Budapest University of Technology and Economics
Institute of Mathematics
Department of Algebra

Some questions on the representations of finite groups

Depth, Vanishing properties, Expansiveness

PhD thesis - Outline

Franciska Petényi

Supervisor: Dr. Erzsébet Horváth

2016

Contents

Chapter 1

Introduction

1.1 Depth

The concept of ordinary depth has its origins in von-Neumann algebras, see [19]. Later depth was studied in the fields of Hopf algebras and Frobenius algebras. The investigation of depth for group algebras started around 2008. First only the notion of depth 1 and 2 was defined in [6] and in [5], respectively. Let us illustrate, what depth 1 and 2 mean for complex group algebras. Let H be a subgroup of G. We say that the group algebra $\mathbb{C}H$ has depth 1 in $\mathbb{C}G$ ($d(\mathbb{C}H, \mathbb{C}G) = 1$) if $\mathbb{C}G$ is a direct summand of a direct power of $\mathbb{C}H$ as $\mathbb{C}H - \mathbb{C}H$-bimodule. Further, $\mathbb{C}H$ has depth 2 in $\mathbb{C}G$, if $\mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}G$ is a direct summand of a direct power of $\mathbb{C}G$ as $\mathbb{C}G$-$\mathbb{C}H$-bimodule. It was very interesting to see that the depth 2 property can be characterized in an algebraic way, namely it is equivalent to H being normal in G. For the definition of depth for general $n \in \mathbb{N}$, see Chapter 2. The *ordinary depth* $d(H, G)$ of the group inclusion $H \leqslant G$ is the depth of the group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$.
In [8] the notion of *combinatorial depth* $d_c(H, G)$ of the group inclusion $H \leqslant G$ was defined. It was shown that $d(RH, RG) \leq d_c(H, G) \leq 2|G : N_G(H)|$ for every finite group G and for every nontrivial commutative ring R. Thus for a finite group, the combinatorial depth is always finite and it is an upper bound for the ordinary depth. This area in intensively studied, both in the group algebra case and in the general context, see [4], [5], [6], [8], [7], [11], [14], [15], [16], [22], [29], [30] and [31]. We will study, whether the depth of some subgroups in a class of groups can be bounded by some constant independent of the size of the group. In [4, Question. 5.4, Ex. 5.5, Prop. 5.6, Rem. 5.7] Boltje, Danz and Külshammer showed that

$$d_c(\times^{p-1}C_2 \ , \ C_2 \wr C_p) = d(\times^{p-1}C_2 \ , \ C_2 \wr C_p) = 2p - 1$$

for every prime p, on the other hand $d_c(\times^{p-1}C_2 \ , \ \times^p C_2) = d(\times^{p-1}C_2 \ , \ \times^p C_2) = 1$ and $d_c(\times^p C_2 \ , \ C_2 \wr C_p) = d(\times^p C_2 \ , \ C_2 \wr C_p) = 2$. Thus the depth does not have transitive property. In these examples some maximal subgroups have constant depth independent of the group order, but not the non-maximal ones. However, in some other examples also non-maximal subgroups can have bounded depth independent of the group order. For example if H is an abelian Sylow subgroup in G, then by Brodkey's Theorem there exists an element $g \in G$ such that $Core_G(H) = H \cap H^g$. Hence $d(H, G) \leq 4$ by Theorem 2.1. Further examples for subgroups with bounded depth can be the normal subgroups (depth 2) and TI subgroups (depth 3).

Our next observation is that in a certain class of group not every maximal subgroup has bounded depth, which is independent of the size of the group. Consider S_n, which is maximal in S_{n+1} and $d_c(S_n, S_{n+1}) = d(S_n, S_{n+1}) = 2n - 1$ (see [4, Prop. 5.1]). Beside [4], the depth in symmetric groups was examined in [11] and in [14].

On the other hand, Fritzshe proved that the depth of the maximal subgroups in $PSL_2(q)$ is at most 5 ([15, Thm. 2.22]). Accordingly, we choose as a topic for our investigation the Suzuki groups $Sz(q)$, which are also Zassenhaus groups, similarly to $PSL_2(q)$ (for even q). We determined the ordinary and combinatorial depth of the maximal subgroups of Suzuki groups $Sz(q)$, (see Theorem 2.4 and Corollary 2.5), which turned out to be independent of the parameter q. Weakening the definition of Zassenhaus groups leads to a wider class, namely to the class of Ree type groups, including Ree groups ${}^2G_2(q)$. As in the previous case we determined the ordinary and combinatorial depth of the maximal subgroups. These are again independent of the parameter q. For more details see Theorem 2.7.

1.2 Vanishing properties of characters

We say that an irreducible character of a finite group vanishes on a conjugacy class or on an element, if it has zero value on it. In the literature zeros of characters are widely investigated, see e.g. [1], [3], [10], [12], [13], [28], [17], [18], [27], [36], [40], [41], [42], [45], [50], [51], [52], [53], and [54]. Our aim was to compare three lower bounds on the number of vanishing elements in a group given by Gallagher (Thm. 3.1), Chillag (Thm. 3.3) and Wilde (Thm. 3.7). Further, we investigated a conjecture by Wilde (Conj. 3.6). Since some of the estimates were originally formulated for conjugacy classes, from the above mentioned bounds we formulated three estimates for conjugacy classes and three estimates for group elements, respectively.

We introduced for these estimates the following notation: the upper index ($\mathfrak{G}$ or $\mathfrak{C}$) indicates if the estimate is for group elements or for conjugacy classes, respectively. The lower index is one of Ga, Ch, W, which refers to the name of the person whose theorem is used.

We proved that the estimates for vanishing group elements: $E^{\mathfrak{G}}_{Ga}$, $E^{\mathfrak{G}}_{Ch}$, $E^{\mathfrak{G}}_{W}$ and for vanishing conjugacy classes $E^{\mathfrak{C}}_{Ga}$, $E^{\mathfrak{C}}_{Ch}$, $E^{\mathfrak{C}}_{W}$ have the property that for each ordered pair from both sets there is a finite group having an irreducible character such that the second estimate is better than the first one. We determined the smallest such group, see Theorem 3.9.

Wilde published a Conjecture (see 3.6), which says that if the order of $g \in G$ does not divide $|G|/\chi(1)$ for some $\chi \in \mathrm{Irr}(G)$, then $\chi(g) = 0$. He proved that for solvable groups, and he also mentioned it is true for the symmetric groups. We proved that this Conjecture holds for symmetric groups (for the sake of selfcontainedness), for Suzuki groups (Thm 3.21) and for Ree groups (Thm 3.23). Since in these special cases the above results are stronger than Wilde's theorem, we may use them to get further estimates, denoted by $E^{\mathfrak{C}}_{Co}$ and by $E^{\mathfrak{G}}_{Co}$. We remark that for the Suzuki groups the converse of the Conjecture also holds, however for Ree groups the converse is false.

In Section 3.1 we consider symmetric groups. We examine which estimate is better than the other for infinitely many irreducible characters. Our main Theorems are 3.18 and 3.19.

In Sections 3.2, 3.3 we present the character tables of Suzuki and Ree groups and compute both the exact numbers and the estimates for the number of vanishing conjugacy classes and vanishing group elements. We have found that for these groups the estimates from the Conjecture are the best for every irreducible character.

1.3 Expansiveness

We say that a finite group G is conjugacy expansive if for any normal subset S and any conjugacy class C of G the normal set SC consists of at least as many conjugacy classes of G as S does. Halasi, Maróti, Sidki and Bezerra have shown that a group is conjugacy expansive if and only if it is a direct product of conjugacy expansive simple or abelian groups.

By considering a character analogue of the above, we say that a finite group G is *character expansive* if for any complex character α and irreducible character χ of G the character $\alpha\chi$ has at least as many irreducible constituents, counting without multiplicity, as α does. We take some initial steps in determining character expansive groups. Our conjecture was that a group is character expansive, if and only if it is a direct product of abelian or character expansive simple groups. One direction we showed in Theorem 4.3, namely the direct product of abelian or character expansive simple groups are character expansive. For the other direction we showed that a minimal counterexample, i.e. a minimal character expansive group, which is not a direct product of abelian or character expansive simple groups, has a unique minimal normal subgroups, which is abelian and noncentral, see Theorem 4.5.
As the corollary of Theorem 4.1 we get that the symmetric groups ($n > 4$) are not character expansive. Furthermore, using Mathematica programs ([38]) we showed that the Suzuki groups and the Ree groups are character expansive. The exact program codes one can find in [43].

