Budapest University of Technology and Economics
Department of Analysis

Attila Lovas

Summary of the

Application of information geometry to quantum mechanical systems

PhD thesis

Supervisor: Attila Andai
PhD, Associate Professor

2017

1 Introduction

Quantum information geometry is a relatively young discipline evolved from quantum information theory and classical information geometry. Similarly to classical information geometry, quantum information geometry endows the space of (quantum) probability distributions with Riemannian metric and establishes connections between differential geometrical properties of the model and their probabilistic features.

Achievements in quantum information geometry are frequently applied in quantum information theory and quantum statistical physics. Quantum information geometry strongly uses results of functional analysis and differential geometrical techniques. Contrary to classical probability theory, quantum event algebra is non-distributive and it has just an orthomodular lattice structure [12]. This fact has numerous important consequences to quantum information theory and quantum information geometry. We can mention here the uncertainty principles, the existence of entangled states and the non-uniqueness of quantum Fisher information. The usual model of a quantum event algebra is projection lattice of a Hilbert space. According to Gleason's theorem, for a separable Hilbert space of complex dimension greater than two, there is a bijection between states of projection lattice and positive operators of trace one, therefore the quantum mechanical state space can be identified with the intersection of the positive cone and the hyperplane of operators with trace one, which is is a compact and convex manifold. If we endow this manifold with monotone metrics characterized by Petz's classification theorem, then we get different quantum statistical manifolds that are the subject of our investigations. For further details the reader is referred to Amari's [6] book, and Petz's paper [14], which can be considered as standard in quantum information geometry.

2 Results

The following references concern to the thesis.

1/a. The precise reformulation of Heisenberg's uncertainty principle given by Schrödinger [16] was generalized to arbitrary number of observables by Robertson in 1934 [15]. Robertson's theorem says that for arbitrary system of observables $(A_k)_{k=1,\dots,N} \subset \mathcal{M}_{n,\mathbb{K}}^{\mathrm{sa}}$ the following determinant inequality holds.

$$\det\left(\left[\mathrm{Cov}_\rho(A_k, A_l)\right]_{k,l=1,\dots N}\right) \geq \det\left(\left[\mathbb{E}_\rho\left(\tfrac{\mathrm{i}}{2}[A_k, A_l]\right)\right]_{k,l=1,\dots,N}\right) \quad \rho \in \mathcal{D}_{n,\mathbb{K}}$$

The main disadvantage of this inequality is that the right-hand-side is zero whenever N is odd, thus we get back just the classical lower bound. Dynamical uncertainty principles conjectured by Gibilisco and Isola [8] and proved by Andai get around this failure moreover, the right-hand-side has a clear geometric meaning.

I show that dynamical uncertainty principles can be originated to the point-wise order of operator

monotone functions that define Riemannian metric on the state space (Theorem 2.2.2).

1/b. Gibilisco, Hiai and Petz studied first the possible non-commutative generalizations of classical covariance to statistical manifolds different from quantum covariance introduced by Schrödinger [9].

I introduce the concept of symmetric quantum f-covariances (Definition 2.3.1), which are another generalization of quantum covariances. The main advantage of this concept is that covariance matrices defined from them have clear geometric interpretation similar to the right-hand-side. Using Brunn–Minkowski determinant inequality, I prove dynamical uncertainty principles for the introduced symmetric quantum f-covariances (Theorem 2.4.1). These new uncertainty principles result sharper inequalities than those, which were studied by Gibilisco (Corollary 2.4.1).

1/c. I show that for a given operator monotone function f the determinant of covariance matrix associated to symmetric f-covariance estimate above the determinant of covariance matrix associated to anti-symmetric f-covariance and the gap between them can be estimated by the determinant of Petz covariance induced by the function $f(x) = \frac{2x}{1+x}$ (Theorem 2.4.2).

