Polytopic Decomposition of Linear Parameter-Varying Models by Tensor-Product Model Transformation

Ph.D. Thesis Booklet

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1 Introduction

1.1 Preliminaries and scientific background of the research work

The research work leading to the results presented in this dissertation is based on the significant changes in control theory, mathematics and system theory, appearing almost simultaneously in the last decade.

In the last decade, the representation of identification models in system theory has changed significantly. The origins of the paradigm shift can be linked with the famous speech given by D. Hilbert in Paris in 1900. Hilbert listed 23 conjectures, hypotheses concerning unsolved problems which he believed would provide the biggest challenge in the 20th century. According to the 13th conjecture there exist continuous multi-variable functions which cannot be decomposed as the finite superposition of continuous functions of less variables [25–27, 33]. In 1957 Arnold disproved this hypothesis [3], moreover, in the same year, Kolmogorov [35] formulated a general representation theorem, along with a constructive proof, where the functions in the decomposition were one dimensional. This proof justified the existence of universal approximators. Kolmogorov’s representation theorem was further improved by several authors (Sprecher [48] and Lorentz [40]). Based on these results, starting from the 1980s, it has been proved that universal approximators exist among the approximation tools of biologically inspired neural networks and genetic algorithms, as well as fuzzy logic [8, 12, 14, 29, 36, 45, 52, 54]. In this manner, these approximators have appeared in the identification models of system theory, and turned out to be effective tools even for systems that can hardly be described in an analytical way.

One of the most fruitful developments in the world of linear algebra and linear algebra-based signal processing is the concept of the Singular Value Decomposition (SVD) of matrices. The history of matrix decomposition goes back to the 1850s. During the last 150 years several mathematicians—Eugenio Beltrami (1835–1899), Camille Jordan (1838–1921), James Joseph Sylvester (1814–1897), Erhard Schmidt (1876–1959), and Hermann Weyl (1885–1955), to name a few of the more important ones—were responsible for establishing the existence of the singular value decomposition and developing its theory [50]. Thanks to the pioneering efforts of Gene Golub, there exist efficient, stable algorithms to compute the singular value decomposition [24]. More recently, SVD started to play an important role in several scientific fields [15, 42, 53]. Its popularity also grew in parallel with the more and more efficient numerical methods. Due to the development of personal computers it became possible to handle larger-scale, multi-dimensional problems, and there is a greater demand for the higher-order generalization of SVD for tensors. Higher Order SVD (HOSVD) is used efficiently in independent component analysis (ICA) [38], as well as in the dimensionality reduction for higher-order factor analysis-type problems—thus reducing the computational complexity [37]—to name a few examples. The HOSVD concept was first published as a whole multi-dimensional SVD concept in 2000 [39], and the Workshop on Tensor Decompositions and Applications held in Luminy, Marseille, France, August 29–September 2, 2005 was the first event where the key topic was HOSVD. Its very unique power in linear algebra comes from the fact that it can decompose a given $N$-dimensional tensor into a full orthonormal system in a special ordering of singular
values, expressing the rank properties of the tensor in order of $L_2$-norm. In effect, the HOSVD is capable of extracting the very clear and unique structure underlying the given tensor. The Tensor Product (TP) model transformation is a further extension to continuous N-variable functions. It is capable of extracting the fully orthonormal and singular value ordered structure of the given function. Note that this structure cannot be analytically achieved, since there is no general analytic solution for the HOSVD. The TP model transformation was also extended to linear parameter-varying (LPV) models in 2003. It generates the HOSVD of LPV models. To be specific: it generates the parameter-varying combination of Linear Time-Invariant (LTI) models that represents the given LPV model in such a way that: i) the number of the LTI components are minimized; ii) the weighting functions are univariate functions of the parameter vector; iii) the weighting functions are in an orthonormal system for each parameter; iv) the LTI systems are also in orthogonal position; v) the LTI systems and the weighting functions are ordered by the singular values.

In conclusion, the TP model transformation finds the clear well defined and unique structure of the given LPV model. This cannot be achieved via analytical derivations. Thus the result of the TP model transformation was termed as the HOSVD-based canonical form of polytopic or LPV models in 2006 [5, 6].