Chapter 2

Depth

We say that the *depth of the group algebra extension* $\mathbb{C}H \subseteq \mathbb{C}G$ is $2n$ if $\mathbb{C}G \otimes_{\mathbb{C}H} \cdots \otimes_{\mathbb{C}H} \mathbb{C}G$ ($n+1$-times $\mathbb{C}G$) is isomorphic to a direct summand of $\oplus_{i=1}^{a} \mathbb{C}G \otimes_{\mathbb{C}H} \cdots \otimes_{\mathbb{C}H} \mathbb{C}G$ (n times $\mathbb{C}G$) as $\mathbb{C}G - \mathbb{C}H$-bimodules for some positive integer a. Furthermore $\mathbb{C}H$ is said to have depth $2n+1$ in $\mathbb{C}G$ if the same assertion holds for $\mathbb{C}H - \mathbb{C}H$-bimodules. Finally $\mathbb{C}H$ has depth 1 in $\mathbb{C}G$ if $\mathbb{C}G$ is isomorphic to a direct summand of $\oplus_{i=1}^{a} \mathbb{C}H$ as $\mathbb{C}H - \mathbb{C}H$ bimodules.
The depth is an ascending property, namely if the group algebra $\mathbb{C}H \leqslant \mathbb{C}G$ has depth n, then it also has depth $n+1$, $n+2$ and so on. Accordingly, in the literature depth is mostly identified with the minimal depth and further we also will use this concept.
The depth of a group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$ is called the *ordinary depth* of H in G, which we denote by $d(H,G)$. This is well defined.

The corollary of the following theorem is that the subgroup, which is a TI set in G, has depth 3 in G.

Theorem 2.1. *[8, Thm 6.9] Suppose that H is a subgroup of a finite group G and $N = Core_G(H)$ is the intersection of m conjugates of H. Then H has ordinary depth $\leq 2m$ in G. If $N \leq Z(G)$ then $d(H,G) \leq 2m-1$.*

The notion of *combinatorial depth* was defined in [4]. We remind the reader to the basic definitions, results and notation on combinatorial depth. Let G be a finite group, H a subgroup of G. To define the combinatorial depth of H in G, we need some facts about bisets.

Let J, K, L be finite groups, let X be a (J,K)-biset, let Y be a (K,L)-biset. Then $X \times Y$ will be a (J,L)-biset. The group K also acts on $X \times Y$ by $k \cdot (x,y) := (xk^{-1}, ky)$ for $x \in X, y \in Y, k \in K$. The set of K-orbits of this action is denoted by $X \times_K Y$. This set also inherits a (J,L)-biset structure. Let $\Theta_1(H,G)$ be the (G,G)-biset G, using left and right multiplication of G on itself. Let $\Theta_{i+1}(H,G) := \Theta_i(H,G) \times_H G$ for $i \geq 1$. We denote by $\Theta'_i(H,G)$ the set $\Theta_i(H,G)$ considered as an (H,H)-biset. We denote by $\Theta^l_i(H,G)$ and $\Theta^r_i(H,G)$, the set $\Theta_i(H,G)$ considered as an (H,G)-biset or (G,H)-biset, respectively. Furthermore, $\Theta'_0(H,G) := H$, as an (H,H)-biset. The subgroup H is said to be of combinatorial depth $2i$ in G if natural numbers a_1, a_2 exist such that $\Theta^r_{i+1}(H,G)$ is a direct summand of $a_1 \cdot \Theta^r_i(H,G)$ for some $i \geq 1$ and $\Theta^l_{i+1}(H,G)$ is a direct summand of $a_2 \cdot \Theta^l_i(H,G)$, respectively. Moreover H has combinatorial depth $2i+1$ if ${\Theta'}_{i+1}(H,G)$ is a direct summand of $a \cdot {\Theta'}_i(H,G)$ for some natural number a. The minimal combinatorial depth $d_c(H,G)$ is the smallest positive integer i such that H has combinatorial depth i in G. This number is well defined.

Both for combinatorial and ordinary depth, $d_c(H,G) \leq 2$ (or $d(H,G) \leq 2$) if and only if H is normal in G. By [4, Remark 4.5], the ordinary depth does not change if we replace the complex number field by any field of characteristic zero, and the ordinary depth $d(H,G)$ is bounded from above by the combinatorial depth $d_c(H,G)$.
Let us denote by $H^x := x^{-1}Hx$ and $H^{(x)} = H \cap H^x$, furthermore $H^{(x_1,\ldots,x_n)} := H \cap H^{x_1} \cap \ldots \cap H^{x_n}$, for $x, x_1, \ldots, x_n \in G$. Let $\mathcal{U}_i := \mathcal{U}_i(H,G) := \{H^{(x_1,\ldots,x_i)} | x_1, \ldots, x_i \in G\}$ and $\mathcal{U}_\infty := \mathcal{U}_\infty(H,G) := \cup_{i \geq 0} \mathcal{U}_i$, where $\mathcal{U}_0 := \{H\}$. In [4] the following characterization of combinatorial depth $d_c(H,G)$ is proved:

Theorem 2.2. *[4, Thm 3.9] Let H be a subgroup of the finite group G and let $i \geq 1$. Then:*

(i) $d_c(H,G) \leq 2i \Leftrightarrow \mathcal{U}_{i-1} = \mathcal{U}_i \Leftrightarrow \mathcal{U}_{i-1} = \mathcal{U}_\infty \Leftrightarrow$ for every $x_1,\ldots,x_i \in G$, there exist some $y_1,\ldots,y_{i-1} \in G$ with $H^{(x_1,\ldots,x_i)} = H^{(y_1,\ldots,y_{i-1})}$.

(ii) Let $i > 1$. Then $d_c(H,G) \leq 2i-1 \Leftrightarrow$ for every $x_1,\ldots,x_i \in G$ there exist some $y_1,\ldots,y_{i-1} \in G$ with $H^{(x_1,\ldots,x_i)} = H^{(y_1,\ldots,y_{i-1})}$ and $x_1hx_1^{-1} = y_1hy_1^{-1}$ for all $h \in H^{(x_1,\ldots,x_i)}$.

(iii) $d_c(H,G) = 1 \Leftrightarrow$ for every $x \in G$ there exists some $y \in H$ with $xhx^{-1} = yhy^{-1}$ for all $h \in H \Leftrightarrow G = HC_G(H)$.

2.1 Depth in Suzuki groups

Zassenhaus groups are doubly transitive permutation groups without any regular normal subgroups, where any non-identity element has at most two fixed points, and there are elements with exactly two fixed points. Accordingly to a well known of Feit, see [25, Thm. 6.1 in Ch. XI], the degree of a Zassenhaus group is always $q_1 + 1$, where q_1 is a prime power. For even q_1, there are two series of such groups, $PSL(2, 2^m)$ and $Sz(2^{2m+1})$. Suzuki proved that these are all the Zassenhaus groups for even q_1. See e. g. [46, 47]. The Suzuki groups can also be defined as subgroups of $GL(4, q)$, see e.g. [25, Ch XI. p. 182] and these are also twisted simple groups of Lie type ${}^2B_2(q)$ for $q \geq 8$. The group $Sz(2)$ is not simple, it is a Frobenius group of order 20. Suzuki also determined the subgroups of the Suzuki group.