1/d. I determine the optimal operator monotone function f_{opt} for which in every state $\rho \in \mathcal{D}_{n,\mathbb{K}}$ and for

arbitrary set of observables $A = (A_k)_{k=1,\dots,N}$

$$\det\left(\mathrm{Cov}^s_{f_{\mathrm{opt}}}(\rho)(A)\right) \geq \det\left(\mathrm{Cov}^{as}_f(\rho)(A)\right) \quad \forall f \in \mathcal{F}_{\mathrm{op}}$$

holds (Theorem 2.4.3).

2/a. The state space of a composite quantum system is a disjoint union of separable (or classically correlated) and entangled states. In general to distinguish separable states from entangled states is an NP hard problem. If a Borel measure is given on the state space, then we can study the ratio of the volume of separable states and the volume of the whole state space which quantity is called separability probability. Investigations of separability probability has a quite long (in about 20 years long) history. This question was posed first by Zyczkowski, Horodecki, Sanpera and Lewenstein [10]. Using numerical techniques Slater have got $\frac{29}{64}$ for the separability probability in the rebit-rebit system [17]. The critical observation for us was Milz and Strunz's conjecture which asserts that the separability probability with respect to the Hilbert–Schmidt metric is independent from the reduced state and the volume of separable states over a fixed reduced state is a simple polynomial expression of the Bloch radii of the reduced state in question.

I prove Milz ans Strunz's conjecture for rebit-rebit and qubit-qubit systems (Theorem 3.2.1 and Corollary 3.2.1).

2/b. I Introduced separability functions $\tilde{\chi}_d$ $d = 1, 2$. **Using them I write the separability probability on**

the state space of 4×4 real and complex density matrices in an explicit integral form (Theorem 3.2.1 and Corollary 3.2.2).

2/c. Using a special parametrization of real 2×2 matrices that reflects a special semigroup property I calculate the separability function $\tilde{\chi}_1$ (Lemma 3.2.1). I calculate the separability probability in the rebit-rebit system and the result coincides with the value conjectured by Slater (Theorem 3.2.2).

3/a. I show that the conjecture of Milz and Strunz is true on the statistical manifold of 4×4 density matrices endowed with the monotone metric associated to the geometric mean (Theorem 3.2.3 and Corollary 3.2.3).

3/b. I prove that separability functions $\tilde{\chi}_1$ and $\tilde{\eta}_1$ are equal to each other. I calculate the separability probability for the real case and monotone metric associated to the geometric mean (Corollary 3.2.3 and Theorem 3.2.4).

4/a. I prove that the state space $\mathcal{D}_{2n,\mathbb{K}}$ is diffeomorphic to the product

$$\mathcal{D}_{n,\mathbb{K}} \times \mathcal{E}_{2^k,\mathbb{K}} \times \mathrm{B}_1\left(\mathbb{K}^{n\times n}\right),$$

where $\mathcal{E}_{2^k,\mathbb{K}}$ stands for the $]-I, I[$ operator interval of $n \times n$ self-adjoint matrices and $\mathrm{B}_1\left(\mathbb{K}^{n\times n}\right)$ stands for the unit ball of $n \times n$ matrices with respect to the canonical operator norm (Definition 3.3.1 and Theorem 3.3.1).

4/b. I show that the interior of the state space of n qubit (or rebit) is diffeomorphic to the product manifold

$$\prod_{k=0}^{n-1}\left(\mathcal{E}_{2^k,\mathbb{K}}\times \mathrm{B}_1\left(\mathbb{K}^{2^k\times 2^k}\right)\right)$$

(Corollary 3.3.1).

4/c. The pullback of a volume form associated to a monotone metrics is calculated (Theorem 3.3.2 and Corollary 3.3.2).

5/a. I show that the manifold of the PPT states $\mathcal{D}_{2n,\mathbb{K}}\subset(\mathcal{M}^{\mathbf{sa}}_{2,\mathbb{K}})^+\otimes(\mathcal{M}^{\mathbf{sa}}_{n,\mathbb{K}})^+$ is diffeomorphic to

$$\Pi^{\mathrm{PPT}}_{n,\mathbb{K}}=\mathcal{D}_{n,\mathbb{K}}\times\left\{(Z,X)\in\mathcal{E}_{n,\mathbb{K}}\times\mathrm{B}_1\left(\mathbb{K}^{n\times n}\right)\middle|1>\left\|\Sigma\left(\sqrt{\tfrac{I-Z}{I+Z}}\right)^{-1}X\Sigma\left(\sqrt{\tfrac{I-Z}{I+Z}}\right)\right\|\right\},$$

