2 &= \tfrac{ax^2-b^2y+\sqrt{ab^2x^2(ax^2+b(y^2-1)+(y-1)^2)}}{a(b-1)x^2+b^2y^2}. \end{aligned} \tag{3.11}$$

3.4.3.3 Isoptic curves of semi-hyperbola

Theorem 3.4.9 ([14]) *Let the semi-hyperbola be given by its equation* $ax^2+2by^2-2y=0$*, where* $|b|<1$ *(see Theorem 3.4.5). Then the compound* α*-isoptic curve* $(0<\alpha<\pi)$ *of the considered conic has the equation*

$$\left(2a\left(b\left(x^2+y^2\right)-y\right)+y^2-1\right)^2\left|y^4+4a^2\left(x^2+y^2\right)\left(\left(b^2-1\right)x^2+(by-1)^2\right)-\right.$$
$$\left.-4a\left(y-\left(2x^2+y^2\right)y+b\left(y^4+\left(x^2-1\right)y^2+x^2\right)\right)-2y^2+1\right|^{-1}=$$

$$= \begin{cases} \cosh^2(\alpha), & ax^2\left(a+2y(by-1)\right) \geq 0 \wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right) > 0 \wedge \\ & x^2+y^2>1 \\ \\ \cos^2(\alpha), & ax^2\left(a+2y(by-1)\right) \geq 0 \wedge \\ & x^2+y^2<1 \\ \\ \sinh^2(\alpha), & ax^2\left(a+2y(by-1)\right) \geq 0 \wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right) < 0, \end{cases}$$

wherein

$$\begin{aligned} u_1 &= -\tfrac{ax+\sqrt{a(ax^2+2y(by-1))}}{y} \\ u_2 &= \tfrac{ax^2-y+x\sqrt{a(ax^2+2y(by+1))}}{y^2} \\ \\ v_1 &= \tfrac{-ax+\sqrt{a(ax^2+2y(by-1))}}{y} \\ v_2 &= \tfrac{ax^2-y-x\sqrt{a(ax^2+2y(by+1))}}{y^2}. \end{aligned}$$

3.4.3.4 Isoptic curves of the osculating parabola

Theorem 3.4.10 ([14]) *Let the osculating parabola be given by its equation* $(1-x^2-y^2)+2a(x+1)y=0$ *(see Theorem 3.4.6). Then the compound* α*-isoptic curve* $(0<\alpha<\pi)$ *of the considered conic has the* *equation*

$$\frac{\left(-2\left(x^2+y^2-1\right)+2a(x+1)y+a^2(x+1)^2\right)^2}{\left|a^2(x+1)^3\left(4(1-x)+4ay+a^2(x+1)\right)\right|}=$$

$$=\begin{cases} \cosh^2(\alpha), & \left(x^2+y^2-1-2a(x+1)y\right)\geq 0\wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right)>0\wedge \\ & x^2+y^2>1 \\ \\ \cos^2(\alpha), & \left(x^2+y^2-1-2a(x+1)y\right)\geq 0\wedge \\ & x^2+y^2<1 \\ \\ \sinh^2(\alpha), & \left(x^2+y^2-1-2a(x+1)y\right)\geq 0\wedge \\ & \left(1-u_1^2-u_2^2\right)\left(1-v_1^2-v_2^2\right)<0, \end{cases}$$

wherein

$$u_1=\frac{-(1+ay)(x-ay)+\sqrt{y^2(x^2+y^2-1-2a(x+1)y)}}{(x^2+y^2)-2axy+a^2y^2}$$
$$u_2=-\frac{y^2-ax(x+1)y+a^2(x+1)y^2+x\sqrt{y^2(x^2+y^2-1-2a(x+1)y)}}{y((x^2+y^2)-2axy+a^2y^2)}$$

$$v_1=-\frac{(1+ay)(x-ay)+\sqrt{y^2(x^2+y^2-1-2a(x+1)y)}}{(x^2+y^2)-2axy+a^2y^2}$$
$$v_2=-\frac{y^2-ax(x+1)y+a^2(x+1)y^2-x\sqrt{y^2(x^2+y^2-1-2a(x+1)y)}}{y((x^2+y^2)-2axy+a^2y^2)}.$$

Our method is suited for determining the isoptic curves to generalized conic sections for all possible parameters. Moreover with this procedure above we may be able to determine the isoptics for other curves as well.

