Local limit theorems and recurrence for the planar Lorentz process

PhD thesis

Tamás Varjú

Institute of Mathematics, Technical University of Budapest

advisor:
Domokos Szász

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Ezen értekezés bírálatai és a védésről készült jegyzőkönyv a későbbiekben a Budapesti Műszaki és Gazdaságtudományi Egyetem Természettudományi Karának Dékáni Hivatalában elérhetők.

Alulírott Varjú Tamás kijelentem, hogy ezt a doktori értekezést magam készítettem és abban csak a megadott forrásokat használtam fel. Minden olyan részt, amelyet szó szerint vagy azonos tartalommal, de átfogalmazva más forrásból átvettem, egyértelműen, a forrás megadásával megjelöltem.

Varjú Tamás

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Chapter 1

Introduction

1.1 Introducing the model and the questions

1.1.1 History preceding Sinai-billiard

The investigation of chaotic, long term behaviour arose in several ways inside mathematics. In the theory of differential-equations, stability is one of the main concepts, and, of course, it is one of the most frequently applied notions in engineering applications. A well understood part of dynamical systems can be described by stable and unstable fixed points, stable and unstable periodic and preperiodic points, saddles and limit cycles. Approaching dynamical systems from this direction, Stephen Smale had the idea in 1960 that the most important systems (the structurally stable ones) could be described as above, in terms of finite graphs. The source of this idea was a particularly fruitful application of gradient flows in the study of manifold-topology by Morse [Morse 34]. Smale extended the results of Morse to non-gradient flows and created the idea of the Morse-Smale flow [Smale 60].

Three years later Smale himself could construct his famous example of the horseshoe ([Smale 63] and [Smale 65]), where periodic points are dense, and both are of saddle type. In this system the vast majority of phase points follows unpredictable trajectories, a huge contrast with Morse-Smale flows where limit cycles rule the phase-space. This example did not fit the idea of describing a flow with a Morse-Smale graph. Moreover, it could not be neglected since the structure of the example is stable under perturbations proving the robustness of such a behaviour (structural stability). The idea of chaotic behaviour has widely spread, and the structure of such systems has been studied with huge efforts.

One other source of chaotic and unpredictable behaviour in dynamical systems came from statistical physics. The idea that a large system forgets any special information about its state in the long run was expressed in the notion of ergodicity, and this information loss was quantified in the entropy production formulae. Statistical physics considers such a system from a probabilistic viewpoint, and mathematical efforts of understanding the fundamentals of this physical discipline focused on how this randomness appears in physical systems. The physical intuition behind this effect is that in a large system every interaction spreads through the whole, and balances between the large number of components of the system. This case turned out to be technically the most challenging. Each study which has considered large systems either had some additional randomisation in the time evolution of the model itself, or made the components forgetting information in some other artificially forced way.

Approaching dynamical systems from the physical side, Yakov Sinai observed that even low-dimensional systems such as the gradient flows on a surface with constant negative curvature shows “stochastic” behaviour. For establishing a physical approach Kolmogorov ([Kol 58] and [Kol 59]) and Sinai [Sinai 59] formulated the notion of entropy suitable for dynamical systems, and calculated the (positive) entropy for the above example [Sinai 60] in the same year 1960, when Smale formulated his false conjecture about the “simplicity” of most dynamical systems.

Sinai was interested in a more “physical” system with positive entropy. He had shown that two circular discs on a flat torus bouncing with elastic collisions (a mechanical model) has positive entropy [Sinai 63]. This example led to the famous model-class of the Sinai-billiards [Sinai 70]. In this system there is a single moving particle which collides elastically with the completely passive other particles, and –in order to create a system with compact phase-space– periodic boundary conditions are applied.

Before introducing the models in mathematical formalism, we note here that in all of these dynamical systems the engine of entropy production is not the balancing effect of the large number of components, but rather hyperbolicity: a phenomenon which made Smale’s horseshoe work.

1.1.2 The Sinai-billiard, and the Lorentz-process

Consider finitely many scatterers $\mathcal{O}_i$ (also called obstacles) on the 2-torus, $\mathbb{T}^2 \supset \mathcal{O} = \cup \mathcal{O}_i$ such that each of the scatterers is strictly convex with a $\mathcal{C}^3$-smooth boundary. Let $n(q)$ denote the unit normal vector of the boundary $\partial\mathcal{O}$ at the point q, directed outwards $\mathcal{O}$. The phase space of the system is:

$$X = \{(q \in \partial\mathcal{O}, v \in \mathbb{R}^2) \mid |v| = 1, \langle v, n(q)\rangle \geq 0\}.$$

The dynamics $T : X \to X$ is uniform motion with v velocity vector followed by an elastic collision (v is mirrored to the tangent line at the point of impact). This system has a natural invariant measure: if we denote by l the total length of $\partial\mathcal{O}$, then $d\mu = \frac{1}{2l}\langle v, n(q)\rangle\, dqdv$, is an invariant probability measure, since $\int_X \langle v, n(q)\rangle\, dqdv = 2l$. The normalising constant $\frac{1}{2l}$ will be denoted by c_μ.

This phase space will be identified with a finite number of cylinders $\partial\mathcal{O} \times \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. So throughout this thesis if v denotes a velocity of a phase point it is meant as $v \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

The boundary of this phase space consists of tangential collisions denoted by S_0. The dynamics resp. the inverse dynamics is non-continuous in backward resp. forward images of this set. We will denote $S_i = T^i S_0,\ i \in \mathbb{Z}$.

The *planar Lorentz process* is the natural $\mathbb{Z}^2$ cover of the above-described toric billiard. More precisely: consider $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$ the factorisation by $\mathbb{Z}^2$. Its fundamental domain D is a square (semi-open, semi-closed) in $\mathbb{R}^2$, so $\mathbb{R}^2 = \cup_{z\in\mathbb{Z}^2}(D + z)$, where $D + z$ is the translated fundamental domain.

We lift the obstacles to the plane (i. e. we take $\tilde{\mathcal{O}} = \Pi^{-1}\mathcal{O}$), and define the phase space $\tilde{X}$, and the dynamics $\tilde{T}$ exactly the same way as above. The *free flight function* $\tilde{\psi} : \tilde{X} \to \mathbb{R}^2$ is defined as follows: $\tilde{\psi}(\tilde{x}) = \tilde{q}(\tilde{T}\tilde{x}) - \tilde{q}(\tilde{x})$. The *discrete free flight function* $\tilde{\kappa} : \tilde{X} \to \mathbb{Z}^2$ is defined as follows: $\tilde{\kappa}(\tilde{x}) = \iota(\tilde{T}\tilde{x}) - \iota(\tilde{x})$, where $\iota(\tilde{x}) = z$ if $\tilde{x} \in D + z$. Observe

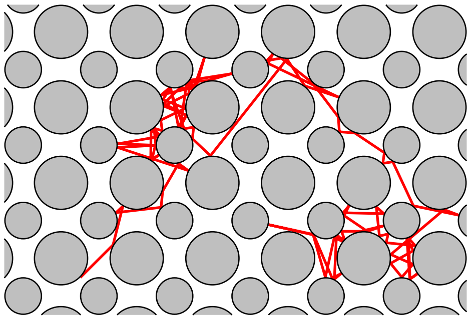


Figure 1.1: 100 collisions Lorentz trajectory segment with circular scatterers and finite horizon

finally that $\tilde{\psi}$ and $\tilde{\kappa}$ are invariant under the $\mathbb{Z}^2$ action, so there are ψ and κ functions defined on X, such that $\tilde{\psi} = \Pi^*\psi$ and $\tilde{\kappa} = \Pi^*\kappa$. Actually for our purposes it will be more convenient to choose the fundamental domain in such a way that $\partial\tilde{\mathcal{O}} \cap \partial D = \emptyset$. In this way κ will be continuous.

One can also consider the Lorentz process, as a skew product over the toric billiard $\tilde{T} : (x, a) \mapsto (Tx, a + \kappa(x))$ where $x \in X$ is the phase-point inside the cell, and $a \in \mathbb{Z}^2$ is the cell-index.

Definition 1. *The system is said to have* finite horizon *if the free flight function is bounded. Otherwise the system is said to have* infinite horizon.

Despite of the (physical) simplicity of the model the number of interpretations is large. Originally this system was considered to model an electron moving in a crystal [Lor 05], but one can also say that this is a model of a gas, where we follow the motion of a single particle, and are not interested in the interaction between the others. This latter approach can not justify periodicity, which is considered only as a technical assumption.

When [BS 81] (see also [BChS 91]) proved weak convergence of the trajectory to the Brownian motion the gas model became justified. The Brownian motion [Brown] was basically known as the trajectory of a single particle inside a positive temperature medium, where the dynamics of the motion is driven by small tosses from the particles of the medium. The result proved that the type of the medium is not an important matter it can be even a frozen crystal as long as the density of the medium does not allow arbitrary long free flights.

1.1.3 The question of recurrence, and local limit theorems

The basic mathematical microscopic model of the Brownian motion is the simple symmetric random walk. A famous result of the field is Pólya's theorem stating recurrence of the moving particle. Here recurrence means that the process almost surely returns to any fixed bounded domain of the configuration space. Once it had been established that the diffusion limit of the planar Lorentz process is, indeed, the Wiener process, the question of its *recurrence* was immediately raised by Ya. G. Sinai in 1979.

The first positive result was obtained in [KSz 85], where a slightly weaker form of recurrence was demonstrated: the process almost surely returns infinitely often to a moderately (actually logarithmically) increasing sequence of domains. The authors used a probabilistic method combined with the dynamical tools of Markov approximations. The weaker form of the recurrence was the consequence of the weaker form of their *local limit theorem*: they could only control the probabilities that the Lorentz process S_n in the moment of n^{th} collision falls into a sequence of moderately increasing domain rather than into a domain of fixed size. These results, moreover, were restricted to the finite horizon case, i. e. to the case when there is no orbit without any collision.

A novel –and surprising– approach appeared in 1998-1999, when independently Schmidt [Sch 98] and Conze [Conze 99] were, indeed, able to deduce the recurrence from the global central limit theorem (CLT) of [BS 81] by adding (abstract) ergodic theoretic ideas. Their approach seems to be essentially restricted to the finite horizon case and to $d = 2$. Our main aim is to return to the probabilistic-dynamical approach and we can first prove a true local central limit theorem (LCLT) for the planar Lorentz process S_n.