Theorem 2.3 (Suzuki)**.** *Let $G = Sz(q)$, where $q = 2^{2m+1}(= 2r^2)$, for some positive integer m. Then G has the following subgroups:*

1. *the Hall subgroup $N_G(F) = FH$, which is a Frobenius group of order $q^2(q-1)$.*
2. *the dihedral group $B_0 = N_G(H)$ of order $2(q-1)$.*
3. *the cyclic Hall subgroups A_1, A_2 of orders $q + 2r + 1, q - 2r + 1$, respectively, where $r = 2^m$ and $|A_1||A_2| = q^2 + 1$.*
4. *the Frobenius subgroups $B_1 = N_G(A_1), B_2 = N_G(A_2)$ of orders $4|A_1|, 4|A_2|$, respectively.*
5. *the subgroups of the form $Sz(s)$, where s is an odd power of 2, $s \geq 8$, and $q = s^n$ for some positive integer n. Moreover, for every odd 2-power s, where $s^n = q$ for some positive integer n, there exists a subgroup isomorphic to $Sz(s)$.*
6. *Subgroups (and their conjugates) of the above groups.*

Using series of Proposition we determined the combinatorial and the ordinary depth of the maximal subgroups in a Suzuki group.

Theorem 2.4 (Héthelyi, Horváth, P [24, Thm. 4.1])**.** *Let us list the representatives of conjugacy classes of the maximal subgroups of the Suzuki group $G = Sz(q)$. By Theorem 2.3 these are the following: $N_G(F)$, $B_0 = N_G(H)$, $B_1 = N_G(A_1)$, $B_2 = N_G(A_2)$ and $G_1 \simeq Sz(s)$ for maximal s such that $s^t = q$ for some positive integer $t > 1$. The combinatorial depths of these subgroups are $d_c(N_G(F), G) = 5$, $d_c(B_0, G) = 4$, $d_c(B_1, G) = 4$, $d_c(B_2, G) = 4$, and $d_c(G_1, G) = 4$.*

Corollary 2.5 (Héthelyi, Horváth, P [24, Cor. 5.1])**.** *The ordinary depth of all subgroups of $G = Sz(q)$ mentioned in Theorem 2.4 is 3 except for $d(N_G(F), G)$ which is 5.*

2.2 Depth in Ree groups

Let $q = 3^{2n+1}$, where $n > 0$. Ree groups can be defined in various ways. These are twisted simple groups of Lie type ${}^2G_2(q)$, see [9, p. 43]. Here we present the maximal subgroups of G. For further information about Ree groups, see [25, XI. 13.2],[32], [34],[35],[49] and [55][Lem. 2, Lem. 3]

Theorem 2.6 (Kleidman). *Let $G = R(q)$ be a Ree group with the parameter $q = 3^{2n+1} = 3m^2 > 3$. Then G has the following maximal subgroups:*

1. *the subgroup $N_G(P)$, where P is a 3-Sylow subgroup;*
2. *the subgroups $N_G(M^{\pm 1})$, where $M^{\pm 1} \in Hall_{q \pm 3m+1}$;*
3. *the subgroup $N_G(M)$, where $M \in Hall_{(q+1)/4}$;*
4. *the centralizer of an involution $C_G(i)$*
5. *the subgroups of form $R(q_0)$, where $q_0^p = q$ and p prime;*
6. *conjugates of the above groups.*

Using series of Propositions and Lemmas we determined the combinatorial and ordinary depth of the maximal subgroups in a Ree group.

Theorem 2.7 (Héthelyi, Horváth, P [23]). *Let G be a Ree group $R(q)$, where $q = 3^{2n+1}$ and $n \geq 1$. The ordinary and combinatorial depths of the maximal subgroups in G are the following:*

- $d_c(N_G(P), G) = d(N_G(P), G) = 5$,
- $d_c(N_G(M^1), G) = d_c(N_G(M^{-1}), G) = 4$,
- $d(N_G(M^1), G) = d(N_G(M^{-1}), G) = 3$,
- $d_c(N_G(M), G) = d_c(C_G(i), G) = 6$,
- $d(N_G(M), G) = d(C_G(i), G) = 3$, *and*
- $d_c(G_0, G) = 4$, $d(G_0, G) = 3$.

Chapter 3

Vanishing properties of characters

Let G be a finite group and χ one of its complex irreducible characters. We say that an element g is *vanishing* with respect to χ, if $\chi(g) = 0$. Our aim is to investigate and compare lower estimates for the number of vanishing conjugacy classes or elements. The first well-known result is due to *Burnside* (see [26, Thm. 3.15]), which states that every nonlinear irreducible character of a finite group vanishes on some element of the group. We constructed three estimates for the number of vanishing conjugacy classes and three for the number of vanishing elements of a fixed irreducible character of the group, respectively. We used the results of *Chillag 3.3, Gallagher 3.1, and Wilde 3.7* to construct these lower bounds.

We introduce the following notations. Denote by $n^{\mathfrak{G}}(\chi)$ and $n^{\mathfrak{C}}(\chi)$ the number of vanishing group elements and conjugacy classes of χ, respectively, i.e.,

$$n^{\mathfrak{G}}(\chi) = |\{g \in G \mid \chi(g) = 0\}| \text{ and } n^{\mathfrak{C}}(\chi) = |\{C \in \mathrm{Cl}(G) \mid \chi|_C = 0\}|.$$

Gallagher improved the result of Burnside as follows; see [26, Thm. 3.15].

Theorem 3.1 (Gallagher [17, Thm. 4])**.** *For every irreducible character χ of a finite group, we have*

$$n^{\mathfrak{G}}(\chi) \geqslant (\chi(1)^2 - 1)|Z(\chi)|,$$

where $Z(\chi)$ is the center of χ. If $|\chi|$ takes only the values 0, 1, and $\chi(1)$, then equality holds.

Let us introduce the notation $E^{\mathfrak{G}}_{Ga}(\chi) = (\chi(1)^2 - 1)|Z(\chi)|$.

Remark 3.2. *There are infinitely many groups where $n^{\mathfrak{G}}(\chi) = E^{\mathfrak{G}}_{Ga}(\chi)$ for some nonlinear irreducible character χ. An example is the case of groups of nilpotency class* 2*, see [26, Cor. 2.30, Thm. 2.31].*

Let us recall the following result.

Theorem 3.3 (Chillag [10, Thm. 1.1])**.** *Let G be a finite non-abelian group such that $G \neq G'$ and χ a non-linear irreducible character of G. Then one of the following holds:*

(i) G has an element x such that $|C_G(x)| \leqslant 2n^C(\chi)$. In fact, one can choose such an x from any conjugacy class of maximal size among the classes on which χ vanishes.

(ii) $\phi = \chi_{G'} \in \mathrm{Irr}(G')$ and $\phi^G = \chi \sum\{\lambda \mid \lambda \in \mathrm{Irr}(G),\ \lambda(1) = 1\}$. Furthermore the set $\{\chi\lambda \mid \lambda \in \mathrm{Lin}(G)\}$ consists of exactly $|G : G'|$ extensions of ϕ, which are all the extensions of ϕ. In particular, ϕ is not linear and $G'' \neq 1$.

We can use this theorem to give a lower bound for $n^{\mathfrak{C}}(\chi)$. This lower bound will be denoted by $E^{\mathfrak{C}}_{Ch}(\chi)$, which is given by

$$E^{\mathfrak{C}}_{Ch}(\chi) = \begin{cases} 0, & \text{if } \chi \text{ linear,} \\ \min_{\{x\in G \mid \chi(x)=0\}} \lceil |C_G(x)|/2 \rceil, & \text{if } G \neq G' \text{ and } \chi_{G'} \notin \mathrm{Irr}(G'), \\ 1, & \text{otherwise.} \end{cases}$$

Let us consider the case where $E^{\mathfrak{C}}_{Ch}(\chi) = 1$.

Proposition 3.4 (P [44, Prop. 2.4])**.** *Let G be a finite group with $G \neq G'$, and let $\chi \in \mathrm{Irr}(G)$ be nonlinear and $\chi_{G'} \notin \mathrm{Irr}(G')$. Let $E^{\mathfrak{C}}_{Ch}(\chi) = 1$. Then $n^{\mathfrak{G}}(\chi) = \frac{|G|}{2}$ and $n^{\mathfrak{C}}(\chi) = 1$. Moreover, G is a Frobenius group with a complement of order 2.*

Suppose we know the sizes of the conjugacy classes of G. Then, using $E^{\mathfrak{G}}_{Ga}(\chi)$, we can construct another lower bound for $n^{\mathfrak{C}}(\chi)$. This lower bound will be denoted by

$$E^{\mathfrak{C}}_{Ga}(\chi) = \min\{|J| \mid J \subseteq I, \sum_{j\in J} |C_j| \geqslant E^{\mathfrak{G}}_{Ga}(\chi)\},$$

where $\mathrm{Cl}(G) = \{C_i\}_{i\in I}$.