Chapter 4

On the geometry of $\widetilde{\mathbf{SL_2R}}$

The $\widetilde{\mathbf{SL_2R}}$ space is the universal cover of $\mathbf{SL_2R}$. Recall that $\widetilde{\mathbf{SL_2R}}$ is one of the eight Thurston geometries [56, 62] that can be derived from the 3-dimensional Lie group of all 2×2 real matrices with unit determinant. According to [40], the space of left invariant Riemannian metrics on the group $\widetilde{\mathbf{SL_2R}}$ is 3-dimensional.

4.1 The projective model for $\widetilde{\mathbf{SL_2R}}$

This geometry can be derived by real 2×2 matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with the unit determinant $ad - bc = 1$ in the projective sphere $\mathcal{PS}^3$ and in the projective space $\mathcal{P}^3$ (see [43]). We introduce the new projective coordinates $(x^0 : x^1 : x^2 : x^3)$ where

$$a := x^0 + x^3, \quad b := x^1 + x^2, \quad c := -x^1 + x^2, \quad d := x^0 - x^3.$$

Then it follows that

$$0 > bc - ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 \tag{4.1}$$

describes the interior of the one-sheeted hyperboloid solid $\mathcal{H}$ in the usual Euclidean coordinate simplex, with the origin $E_0(1 : 0 : 0 : 0)$ and the ideal points of the axis $E_1^\infty(0 : 1 : 0 : 0)$, $E_2^\infty(0 : 0 : 1 : 0)$, $E_3^\infty(0 : 0 : 0 : 1)$. The elements of the isometry group of $\mathrm{SL}_2(\mathbb{R})$ (and so by the above extension the isometries of $\widetilde{\mathbf{SL_2R}}$) can be described by the matrix (a_i^j) (see [43, 45])

$$(a_i^j) = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ \mp a_0^1 & \pm a_0^0 & \pm a_0^3 & \mp a_0^2 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ \pm a_2^1 & \mp a_2^0 & \mp a_2^3 & \pm a_2^2 \end{pmatrix}, \tag{4.2}$$

where

$$\begin{cases} -(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 & = & -1, \\ -(a_2^0)^2 - (a_2^1)^2 + (a_2^2)^2 + (a_2^3)^2 & = & 1, \\ -a_0^0 a_2^0 - a_0^1 a_2^1 + a_0^2 a_2^2 + a_0^3 a_2^3 & = & 0, \\ -a_0^0 a_2^1 + a_0^1 a_2^0 - a_0^2 a_2^3 + a_0^3 a_2^2 & = & 0, \end{cases}$$

and we allow positive proportionality, of course, with projective freedom.

We define the *translation group* $\mathbf{G}_T$, as a subgroup of the isometry group of $\mathrm{SL}_2(\mathbb{R})$, those isometries act transitively on the points of $\mathcal{H}$. $\mathbf{G}_T$ maps the origin $E_0(1 : 0 : 0 : 0)$ onto $X(x^0 : x^1 : x^2 : x^3) \in \mathcal{H}$. These isometries and their inverses (up to a positive determinant factor) can be given by

$$\mathbf{T} = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}, \quad \mathbf{T}^{-1} = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix}. \tag{4.3}$$

We introduce a so-called hyperboloid parametrization by [43] as follows

$$\begin{cases} x^0 = \cosh r \cos\phi, \\ x^1 = \cosh r \sin\phi, \\ x^2 = \sinh r \cos(\theta - \phi), \\ x^3 = \sinh r \sin(\theta - \phi), \end{cases} \tag{4.4}$$

where (r, θ) are the polar coordinates of the base plane, and ϕ is the fibre coordinate $\big(r \in \mathbb{R}^+,\ \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \theta \in [0, 2\pi)\big)$. We note that