LCLT's for functions of a Markov chain were first obtained by Kolmogorov in 1949 [Kol 49] using probabilistic ideas. Then, in 1957, Nagaev, [Nag 57] –by using operator valued Fourier transforms and perturbation theory– could find a general form of LCLT's for functions of a Markov chain. Independently, variants of this method got later rediscovered and/or applied A) by Krámli and Szász [KSz 83] to prove a LCLT for random walks with internal states, B) by Guivarch and Hardy [GH 88] in the setting of Anosov diffeomorphisms C) by Roussean-Egele, [R-E 83], Morita [Mor 94] and Broise [Bro 96] for expanding maps of the interval and finally D) by Aaronson and Denker, [AD 01] in the setting of Gibbs-Markov maps.

The recurrence itself also has an additional interesting conclusion. Let us note first that for the Lorentz process strong stochastic properties, like correlation decay, limit laws, etc. could only be obtained in the case of a periodic configuration of scatterers for then its factor is a Sinai billiard. For this same case, however, it is an interesting question whether the Lorentz dynamics is ergodic without this factorisation as well (N. B. in this case the invariant measure is infinite!). An old result of Simányi, [Sim 89] states equivalence of recurrence and ergodicity of the infinite measure in the case of the Lorentz-process.

1.1.4 The case of infinite horizon

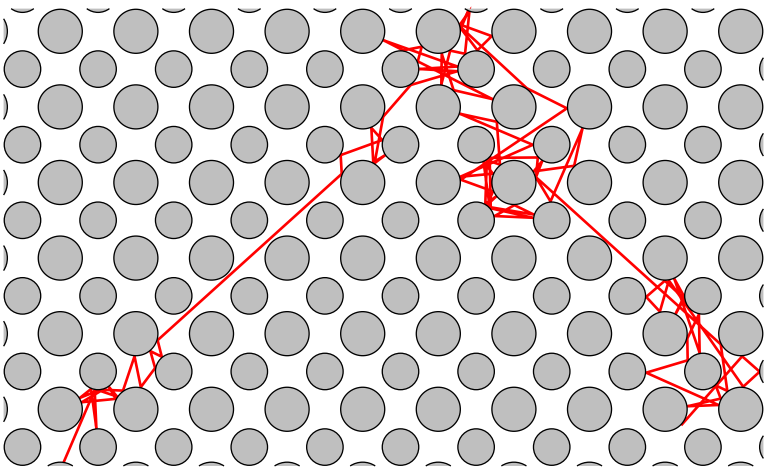


Figure 1.2: 100 collisions Lorentz trajectory-segment with circular scatterers and infinite horizon

Turning to the infinite horizon case it had been observed in the physical literature (cf. [FM 84], [BD 85], [Bun 85] and [ZGNR 86]) that, an anomalous diffusive behaviour can appear. The possibility that the moving particle can flight freely arbitrarily long, causes these free flights to dominate the displacement: the sum of long free-flights grow slightly faster then the sum of bounded ones causing a *superdiffusive* effect. Quantitatively this is related to its tail behaviour, that is to the behaviour of the free flight vector in the neighbourhood of collision-free orbits.

Bleher in 1992 showed in his partially rigorous, partially heuristic paper [Bleher] that the asymptotic behaviour of the displacement S_n, taken in the moment of the n^{th} reflection from a scatterer, is slightly superdiffusive and $\frac{S_n}{\sqrt{n \log n}}$ was expected to possess a limiting Gaussian distribution. It also worth mentioning that the discrete-time behaviour is now not obviously connected to the physical time, since the ratio (the length of the free flight vector) is unbounded. Bleher also showed (relying on the global limit theorem for the discretised system) that in physical time the same scaling $\sqrt{t \log t}$ has to be applied in order to get a Gaussian limit.

In this case not even the global limit theorem was established rigorously. Our first aim is this global limit theorem, and then –as in the finite horizon case, but with a different scaling– to deduce a local limit theorem and establish recurrence.

1.2 Results

1.2.1 Limit theorems for finite horizon

In the case of the finite-horizon Lorentz process for the discretised displacement $S_n = \sum_{i=0}^{n-1} \kappa \circ T^i$ the following holds:

Theorem 1. *Let $k_n \in \mathbb{Z}^2$ be such that $\frac{k_n}{\sqrt{n}} \to k \in \mathbb{R}^2$. If the horizon is finite then the following holds:*

$$\lim_{n\to\infty} n \cdot \mu\{S_n = k_n\} = \frac{e^{-\frac{1}{2}k\Sigma^{-1}k^T}}{2\pi \det \Sigma}$$

for a nondegenerate covariance-matrix Σ.

For the non-discretised displacement the following statement holds. Let ν_n denote the distribution of the Birkhoff-sum $\sum_{i=0}^{n-1} \psi \circ T^i$.

Theorem 2. *Let $v_n \in \mathbb{R}^2$ be such that $\frac{v_n}{\sqrt{n}} \to v \in \mathbb{R}^2$. If the horizon is finite then the following holds:*

$$\lim_{n\to\infty} n \cdot \nu_n = \frac{e^{-\frac{1}{2}v\Sigma^{-1}v^T}}{2\pi \det \Sigma} \; c_\mu^2 \cdot \sharp \star m_+ \star m_-$$

where $\sharp$ is the counting measure on $\mathbb{Z}^2$, m_+ is arclength measure on the boundary of the scatterers inside the fundamental domain $\partial\mathcal{O} \cap D$, m_- is the distribution of the opposite of an m_+ distributed vector. The convergence is meant in the weak topology of measures, and $\star$ stands for convolution.

These two theorems are deduced from a version of the local limit theorem stating even more than that:

Theorem 3. *Let $k_n \in \mathbb{Z}^2$ be such that $\frac{k_n}{\sqrt{n}} \to k \in \mathbb{R}^2$. Let Υ_n denote the joint distribution of the triple $(S_n(x) - k_n, x, T^n x)$. If the horizon is finite then the following holds:*

$$\lim_{n\to\infty} n \cdot \Upsilon_n = \frac{e^{-\frac{1}{2}k\Sigma^{-1}k^T}}{2\pi \det \Sigma} \; \sharp \cdot \mu \cdot \mu$$

where $\sharp$ is counting measure on $\mathbb{Z}^2$, and the convergence is meant in the weak topology of measures.

For to prove the recurrence we had to prove also an asymptotic independence statement, which was given in the form of a joint limit theorem.

Theorem 4. *Let $j_n \in \mathbb{Z}^2$ be such that $\frac{j_n}{\sqrt{n}} \to j \in \mathbb{R}^2$, and $k_n \in \mathbb{Z}^2$ be such that $\frac{k_n}{\sqrt{n}} \to k \in \mathbb{R}^2$. If the horizon is finite, then*

$$\lim_{m,n-m\to\infty} m \cdot (n-m) \cdot \mu\{S_m = j_m, S_n = j_m + k_{n-m}\} = \frac{e^{-\frac{1}{2}\left(j\Sigma^{-1}j^T + k\Sigma^{-1}k^T\right)}}{4\pi^2 \det^2 \Sigma}$$

This last theorem also has the form involving the independence of the phase points $x, T^m x, T^n x$ but stating that would be superfluous.

1.2.2 Limit theorems for infinite horizon

As mentioned before the free flights in the neighbourhood of collision-free orbits dominate the Birkhoff-sum, so we can describe the covariance matrix of the limiting Gaussian law in terms of geometric constants. For this purpose let us define the set of corridor points as those periodic points, whose trajectory is always tangent.

$$\mathcal{C} = \{x \in \partial X \mid \exists i \quad T^i x = x \quad \forall j \quad T^j x \in \partial X\}$$

Let us define the matrix Σ as

$$\sum_{x\in\mathcal{C}} \frac{c_\mu d_x^2}{2|\psi(x)|} \begin{pmatrix} \psi_1^2(x) & \psi_1(x)\psi_2(x) \\ \psi_1(x)\psi_2(x) & \psi_2^2(x) \end{pmatrix} \tag{1.1}$$

where $\psi = (\psi_1, \psi_2)$ is the notation for the component functions, and d_x is the width of the corridor described in section 2.1 (cf. figure 2.1).

For both the discretised and the non-discretised free flight vector the Birkhoff-sum i. e. the displacement S_n satisfies the global limit theorem with $\sqrt{n \log n}$ scaling.

Theorem 5. *Suppose that the corridor free flights $\{\psi(x) \mid x \in \mathcal{C}\}$ span the plane. Then*

$$\mu\left\{\frac{S_n}{\sqrt{n\log n}} \in A\right\} \to \int_A \varphi(k)$$

where φ is a nondegenerate normal density function with zero expectation and covariance matrix Σ

If the corridor free flights does not span the plane, but the horizon is infinite one should apply anisotropic scaling in order to get a nondegenerate Gaussian limit. Here we do not know the exact form of the limiting covariance matrix.

Theorem 6. *Suppose that all the corridor free flight vectors are falling in the same direction* $\psi(x) \parallel \vec{v}$ $(\forall x \in \mathcal{C})$*. Define the matrix* $B_n = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{n\log n} \end{pmatrix}$ *in the orthogonal basis* $(\vec{v}^\perp, \vec{v})$*. Then*

$$\mu\{S_n B_n \in A\} \to \int_A \varphi(k)$$

where φ *is a nondegenerate normal density function with zero expectation.*

Turning to the local limit theorems:

Theorem 7. *Suppose that the corridor free flights* $\{\psi(x) \mid x \in \mathcal{C}\}$ *span the plane. Let* $k_n \in \mathbb{Z}^2$ *be such that* $\frac{k_n}{\sqrt{n\log n}} \to k \in \mathbb{R}^2$*. Then*

$$n\log n \;\; \mu\{S_n = k_n\} \to \varphi(k)$$

where φ *is a nondegenerate normal density function with zero expectation and covariance matrix* Σ*.*

As in the finite horizon case we also need an asymptotic independence statement.