Similarly, from $E^{\mathfrak{C}}_{Ch}(\chi)$ we can construct a lower estimate $E^{\mathfrak{G}}_{Ch}(\chi)$ of $n^{\mathfrak{G}}(\chi)$. If we use the result of Proposition 3.4, as well as the fact that χ cannot vanish on $Z(G)$, then we obtain

$$E^{\mathfrak{G}}_{Ch}(\chi) = \begin{cases} 0, & \text{if } \chi \text{ is linear,} \\ \dfrac{|G|}{2}, & \text{if } E^{\mathfrak{C}}_{Ch}(\chi) = 1,\ G \neq G', \\ & \quad \text{and } \chi_{G'} \notin \mathrm{Irr}(G'), \\ \min_{J\subseteq I}\{\sum_{j\in J} |C_i| \mid & \\ \quad C_j \in \mathrm{Cl}(G),\ C_j \nsubseteq Z(G) \text{ and } |J| = E^{\mathfrak{C}}_{Ch}(\chi)\}, & \text{otherwise.} \end{cases}$$

We remark that, if $\chi_{G'} \in \mathrm{Irr}(G')$, then $E^{\mathfrak{G}}_{Ch}(\chi)$ is the size of the smallest non-central conjugacy class.

Before formulating the lower bound by Wilde, we mention the following fact.

Theorem 3.5 (Brauer, Nesbitt [26, Thm. 8.17])**.** *Let $\chi \in \mathrm{Irr}(G)$ and suppose $p \nmid \frac{|G|}{\chi(1)}$ for some prime p. Then $\chi(g) = 0$ whenever $p \mid o(g)$.*

This is equivalent to saying that, whenever $\chi(g) \neq 0$, then the squarefree part $o(g)_0$ of $o(g)$ divides $\frac{|G|}{\chi(1)}$.

In [50, Conj. 1.1], Wilde proposed the following conjecture.

Conjecture 3.6 (Wilde)**.** *Let G be a finite group. Furthermore, let $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and suppose that $\chi(g) \neq 0$. Then $o(g)$ divides $\frac{|G|}{\chi(1)}$.*

Wilde proved in [50] that Conjecture 3.6 holds for solvable groups, and he mentioned that it is true also for S_n, which we proved for the sake of selfcontainedness. We also showed that the Conjecture is true for Suzuki groups and Ree groups, see Proposition 3.21 and Proposition 3.23.

Consequently we can use Burnside's Theorem (see [26, Thm. 3.15]) and this Conjecture to give two other lower estimates for $n^{\mathfrak{G}}(\chi)$ and $n^{\mathfrak{C}}(\chi)$, for a nonlinear, irreducible character χ of a symmetric group or a Suzuki group or a Ree group:

$$E^{\mathfrak{G}}_{Co}(\chi) = \max\left\{\left|\left\{g \in G \mid o(g) \nmid \frac{|G|}{\chi(1)}\right\}\right|, \min_{C\in \mathrm{Cl}(G), |C|\neq 1} |C|\right\},$$

$$E^{\mathfrak{C}}_{Co}(\chi) = \max\left\{\left|\left\{C \in \mathrm{Cl}(G) \mid o(g) \nmid \frac{|G|}{\chi(1)} \text{ for all } g \in C\right\}\right|, 1\right\}.$$

For linear characters, we define $E^{\mathfrak{G}}_{Co}(\chi) = E^{\mathfrak{C}}_{Co}(\chi) = 0$.

Wilde proved the following partial result.

Theorem 3.7 (Wilde [50, Thm. 2.1])**.** *Let χ be an irreducible character of G and $g \in G$ a group element with $\chi(g) \neq 0$. Then $o(g)o(g)_0 \mid (\frac{|G|}{\chi(1)})^2$ and $o(g)^3 \mid \frac{|G|^3}{\chi(1)^2}$.*

We use Theorem 3.7 and Burnside's theorem, see [26, Thm. 3.15], to give lower estimates for $n^{\mathfrak{G}}(\chi)$ and $n^{\mathfrak{C}}(\chi)$, which we will denote by $E^{\mathfrak{G}}_{W}(\chi)$, $E^{\mathfrak{C}}_{W}(\chi)$, respectively:

$$E^{\mathfrak{G}}_{W}(\chi) = \max\left\{\left|\left\{g \in G \mid o(g)o(g)_0 \nmid \frac{|G|^2}{\chi(1)^2} \text{ or } o(g)^3 \nmid \frac{|G|^3|}{\chi(1)^2}\right\}\right|, \min_{C\in \mathrm{Cl}(G), C\nsubseteq Z(G)} |C|\right\},$$

$$E^{\mathfrak{C}}_{W}(\chi) = \max\left\{\left|\left\{C \in \mathrm{Cl}(G) \mid o(g)o(g)_0 \nmid \frac{|G|^2}{\chi(1)^2} \text{ or } o(g)^3 \nmid \frac{|G|^3}{\chi(1)^2} \text{ for all } g \in C\right\}\right|, 1\right\}.$$

Of course, the inequalities $E^{\mathfrak{G}}_{Co}(\chi) \geq E^{\mathfrak{G}}_{W}(\chi)$ and $E^{\mathfrak{C}}_{Co}(\chi) \geq E^{\mathfrak{C}}_{W}(\chi)$ always hold. Thus we use $E^{\mathfrak{C}}_{Co}(\,), E^{\mathfrak{G}}_{Co}(\,)$ instead of $E^{\mathfrak{C}}_{W}(\,), E^{\mathfrak{G}}_{W}(\,)$ for symmetric groups, for Suzuki groups and for Ree groups.

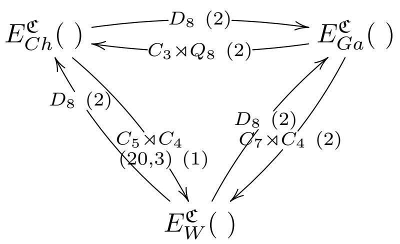
We know that Conjecture 3.6 is true for p-groups. In fact one can prove more.

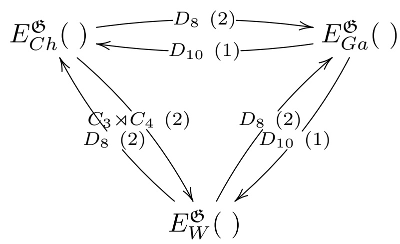
Proposition 3.8 (P [44, Prop. 2.8])**.** *If G is a p-group, then for every $g \in G$ and for every $\chi \in \mathrm{Irr}(G)$, we have $o(g) \mid \frac{|G|}{\chi(1)}$. In particular, for every $\chi \in \mathrm{Irr}(G)$, we have $E^{\mathfrak{C}}_{W}(\chi) = 1$ and $E^{\mathfrak{G}}_{W}(\chi) = \min_{C\in \mathrm{Cl}(G), C\nsubseteq Z(G)} |C|$.*

For both $n^{\mathfrak{G}}(\chi)$ and $n^{\mathfrak{C}}(\chi)$ we have the following results on the above lower bounds. We checked the examples with the help of the GAP system [48], see the used programs in [43].

Theorem 3.9 (P [44, Thm. 2.9])**.** *For each ordered pair of the above estimates on conjugacy classes $\{E^{\mathfrak{C}}_{Ch}(\,), E^{\mathfrak{C}}_{Ga}(\,), E^{\mathfrak{C}}_{W}(\,)\}$ and group elements $\{E^{\mathfrak{G}}_{Ch}(\,), E^{\mathfrak{G}}_{Ga}(\,), E^{\mathfrak{G}}_{W}(\,)\}$, there exists a group and an irreducible character of that group such that the first estimate is better than the second.*

We demonstrate the result using the following diagrams:





Here, an arrow points to the estimate which is better on some irreducible character of the group corresponding to it. We labelled the arrows with the smallest possible groups having suitable character. When the semidirect product is not unique, we included the group ID of the GAP system. In brackets you can find the size of the centre of the character. With this data, the group and set of suitable characters are uniquely determined.

