Numerical methods for the stability and stabilizability analysis of delayed dynamical systems

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BOOKLET

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Introduction

In the recent decades an increasing number of research papers and books have dealt with time-delay systems in applied sciences. Time delay plays an important role in population dynamics, epidemics, control systems, human balancing, traffic dynamics, wheel dynamics or machine tool vibrations, just to mention a few fields of application. In the above examples, time delay typically has a destabilizing effect, which is manifested in unwanted vibrations or oscillations around the desired steady-state motion. The local stability analysis of these time-delay systems provides the primary characteristics of their behavior around stationary states. As a result, several analytical and numerical methods have been developed for local stability analysis. While the stability boundaries in the plane of system parameters can be derived in closed form for some autonomous systems, the stability analysis of time-periodic systems usually requires numerical approximation techniques, especially in the presence of time-delays. The related literature provides several numerical methods for the stability analysis of time-periodic time-delay systems, such as the semi-discretization method, full discretization method, spectral element method or the pseudospectral collocation method.

The optimal selection of system parameters plays an important role during the process of design of engineering applications. Stability diagrams show the stable domains of linearized systems in the space of system parameters. For the calculation of stability diagrams the stability of the linearized system has to be determined for different combinations of system parameters. As a result, the computational properties of the different numerical methods for stability analysis and the development of new, computationally more efficient methods are still important tasks in engineering.

In addition to the stability analysis of time-delay systems, the condition of stabilizability is often required in many applications. Stabilizability properties describe whether the locally linearized dynamical system can be made stable by the proper choice of control or system parameters. Stabilizability plays an important role
in parameter optimization, where, besides keeping the stability of the system, a cost function has to be minimized. In human balancing the loss of balance is also often associated with the loss of stabilizability of the mathematical model.

### Aims of the work

The thesis deals with the numerical stability analysis and the stabilizability of dynamical systems governed by delay-differential equations (DDEs). In Chapter 1 the mathematical basis is briefly presented, which is necessary for the stability analysis of linear DDEs.

In Chapter 2, two novel numerical methods are described for the finite dimensional approximation of DDEs: the pseudospectral tau (PsT) and spectral element (SE) methods. The derivation of these methods are presented with computational examples and a comparison is carried out with well-known numerical methods from the literature having high convergence rates. Furthermore, the extension of the PsT and the SE methods is presented for hybrid feedback systems subjected to delay and numerical integration in the feedback loop.

In Chapter 3, the presented numerical methods are applied to different machine tool chatter models. These mathematical models take into account the feedback loop of the linear drive of the tool holder and the workpiece holder, and the feedback loop of the active damper placed close to the tool tip. Stability diagrams are determined in the plane of machining parameters and the effect of control parameters on the stability of machining is demonstrated.

In Chapter 4, the stabilizability of delayed dynamical systems is investigated. Two particular problems are studied: the optimization of control parameters for the increase of maximum admissible depth of cut in turning processes subjected to digitally controlled active damper and the loss of balance in human balancing where the balancing process is modeled with a delayed proportional-integral-derivative-acceleration feedback controller.
Theses

I have introduced the pseudospectral tau method for the finite-dimensional approximation of linear delay-differential equations. The results are summarized as follows.

Thesis 1

The pseudospectral tau (PsT) method can be used for the finite-dimensional approximation of delay-differential equations of the form

\[ \dot{\xi}(t) = A(t)\xi(t) + \sum_{p=1}^{v} B_p(t)\xi(t - \tau_p(t)) + \sum_{b=1}^{m} \int_{-\sigma_b}^{-\sigma_{b-1}} \gamma_b(t, \theta) \xi(t + \theta) d\theta, \]

with a system of ordinary differential equations. Based on the results of numerical experiments, the following statements can be made.

1) The stability properties of the approximating system of ordinary differential equations converge to the stability properties of the above equation with respect to the increase of polynomial order: the stability boundaries converge to the exact ones for both autonomous and non-autonomous systems; the real part of the rightmost root converges for autonomous systems.

2) For the Hayes equation and for the oscillator with distributed delay, the convergence rate of the PsT method with respect to the increase of the polynomial degree shows the same convergence order as the spectral element (SE) and the spectral Legendre-tau (SLT) methods, while it has better convergence order than that of the pseudospectral collocation (PsC) method. For the oscillator with two delays, the SE has better convergence order than that of the PsC, SLT and PsT methods.

3) Considering the computation time of the stability diagrams of the Hayes equation, oscillator with two delays, and oscillator with distributed delay, the time-need of the PsT method is less than that of the SE method, same as that of the SLT method and higher than that of the PsC method.

Related publications: [1, 3, 9]
I have generalized the spectral element method for linear systems with time-periodic coefficients and distributed delays and derived explicit formulas for the matrix approximation of the monodromy operator. The results are summarized as follows.

**Thesis 2**

The spectral element method can be extended for the stability analysis of time-periodic delay-differential equations of the form

\[
\dot{\xi}(t) = A(t)\xi(t) + \sum_{p=1}^{v} B_p(t)\xi(t-\tau_p) + \sum_{b=1}^{m} \int_{-\sigma_b}^{-\sigma_{b-1}} \gamma_b(t, \theta) \xi(t + \theta) d\theta.
\]

Explicit formulas can be determined for the calculation of the matrix approximation \( U \) of the monodromy operator of this dynamical system. Using these formulas, \( U \) can be composed for arbitrary time-periodic coefficient matrices \( A(t), B_p(t), \gamma_b(t, \theta) \). According to numerical experiments, the stability boundaries determined by the method converge with respect to the increase of the polynomial order.