Theorem 8. *Suppose that the corridor free flights* $\{\psi(x) \mid x \in \mathcal{C}\}$ *span the plane. Let* $j_n \in \mathbb{Z}^2$ *be such that* $\frac{j_n}{\sqrt{n\log n}} \to j \in \mathbb{R}^2$*, and* $k_n \in \mathbb{Z}^2$ *be such that* $\frac{k_n}{\sqrt{n\log n}} \to k \in \mathbb{R}^2$*. If the corridor free flights span the plane, then*

$$\lim_{m,n-m\to\infty} m\log m \;\; (n-m)\log(n-m) \;\; \mu\{S_m = j_m, S_n = j_m + k_{n-m}\} = \varphi(j)\varphi(k)$$

where ϕ *is a Gaussian density with zero expectation and covariance matrix* Σ*.*

We have all the other forms of the local limit theorem i. e. for the joint distribution of $(S_n(x), x, T^n x)$, for the non-discretised free-flight and so on.

1.2.3 Recurrence and ergodicity

For any planar Lorentz-process (both finite and infinite horizon) for the discretised displacement S_n the following holds:

Theorem 9 (Recurrence)**.**

$$\mu(\exists n_k \to \infty \quad S_{n_k} = 0) = 1$$

As a corollary of this theorem and [Sim 89]:

Theorem 10. *The invariant (infinite) measure of the Lorentz process* $d\tilde{\mu} = \cos v dq dv$ *(where* dq *is the arclength measure on the boundary of infinitely many scatterers) is ergodic i. e. for any invariant set* A *either* $\tilde{\mu}(A) = 0$ *or* $\tilde{\mu}(\tilde{X} \setminus A) = 0$*.*

1.2.4 Structure of the thesis

Here we give a concise summary or strategy of the proofs. In the infinite horizon case one has to understand the behaviour of the free-flight function in that part of the phase-space where it becomes unbounded. Chapter 2 is devoted to this geometrical study.

The general aim in both cases is to handle the Fourier transform of the Birkhoff-sum. For that purpose we will need a symbolic dynamics with a spectral gap in the transfer-operator. This was done by Young, who introduced her famous tower construction. In the tower construction there is the definition of function spaces, which contain all functions associated with Hölder observables on the original phase-space, and the transfer operator has a gap on this function-space. These techniques are described in chapter 3.

In chapter 4 we define the Fourier-transform operator P_t, which has the property that acting on the invariant density ρ the integral $\int P_t^n(\rho)$ is the Fourier transform of the Birkhoff-sum at t. For small values of t, P_t can be considered as a perturbation of P, since $P_0 = P$. Then one, in general, proves that P_t possesses a gap between the leading simple eigenvalue λ_t and the rest of the spectrum and the gap is uniformly bounded away from zero.

For large values of t one needs to know exactly for which t values will the unit circle intersect the spectrum. In our case for the continuous free flight function this occurs when $t \in 2\pi\mathbb{Z}^2$. For that reason we switched to the discretised function κ, since the latter one is an integer valued vector function, and thus $P_t = P$ when $t \in 2\pi\mathbb{Z}^2$, so we can factorise, and consider $t \in 2\pi\mathbb{T}^2$. This is the question of minimality which is completely described in section 4.2.

Moreover after proving the spectral gap for small values of t, we have a simple asymptotics $P_t^n = \lambda_t^n + O(\vartheta^n)$, for a suitable $\vartheta < 1$. Consequently, the characteristic function of the dependent sum can be approximated with a power. Thus it is only the asymptotics of λ_t that has remained to be investigated. In the finite horizon case this is done via the second order (operator coefficient) Taylor-polynomial of P_t in section 4.3.1.

The second order Taylor-expansion of P_t does not exist in the infinite horizon case, one needs a completely different approach. A mostly geometric approach of the problem was sketched in [SzV 04b]. However, before we had completed our work with the technical proof of limit theorems in the infinite horizon case, there appeared a much interesting work of Bálint and Gouëzel, [BG 06]: for the stadium billiard they gave a quite analytic proof of a global limit theorem which also uses the $\sqrt{n \log n}$ scaling. The coincidence of scalings is explained by the analogous behaviour of long free flights in our model (i. e. in corridors of the Lorentz process) and that of the quasi integrable trajectories between the linear sides of the stadium billiard. The arguments of [BG 06] also helped us to simplify our approach substantially at essentially three points:

1. once one has a tower construction à la Young, their Lemma 3.5 (cf. our Theorem 29) provides a general, concise condition for the validity of a non-standard Gaussian limit law;

2. they reduce the "tower-sums" to a more tractable, still dominant part;

3. for describing excursions from the tail they use a delicate result of Chernov, [Ch 99] (see in our Lemma 32).

It is a question, though, whether Chernov's result, a beautiful but quite strong tool, is indeed, necessary in this proof. We formulate the main theorem of [BG 06], and check the conditions in section 4.3.2. These results prove theorem 5, and give also the expansion of λ_t, a key ingredient for the local limit theorems.

Finally we give the proofs of our local limit theorems and recurrence in chapter 5.

Chapter 2

Geometrical study of infinite horizon

We are going to describe that part of the phase space, where the free-flight vector becomes unbounded.

2.1 The corridors, their geometry and the tail of the free flight

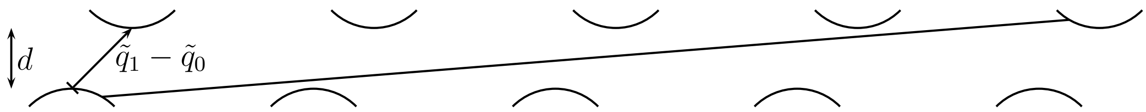


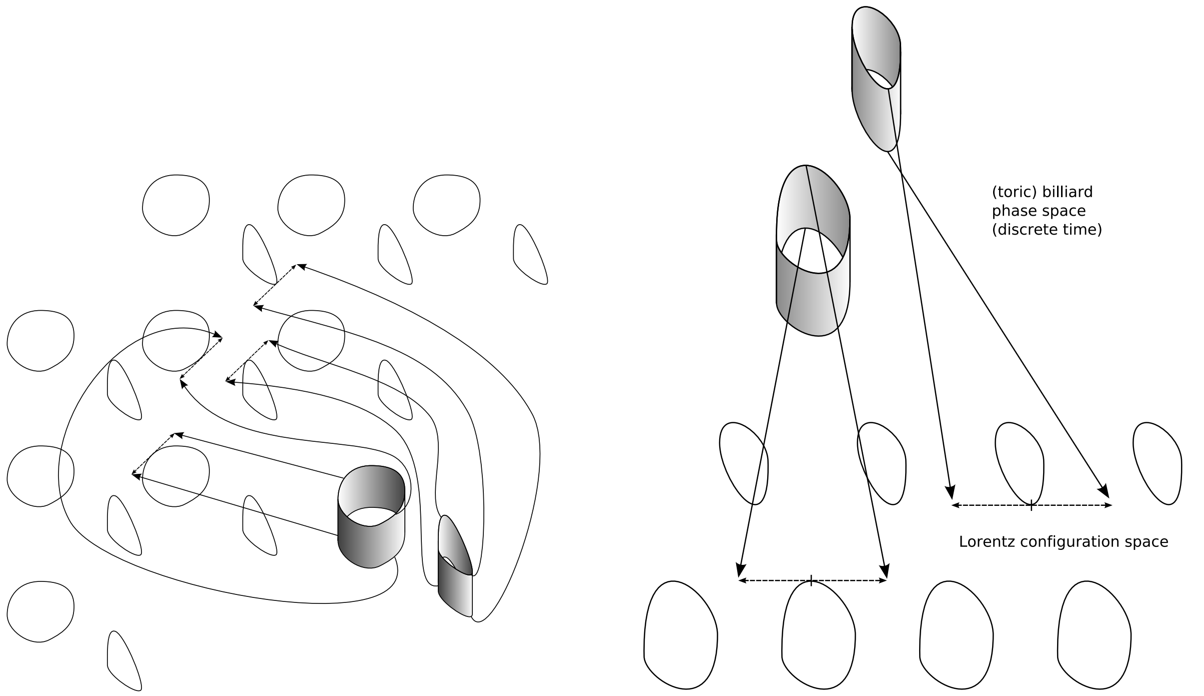
Figure 2.1: Free-flight crossing a corridor, and some geometric constants

In the infinite horizon case the only reason for the unboundedness of the free flight is the presence of *corridors*. These are bi-infinite strips in the billiard-table $\mathbb{R}^2 \setminus \tilde{\mathcal{O}}$. The strips are tangent to the obstacles, and their slope is necessarily rational, and, moreover, - up to $\mathbb{Z}^2$ translations - there are finitely many of them. We will suppose the - geometrically generic - condition: for each corridor, and each side of the corridor the tangent obstacles are the images of a *single* scatterer under $\mathbb{Z}^2$ translations. On figure 2.2(a) is a counterexample. Our results are also valid in the excluded cases, but the geometric constants, which we will calculate would have a more complicated form.

For such a corridor there are four corresponding points in the phase-space, as shown on figure 2.2(b). These points are on the boundary of X, and are fixed by the dynamics. (Without the previous condition these would be only periodic points.) Outside of the neighbourhood of these points the free-flight is bounded.

Let us fix such a fixed point on the boundary as $x_0 = (q_0; v_0)$, where v_0 is either $\pi/2$ or $-\pi/2$. Let us denote by $\mathcal{O}_0$ the obstacle on which q_0 is placed. The free flight $\psi(x_0)$ is a lattice vector $\psi(x_0) = \kappa(x_0)$, since $Tx_0 = x_0$. Denote $\kappa(x_0) = w_0$, and the curvature of $\mathcal{O}_0$ at q_0 by ξ_0. Denote the considered (small enough) neighbourhood of x_0 by U_0.

In U_0 (means close enough to x_0) two types of nonsingular collisions can happen. First when the moving particle is "crossing" the corridor (see figure 2.1). In this case the free-flight is long (actually this is the only case), the closer the phase point lies to x_0 the longer the free-flight can be. The next collision happens on the "other side" of the corridor. To



(a) A corridor failing the fixed-point condition. Both side consists of corridor points of period two.

(b) A corridor, and the corresponding four phase points.

make it precise let us denote by $x_1 = (q_1; v_1)$ the phase point which corresponds to the same corridor as x_0, but $v_1 = -v_0$ and $q_1 \neq q_0$ (see figure 2.2(b)). Denote by $\vec{Q}_0$ the planar lattice vector $\iota(\tilde{q}_1) - \iota(\tilde{q}_0)$, where the lifting is such that $\tilde{q}_1 - \tilde{q}_0$ "crosses" the corridor (see figure 2.1). We also need to define the "width" of the corridor d which is the length of that component of $\tilde{q}_1 - \tilde{q}_0$ which is perpendicular to w_0.