3.1 Estimates for symmetric groups

Let consider the estimates of the previous section for symmetric groups.

Lemma 3.10 (P [44, Lem. 3.1]). *Let χ be an irreducible character of the symmetric group S_n for $n \geq 5$. Then $E^{\mathfrak{G}}_{Ga}(\chi) = \chi(1)^2 - 1$. Thus $n^{\mathfrak{G}}(\chi) \geq \chi(1)^2 - 1$.*

Lemma 3.11 (P [44, Lem. 3.2]). *Let $\chi \in \mathrm{Irr}(S_n)$ be the character corresponding to the partition λ. Then $E^{\mathfrak{C}}_{Ch}(\chi) = 1$ if and only if λ is not symmetric and χ is not linear, or $n = 3$ and χ is corresponding to the partition $(2,1)$.*

Lemma 3.12 (P [44, Lem. 3.4]). *If χ is an irreducible character of S_n corresponding to a symmetric partition, then $E^{\mathfrak{C}}_{Ch}(\chi) = \lceil \frac{n-1}{2} \rceil$.*

Lemma 3.13 (P [44, Lem. 3.5]). *If $\chi \in \mathrm{Irr}(S_n)$ and $n \geq 851$, then $E^{\mathfrak{C}}_{Ga}(\chi) \leq 1$.*

Moreover, we have the following conjecture.

Conjecture 3.14 (P [44, Conj. 3.6]). *Let χ be a non-linear irreducible character of a symmetric group. The above defined $E^{\mathfrak{C}}_{Ga}(\chi)$ is two exactly if χ corresponds to one of the following partitions:*

$$(3,1^2),\ (3,2,1),\ (3,2,1^2),\ (4,2,1),\ (4,2,1^2),\ (4,2^2,1),\ (4,3,1^2),$$
$$(4,3,2,1),\ (4,3,2,1^2),\ (5,3,2,1),\ (5,3,2,1,1),\ (6,4,3,2,1,1).$$

In all other cases, $E^{\mathfrak{C}}_{Ga}(\chi)$ is exactly 1.

Remark 3.15. *Using results of McKay in [39], we can see that this conjecture is true if $n \leq 75$. It is easy to check (for example with the help of GAP [48]) that the conjecture holds for 2-part and 3-part partitions. One can also prove by induction that the conjecture is true for hook-partitions. The conjecture shows for symmetric groups that Gallagher's estimate for conjugacy classes ($E^{\mathfrak{C}}_{Ga}(\ )$) is not really better than Burnside's, see [26, Thm. 3.15].*

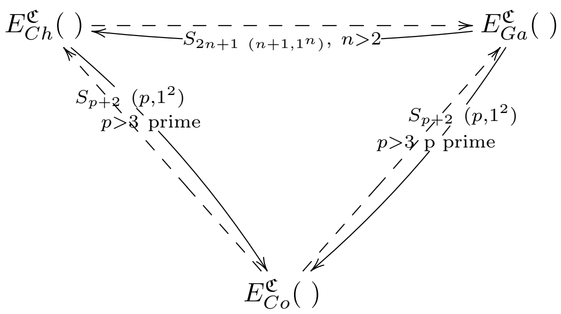
Proposition 3.16 (P [44, Lem. 3.8]).

1. *The minimal degrees of non-linear irreducible characters of S_n are $n-1$, if $n > 4$.*
2. *The minimal degrees of those non-linear irreducible characters of S_n which correspond to symmetric partitions are at least $2^{n/4}$.*

Corollary 3.17 (P [44, Cor. 3.9]). *If χ is a non-linear, irreducible character of S_n for $n > 4$, then $E^{\mathfrak{G}}_{Ga}(\chi) \geq (n-1)^2 - 1$ and, if χ corresponds to a symmetric partition, then $E^{\mathfrak{G}}_{Ga}(\chi_\lambda) \geq 2^{n/2} - 1$ for $n > 2$.*

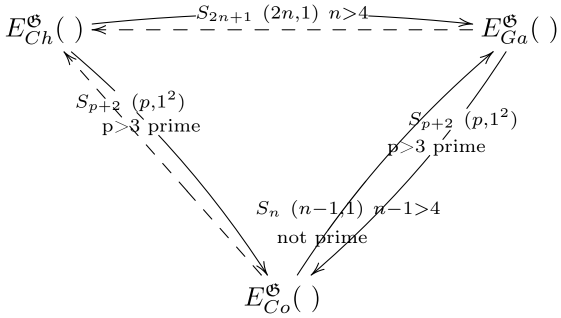
Theorem 3.18 (P [44, Thm. 4.1]). *For each ordered pair of elements of the set $\{E^{\mathfrak{C}}_{Ch}, E^{\mathfrak{C}}_{Ga}, E^{\mathfrak{C}}_{Co}\}$, except for the pairs $(E^{\mathfrak{C}}_{Ga}, E^{\mathfrak{C}}_{Co})$, $(E^{\mathfrak{C}}_{Ga}, E^{\mathfrak{C}}_{Ch})$, and $(E^{\mathfrak{C}}_{Ch}, E^{\mathfrak{C}}_{Co})$, there are infinitely many symmetric groups having a non-linear, irreducible character, with the property that the first estimate is better then the second. In the remaining cases, there are just finitely many symmetric groups with this property.*

We illustrate the theorem in the following diagram.



The broken arrow notation means that there are just finitely many symmetric groups where either $E^{\mathfrak{C}}_{Ch}(\chi) > E^{\mathfrak{C}}_{Co}(\chi)$ or $E^{\mathfrak{C}}_{Ga}(\chi) > E^{\mathfrak{C}}_{Ch}(\chi)$ or $E^{\mathfrak{C}}_{Ga}(\chi) > E^{\mathfrak{C}}_{Co}(\chi)$ for some irreducible character χ.
However, there are infinitely many symmetric groups having an irreducible character χ such that $E^{\mathfrak{C}}_{Ch}(\chi) = E^{\mathfrak{C}}_{Ga}(\chi) = E^{\mathfrak{C}}_{Co}(\chi)$. For example, if $n > 5$ with $n-1$ being not prime, and, if in addition χ corresponds to the partition $(n-1,1)$. In this case $E^{\mathfrak{C}}_{Ch}(\chi) = E^{\mathfrak{C}}_{Ga}(\chi) = E^{\mathfrak{C}}_{Co}(\chi) = 1$. In particular this value is smaller than $n^{\mathfrak{C}}(\chi)$.

Theorem 3.19. *[*P [44, Thm. 4.10]*] For each ordered pair from $\{E^{\mathfrak{G}}_{Ch}, E^{\mathfrak{G}}_{Ga}, E^{\mathfrak{G}}_{Co}\}$ there are infinitely many groups, having an irreducible character, for which the second estimate is better then the first, except for the pairs $(E^{\mathfrak{G}}_{Ga}, E^{\mathfrak{G}}_{Ch})$, $(E^{\mathfrak{G}}_{Co}, E^{\mathfrak{G}}_{Ch})$.*



3.2 Estimates for Suzuki groups

Let G be a group isomorphic to $Sz(q)$ for some $q = 2^{2n+1} = r^2/2$ and $n \in \mathbb{N}^+$. In the following we will describe the character table of G. For this we need some more notation. Let us remind the reader that the normalizers both of $A_1 = \langle y \rangle$ and of $A_2 = \langle z \rangle$ contain (at least) one element of order 4, denote them by s_1 and s_2, respectively. We know that $y^{s_1} = y^q$ and $z^{s_2} = z^q$. Thus we can consider an equivalence relation $\sim$ generated by $k \sim kq$ both on $\mathbb{Z}_{q+r+1}$ and on $\mathbb{Z}_{q-r+1}$. We remark that $q^2 \equiv -1 \bmod q \pm r + 1$ and hence we can consider this equivalence relation as an equivalence relation on $\mathbb{Z}_{q^2+1}$. In $\mathbb{Z}_{q+r+1}$ for example, there are $\frac{q+r}{4} + 1$ equivalence classes, these are $\{1\}$, $\{2, 2q, -2, -2q\}$, $\{3, 3q, -3, -3q\}$, ….