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.$$

The inhomogeneous coordinates, which will play an important role in the $\mathbf{E}^3$-visualizations of $\widetilde{\mathbf{SL_2R}}$, are given by

$$\begin{cases} x := \dfrac{x^1}{x^0} = \tan\phi, \\ y := \dfrac{x^2}{x^0} = \tanh r \dfrac{\cos(\theta - \phi)}{\cos\phi}, \\ z := \dfrac{x^3}{x^0} = \tanh r \dfrac{\sin(\theta - \phi)}{\cos\phi}. \end{cases} \tag{4.5}$$

4.2 Geodesic and translation curves

The infinitesimal arc-length-square can be derived by the standard pull back method (see [42, 46, 61]).

$$(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + \cosh^2 r \sinh^2 r (\mathrm{d}\theta)^2 + \big[(\mathrm{d}\phi) + \sinh^2 r (\mathrm{d}\theta)\big]^2.$$

Hence we obtain the symmetric metric tensor field g_{ij} on $\widetilde{\mathbf{SL_2R}}$ by components:

$$g_{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r(\sinh^2 r + \cosh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}. \tag{4.6}$$

Similarly to the above computations we obtain the metric tensor for coordinates (x^1, x^2, x^3):

$$g_{ij} := \begin{pmatrix} \frac{1+(x^2)^2+(x^3)^2}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} & \frac{-x^1x^2-2x^3}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} & \frac{-x^1x^3+2x^2}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} \\ \frac{-x^1x^2-2x^3}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} & \frac{1+(x^1)^2+(x^3)^2}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} & \frac{x^2x^3}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} \\ \frac{-x^1x^3+2x^2}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} & \frac{x^2x^3}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} & \frac{1+(x^1)^2+(x^2)^2}{(-1-(x^1)^2+(x^2)^2+(x^3)^2)^2} \end{pmatrix}. \tag{4.7}$$

4.2.1 Geodesic curves

The geodesic curves of $\widetilde{\mathbf{SL_2R}}$ are generally defined as having locally minimal arc length between any two of their (sufficiently close) points.

As is standard, the *geodesic distance* $d(P, Q)$ between points $P, Q \in \widetilde{\mathbf{SL_2R}}$ is defined as the arc length of the geodesic curve from P to Q.

4.2.2 Translation curves

Definition 4.2.1 *The curve $\mathcal{C}(t)$, $t \geqslant 0$, is said to be a translation curve if*

$$\dot{\mathcal{C}}(0) \cdot T(t) = \dot{\mathcal{C}}(t), \quad t \geqslant 0.$$

Definition 4.2.2 *The translation distance $\rho(E_0, X)$ between the origin $E_0(1:0:0:0)$ and the point $X(1:x:y:z)$ is the length of the translation curve connecting them.*

Table 4.1: Geodesic curves.

direction	parametrization of a geodesic curve
$-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$ ($\mathbf{H}^2 -$ like)	$r(s,\alpha) = \text{arsinh}\Big(\frac{\cos\alpha}{\sqrt{\cos 2\alpha}} \sinh(s\sqrt{\cos 2\alpha})\Big)$ $\theta(s,\alpha) = -\text{arctan}\Big(\frac{\sin\alpha}{\sqrt{\cos 2\alpha}} \tanh(s\sqrt{\cos 2\alpha})\Big)$ $\phi(s,\alpha) = 2s\sin\alpha + \theta(s,\alpha)$
$\alpha = \pm\frac{\pi}{4}$ (light $-$ like)	$r(s,\alpha) = \text{arsinh}\Big(\frac{\sqrt{2}}{2}s\Big)$ $\theta(s,\alpha) = -\text{arctan}\Big(\frac{\sqrt{2}}{2}s\Big)$ $\phi(s,\alpha) = \sqrt{2}s + \theta(s,\alpha)$
$-\frac{\pi}{2} \leq \alpha < -\frac{\pi}{4}$ $\frac{\pi}{4} < \alpha \leq \frac{\pi}{2}$ (fibre $-$ like)	$r(s,\alpha) = \text{arsinh}\Big(\frac{\cos\alpha}{\sqrt{-\cos 2\alpha}} \sin(s\sqrt{-\cos 2\alpha})\Big)$ $\theta(s,\alpha) = -\text{arctan}\Big(\frac{\sin\alpha}{\sqrt{-\cos 2\alpha}} \tan(s\sqrt{-\cos 2\alpha})\Big)$ $\phi(s,\alpha) = 2s\sin\alpha + \theta(s,\alpha)$