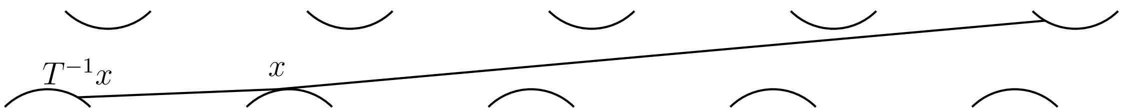


Figure 2.2: If phase point $T^{-1}x$ is such that $\kappa(x) = w_0$, then if $x \in U_0$ the free flight starting in $q(x)$ "crosses" the corridor.

The other type of nonsingular trajectory in U_0 is when the next collision is on the "same side" of the corridor i. e. $\kappa = w_0$. The phase point Tx is then again close to x_0, but this time it is of the first type, so consecutive "same side" collisions cannot happen. This can be seen on figure 2.2.

Proposition 11. *Let $U_0 \subset X$ be a sufficiently small neighbourhood of x_0 then*

$$\mu\{x \in U_0 \mid \kappa(x) = Nw_0 + \vec{Q}_0\} \sim c_\mu d^2 |w_0|^{-1} N^{-3}$$

The range of κ in U_0 is $\vec{Q}_0 + w_0\mathbb{Z}^+$ with possibly finitely many exceptions.

We postpone the sketchy proof for the first fact to section 2.1.2. The second statement was essentially proved in the above text.

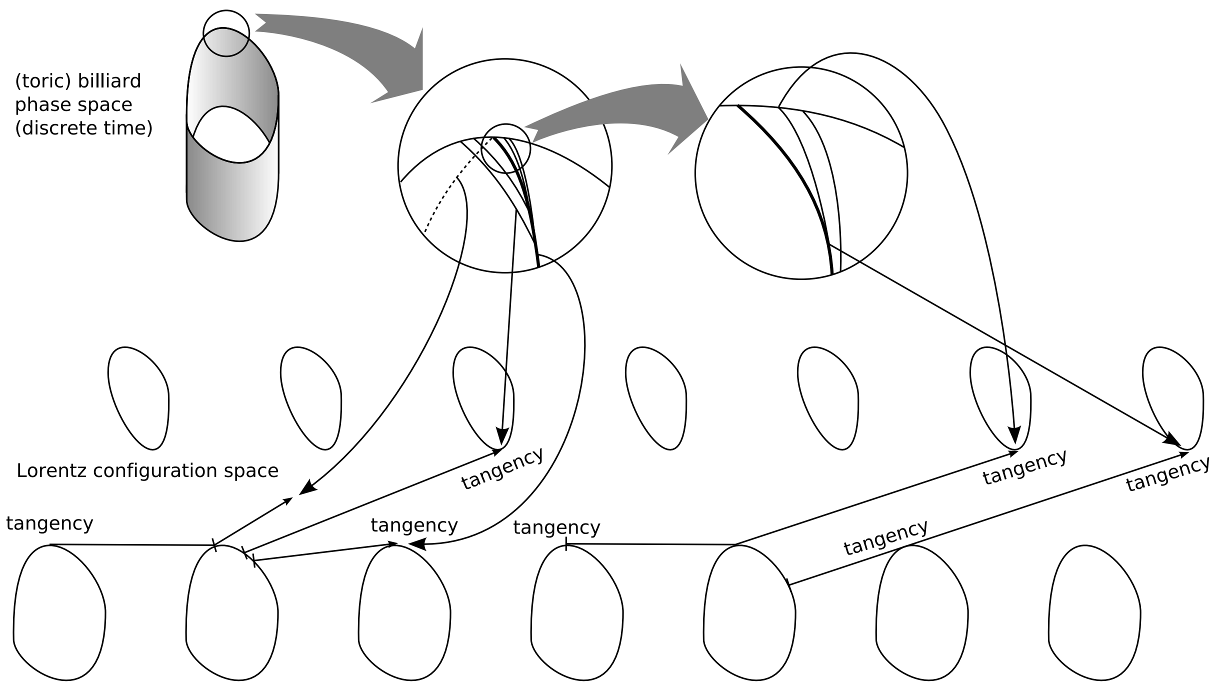


Figure 2.3: Singularity structure and singular trajectories near x_0

2.1.1 The singularity structure

We are going to describe the singularity structure, and the type of singular trajectories in U_0. The importance of this is that singularities bound the sets for which we want to derive measure estimates. On figure 2.3 we also plot some trajectory-segments, the configuration component of the corresponding phase point is denoted by a small tick perpendicular to the trajectory.

There is a singularity curve from S_{-1} which is a preimage of tangential collisions, denoted by thick line in figure 2.3, starting from x_0. This consists of phase points where the next collision will be tangential on $\mathcal{O}_0 + w_0$. This is called the "main" singularity.

There are singularity curves printed on the left of this one, starting on the boundary of the phase space and ending on the main singularity. These consist of phase points where the free flight "crosses" the corridor, and the next collision is tangential. Therefore this is also a part of S_{-1}. A level-set of the discretised free-flight function κ consists of a curvilinear rectangle bounded by two neighbouring curves from this singularity family, the edge of the phase space and the main singularity.

On the right of the main curve there are some curves from S_{-2}. These also start on the boundary and end on the main curve, but unlike the previous ones, these curves have zero angle with the main line. These consist of phase points, for which the first collision is on $\mathcal{O}_0 + w_0$, and the next one is tangential after crossing the corridor. On the right of the main singularity the next collision for any phase point occurs on $\mathcal{O}_0 + w_0$, therefore κ is constant w_0 in this half of U_0.

These two families of singularity curves have an infinite number of pieces accumulating in x_0. The closer the curve is to x_0 the further the tangential collision occurs, after the moving particle has crossed the corridor. All these singularity curves (including the main one) have $-\xi_0$ as slope on the boundary of the phase space.

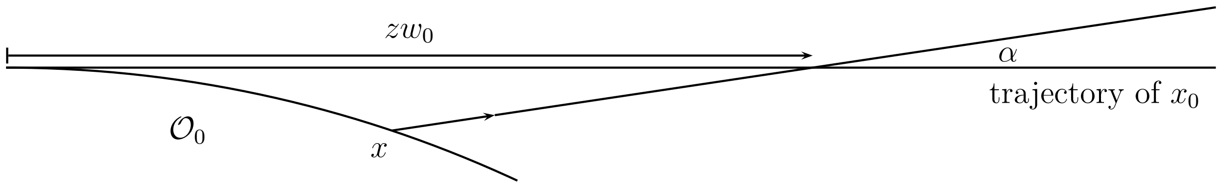


Figure 2.4: The new coordinates z, α.

The last singularity curve we want to describe is from S_1. It consists of phase points, where the previous collision was tangential on $\mathcal{O}_0 - w_0$. This is the image of the half of $\partial X \cap U_0$, namely that half where $\kappa = w_0$ (on the right of x_0 on figure 2.3). Consequently (since x_0 is fixed) this curve starts at x_0. This is drawn with a dotted line on figure 2.3. This curve has slope 2 on the boundary. The sign of the second derivatives of all the singularity curves can be read from the picture.

2.1.2 New coordinates, and the joint distribution of κ, $\kappa \circ T^{-1}$

In this subsection some proofs will be omitted, some will be sketchy or require further estimates. However, since the missing parts rely on simple but tedious geometrical calculations and the application of these results in later sections does not need sharp estimates, we intended to keep this section not too long.

Instead of using the original (q_0, v_0) coordinates we are going to introduce new (α, z) coordinates in U_0. The *new coordinates* z and α are shown on figure 2.4 (these coordinates are also different from those of [Bleher]). During the free-flight, bouncing off the scatterer $\mathcal{O}_0$, the trajectory of x crosses the trajectory of x_0. This crossing point is therefore $q_0 + zw_0$ for some $z \in \mathbb{R}$. The reader should convince himself that the coordinate z is, in fact, a periodic one with period 1. The other coordinate α is the angle $\measuredangle(w_0, \psi(x))$.

The reason for this change is that these coordinates are more suitable for computations in relation with the free-flight since they are more intrinsically related to the geometry of the model, especially asymptotically (when the free flight goes to infinity). For example, the free flight has the asymptotic form $|\kappa| \sim \frac{d}{\alpha}$ where d is the 'width' of the corridor (cf. fig. 2.1). Also, the invariant measure is asymptotically equal to $c_\mu |\alpha||w_0| dz d\alpha$.

We note that the crossing point (which was the base of this coordinatisation) does not exist when the next collision occurs on $\mathcal{O}_0 + w_0$. So these new coordinates map only the half of U_0 to the (z, α) plane. Namely that half which is drawn on the left of the main singularity on figure 2.3. We will denote this part by U_0' (remember that x_0 and U_0 are fixed). However, this restriction does not influence the study of asymptotics, since we miss only the w_0-level set of κ inside U_0.

Proof of Proposition 11. The level-set of κ is a curvilinear rectangle in the (z, α) plane (cf. figure 2.5). We are going to multiply the height, the width and the density to get the measure. The width is simply 1. The height can be obtained from the formula $\alpha \sim \frac{d}{|\kappa|}$. Writing $\alpha' \sim \frac{d}{|\kappa| + |w_0|}$ we get $\alpha - \alpha' \sim \frac{d|w_0|}{|\kappa|^2}$. The density is $c_\mu |\alpha||w_0|$ and by substituting α we get $c_\mu d \frac{|w_0|}{|\kappa|}$. So the measure is $\sim \frac{c_\mu d^2 |w_0|^2}{|\kappa|^3}$. □

Let us explain how this (z, α) image on figure 2.5 is related to the phase portrait on figure 2.3 explained before. The largest curvilinear rectangle on figure 2.5 is the image

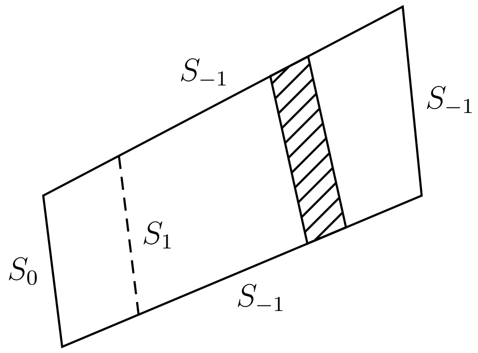


Figure 2.5: Level set of κ inside U_0, and its intersection with a level set of $\kappa \circ T^{-1}$ (in the (z, α) coordinate plane)

of a level-set of κ under the (z, α) coordinate-mapping. This level set is bounded by the boundary of the phase-space on the left, two singularity curves from the first family on the top, and on the bottom, and the main singularity on the right. There is also the dashed line, which is the singularity line from S_1, already explained before, too.