The appearance of Lyapunov-based stability criteria made a significant improvement in the control theory of nonlinear systems. This change of the viewpoint was invoked by the reformulation of these criteria in the form of linear matrix inequalities, in the early 1990s. Herewith, the stability questions of control theory were given in a new representation, and the feasibility of Lyapunov-based criteria was reinterpreted as a convex optimization problem, as well as, extended to an extensive model class. The pioneers GAHIYET, BOKOR, CHILAI, BOYD, and APKARIAN were responsible for establishing this new concept [1,2,9,16,18,19,22,32,43,47]. The geometrical meaning and the methodology of this new representation were developed in the research group of Prof. József BOKOR. Soon, it was also proved that this new representation could be used for the formulation of different control performances—in the form of linear matrix inequalities—beyond the stability issues together with the optimization problem. Ever since, the number of papers about linear matrix inequalities guaranteeing different stability and control properties are increasing drastically. BOYD’s paper [10] states that it is true of a wide class of control problems that if the problem is formulated in the form of linear matrix inequalities, then the problem is practically solved.

In parallel with the above research and thanks to the significant increase in the computational performance of computers, efficient numerical mathematical methods and algorithms were developed for solving convex optimization problems—thus linear matrix inequalities. The breakthrough in the use of convex optimization in practical applications dates back to the introduction of interior point methods. These methods were developed in series of papers [31], and have real importance in connection with linear matrix inequality problems in the work of Yuriii NESTEROV and Arkadii NEMIROVSKI [44]. Today, these methods are used in “everyday” engineering work, and it turns out to be equally efficient in cases when the closed formulation is unknown. In consequence, the formulation of analytical problems has gained a new meaning.
It is well-known that a considerable part of the problems in modern control theory necessitate the solution of *Riccati-equations*. However, the general analytical (closed formulation) solution of multiple Riccati-equations is unknown. In turn—with the usage of numerical methods of convex optimization—we consider solved those problems today that require the resolution of a large number of convex algebraic Ricatti-equations, in spite of the fact that a result of the obtained solution is not a closed (in classical sense) analytical equation.

In conclusion, the most advantageous property of the new, convex optimization based representation in control theory is that it is possible to easily combine different controller design conditions and goals in the form of numerically manageable linear matrix inequalities [10]. This makes it possible to solve numerous (complex) control theory problems with remarkable efficiency.

This is especially true of Lyapunov-based analysis and synthesis, but also of optimal LQ control, $H_\infty$ control [17,23,51], as well as minimal variance control. The linear matrix inequality-based design also appeared in other areas such as estimation, identification, optimal design, structural design, and matrix-sizing problems. The following enumeration lists further problems that can be managed and solved in a representation using linear matrix inequalities: robust stability of linear time-invariant systems with uncertainty ($\mu$-analysis) [46,49,55], quadratic stability [11,28], Lyapunov-based stability of parameter-dependent systems [21], the guarantee of constraints on linear time-invariant system inputs, state variables, and outputs, or other goals [10], multi-model and multi-objective state-feedback control [4,7,10,13,34], robust pole-placement, optimal LQ control [10], robust $H_\infty$ control [20, 30], multi-goal $H_\infty$ synthesis [13,34,41], control of stochastic systems [10], weighted interpolation problems [10].

### 1.2 General description of the scientific problem

According to the above, a model, given either by a closed analytical formulation, or as the output of an identification using soft-computing techniques such as fuzzy logic, neural networks, or genetic algorithms, can be transformed to HOSVD-based canonical form by TP model transformation. The only requirement is that the model must be able to be discretized over a grid. From the HOSVD-based canonical form, different convex polytopic models can be generated by simple numerical transformations depending on the further application of the model. The control goals also can be given as a convex optimization problem in the form of linear matrix inequalities, and can be solved using modern numerical convex optimization algorithms. The convex polytopic models generated by the TP model transformation can be immediately applied to the linear matrix inequalities, and this makes it possible to solve a wide class of problems irrespective of the existence or non-existence of an analytical solution in a closed form. The TP model transformation based controller design methodology offers a uniform, tractable, straightforward numerical controller design framework. In many cases, the analytical derivation, affine decomposition, or the convex optimization can be really troublesome, time consuming or even impossible even with state-of-the-art analytical tools, while in many applications it can be proved that
numerical methods can easily overcome on these difficulties. The intention of the presented
dissertation is to analyze the applicability and feasibility of TP model transformation in
control theory problems.