Table 4.2: Translation curves.

direction	parametrization of a translation curve
$-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$ ($\mathbf{H}^2 -$ like)	$\begin{pmatrix} x(s,\alpha,\lambda) \\ y(s,\alpha,\lambda) \\ z(s,\alpha,\lambda) \end{pmatrix} = \frac{\tanh(s\sqrt{\cos 2\alpha})}{\sqrt{\cos 2\alpha}} \begin{pmatrix} \sin\alpha \\ \cos\alpha\cos\lambda \\ \cos\alpha\sin\lambda \end{pmatrix}$
$\alpha = \pm\frac{\pi}{4}$ (light $-$ like)	$\begin{pmatrix} x(s,\alpha,\lambda) \\ y(s,\alpha,\lambda) \\ z(s,\alpha,\lambda) \end{pmatrix} = \frac{\sqrt{2}s}{2} \begin{pmatrix} 1 \\ \cos\lambda \\ \sin\lambda \end{pmatrix}$
$-\frac{\pi}{2} \leq \alpha < -\frac{\pi}{4}$ $\frac{\pi}{4} < \alpha \leq \frac{\pi}{2}$ (fibre $-$ like)	$\begin{pmatrix} x(s,\alpha,\lambda) \\ y(s,\alpha,\lambda) \\ z(s,\alpha,\lambda) \end{pmatrix} = \frac{\tan(s\sqrt{-\cos 2\alpha})}{\sqrt{-\cos 2\alpha}} \begin{pmatrix} \sin\alpha \\ \cos\alpha\cos\lambda \\ \cos\alpha\sin\lambda \end{pmatrix}$

4.3 Geodesic and translation triangles

4.3.1 Geodesic triangles

We consider three points A_1, A_2, A_3 in the projective model of $\widetilde{\mathbf{SL_2R}}$ space. The *geodesic segments* a_k between the points A_i and A_j $(i < j,\ i,j,k \in \{1,2,3\}, k \neq i,j)$ are called sides of the *geodesic triangle* T_g with vertices A_1, A_2, A_3.

4.3.1.1 Fibre-like right angled triangles

A geodesic triangle is called fibre-like if one of its edges lies on a fibre line.

Lemma 4.3.1 ([15]) *The sum of the interior angles of a fibre-like right angled geodesic triangle is greater or equal to* π.

In Table 4.3 we summarize some numerical data of geodesic triangles for given parameters:

4.3.1.2 Hyperbolic-like right angled geodesic triangles

A geodesic triangle is hyperbolic-like if its vertices lie in the base plane (i.e. $[y,z]$ coordinate plane) of the model.

Lemma 4.3.2 ([15]) *The interior angle sums of hyperbolic-like geodesic triangles can be less than or equal to* π.