Related publications: [2, 7]
I have extended both the pseudospectral tau and the spectral element methods for the stability analysis of linear time-periodic hybrid systems with both continuous-time delay (point delay) and discrete-time delay (terms with piecewise constant arguments). The results are summarized as follows.

**Thesis 3**

The pseudospectral tau and the spectral element methods can be applied for the calculation of the matrix approximation of the monodromy operator of time-periodic hybrid time delay systems of the form

\[
\dot{\xi}(t) = A(t)\xi(t) + \sum_{p=1}^{v} B_p(t)\xi(t - \tau_p) + C\xi(t_l - \Delta t) + E\chi_l, \quad t \in [t_l, t_{l+1}),
\]

\[
\chi_l = \chi_{l-1} + \sum_{q=1}^{\tilde{n}} W_b \xi(t_l - b\Delta t),
\]

where \(t_l = l\tilde{n}\Delta t\) and \(l \in \mathbb{N}\). Based on numerical experiments, both methods converge to the same stability boundaries with respect to the increase of polynomial order.

Related publications: [8]
On the example of the mathematical model of milling operations, I have generalized the spectral element method for the analysis of time-periodic delay-differential equations with discontinuous time-periodic coefficients. The advantage of this generalization is that no adjustment of the element length is necessary in order to guarantee results with exponential convergence rates. The results are summarized as follows.

**Thesis 4**

Convergent stability boundaries can be achieved by the spectral element method in the stability lobe diagrams of milling operations without the adjustment of the element length. In order to do so, the integral terms of the approximation scheme should be split at the discontinuity fronts of the time-periodic coefficients. The comparison of the spectral element method to the time-domain methods of the machining literature shows that the spectral element method provides converged stability diagrams with smaller computational time. The critical characteristic multipliers of the matrix approximation of the monodromy operator of the governing equations converge faster with respect to the polynomial order, than that of the time-domain methods of the machining literature.

Related publication: [13]
I have analyzed the effect of three different digital feedback control mechanisms on the stability of milling operations. In particular, I have applied a digital control scheme to two existing models of milling processes subjected to feedback control. In addition, I have proposed a new mechanical model, in which the control loop of the drive of the workpiece holder is involved using the same digital control scheme. This control scheme considers the sampling and actuation periods separately and assumes piecewise constant control force. The stability analysis of these systems with fixed sampling period has been carried out for the first time. The results are summarized as follows.

**Thesis 5**

![Figure 1: Model of milling operations with digitally controlled workpiece holder](image)

The mechanical model shown in Figure 1 can be used to analyze the effect of the feed drive of the workpiece holder on the stability of milling operations. In the figure, \( m_t \), \( c \) and \( k \) are the modal mass, damping and stiffness of the tool, respectively, whose displacement is measured by \( x_1 \). The spindle speed is \( \Omega \), the horizontal cutting force component is \( F_c \), the mass of the workpiece holder together with the workpiece is \( m_w \), whose displacement is measured by \( x_2 \). The control force is \( Q \), the desired position of the workpiece holder is \( x_d \), the sampling period is \( \Delta t \) and the actuation period is \( \Delta T = \tilde{n}\Delta t \), where \( \tilde{n} \in \mathbb{Z}^+ \) is the number of samples per
actuation period. With the application of a digital proportional-integral-differential controller with piecewise constant actuation and numerical integration, the stability lobe diagrams can be determined for this model, for the model of milling operations subjected to active damping and for the model which incorporates the control loop of the feed drive of the tool holder. Depending on the selection of control parameters, these stability diagrams show significant differences from that of the corresponding standard models of the machining literature.

Related publications: [8, 10]
I have investigated the effect of active damper on the stability and stabilizability of turning operations. The active damper was assumed to be controlled by a digital proportional-differential feedback controller, which generates a piecewise constant force acting on the tool. The results are summarized as follows.

**Thesis 6**

*In the mechanical model of turning operations subjected to active damping, the material removal rate can be increased by the proper tuning of the control gains of the active damper. The proper tuning of the control parameters can be carried out using stability diagrams. In case of an active damper controlled by a digital proportional-differential feedback loop with piecewise constant control force, the omission of the delay, caused by sampling, and the piecewise constant nature of the control force can lead to significant differences in the stability diagrams. The sampling effect of the digital controller limits the maximum achievable material removal rate. This limitation is captured by the stabilizability diagram which shows the maximum achievable specific cutting force coefficient versus the spindle speed.*

Related publications: [5, 6, 11, 12]
I have modeled two human balancing tasks: stick balancing and quiet stance. The balancing activity of humans was considered by a proportional-integral-differential-acceleration delayed feedback controller for both models. Stabilizability diagrams were determined where the loss of stabilizability is associated with the loss of balance of humans. I carried out comparison between the calculated and the experimental results.

**Thesis 7**

By modeling the human balancing process with a proportional-integral-differential-acceleration (PIDA) delayed feedback controller in stick balancing and quiet stance, stabilizability diagrams can be computed. These diagrams present the domain of system and control parameters, for which the unstable equilibrium of the open-loop system becomes stable in the closed-loop system. The comparison of the results to those of the literature shows, that in case of the quiet stance model, there always exists a stabilizing set of control parameters. In contrast, for the stick balancing model, there always exists a critical stick length below which the stick cannot be stabilized for any set of control parameters. The integral gain of the control loop cannot improve the stabilizability properties of the investigated models of the balancing tasks.

Related publications: [4]
Bibliography