where $\Sigma\left(\sqrt{\frac{I-Z}{I+Z}}\right)$ is a diagonal matrix, which contains the eigenvalues of $\sqrt{\frac{I-Z}{I+Z}}$ in its diagonal in a decreasing order (Theorem 3.3.3). This theorem implies that Milz and Strunz's conjecture remains true in a much stronger form on the state space $\mathcal{D}_{2n,\mathbb{K}}$ (Corollary 3.3.3).

5/b. I show that the submanifold $\mathcal{D}^{\mathrm{PPT}}_{2n,\mathbb{K}}\subset\mathcal{D}_{2n,\mathbb{K}}$ is diffeomorphic to a $\mathcal{D}_{n,\mathbb{K}}\times Fl_{n,\mathbb{K}}\times\mathbb{R}^+$-bundle, which base manifold is the open epigraph of the separability function $r:\Delta_{n-1,\geq}\times\partial\mathrm{B}_1\left(\mathbb{K}^{n\times n}\right)\to[0,1]$, where $Fl_{n,\mathbb{K}}=\mathcal{U}(\mathbb{K}^n)/\mathcal{U}(\mathbb{K})^n$ denotes the so-called flag manifold (Theorem 3.3.4). Using this diffeomorphism, I

present an explicit integral form for the geometric probability of PPT states on $\mathcal{D}_{2n,\mathbb{K}}$ with respect to the Hilbert–Schmidt metric and the monotone metric $g_{f_{GM}}$ (Theorem 3.3.5).

5/c. For a given entangled quantum state the distance from the set of separable states quantifies how much entanglement is contained in it. Different metrics define different entanglement measures.

Using the congruence invariant Thompson metric I introduce the Thompson entanglement measure for entangled states in $\mathcal{D}^{\text{ent}}_{2n,\mathbb{K}}$ (Definition 3.3.3). I calculate the Thompson entanglement measure of a state $\rho = \phi(D, Z, X)$ which turned to be independent from the reduced state D (Theorem 3.3.6).

6/a. Quantum channels play a key role in quantum information processing and communication because to every quantum operation a quantum channel can be associated.

Choi representation identifies the space of general (unit preserving) qubit channels with a compact and convex submanifold of $\mathbb{R}^{12}$ ($\mathbb{R}^9$). This makes possible the information geometrical investigation of qubit channels.

The classical trace (or underlying classical channel) of a given quantum channel is defined (Definition 4.1.2) and the distribution of uniformly distributed qubit channels over classical channels is calculated. Furthermore we compute the volume of the space of general qubit channels with respect to the canonical

Lebesgue measure (Theorem 4.2.1).

6/b. Unit preserving quantum channels are non commutative analogues of doubly stochastic matrices. The famous theorem of Birkhoff and von Neumann asserts that the space of $n \times n$ convex matrices is exactly the convex hull of $n \times n$ permutation matrices hence it is polytope which is often called Birkhoff polytope. The volume of this polytope is known just for $n = 1, 2, \dots, 10$ and for general n we have only asymptotic formulas [13].

Explicit formula is presented for the distribution of uniformly distributed unital qubit channels over classical channels and the volume of the space of unital qubit channels with respect to the canonical Lebesgue measure is computed (Theorem 4.2.2).

7/a. Random qubit channels has considerable scientific interest. A quantum algorithm is actually a sequence of quantum channels where the error of each step is cumulated in the output. We can model the error of steps by random quantum channels and thus the error of the output can be controlled. Random quantum operation is an appropriate framework to describe external noise that leads to decoherence and the lost of information stored in the quantum register. Bruzda, Cappellini, Sommers és Zyczkowski [7] studied the spectral properties of random quantum maps [7]. The have proved the quantum analogue of the Perron–Frobenius theorem known from linear algebra and investigated numerically the action of special iterated quantum channels on random quantum states.