This latter line plays an important role in the joint distribution. On the left of this line $\kappa \circ T^{-1} = w_0$. On the right of this line the mapping T^{-1} takes values in the neighbourhood of another corridor-phase-point x_1 (see subsection 2.1). We are going to use w_1, ξ_1, U_1, U_1' for the point x_1 in the same sense as w_0, ξ_0, U_0, U_0' have been used for the point x_0. Using this notation, on the right of the line we are describing now, there lies TU_1'.

Talking about the joint distribution in terms of our new coordinate functions, note that the α coordinate function is defined in U_0'. Therefore $\alpha \circ T^{-1}$ in the domain $U_0' \cap TU_1'$ has to be meant as applying the same coordinatisation rule in U_1. Since $w_1 = w_0$, the α coordinate functions in U_0 and U_1 are comparable also as absolute angles: the observer only has to change signs.

By definition the sign of α is positive, and z is mostly positive. More precisely remember that the asymptotic form of the invariant measure does not depend on z meaning that in the $|\kappa| \to \infty$ limit the distribution of z is uniform. Now consider the range of z in the domain $\kappa = \vec{N}$:

$$z_{\min} \stackrel{\text{def}}{=} \min\{z(x) \mid x \in U_0, \quad \kappa(x) = \vec{N}\}$$

and respectively

$$z_{\max} \stackrel{\text{def}}{=} \max\{z(x) \mid x \in U_0, \quad \kappa(x) = \vec{N}\}$$

Proposition 12. *The asymptotics of the range of z in the domain $\kappa = \vec{N}$ when $\vec{N}$ is in the range of κ (see proposition 11) and $|\vec{N}| \to \infty$ is*

$$z_{\min} \sim 1 - z_{\max} \sim \frac{d}{2|w_0|\xi_0|\vec{N}|}$$

Easy geometrical calculations yield the collision equation:

Proposition 13. *On $U_0' \cap TU_1'$:*

$$\alpha \circ T^{-1} \sim -\alpha + 2\sqrt{\alpha^2 + 2\alpha z|w_0|\xi_0} \qquad (\alpha \to 0)$$

Proposition 14. *In U_0 the following holds:*

$$\min|\kappa|\circ T^{-1}\sim\sqrt{\frac{d|\kappa|}{8\xi_0|w_0|}}\quad \textit{and}\quad \max|\kappa|\circ T^{-1}\sim\frac{8\xi_0|w_0||\kappa|^2}{d}\qquad |\kappa|\to\infty$$

Proof. Substituting the asymptotic maximum of z ($\max z\to 1$ as $\alpha\to 0$) to the collision equation and omitting non-dominant terms we get $\max\alpha\circ T^{-1}\sim 2\sqrt{2\alpha|w_0|\xi_0}$. Since $\alpha\sim\frac{d}{|\kappa|}$ and $\alpha\circ T^{-1}\sim\frac{d}{|\kappa\circ T^{-1}|}$, substituting α and $\alpha\circ T^{-1}$ and then rearranging yields the first statement of the proposition. Using the time-reversion symmetry for this formula we get the second one. □

We can also compute the joint distribution of $(\kappa,\kappa\circ T^{-1})$.

Proposition 15.

$$\mu(\{\kappa=\vec{N}\}\cap\{\kappa\circ T^{-1}=\vec{M}\}\cap U_0)\lesssim\frac{c_\mu d^3|w_0|^2}{4\xi_0}\;\frac{|\vec{N}|+|\vec{M}|}{|\vec{N}|^3|\vec{M}|^3}$$

Proof. For a nonempty intersection we are going to multiply the height the width and the density to get the measure. The density and the height are the same as in the proof of proposition 11. For the width consider the derivative of the collision equation

$$\frac{\partial\alpha\circ T^{-1}}{\partial z}\sim\frac{2\alpha|w_0|\xi_0}{\sqrt{\alpha^2+2\alpha z|w_0|\xi_0}}$$

To express the square root in terms of κ and $\kappa\circ T^{-1}$ we can rearrange the collision equation, and substitute α and $\alpha\circ T^{-1}$:

$$\frac{d}{|\kappa\circ T^{-1}|}+\frac{d}{|\kappa|}\sim 2\sqrt{\alpha^2+2\alpha z|w_0|\xi_0}$$

We can express the increment of z as the inverse of the derivative multiplied by the increment of $\alpha\circ T^{-1}$. That is

$$\frac{\frac{d}{|\kappa\circ T^{-1}|}+\frac{d}{|\kappa|}}{4\frac{d}{\kappa}|w_0|\xi_0}\;\frac{d|w_0|}{(\kappa\circ T^{-1})^2}\sim d\frac{\kappa+\kappa\circ T^{-1}}{4\xi_0(\kappa\circ T^{-1})^3}$$

Hence the proposition. □

The measure of the set $\{\kappa=\vec{N}\}\cap\{\kappa\circ T^{-1}=\vec{M}\}\cap U_0$ can be zero when $\vec{N}$ or $\vec{M}$ is not in the range of κ inside U_0 (see proposition 11) or they fail the range inequality (we only gave the asymptotics of this in proposition 14 roughly $c_3\sqrt{|\vec{N}|}<|\vec{M}|<c_4|\vec{N}|^2$). It can be also smaller than the expression given in proposition 15 when the pair $(\vec{N},\vec{M})$ is close to the boundary of the range inequality, but we do not want to formulate the validity precisely, we just mention that in most part of the domain the inequality is sharp. It can also be checked by summing the right hand side, and getting 1 in the limit.

What this essentially means is that the previous free-flight $|\kappa\circ T^{-1}|$ is mostly in the range of $\sqrt{|\kappa|}$. The measure of being in any other range can be estimated from above with $|\kappa|$ powers. To formulate precisely what we will use later in subsection 4.3.3:

Proposition 16.

$$\mu(\{\kappa = \vec{N}\} \cap \{\kappa \circ T^{-1} > |\kappa|^{\frac{3}{4}}\} \cap U_0) = O(|\vec{N}|^{-3.5})$$

Proof. $\sum_{M=N^{\frac{3}{4}}} \frac{N+M}{N^3 M^3} = O(N^{-3.5})$ □

The other level set of $\kappa \circ T^{-1}$ which intersects U_0 is the $\kappa \circ T^{-1} = w_0$ set.

Proposition 17.

$$\mu(\{\kappa = \vec{N}\} \cap \{\kappa \circ T^{-1} = w_0\} \cap U_0) = O(|\vec{N}|^{-4})$$

Proof. As before the measure will be estimated with the product of the height, density and the width. It remained to estimate only the width. The domain is the left-hand side of the dashed line on figure 2.5. Going from right to left inside the level set in figure 2.5, the dashed line is reached exactly when $\alpha \circ T^{-1}$ reaches its minimum. According to proposition 14 we have to estimate what is the value of z for which $\alpha \circ T^{-1}$ reaches const $\cdot \, \alpha^2$. Exact calculations based on the derivative of the mapping would give $O(|\vec{N}|^{-5})$ in the right hand side of the proposition, but here it is sufficient to observe that this singularity line denoted by the dashed curve is on the left of the $z = 0$ line, where $\alpha = \alpha \circ T^{-1}$. So the width can be estimated by $z_{\min}$ which was given in proposition 12. □

2.2 Non-normal domain of attraction of the normal law

It is well known (cf. [F 66]) that a random variable R with distribution function $\mathbb{P}_R$ belongs to the domain of attraction of a normal distribution if its characteristic function satisfies

$$\log \int e^{itu} d\mathbb{P}_R(u) = it\nu - \frac{1}{2} t^2 L(1/|t|)$$

for some constant $\nu \in \mathbb{R}$ and a slowly varying function $L : \mathbb{R}^+ \to \mathbb{R}^+$ which is bounded below.

The normal (or classical) domain of attraction consists of the class L_2, and is characterised by the boundedness of the slowly varying function L. In the "non-normal" domain of attraction the function L is unbounded and is determined (up to asymptotic equivalence) by the tails of the distribution:

$$\begin{aligned} 1 - \mathbb{P}_R(x) &\sim c_1 x^{-2} l(x) \\ \mathbb{P}_R(x) &\sim c_2 x^{-2} l(x) \quad x \to \infty \end{aligned}$$

for some constants $c_1, c_2 > 0, c_1 + c_2 = 1$ and some slowly varying function l, which in turn determines L by

$$L(x) = \int_{-x}^{x} u^2 \; d\mathbb{P}_R u.$$

and it follows that $l(x) = o(L(x))$.

For a random variable R in the non-normal domain of attraction of the normal law, the independent sum of $\mathbb{P}_R$-distributed random variables S_n^* satisfies the limit theorem:

$\frac{S_n^* - n\nu}{B_n} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1)$ as $n \to \infty$ in distribution, where B_n is the normalising sequence defined by the asymptotics $\frac{nL(B_n)}{B_n^2} \to 1$.

The random variable κ is a vector. It was shown in proposition 11 that its component function belongs to the non-normal domain of attraction of the normal law, if that component is not perpendicular to all of the corridor-free-flights w_i. In this case $l(x)$ is a constant function $l(x) \equiv \mathbf{c}$. Since the free-flight is symmetric, we have $c_1 = c_2 = \frac{1}{2}$, and $\nu = 0$. Consequently $L(x) \sim 2\mathbf{c} \log x$, and $B_n = \sqrt{\mathbf{c} n \log n}$ is a normalising sequence. The constant $\mathbf{c}$ depends on which component we are looking at. If we choose $\vec{v} \in \mathbb{R}^2$ a unit vector, then the constant of the $\vec{v}$ component has the following expression in the terms of the geometric constants:

$$\mathbf{c} = \sum_{x \in \partial X | Tx = x} c_\mu d_x^2 \frac{\langle \psi(x), \vec{v} \rangle^2}{2|\psi(x)|}. \tag{2.1}$$

Remember that for such points $\psi(x) = \kappa(x)$, and we used the notation $\psi(x_0) = w_0$ for the fixed corridor we were investigating. Also note, that every term in the above sum appears exactly four times (cf. figure 2.2(b) in section 2.1)!