Table 4.3: Fibre-like right angled geodesic triangles for $x^3 = 1/5$

y^2	$\|\alpha_2^3\| = \|\alpha_3^2\|$	$d(A_2A_3)$	ω_2	ω_3	$\sum_{i=1}^3(\omega_i)$
$\to 0$	$\to \pi/2$	$\to$ $\arctan(1/5) \approx$ ≈ 0.1974	$\to \pi/2$	$\to 0$	$\to \pi$
1/1000	1.5657	0.1974	1.5658	0.0051	3.1417
1/3	0.4993	0.3970	0.3560	1.0715	3.1806
1/2	0.3170	0.5809	0.3560	1.2538	3.1806
3/4	0.1630	0.9891	0.2043	1.4078	3.1829
999/1000	0.0299	3.8032	0.0422	1.5409	3.1540
$\to 1$	$\to 0$	$\to \infty$	$\to 0$	$\to \pi/2$	$\to \pi$

Table 4.4: Hyperbolic-like geodesic triangles for $y^2 = 1/2$

z^3	$\|\alpha_2^3\| = \|\alpha_3^2\|$	$d(A_2A_3)$	ω_1	ω_2	$\sum_{i=1}^3(\omega_i)$
$\to 0$	$\to 0$	$\to$ $\operatorname{arctanh}(1/2) \approx$ ≈ 0.5493	$\to 0$	$\to \pi/2$	$\to \pi$
1/10	0.0811	0.5638	0.1334	1.2830	2.9872
1/3	0.2103	0.6994	0.3613	0.7170	2.6491
$\frac{999}{1000}$	0.0649	4.0707	0.5817	0.0913	2.2438
$\frac{(10^6-1)}{10^6}$	0.0330	7.5174	0.0467	0.6112	2.2288

Conjecture 4.3.3 ([15]) *The sum of the interior angles of any hyperbolic-like right angled geodesic triangle is less than or equal to π.*

In Table 4.4 we summarize some numerical data of geodesic triangles for given parameters:

4.3.1.3 Geodesic triangles with interior angle sum π

Lemma 4.3.4 ([15]) *There are geodesic triangles $A_1A_2A_3$ with interior angle sum π where all the vertices are proper (i.e. $A_i \in \mathbf{SL_2R}$).*

Theorem 4.3.5 ([15]) *The sum of the interior angles of a geodesic triangle of $\widetilde{\mathbf{SL_2R}}$ space can be greater than, less than or equal to π.*

4.3.2 Translation triangles

Considering a translation triangle we can assume that one of its vertices is the origin (translate one if necessary) and the other two points are $A(1 : x_1 : y_1 : z_1)$ and $B(1 : x_2 : y_2 : z_2)$.

Lemma 4.3.6 ([15]) *A plane σ trough the origin with Euclidean normal vector $\mathbf{v}$ is invariant for translations perpendicular to $\mathbf{v}$ (in the Euclidean sense) if and only if $\mathbf{v}$ is light–like.*

Theorem 4.3.7 ([15]) *The sum of the interior angles of a translation triangle in the $\widetilde{\mathbf{SL_2R}}$ geometry is greater than or equal to π.*

Remark 4.3.8 *The sum of the interior angle of the translation triangle is π if and only if the normal vector of the plane containing the triangle is light–like.*

4.4 Isoptic surface of a line segment with translation curves

Theorem 4.4.1 (CsG) *Given a translation line segment in the* $\widetilde{\mathbf{SL_2R}}$ *geometry by its endpoints* $A = (1:0:a:0)$ *and* $B = (1:0:-a:0)$ *where* $a \in (0,1)$*. Then the* α *and* $\pi - \alpha$ *isoptic surfaces of a line segment* $\overline{AB}$ *have the equation:*

$$\cos^2(\alpha) = \frac{\left(x^2+y^2+z^2-a^2(1+x^2+z^2)\right)^2}{\left(x^2+y^2+z^2+a^2(1+x^2+z^2)\right)^2-(4axz-2ay)^2} \tag{4.8}$$

Remark 4.4.2 *If we set* $\alpha = \frac{\pi}{2}$*, then we get the so called* $\widetilde{\mathbf{SL_2R}}$ *Thales surface, which has the equation of a spheroid:*

$$\frac{x^2(1-a^2)}{a^2} + \frac{y^2}{a^2} + \frac{z^2(1-a^2)}{a^2} = 1$$

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