I prove that the image of an arbitrary qubit under

the action of a uniformly distributed random qubit channel has a spherical symmetric distribution and the radial distribution depends only on the Bloch radius of the initial qubit (Lemma 4.3.1). I show that the Bloch radii of the image is $\sqrt{Y}$ times the initial Bloch radii, where Y is a random variable coming from the $\beta\left(\frac{3}{2},4\right)$ distribution (Theorem 4.3.1, Corollary 4.3.1).

7/b. I show that there is a typical radius $r>0$ to which the maximally mixed state is mapped under the action of a uniformly distributed random qubit channel (Theorem 4.3.2).

7/c. External noise influencing a single qubit state is modelled by a sequence of independent uniformly distributed qubit channels. To measure the information loss residual information of a qubit and information loss compared to the initial state are introduced (Definition 4.4.2), moreover I show that the residual information of the qubit tends exponentially fast to zero as the number of random qubit channels applied goes to infinity. Using the law of iterated logarithm I calculate the rate of the decrease.

3 References

Own articles

References

[1] Andai A. and Lovas A. On robertson-type uncertainty principles. In Václav Kratochvíl, editor, *Information Geometry and its Applications IV*, pages 28–29, Liblice, Czech Republic, June 2016.

[2] Lovas A. and Andai A. Refinement of robertson-type uncertainty principles with geometric interpretation. *International Journal of Quantum Information*, 14(02):1650013, 2016.

[3] Lovas A. and Andai A. Volume of the space of qubit channels and some new results about the distribution of the quantum dobrushin coefficient. *arXiv preprint arXiv:1607.01215*, 2016.

[4] Lovas A. and Andai A. Volume of the space of qubit channels and the distribution of some scalar quantities on it. In Václav Kratochvíl, editor, *Information Geometry and its Applications IV*, pages 48–49, Liblice, Czech Republic, June 2016.

[5] Lovas A. and Andai A. Invariance of separability probability over reduced states in 4×4 bipartite systems. *Journal of Physics A: Mathematical and Theoretical*, 2017.

Other articles

[6] S. Amari. *Differential-geometrical methods in statistics*, volume 28 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 1985.

[7] W. Bruzda, V. Cappellini, H. J. Sommers, and K. Życzkowski. Random quantum operations. *Physics Letters A*, 373(3):320–324, 2009.

[8] P. Gibilisco and T. Isola. Uncertainty principle and quantum Fisher information. *Ann. Inst. Statist. Math.*, 59(1):147–159, 2007.

[9] P. Gibilisco, F. Hiai, and D. Petz. Quantum covariance, quantum Fisher information, and the uncertainty relations. *IEEE Trans. Inform. Theory*, 55(1):439–443, 2009.

[10] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein. Volume of the set of separable states. *Phys. Rev. A*, 58:883–892, Aug 1998.

[11] S. Milz and W. T. Strunz. Volumes of conditioned bipartite state spaces. *J. Phys. A*, 48(3):035306, 16, 2015.

[12] János Neumann. *A kvantummechanika matematikai alapjai*. Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1980. Translated from the 1964 Russian edition by Ákos Sebestyén.

[13] I. Pak. Four questions on birkhoff polytope. *Annals of Combinatorics*, 4(1):83–90, 2000.

[14] D. Petz. Geometry of canonical correlation on the state space of a quantum system. *J. Math. Phys.*, 35(2):780–795, 1994.

[15] H. P. Robertson. An indeterminacy relation for several observables and its classical interpretation. *Phys. Rev.*, 46:794–801, Nov 1934.

[16] E. Schrödinger. About Heisenberg uncertainty relation (original annotation by A. Angelow and M.-C. Batoni). *Bulgar. J. Phys.*, 26(5-6):193–203 (2000), 1999. Translation of Proc. Prussian Acad. Sci. Phys. Math. Sect. **19** (1930), 296–303.

[17] P. B. Slater. A concise formula for generalized two-qubit Hilbert–Schmidt separability probabilities. *Journal of Physics A: Mathematical and Theoretical*, 46(44):445302, 2013.