Since the configuration is planar, the following is true: if the corridor free-flight vectors span the plane, then every component of κ, hence the vector itself is in the non-normal domain. If this is not the case, then one has to apply anisotropic scaling to get a nondegenerate limit distribution. Namely it should be $\begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{n \log n} \end{pmatrix}$ in a basis, where the first element is perpendicular to the corridor free-flights.

The goal of the forthcoming arguments is to establish Bleher's hypothesis: though the (stationary) process of the free flights of our model is not an independent process, nevertheless in many respects the partial sums behave asymptotically the same way as if the variables were independent.

Chapter 3

Young towers

3.1 Construction and basic properties

According to our recent understanding the most efficient way for constructing Markov partitions for billiards is to use Young towers, cf. [You 98]. We are going to give a detailed description about the properties of the tower. Also we are going to give some explanation about the construction. Further details can be found in [You 98] and [Ch 99]. In the latter one, towers for the infinite horizon case are constructed also.

The presence of singularities prevent stable and unstable curves to possess a lower bound for their size in any part of the phase-space. Therefore the product structure - the key ingredient of several hyperbolic argument - can only be introduced in a complicated set.

Let T be a $C^{1+\epsilon}$ diffeomorphism with singularities of a compact Riemannian manifold X with boundary. More precisely, there exists a finite or countably infinite number of pairwise disjoint open regions $\{X_i\}$ whose boundaries are C^1 submanifolds of codimension 1, and finite volume such that $\cup X_i = X$, $T\big|_{\cup X_i}$ is $1-1$ and $T\big|_{X_i}$ can be extended to a $C^{1+\epsilon}$-diffeomorphism of $\bar{X}_i$ onto its image. Then $S_{-1} = X \setminus \cup X_i$ is the *singularity set*. The Riemannian measure will be denoted by m, and if $W \subset X$ is a submanifold, then m_W will denote the induced measure. The invariant Borel probability measure will be denoted by μ.

Definition 2. *An embedded disk $\gamma \subset X$ is called an* unstable manifold *or an* unstable disk *if $\forall x, y \in \gamma$, $d(T^{-n}x, T^{-n}y) \to 0$ exponentially fast as $n \to \infty$; it is called a* stable manifold *or a* stable disk *if $\forall x, y \in \gamma$, $d(T^n x, T^n y) \to 0$ exponentially fast as $n \to \infty$. We say that $\Gamma^u = \{\gamma^u\}$ is a* continuous family of C^1 unstable disks *if the following hold:*

- *K^s is an arbitrary compact set; D^u is the unit disk of some $\mathbb{R}^n$;*
- *$\Phi^u \colon K^s \times D^u \to X$ is a map with the property that*
  - *Φ^u maps $K^s \times D^u$ homeomorphically onto its image,*
  - *$x \to \Phi^u \mid (\{x\} \times D^u)$ is a continuous map from K^s into the space of C^1 embeddings of D^u into X,*
  - *γ^u, the image of each $\{x\} \times D^u$, is an unstable disk.*

Continuous families of C^1 stable disks *are defined similarly.*

Definition 3. *We say that* $\Lambda \subset X$ *has a* hyperbolic product structure *if there exist a continuous family of unstable disks* $\Gamma^u = \{\gamma^u\}$ *and a continuous family of stable disks* $\Gamma^s = \{\gamma^s\}$ *such that*

(i) $\dim \gamma^u + \dim \gamma^s = \dim X$

(ii) the γ^u*-disks are transversal to the* γ^s*-disks with the angles between them bounded away from* 0*;*

(iii) each γ^u*-disk meets each* γ^s*-disk in exactly one point;*

(iv) $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$.

The construction of such a hyperbolic set in the billiard phase space is roughly the following. First of all we have to mention that the singularity set S_{-1} contains not only the preimage of the boundary, but also, the so called secondary singularities. These are introduced around $T^{-1}\partial X$ to cut the phase space into infinitely many domains. Inside these domains the distortion bounds will hold, which would not be the case without them.

First choose an unstable curve W, which is short enough to ensure: a high amount of the points possesses unstable curve of this length. Then define a subset of this curve consisting of points, which remain a certain (exponentially shrinking) distance apart from S_{-1}.

$$\Omega_\infty := \{y \in W \mid d(T^n y, S_{-1}) > \delta_1 \lambda^{-n} \quad \forall n \geq 0\}$$

where λ is the hyperbolicity constant. If δ_1 is chosen small enough this set has positive measure. By construction each point in Ω_∞ possesses a stable curve of length δ_1.

So far we have one unstable curve W, and a family of stable curves $\{\gamma^s\}$. Let us consider all the nearby unstable curves, which are long enough, and intersect all the stable curves in the previous family. These two families of curves $\{\gamma^s\}$ and $\{\gamma^u\}$ define the hyperbolic product-set $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$.

This set is going to be the base of the hyperbolic Young-tower. To continue the construction of the tower we are going to focus on recurring subsets of Λ. On figure 3.1 we can see that some parts of Λ are mapped to Λ. However we are only interested in those returns, which respect the product structure.

Definition 4. *Suppose* Λ *has a hyperbolic product structure. Let* Γ^u *and* Γ^s *be the defining families for* Λ*. A subset* $\Lambda_0 \subset \Lambda$ *is called an* s-subset *if* Λ_0 *also has a hyperbolic product structure and its defining families can be chosen to be* Γ^u *and* Γ^s_0 *with* $\Gamma^s_0 \subset \Gamma^s$*;* u-subsets *are defined analogously. For* $x \in \Lambda$*, let* $\gamma^u(x)$ *denote the element of* Γ^u *containing* x*.*

We can see three intersections on the figure, the lower and upper ones are u-subsets. Talking about these intersections black covers grey in the unstable direction (when the reader sees black in these intersections, then on that unstable curve black covers grey). On the contrary grey covers black in the stable direction (on each stable line black can appear only where grey is already there). The inverse image of each of these two intersections is an s-subset.

A Markov-return is an event when some $T^n\Lambda \cap \Lambda$ is a u-subset, and it's inverse image under T^{-n} is an s-subset. The possible non-Markov returns are when the intersection is not a u-subset (this is printed as the middle intersection), or when the inverse image is not an s-subset. This latter event occurs when a recurring part goes over the edge of Λ in the stable direction.

The inverse image of the Markov-recurring part is not necessarily a solid rectangle intersected with Λ. It can have infinitely many "holes" in it, as demonstrated on figure 3.1.

Now we are going to enumerate the properties which are needed to build the (exponentially shrinking, hyperbolic) Young-tower. First a notation: in general a measurable bijection $M : (X_1, m_1) \to (X_2, m_2)$ between two finite measure spaces is called *nonsingular* if it maps sets of m_1-measure 0 to sets of m_2-measure 0. If M is nonsingular, we define the Jacobian of M wrt m_1 and m_2, written $J_{m_1,m_2}(M)$ or simply $J(M)$, to be the Radon-Nikodym derivative $\frac{d(M_*^{-1}m_2)}{dm_1}$. To denote $J(T)$ wrt m_{γ^u} we will use $\det DT^u$.

Definition 5. *We call* (X, T, μ) *a* Young system, *if the following Properties* **(P1)-(P8)** *are true:*

(P1) There exists a $\Lambda \subset X$ with a hyperbolic product structure and with $m_\gamma\{\gamma \cap \Lambda\} > 0$ for every $\gamma \in \Gamma^u$.

(P2) There is a countable number of disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that

- on each γ^u-disk $m_{\gamma^u}\{(\Lambda \setminus \cup\Lambda_i) \cap \gamma^u\} = 0$;
- for each i, $\exists R_i \in \mathbb{Z}^+$ such that $T^{R_i}\Lambda_i$ is a u-subset of Λ;
- for each n there are at most finitely many i's with $R_i = n$;
- $\min R_i \geq$ some R_0 depending only on T

(P3) For every pair $x, y \in \Lambda$, we have a notion of *separation time* denoted by $s_0(x, y)$. If $s_0(x, y) = n$, then the orbits of x and y are thought of as being "indistinguishable" or "together" through their n^{th} iterates, while $T^{n+1}x$ and $T^{n+1}y$ are thought of as having been "separated." (This could mean that the points have moved a certain distance apart, or have landed on opposite sides of a discontinuity manifold, or that their derivatives have ceased to be comparable.) We assume:

(i) $s_0 \geq 0$ and depends only on the γ^s-disks containing the two points;
(ii) the number of "distinguishable" n-orbits starting from Λ is finite for each n;
(iii) for $x, y \in \Lambda_i$, $s_0(x, y) \geq R_i + s_0(T^{R_i}x, T^{R_i}y)$;

(P4) Contraction along γ^s disks. There exist $C > 0$ and $\alpha < 1$ such that for $y \in \gamma^s(x)$, $d(T^n x, T^n y) \leq C\alpha^n \ \forall n \geq 0$.

(P5) Backward contraction and distortion along γ^u. For $y \in \gamma^u(x)$ and $0 \leq k \leq n < s_0(x, y)$, we have

(a) $d(T^n x, T^n y) \leq C\alpha^{s_0(x,y)-n}$;
(b)

$$\log \prod_{i=k}^{n} \frac{\det DT^u(T^i x)}{\det DT^u(T^i y)} \leq C\alpha^{s_0(x,y)-n}.$$

(P6) Convergence of $D(T^i|\gamma^u)$ and absolute continuity of Γ^s.

(a) for $y \in \gamma^s(x)$,

$$\log \prod_{i=n}^{\infty} \frac{\det T^u(T^i x)}{\det T^u(T^i y)} \leq C\alpha^n \quad \forall n \geq 0.$$

(b) for $\gamma, \gamma' \in \Gamma^u$, if $\Theta\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ is defined by $\Theta(x) = \gamma^s(x) \cap \gamma'$, then Θ is absolutely continuous and

$$\frac{d(\Theta_*^{-1} m_{\gamma'})}{dm_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det DT^u(T^i x)}{\det DT^u(T^i \Theta x)}.$$

(P7) $\exists C_0 > 0$ and $\theta_0 < 1$ such that for some $\gamma \in \Gamma^u$,

$$m_\gamma\{x \in \gamma \cap \Lambda : R(x) > n\} \leq C_0 \theta_0^n \quad \forall n \geq 0;$$

(P8) (T^n, μ) is ergodic $\forall n \geq 1$.