The linear matrix inequality based controller design was well-researched in the last
decade, therefore its validity and applicability analysis is not set as a goal in this disserta-
tion.

1.3 The goal of the dissertation

As a result of the paradigm shift outlined in Section 1.1 several efficient system theory
tools were developed using different representations. Even with the joint usage of the
tools, the adaptation of different representations—in the point of view of design methods
and the mathematical tools, especially the analytical transformations—are difficult, and
in many cases this problem is not solved. During my research work, my main goal was
to investigate and analyze the applicability and feasibility, and to extend the usability of
the Tensor Product model transformation based controller design methodology, a tractable
and uniform controller design methodology for a wide class of complex control theory
problems.

Therefore, my comprehensive goals in details are as follows

• Investigate whether the Tensor Product model transformation for complex, bench-
mark and real-world type dynamic systems yields models that are interpretable
and can be formulated for convex optimization problems composed in the form
of linear matrix inequalities. An academic benchmark problem is chosen for the
theoretical evaluation, and valid comparison with other published methodologies.
Besides, industry related experimental analysis is also an important aspect of the
investigation.

• Show that the finite element HOSVD-based canonical form and the convex finite
element TP models of the chosen models exist and the TP model transformation
generates the minimal number of linear time-invariant systems.

• Since a crucial point of the Tensor Product model transformation is the computational
complexity explosion for higher dimensional problems, my goal is to investigate the
applicability of the Tensor Product model transformation and to suggest algorithms
and methods for decreasing the computational complexity load of the transformation
for higher dimensional problems.

2 Brief introduction to Tensor Product model transfor-
mation

The Tensor Product model transformation is a uniform, numerical method that is capable
of transforming uniformly both in a theoretical and algorithm execution aspect the linear
Consider the following linear parameter-varying (LPV) state-space model:

\[
\begin{align*}
\dot{x}(t) &= A(p(t))x(t) + B(p(t))u(t), \\
y(t) &= C(p(t))x(t) + D(p(t))u(t),
\end{align*}
\]  

(1)

with input \(u(t)\), output \(y(t)\) and state vector \(x(t)\). The system matrix

\[
S(p(t)) = \begin{pmatrix}
A(p(t)) & B(p(t)) \\
C(p(t)) & D(p(t))
\end{pmatrix} \in \mathbb{R}^{O \times I}
\]

(2)

is a parameter-varying object, where \(p(t) \in \Omega\) is a time varying \(N\)-dimensional parameter vector, and is an element of the closed hypercube \(\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N\). Parameter \(p(t)\) can also include some elements of \(x(t)\). Therefore this type of model is considered to belong to the class of non-linear models.

**Definition 2.1** (Linear Parameter-Varying state-space model). Consider the following linear parameter-varying (LPV) state-space model:

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**Definition 2.2** (Finite element TP model). \(S(p(t))\) of (1) is given for any parameter \(p(t)\) as the parameter-varying combination of linear time-invariant (LTI) system matrices \(S\) also called vertex systems

\[
\begin{pmatrix}
\dot{x}(t) \\
y(t)
\end{pmatrix} = \mathcal{S} \bigotimes_{n=1}^{N} w_n(p_n(t)) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix},
\]

(3)

where row vector \(w_n(p_n) \in \mathbb{R}^{I_n}\) \(n = 1, \ldots, N\) contains one bounded variable and continuous weighting functions \(w_{n,i_n}(p_n), (i_n = 1 \ldots I_n)\). The weighting function \(w_{n,i_n}(p_n(t))\) is the \(i_n\)th weighting function defined on the \(n\)th dimension of \(\Omega\), and \(p_n(t)\) is the \(n\)th element of vector \(p(t)\). \(I_n < \infty\) denotes the number of the weighting functions used in the \(n\)th dimension of \(\Omega\). Note that the dimensions of \(\Omega\) are respectively assigned to the elements of the parameter vector \(p(t)\). The \((N + 2)\)-dimensional coefficient tensor \(\mathcal{S} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times O \times I}\) is constructed from LTI vertex systems \(\mathcal{S}_{i_1 i_2 \ldots i_N} \in \mathbb{R}^{O \times I}\).