Theorem 3.20 (Suzuki[46, Thm. 13] and [37])**.** *Let x, y, z, f and t be elements of G of order $q-1$, $q+r+1$, $q-r+1$, 4 and 2, respectively. Moreover ω, ζ and θ be primitive $q-1$-st, $q+r+1$-st and*

$q-r+1$-st roots of unity, respectively. Then the character table of G is as follows:

	1	x^a	y^b	z^c	t	f	f^{-1}
$\mathbb{1}$	1	1	1	1	1	1	1
α	q^2	1	-1	-1	0	0	0
β_1	$r(q-1)/2$	0	1	-1	$-r/2$	$ri/2$	$-ri/2$
β_2	$r(q-1)/2$	0	1	-1	$-r/2$	$-ri/2$	$ri/2$
γ_j	q^2+1	$\omega^{ja}+\omega^{-ja}$	0	0	1	1	1
δ_k	$(q-1)(q-r+1)$	0	$-\psi_k^b$	0	$r-1$	-1	-1
η_l	$(q-1)(q+r+1)$	0	0	$-\psi_l^c$	$-r-1$	-1	-1

,

where $a,j \in \mathbb{Z}^*_{(q-2)/2}$, $b,k \in \mathbb{Z}^*_{q+r+1}/\sim$, $c,l \in \mathbb{Z}^*_{q-r+1}/\sim$, i is the imaginary unit and $\psi_k^b = \zeta^{kb}+\zeta^{-kb}+\zeta^{qkb}+\zeta^{-qkb}$, $\psi_l^c = \theta^{lc}+\theta^{-lc}+\theta^{qlc}+\theta^{-qlc}$.

Proposition 3.21. *Conjecture 3.6 holds for Suzuki groups, i.e. if $\chi(g) \neq 0$ for some irreducible character χ, then $o(g)$ divides $\frac{|G|}{\chi(1)}$. Furthermore the opposite direction also holds. Thus $n^{\mathfrak{G}}(\chi) = E^{\mathfrak{G}}_{Co}(\chi)$ and $n^{\mathfrak{C}}(\chi) = E^{\mathfrak{C}}_{Co}(\chi)$.*

Since G is a simple and non-Abelian group, we know that $E^{\mathfrak{C}}_{Ch}(\chi) = 1$ and $E^{\mathfrak{G}}_{Ch}(\chi) = \min_{C\in \mathrm{Cl}(G)} |C| = (q^2+1)(q-1)$. Comparing the different estimates we get that not just $E^{\mathfrak{G}}_{Co}$ and $E^{\mathfrak{C}}_{Co}$ are strict, but in one case $E^{\mathfrak{G}}_{Ga}$ is also strict. Let us remind the reader that by Theorem 3.14 the estimate $E^{\mathfrak{G}}_{Ga}(\chi)$ is strict if and only if $|\chi|$ takes only the values 0, 1, $\chi(1)$.

3.3 Estimates for Ree groups

Let G be a group isomorphic to $R(q)$, for some $q = 3^{2n+1} = 3m^2$ and $n \in \mathbb{N}^+$. We are interested in the character table of G (see [49]).

Let R, S, V, W, J be elements of order $\frac{q-1}{2}$, $\frac{q+1}{4}$, $q-3m+1$, $q+3m+1$ and 2, respectively and let X, Y, T be fixed elements in $Z(P)$, in $P \setminus P'$ and in $P' \setminus Z(P)$, respectively.
The normalizers of $\langle S\rangle$, $\langle V\rangle$ and $\langle W\rangle$ contain (at least) one element of order 3 (Theorem 2.2), denote them by h_S, h_V and h_W, respectively. Since these subgroups are cyclic, there are integers $N_S \in \mathbb{Z}_{(q+1)/4}$, $N_V \in \mathbb{Z}_{q-3m+1}$ and $N_W \in \mathbb{Z}_{q+3m+1}$ such that $S^{h_S} = S^{N_S}$, $V^{h_V} = V^{N_V}$ and $W^{h_W} = W^{N_W}$. The numbers $(q+1)/4$, $q-3m+1$ and $q+3m+1$ are pairwise coprime, so by the Chinese reminder Theorem we get that there is $N \in \mathbb{Z}_{(q^3+1)/4}$ such that $N \equiv N_S \mod (q+1)/4$, $N \equiv N_V \mod q-3m+1$ and $N \equiv N_W \mod q+3m+1$.
We can consider an equivalence relation $\sim$ generated by $s \sim sN \sim -s$ separately on $\mathbb{Z}_{(q+1)/4}$, $\mathbb{Z}_{q-3m+1}$ and $\mathbb{Z}_{q+3m+1}$ or at ones on $\mathbb{Z}_{(q^3+1)/4}$. Since $N^3 \equiv 1 \mod (q^3+1)/4 = (q+1)(q-3m+1)(q+3m+1)/4$, the equivalence classes for example on $\mathbb{Z}_{(q+1)/4}$ are $\{1\}$, $\{2,-2,2N,-2N,2N^2,-2N^2\}$, ...
Now we can present the conjugacy classes of G labelled by an element of it.

$$\{1, \{R^a\}_{a\in A}, \{C^b\}_{b\in B}, \{V^c\}_{c\in C}, \{W^d\}_{d\in D}, X, Y, T, T^{-1}, YT, YT^{-1},$$

$$JT, JT^{-1}, \{JR^a\}_{a\in A}, \{J_k S^b\}_{k\in\mathbb{Z}_3, b\in B}, J\},$$

where $A = \mathbb{Z}^*_{(q-3)/4}$, $B = \mathbb{Z}^*_{(q+1)/4}/\sim$, $C = \mathbb{Z}^*_{q-3m+1}/\sim$ and finally $D = \mathbb{Z}^*_{q+3m+1}/\sim$.

Let τ, ι, and $\sigma^\pm$ be primitive $(q-1)/2$-th, $(q+1)/4$-th and $(q\pm 3m+1)$-th roots of unity. We will index the families of exceptional characters by following collections of roots of unity: $\mathcal{R} = \{\tau^k\}_{k\in\mathbb{Z}^*_{(q-3)/4}}$, $\mathcal{T} = \{\iota^k\}_{k\in\mathbb{Z}^*_{(q+1)/4}/\sim}$, $\mathcal{T}' = \{\iota^k\}_{k\in\mathbb{Z}^*_{(q-3)/8}}$, $\mathcal{J} = \{(\sigma^-)^k\}_{k\in\mathbb{Z}_{q-3m+1}/\sim}$ and $\mathcal{K} = \{(\sigma^+)^k\}_{k\in\mathbb{Z}_{q+3m+1}/\sim}$. The elements of different collections generate different exceptional characters. Usually r, t, t', j and k are chosen from $\mathcal{R}$, $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{J}$ and $\mathcal{K}$, respectively.