Now we will define the *Markov extension*, also known as the hyperbolic Young-tower. Let $R: \Lambda \to \mathbb{Z}_+$ be the function which is R_i on Λ_i, and let

$$\Delta \stackrel{\text{def}}{=} \{(x, \omega) : x \in \Lambda;\ \ \omega = 0, 1, \ldots, R(x) - 1\}$$

and the dynamics on the tower is

$$F(x, \omega) = \begin{cases} (x, \omega + 1) & \text{if} \quad \omega + 1 < R(x) \\ (T^R x, 0) & \text{if} \quad \omega + 1 = R(x) \end{cases}$$

We will refer to Δ_ω as the ω^{th} level of the tower Δ. Young also has a construction for μ_Δ, the SRB-measure of the extension, for which the pushforward is μ, and $J(F) \equiv 1$ except on $F^{-1}(\Delta_0)$.

On the tower a Markov partition $\mathcal{D}$ can be defined, with the following properties:

(a) $\mathcal{D}$ is a refinement of the partition Δ_ω. ($\mathcal{D}_\omega$ denotes $\mathcal{D}|\Delta_\omega$.)

(b) $\mathcal{D}_\omega$ has only a finite number of elements and each one is the union of a collection of Λ_i's;

(c) $\mathcal{D}_\omega$ is a refinement of $F\mathcal{D}_{\omega-1}$;

(d) if x and y belong to the same element of $\mathcal{D}_\omega$, then $s_0(F^{-\omega}x, F^{-\omega}y) \geq \omega$;

(e) if $R_i = R_j$ for some $i \neq j$, then Λ_i and Λ_j belong to different elements of $\mathcal{D}_{R_i - 1}$.

Let $\Delta_{\omega,j}^* = \Delta_{\omega,j} \cap F^{-1}(\Delta_0)$. We think of $\Delta_{\omega,j} \setminus \Delta_{\omega,j}^*$ as "moving upward" under F, while $\Delta_{\omega,j}^*$ returns to the base.

It is natural to *redefine the separation time* to be $s(x, y) \stackrel{\text{def}}{=}$ the largest n such that for all $i \leq n$, $F^i x$ and $F^i y$ lie in the same element of $\{\Delta_{\omega,j}\}$. We claim that **(P5)** is valid for $x, y \in \gamma^u \cap \Delta_{\omega,j}$ with s in the place of s_0. To verify this, first consider $x, y \in \Lambda$. We claim that $s(x, y) \leq s_0(x, y)$. If x, y do not belong to the same Λ_i, then this follows from rule (d) in the construction of $\mathcal{D}_\omega$; if $x, y \in \Lambda_i$, but $T^R x, T^R y$ are not contained in the same Λ_j, then $s(x, y) = R_i + s(T^R x, T^R y)$, which is $\le s_0(x, y)$ by property **(P3)**,(iii) of s_0, and so on. In general, for $x, y \in \Delta_{\omega,j}$, let $x_0 = F^{-\omega}x,\ y_0 = F^{-\omega}y$ be the unique inverse images of x and y in Δ_0. Then by definition $s(x, y) = s(x_0, y_0) - \omega$, and what is said earlier on about x_0 and y_0 is equally valid for x and y.

*From here on s_0 is replaced by s and **(P5)** is modified accordingly.*

This tower is only hyperbolic, and as a usual tool in this field Young has also introduced a factorised version of it $\bar{\Delta}$. Simply collapse the stable direction! This is also demonstrated on figure 3.1. For to be more detailed let us recall an important distortion property of the so called sliding map!

Fix an arbitrary $\hat{\gamma} \in \Gamma^u$. For $x \in \Lambda$, let $\hat{x}$ denote the point in $\gamma^s(x) \cap \hat{\gamma}$, and define

$$u_n(x) = \sum_{i=0}^{n-1} (\varphi(T^i x) - \varphi(T^i \hat{x}))$$

where $\varphi = \log |\det DT^u|$. From **(P6)**(a) it follows that u_n converges uniformly to some function u. On each $\gamma \in \Gamma^u$, we let $\tilde{m}_\gamma$ be the measure, whose density wrt m_γ is $e^u \cdot 1_{\gamma\cap\Lambda}$. Clearly, $T^{R_i}|(\Lambda_i \cap \gamma)$ is nonsingular wrt these reference measures. If $T^{R_i}(\Lambda_i \cap \gamma) \subset \gamma'$, then for $x \in \Lambda_i \cap \gamma$ we write $J(T^R)(x) = J_{\tilde{m}_\gamma, \tilde{m}_{\gamma'}}(T^{R_i}|(\Lambda \cap \gamma))(x)$.

Lemma 18. *(1) Let $\Theta_{\gamma,\gamma'}\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ be the sliding map along Γ^s. Then $\Theta_* \tilde{m}_\gamma = \tilde{m}_{\gamma'}$.*

(2) $J(T^R)(x) = J(T^R)(y)\ \ \forall y \in \gamma^s(x)$.

(3) $\exists C_1 > 0$ such that $\forall i$ and $\forall x, y \in \Lambda_i \cap \gamma$,

$$\left| \frac{J(T^R)(x)}{J(T^R)(y)} - 1 \right| \le C_1 \alpha^{\frac{1}{2} s(T^R x, T^R y)}.$$

Now we are ready to introduce the factorised dynamics with a factorisation along stable manifolds of Δ. The advantage is that this dynamics will behave as an expanding map, a simpler object to study. Let $\bar{\Delta} := \Delta / \sim$ where $x \sim y$ iff $y \in \gamma^s(x)$. Since F takes γ^s-leaves to γ^s-leaves, the quotient dynamical system $\bar{F}\colon \bar{\Delta} \to \bar{\Delta}$ is clearly well defined.

Let us define $\bar{m}$ in the following way: let $\bar{m}|\bar{\Delta}_\omega$ be the measure induced from the natural identification of $\bar{\Delta}_\omega$ with a subset of $\bar{\Delta}_0$, so that $J(\bar{F}) \equiv 1$ except on $\bar{F}^{-1}(\bar{\Delta}_0)$, where $J(\bar{F}) = J(\overline{T^R} \circ \bar{F}^{-(R-1)})$.

We now define $\bar{m}$ on $\bar{\Lambda}$ following the ideas that have been used for Axiom A. Lemma 18 (1) allows us to define $\bar{m}$ on $\bar{\Lambda}$ to be the measure whose representative on each $\gamma \in \Gamma^u$ is $\tilde{m}_\gamma$. Statement (2) says that $J(T^R)$ is well defined wrt $\bar{m}$, and (3) says that $\log J(T^R)$ has a dynamically defined Hölder type property, in the sense that $\alpha^{s(T^R x, T^R y)}$ could be viewed as a notion of distance between $T^R x$ and $T^R y$ (see **(P5)**). By using this lemma Young obtains a distortion property of the factorised map with a weaker constant β. Let β be such that $\alpha^{\frac{1}{2}} \le \beta < 1$, and let C_1 be as in Lemma 18 (3).

(I) Height of tower.

(i) $R \ge N$ for some N satisfying $C_1 e^{C_1} \beta^N \le \frac{1}{100}$;

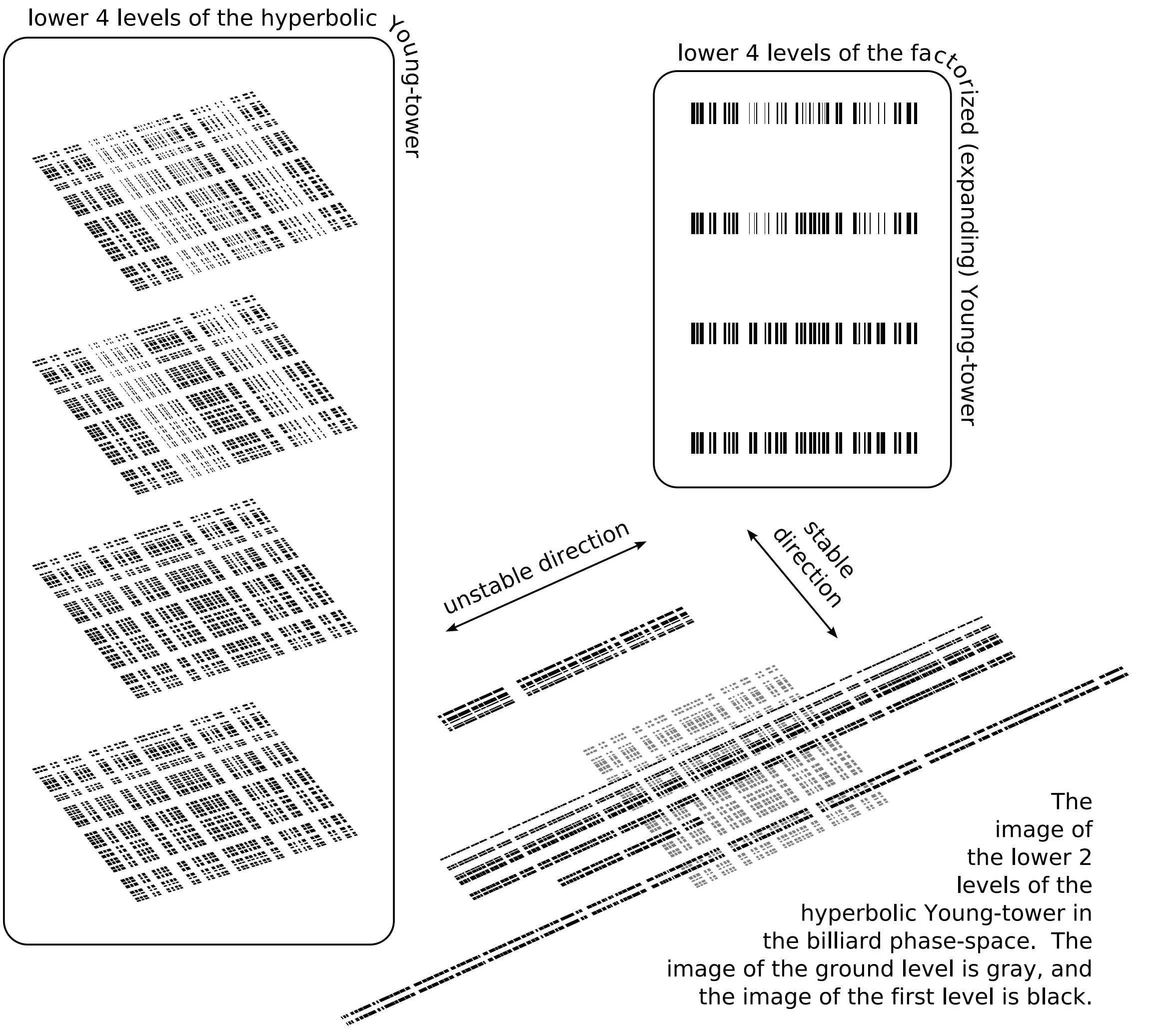


Figure 3.1: Young-towers, and Markov-return

(ii) $\bar{m}\{R \geq n\} \leq C_0' \theta_0^n \ \forall n \geq 0$ for some $C_0' > 0$ and $\theta_0 < 1$.