**Definition 2.3** (HOSVD-based canonical form of finite element TP model). Consider (1) which have the form of (3). Then we can determine:

\[
\begin{pmatrix}
\dot{x}(t) \\
y(t)
\end{pmatrix} = \mathcal{D} \bigotimes_{n=1}^{N} w_n(p_n(t)) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix},
\]

(4)
by generating the Exact Minimized form of HOSVD on the first $N$-dimension of $S$. The resulting tensor $D$ has the size of $r_1 \times \cdots \times r_N \times O \times I$. The weighting functions have the following properties:

1. The $r_n$ number of weighting functions $w_{n,i}(p_n)$ contained in vector $w_n(p_n)$ form an orthonormal system. The weighting function $w_{i,n}(p_n)$ is an $i$th singular function on dimension $n = 1 \ldots N$.

2. Tensor $D$ has the following properties:

   (a) all-orthogonality: select one element of dimension $N + 1$ and $N + 2$ of tensor $D$. The selected $N$-dimensional subtensor $D'$ has all-orthogonality as: two subtensors $D'_{i_n=\alpha}$ and $D'_{i_n=\beta}$ are orthogonal for all possible values of $n, \alpha$ and $\beta$: $\langle D'_{i_n=\alpha}, D'_{i_n=\beta} \rangle = 0$ when $\alpha \neq \beta$.

   (b) ordering: $\|D'_{i_n=1}\| \geq \|D'_{i_n=2}\| \geq \cdots \geq \|D'_{i_n=I_n}\| \geq 0$ for all possible values of $n = 1 \ldots N$.

3. The Frobenius-norms $\|D'_{i_n=i}\|$, symbolized by $\sigma_i^{(n)}$, are $n$-mode singular values of $S$.

4. $D$ is termed core tensor consisting the LTI systems.

**Definition 2.4** (Convex finite element TP model). The finite element TP model (3) is convex only if the weighting functions are

$$\forall n, i, p_n(t) : w_{n,i}(p_n(t)) \in [0, 1]$$

$$\forall n, p_n(t) : \sum_{i=1}^{I_n} w_{n,i}(p_n(t)) = 1$$

### 3 Theses

The scientific results of the following theses are published in [P–1–P–18] (accumulated impact factor is 1.02) and for these publications there are 8 independent citations.

**Thesis 1: Computational complexity relaxation of the Tensor Product model transformation by decreasing the discretization grid density [P–1]**

I proposed a computational complexity relaxed TP model transformation that is executable on a sparse discretization grid of the system matrix that immediately leads to a computational complexity reduction. However, the utilization of the sparse discretization grid may degrade the accuracy of the convex hull. In order to compensate this problem, I also proposed a theoretical modification of the TP model transformation that is capable
of extending the TP model to a dense discretization grid. I also developed a numerical implementation of the modified TP model transformation. The proposed implementation is ready to use in real-world applications, I made it available as a toolbox to MATLAB.

I gave an estimate that the proposed relaxation polynomially reduces the computational load of the TP model transformation. In practical cases the expected number of weighting functions are small, the grid density can be reduced by several orders.

**Thesis 2: Computational complexity relaxation of the Tensor Product model transformation by separating the constant and non-constant elements [P–2]**

I made a suggestion for the relaxation of the computational load in TP model transformation by avoiding the computation of constant elements in the LPV model. The key idea of this approach is the separation of constant and non-constant elements of the discretized system matrix. For the numerical implementation, I developed an algorithm that decompose the system matrix into constant and non-constant matrices, then execute a modified TP model transformation on each discretized set, and finally reconstruct into the TP model form. I must emphasize here that the proposed modification can only be applied to convex TP models, it cannot be used for HOSVD-based canonical form. The proposed method is ready to use in real-world applications, I made it available as a toolbox to MATLAB.