Theorem 3.22 (Ward [49]). *Then the character table of G is*

	1	R^a	S^b	V^c	W^d	X	Y	T	T^{-1}	YT	YT^{-1}	JT	JT^{-1}	JR^a	J_kS^b	J
ξ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_2	q^2-q+1	1	3	0	0	$1-q$	1	1	1	1	1	-1	-1	-1	-1	-1
ξ_3	q^3	1	-1	-1	-1	0	0	0	0	0	0	0	0	1	-1	q
ξ_4	$q(q^2-q-1)$	1	-3	0	0	q	0	0	0	0	0	0	0	-1	1	$-q$
ξ_5	$(q-1)m\frac{q+3m+1}{2}$	0	1	-1	0	$-\frac{q+m}{2}$	m	$\frac{-m+im^2\sqrt{3}}{2}$	$\frac{-m-im^2\sqrt{3}}{2}$	$\frac{-m-im\sqrt{3}}{2}$	$\frac{-m+im\sqrt{3}}{2}$	$\frac{1-im\sqrt{3}}{2}$	$\frac{1+im\sqrt{3}}{2}$	0	1	$-\frac{q-1}{2}$
ξ_6	$(q-1)m\frac{q-3m+1}{2}$	0	-1	0	1	$\frac{q-m}{2}$	m	$\frac{-m+im^2\sqrt{3}}{2}$	$\frac{-m-im^2\sqrt{3}}{2}$	$\frac{-m-im\sqrt{3}}{2}$	$\frac{-m+im\sqrt{3}}{2}$	$\frac{-1+im\sqrt{3}}{2}$	$\frac{-1-im\sqrt{3}}{2}$	0	-1	$\frac{q-1}{2}$
ξ_7	$(q-1)m\frac{q+3m+1}{2}$	0	1	-1	0	$-\frac{q+m}{2}$	m	$\frac{-m-im^2\sqrt{3}}{2}$	$\frac{-m+im^2\sqrt{3}}{2}$	$\frac{-m+im\sqrt{3}}{2}$	$\frac{-m-im\sqrt{3}}{2}$	$\frac{1+im\sqrt{3}}{2}$	$\frac{1-im\sqrt{3}}{2}$	0	1	$-\frac{q-1}{2}$
ξ_8	$(q-1)m\frac{q-3m+1}{2}$	0	-1	0	1	$\frac{q-m}{2}$	m	$\frac{-m-im^2\sqrt{3}}{2}$	$\frac{-m+im^2\sqrt{3}}{2}$	$\frac{-m+im\sqrt{3}}{2}$	$\frac{-m-im\sqrt{3}}{2}$	$\frac{-1-im\sqrt{3}}{2}$	$\frac{-1+im\sqrt{3}}{2}$	0	-1	$\frac{q-1}{2}$
ξ_9	$m(q^2-1)$	0	0	-1	1	$-m$	$-m$	$-m+im^2\sqrt{3}$	$-m-im^2\sqrt{3}$	$\frac{m+im\sqrt{3}}{2}$	$\frac{m-im\sqrt{3}}{2}$	0	0	0	0	0
ξ_{10}	$m(q^2-1)$	0	0	-1	1	$-m$	$-m$	$-m-im^2\sqrt{3}$	$-m+im^2\sqrt{3}$	$\frac{m-im\sqrt{3}}{2}$	$\frac{m+im\sqrt{3}}{2}$	0	0	0	0	0
η_r	q^3+1	$\omega_r(a)$	0	0	0	1	1	1	1	1	1	1	1	$\omega_r(a)$	0	$q+1$
η'_r	q^3+1	$\omega_r(a)$	0	0	0	1	1	1	1	1	1	-1	-1	$-\omega_r(a)$	0	$-q-1$
η_t	$(q-1)(q^2-q+1)$	0	$-\psi_t(b)$	0	0	$2q-1$	-1	-1	-1	-1	-1	-3	-3	0	$-\psi_t(b)$	$3q-3$
$\eta_{t'}$	$(q-1)(q^2-q+1)$	0	$-\psi_t(b)$	0	0	$2q-1$	-1	-1	-1	-1	-1	1	1	0	$-\psi'_{t'}(b)$	$-q+1$
η_j^-	$(q^2-1)(q+3m+1)$	0	0	$-\psi_j(c)$	0	$-(q+3m+1)$	-1	$-3m-1$	$-3m-1$	-1	-1	0	0	0	0	0
η_k^+	$(q^2-1)(q-3m+1)$	0	0	0	$-\psi_k(d)$	$-(q-3m+1)$	-1	$3m-1$	$3m-1$	-1	-1	0	0	0	0	0

,

where $\omega_r(a) = r^a + r^{-a}$, $\psi_r(b) = r^b + r^{-b} + r^{bN} + r^{-bN} + r^{bN^2} + r^{-bN^2}$ *and* $\psi'_r(b) = -r^b - r^{-b} - r^{bN} - r^{-bN} + r^{bN^2} + r^{-bN^2}$.

Originally some parts of the above character table were not completely determined in [49]. For the sake of competence in the thesis we gave a proof for these parts all other parts can be found in [49].

Proposition 3.23. *Conjecture 3.6 holds for Ree groups, i.e. if* $\chi(g) \neq 0$ *for some irreducible character* χ*, then* $o(g)$ *divides* $\frac{|G|}{\chi(1)}$*. Furthermore* $E^{\mathfrak{C}}_{Co}(\chi) = n^{\mathfrak{C}}(\chi)$ *and* $E^{\mathfrak{G}}_{Co}(\chi) = n^{\mathfrak{G}}(\chi)$ *for* $\chi \in \mathrm{Irr}(G) \setminus \{\xi_4, \xi_9, \xi_{10}, \eta_{t'}\eta_j^+, \eta_k^-\}$.

Since G is a simple, non-Abelian group, we know that $E^{\mathfrak{C}}_{Ch}(\chi) = 1$ and $E^{\mathfrak{G}}_{Ch}(\chi) = (q-1)(q^3+1)$ for $\chi \in \mathrm{Irr}(G) \setminus \mathrm{Lin}(G)$. Using Mathematica programs [38] (the used program can be found in [43]) we showed that

$$E^{\mathfrak{C}}_{Co}(\chi) > E^{\mathfrak{C}}_{Ga}(\chi) > E^{\mathfrak{C}}_{Ch}(\chi) \text{ and } E^{\mathfrak{G}}_{Co}(\chi) > E^{\mathfrak{G}}_{Ga}(\chi) > E^{\mathfrak{G}}_{Ch}(\chi),$$

for every $\chi \in \mathrm{Irr}(G)$.

Chapter 4

Expansiveness

In this note we try to find character analogues of the results in [21]. For this we define $n(\alpha)$ to be the number of irreducible constituents, counting without multiplicity, of a character α of G. So G is *character expansive* if for any character α and any irreducible character χ we have $n(\alpha) \leq n(\alpha\chi)$.

Our first observations on character expansive groups are the following.

Theorem 4.1 (Halasi, Maróti, P [20, Thm. 1.1]). *For a character expansive group G we have the following.*

1. *If G is solvable then it is abelian.*
2. *If G is almost simple then it is simple.*
3. *If G is quasisimple then it is simple.*

Remark 4.2. *Since symmetric groups are almost simple ($n > 4$), but not simple, such a symmetric group can not be expansive.*

The ideas of [21, Section 3] can directly be translated to this character context to prove the following.

Theorem 4.3 (Halasi, Maróti, P [20, Thm. 1.2]). *Let G be a direct product of groups. Then G is character expansive if and only if every direct factor of G is character expansive.*

Theorems 4.1, 4.3, and the results on conjugacy classes above suggest us to consider the following.

Problem 4.4 (Halasi, Maróti, P [20, Prob. 1.3]). *Is it true that a character expansive group is a direct product of simple or abelian groups?*

The converse of Problem 4.4 is false. For let $n = k^2$ for some integer k at least 3. By [2, Theorem 5.6], there are four irreducible characters χ_1, $\chi_2 \neq \overline{\chi}_2$, χ_3 of A_n so that $\chi_1\chi_2 = \chi_3 = \chi_1\overline{\chi}_2$. This means that A_n cannot be character expansive for $n = k^2$. Furthermore, for the same reason, none of the sporadic simple groups Co_1, Co_2, Co_3, Fi_{24}', M, M_{12}, M_{24}, and Th can be character expansive. (Using the Hungarian algorithm [33] it can be shown by computer that among the 138 non-abelian simple groups in the Gap [48] library all other groups are character expansive.) Using [38] we also showed that Suzuki groups and Ree groups are character expansive, the exact program codes can be found in [43].

Unfortunately we are unable to solve Problem 4.4. We can only show

Theorem 4.5 (Halasi, Maróti, P [20, Thm. 1.4]). *A minimal counterexample to Problem 4.4 has a unique minimal normal subgroup and that is abelian and non-central.*

Let V be a finite faithful irreducible FG-module for some semisimple finite group G and prime field F. For a complex linear character λ of V let $I_G(\lambda)$ be the stabilizer of λ in G and for a finite group H let $k(H)$ be the number of conjugacy classes of H. An affirmative answer to the following problem would imply Problem 4.4.

Problem 4.6 (Halasi, Maróti, P [20, Prob. 1.5]). *With the notations and assumption above, does there exists $\lambda \in \mathrm{Irr}(V)$ with $k(I_G(\lambda)) < k(G)$?*

Interestingly, Problem 4.6 seems to be close to the $k(GV)$ problem.