I showed that the proposed separation of the constant and non-constant elements of the discretized system matrix can significantly reduce the computational load in cases when the ratio of constant elements in the discretized matrix is high. The proposed method gives a linear complexity reduction. In practical cases the ratio of the constant elements in the system matrix is about 70–80%, thus the proposed method offers a good relaxation in these cases.

**Thesis 3: Solving complex control problems by Tensor Product model transformation based controller design [P–3, 5, 6, 12, 16–18]**

**Thesis 3.1** I proved that the finite element HOSVD-based canonical forms of the TORA and SPG systems exist as a parameter-varying weighting combination of ten, fourteen vertex systems, respectively. These new forms are given as

\[
\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \bigotimes_{n=1}^{2} w_n(p_n(t)) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{7}
\]

In this regards, I proved that different types of convex finite element TP models (Sum-Normalized and Non-Negative, tight convex, Inverted Normalized and Relaxed Normalized) exist and can be given as a convex combination of minimum ten and fourteen vertex systems, respectively for TORA and SPG systems.

These TP model forms are new representations, they have not been published before in the scientific literature. This new representation opens a new way for LMI based controller design methodologies.
Thesis 3.2 I proved by the Lyapunov stability theorems formulated in terms of linear matrix inequalities that the continuous finite element convex polytopic model of TORA and SPG systems are controllable and observable in the space $\Omega$.

I proved that the TP models (7) of the TORA and SPG systems with SN, NN, and NO type weighting functions satisfies the conditions of the Parallel Distributed Compensation based controller design. Utilizing this controller design technique I derived controllers. I proved by the LMI theorems under PDC framework that the derived controllers guarantee multi-objective control performances such as asymptotic stability and given constraints on control value and on the output vector.

I also showed that the tight convex TP model of the TORA system immediately applicable for observer structure based output feedback control design. Utilizing this observer structure we are capable of estimating unmeasurable state vectors, and guarantee that this output feedback control design satisfies multi-objective control performances such as asymptotic stability and given constraints on control value and on the output vector.

Thesis 3.3 I carried out the analysis of the trade-off property between the computational complexity and approximation accuracy of different types of convex (SN, NN, and CNO type weighting functions) polytopic models of the TORA system. I showed by numerical simulations that there is no significant difference in control performance of the different complexity reduced TP models, whilst the computational complexity of the TP model is reduced by 60% and the controller is 76% smaller than the original. It is also important to emphasize that not only the complexity of the controller reduced significantly, but also the complexity load of the whole control design process became much smaller as the decrease of the linear time-invariant systems directly polynomially reduce the number of LMI terms.

4 Application of results

The Tensor Product model transformation offers solutions to the transformation of models used in control problems to a new, TP model form, thus enabling their analysis according to various engineering viewpoints. By transforming the given problem into a convex optimization problem, we can consider such new goals that may have not arisen yet. The dissertation also shows examples by the TORA system, and the Single Pendulum Gantry. In both cases, the Tensor Product model provides the opportunity to optimize further control properties besides stability criterium. The Tensor Product model transformation can numerically and efficiently determine continuous functions satisfying gives properties without the use of analytical deductions. Thus, those design processes can turn out to be a routine task that needed the series of analytical deductions before. The dissertation shows an experimental example, the Single Pendulum Gantry system for this, where the controller with the given control performance was resulted “automatically” by the execution of numerical methods. Several methods based on numerical or soft-computing-based techniques exist which automatically yield the model. The Tensor Product model transformation can easily fit to these models. In case of these control theory problems, it
is possible to reach the controller design (with given control performance) phase starting from the identification, through the use of automatically executable methods. The Tensor Product model transformation approaches and modeling techniques, extended with the proposed complexity relaxation methods for high degree nonlinear systems, can be used successfully for analysis of dynamic models of land vehicles (commercial road vehicle), thus, among others, efficiently for truss analysis of commercial road vehicle as an extension to finite element analysis. Similarly, the TP model transformation can also be used in problems related to braking and steering of vehicles.

**Publications**


References