Theorem 4.7 (Halasi, Maróti, P [20, Thm. 1.6]). *With the notations and assumption above, Problem 4.6 has an affirmative solution if G is simple and $(|G|, |V|) = 1$, or if G is simple and $G = GL(V)$.*

Bibliography

[1] Y. Berkovich and L. Kazarin. Finite groups in which the zeros of every nonlinear irreducible character are conjugate modulo its kernel. *Houston Journal of Mathematics*, 24:619–630, 1998.

[2] C. Bessenrodt and A. Kleshchev. On Kronecker products of complex representations of the symmetric and alternating groups. *Pacific Journal of Mathematics*, 190:201–223, 1999.

[3] M. Bianchi, D. Chillag, and A. Gillio. Finite groups in which every irreducible character vanishes on at most two conjugacy classes. *Houston Journal of Mathematics*, 25:451–461, 2000.

[4] R. Boltje, S. Danz, and B. Külshammer. On the depth of subgroups and group algebra extensions. *Journal of Algebra*, 335:258–281, 2011.

[5] R. Boltje and B. Külshammer. On the depth 2 condition for group algebra and Hopf algebra extensions. *Journal of Algebra*, 323:1783–1796, 2010.

[6] R. Boltje and B. Külshammer. Group algebra extensions of depth one. *Algebra Number Theory*, 5:63–74, 2011.

[7] S. Burciu and L. Kadison. Subgroups of Depth Three, 2009.

[8] S. Burciu, L. Kadison, and B. Külshammer. On subgroup depth. *Int. Electron. J. Algebra*, 9:133–166, 2011.

[9] R. Carter, G. Segal, and I. Macdonald. *Lectures on Lie groups and Lie algebras*, volume LMS Student Texts of *32*. Cambridge University Press, 1995.

[10] D. Chillag. On zeros of characters of finite groups. *Proceedings of the American Mathematical Society*, 127:977–983, 1999.

[11] S. Danz. The depth of some twisted group algebra extensions. *Communications in Algebra*, 39:1631–1645, 2011.

[12] S. Dolfi, G. Navarro, E. Pacifici, P. Sanus, and P. H. Tiep. Non-vanishing elements of finite groups. *Journal of Algebra*, 323:540–545, 2010.

[13] S. Dolfi, E. Pacifici, L. Sanus, and P. Spiga. On the orders of zeros of irreducible characters. *Journal of Algebra*, 321:345–352, 2009.

[14] T. Fritsche, B. Külshammer, and C. Reiche. The depth of Young subgroups of symmetric groups. *Journal of Algebra*, 381:96–109, 2013.

[15] T. Fritzsche. The depth of subgroups of $PSL(2, q)$. *Journal of Algebra*, 349:217–233, 2012.

[16] T. Fritzsche. The depth of subgroups of $PSL(2, q)$ II. *Journal of Algebra*, 381:37–53, 2013.

[17] P. X. Gallagher. Group characters and commutators. *Mathematische Zeitschrift*, 79:122–126, 1962.

[18] P. X. Gallagher. Zeros of characters of finite groups. *Journal of Algebra*, 4:42–45, 1966.

[19] F. Goodman, P. de la Harpe, and V.F.R. Jones. *Coxeter graphs and towers of algebras*. Springer, New York, 1989.

[20] Z. Halasi, A. Maróti, and F. Petényi. Character expansiveness in finite groups. *International Journal of Group Theory*, 2:9–17, 2013.

[21] Z. Halasi, A. Maróti, S. Sidki, and M. Bezerra. Conjugacy expansiveness in finite groups. *Journal of Group Theory*, 15:485–496, 2012.

[22] A. Hernandez, L. Kadison, and M. Szamotulski. Subgroup depth and twisted coefficients. *Communications in Algebra*, 44:3570–3591, 2016.

[23] L. Héthelyi, E. Horváth, and F. Petényi. The depth of the maximal subgroups of Ree groups $({}^2G_2(q))$.

[24] L. Héthelyi, E. Horváth, and F. Petényi. The Depth of Subgroups of Suzuki Groups. *Communications in Algebra*, 43:4553–4569, 2015.

[25] B. Huppert and N. Blackburn. *Finite groups III.* Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[26] I. M. Isaacs. *Character Theory of Finite Groups.* Dover Publications, 1994.

[27] I. M. Isaacs, G. Navarro, and T. R. Wolf. Finite group elements where no irreducible character vanishes. *Journal of Algebra*, 222:413–423, 1999.

[28] S. M. Gagola Jr. Characters vanishing on all but two conjugacy classes. *Pacific Journal of Mathematics*, 109:363–385, 1983.

[29] L. Kadison. Finite depth and Jacobson-Bourbali correspondence. *Journal Pure Application Algebra*, 212:1822–1839, 2008.

[30] L. Kadison. Algebra depth in tensor categories, to appear.

[31] L. Kadison and B. Külshammer. Depth two, normality and a trace ideal condition for Frobenius extensions. *Communications in Algebra*, 34:3103–3122, 2006.

[32] P. B. Kleidman. The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups. *Journal of Algebra*, 117:30–71, 1988.

[33] H. W. Kuhn. The hungarian method for the assignment problem. *Naval Research Logistics Quarterly*, 2:83–97, 1955.

[34] P. Landrock and G. O. Michler. Principal 2-blocks of the simple groups of Ree type. *Transactions of the American Mathematical Society*, 260:83–111, 1980.

[35] V. M. Levchuk and Ya. N. Nuzhin. Structure of Ree groups. *Algebra and Logic*, 24:16–24, 1985.

[36] G. Malle, G. Navarro, and J. B. Olsson. Zeros of characters of finite groups. *Journal of Group Theory*, 3:353–368, 2000.

[37] R. P. Martineau. On representations of the Suzuki groups. *Journal of the London Mathematical Society*, 6:153–160, 1972.

[38] Wolfram Mathematica. http://www.wolfram.com/mathematica.

[39] J. McKay. The largest degrees of irreducible characters of symmetric groups. *Mathematics of Computation*, 30:624–631, 1976.

[40] A. Moretó and J. Sangroniz. On the number of conjugacy classes of zeros of characters. *Israel Journal of Mathematics*, 142:163–167, 2004.

[41] A. Moretó and J. Sangroniz. On the number of zeros in the columns of the character table of a group. *Journal of Algebra*, 279:726–736, 2004.

[42] G. Navarro. Zeros of primitive characters in solvable groups. *Journal of Algebra*, 221:644–650, 1999.

[43] F. Petényi. Programs. www.math.bme.hu/∼pfranci/programs.html, 2012.

[44] F. Petényi. Comparing estimates on the number of zeros of irreducible characters in symmetric groups. *Seminaire Lotharingien de Combinatoire*, 72:B72c, 2015.

[45] G. Qian. Bounding the Fitting height of a solvable group by the number of zeros in a character table. *Proceedings of the American Mathematical Society*, 130:3171–3176, 2002.

[46] M. Suzuki. On a class of doubly transitive groups. *Annals of Mathematics*, 75:105–114, 1962.

[47] M. Suzuki. On a class of doubly transitive groups II. *Annals of Mathematics*, 78:514–589, 1962.

[48] Algorithms The GAP Group, GAP Groups and Programs. http://www.gap-system.org.

[49] H. N. Ward. On Ree's series of simple groups. *Transactions of the American Mathematical Society*, 121:62–89, 1966.

[50] T. Wilde. Orders of elements and zeros and heights of characters in a finite group, 2006.

[51] J. ZHANG, Z. SHEN, and S. WU. A note on the number of zeros in the columns of the character table of a group. *Journal of Algebra and Its Applications*, 12:10p., 2013.

[52] É. M. Zhmud. The zeros of group characters. *Uspekhi Matematicheskikh Nauk*, 32:223–224, 1977.

[53] É. M. Zhmud. On estimates of the number of zeros of irreducible characters of a finite group. *Publicationes Mathematicae Debrecen*, 33:125–146, 1986.

[54] É. M. Zhmud. On finite groups all irreducible characters of which take at most two nonzero values. *Ukrainian Mathematical Journal*, 47:1308–1313, 1995.

[55] S. Zhou, H. Li, and W. Liu. The Ree groups ${}^2G_2(q)$ and $2-(v,k,1)$ block designs. *Discrete Mathematics*, 224:251–258, 2000.