(II) Regularity of the Jacobian.

(i) $J\bar{F} \equiv 1$ on $\bar{\Delta} - \bar{F}^{-1}(\bar{\Delta}_0)$,

(ii)

$$\left| \frac{J\bar{F}(\bar{x})}{J\bar{F}(\bar{y})} - 1 \right| \leq C_1 \beta^{s(\bar{F}\bar{x}, \bar{F}\bar{y})} \quad \forall \bar{x}, \bar{y} \in \bar{\Delta}^*_{\omega,j}.$$

Young proves [You 98], that there exists an invariant probability measure $\bar{\mu}_\Delta$, absolutely continuous wrt $\bar{m}$, such that $\rho = \frac{d\bar{\mu}_\Delta}{d\bar{m}}$ is bounded away from zero and infinity, and is Lipschitz-continuous wrt the distance β^s.

As a brief summary of this section we have the following commutative diagram of measure preserving transformations:

$$\begin{array}{ccccc} (\bar{\Delta},\bar{\mu}_\Delta) & \xleftarrow{\pi_{\bar{\Delta}}} & (\Delta,\mu_\Delta) & \xrightarrow{\pi_X} & (X,\mu) \\ {\scriptstyle \bar{F}}\Big\uparrow & & {\scriptstyle F}\Big\uparrow & & {\scriptstyle T}\Big\uparrow \\ (\bar{\Delta},\bar{\mu}_\Delta) & \xleftarrow{\pi_{\bar{\Delta}}} & (\Delta,\mu_\Delta) & \xrightarrow{\pi_X} & (X,\mu) \end{array} \tag{3.1}$$

The projection to the original phase-space is not 1-1. On figure 3.1 the intersection in the middle has at least two inverse images. One of them is in the ground floor, and the other is on the first floor. Since the return is not Markovian these point are to be considered as different points on the tower.

3.2 Function spaces and the transfer operator

3.2.1 The Doeblin-Fortet inequality and spectral properties

Definition 6. *Let $(\mathcal{C},\mathcal{L})$ be a pair of Banach spaces, such that $\mathcal{L}\le\mathcal{C}$ is a linear subspace, $\|\,.\,\|_{\mathcal{L}}\ge\|\,.\,\|_{\mathcal{C}}$. We call this pair* adapted *if each $\mathcal{L}$-bounded set is precompact in $\mathcal{C}$.*

Definition 7. *Let $(\mathcal{C},\mathcal{L})$ be an adapted pair. We call an $A\colon\mathcal{C}\to\mathcal{C}$ bounded linear operator a* Doeblin-Fortet operator*, if $\exists\tau<1,\exists K>0,\exists n\in\mathbb{N}\quad\forall\varphi\in\mathcal{L}$,*

$$\|A^n\varphi\|_{\mathcal{L}}\le\tau\,\|\varphi\|_{\mathcal{L}}+K\,\|\varphi\|_{\mathcal{C}}\,.$$

This latter is called the Doeblin-Fortet inequality*.*

Theorem 19. *[I-TM 50] If A is a Doeblin-Fortet operator on the adapted pair $(\mathcal{C},\mathcal{L})$, then $\exists\vartheta<1, N\ge1$, projections $E_1,\dots,E_N$ onto finite dimensional subspaces of $\mathcal{L}$, and $\lambda_1,\dots,\lambda_N\in\{z\in\mathbb{C}:|z|=1\}$ such that $\forall\varphi\in\mathcal{L},n\in\mathbb{N}$*

$$\left\|A^n\varphi-\sum_{k=1}^{N}\lambda_k^nE_k\varphi\right\|_{\mathcal{L}}\le K\vartheta^n\,\|\varphi\|_{\mathcal{L}}\,.$$

Now we will define the *function spaces* on the factorised Young tower $\bar{\Delta}$. Let $\epsilon>0$ be such that

(ϵi) $e^{2\epsilon}\theta_0<1$,

(ϵii) $\bar{m}(\bar{\Delta}_0)^{-1}\sum_{\omega,j}\bar{m}(\bar{\Delta}^*_{\omega,j})e^{\omega\epsilon}\le2$.

Now we are ready to define the function spaces. The elements will be functions $\bar{\varphi}\colon\bar{\Delta}\to\mathbb{C}$ and the $\mathcal{C}$ norm is

$$\|\bar{\varphi}\|_{\mathcal{C}}\overset{\text{def}}{=}\sup_{\omega,j}\left\|\bar{\varphi}|_{\bar{\Delta}_{\omega,j}}\right\|_\infty e^{-\omega\epsilon}$$

where $\|\,.\,\|_\infty$ is the essential supremum wrt $\bar{m}$. By (ϵi) it is clear that constant multiple of this norm dominates the L_1-norm wrt $\bar{m}$. Let us introduce

$$\|\bar{\varphi}\|_h\overset{\text{def}}{=}\sup_{\omega,j}\left(\sup_{\bar{x},\bar{y}\in\bar{\Delta}_{\omega,j}}\frac{|\bar{\varphi}(\bar{x})-\bar{\varphi}(\bar{y})|}{\beta^{s(\bar{x},\bar{y})}}\right)e^{-\omega\epsilon};$$

where the inner sup is again essential supremum wrt $\bar{m} \times \bar{m}$ and $\mathcal{L}$-norm is

$$\|\bar{\varphi}\|_{\mathcal{L}} \stackrel{\text{def}}{=} \|\bar{\varphi}\|_{\mathcal{C}} + \|\bar{\varphi}\|_h .$$

$\mathcal{C}$ resp. $\mathcal{L}$ consist of functions for which the $\mathcal{C}$-norm resp. $\mathcal{L}$-norm is finite. The adaptedness is an easy consequence of the Arzela-Ascoli theorem. The Perron-Frobenius operator acting on these spaces is defined as follows:

$$P(\bar{\varphi})(\bar{x}) = \sum_{\bar{x}^{-1}:\bar{F}\bar{x}^{-1}=\bar{x}} \frac{\bar{\varphi}(\bar{x}^{-1})}{J\bar{F}(\bar{x}^{-1})}.$$

This is the adjungate operator of $\bar{\varphi} \mapsto \bar{\varphi} \circ \bar{F}$ on $L_2(\bar{m})$. By (ϵi) both $\mathcal{C}$ and $\mathcal{L}$ is contained in $L_2(\bar{m})$. The fact, that P is a bounded operator on $\mathcal{C}$ follows from (ϵii). The similar statement for $\mathcal{L}$ is proved in [You 98], where Young deduces that

(i) P is a contraction in $\mathcal{L}$.

(ii) it satisfies the D-F inequality,

(iii) by Theorem 19 it has a spectral gap,

(iv) and by **(P8)** its only eigenvalue on the unit circle is 1 and it is simple. (The eigenfunction is the invariant density ρ.)

Later we will need the adjungate of $\bar{\varphi} \mapsto \bar{\varphi} \circ \bar{F}$ on $L_2(\bar{\mu})$, this is $P^\rho(\bar{\varphi}) \stackrel{\text{def}}{=} \frac{1}{\rho}P(\rho\bar{\varphi})$. Note that the spectrum of P and P^ρ is the same, just the eigenfunctions are divided by ρ.

3.2.2 Associated functions on the tower

In this section we are working with Young systems throughout. Let $f\colon X \to \mathbb{R}^d$ be a bounded, piecewise η-Hölder function i. e. $f(x) - f(y) \le C_f d(x,y)^\eta$ whenever $x, y \in X_i$. We are going to associate a function $\bar{f}\colon \bar{\Delta} \to \mathbb{R}^d$ of the symbolic space. First we *pull back* f along the projection map $\pi_X\colon \Delta \to \cup T^n\Lambda$ to a function $\tilde{f}\colon \Delta \to \mathbb{R}^d$. This is clearly bounded and by **(P5)** $\tilde{f}(x) - \tilde{f}(y) \le C_f \left(C\alpha^{s(x,y)}\right)^\eta$ meaning $\tilde{f}$ is η-Hölder wrt the metric α^s. Next we use a standard method described for example in [PP 90]. We choose an unstable manifold in each Markov-rectangle $\Delta_{\omega,j}$, and consider the projection Ξ which sends each point along its stable manifold to our preferred unstable manifold. Consider the function $h \stackrel{\text{def}}{=} \sum_{n=0}^\infty \left(\tilde{f} \circ F^n - \tilde{f} \circ F^n \circ \Xi\right)$! The defining series converges since $\tilde{f}F^n x - \tilde{f}F^n\Xi x \le C_f d(T^n\pi x, T^n\pi\Xi x)^\eta$ and by **(P4)** $\le C_f(C\alpha^n)^\eta$.

$$\begin{aligned} h - h \circ F &= \sum_{n=0}^\infty \left(\tilde{f} \circ F^n - \tilde{f} \circ F^n \circ \Xi\right) - \sum_{n=0}^\infty \left(\tilde{f} \circ F^n + 1 - \tilde{f} \circ F^n \circ \Xi \circ F\right) \\ &= \tilde{f} - \left[\tilde{f} \circ \Xi + \sum_{n=0}^\infty \tilde{f} \circ F^{n+1} \circ \Xi - \tilde{f} \circ F^n \circ \Xi \circ F\right]. \end{aligned}$$

This can be rewritten as $h - h \circ F = \tilde{f} - \bar{f}$, where $\bar{f}$ is defined by the expression in square brackets. Evidently $\bar{f}$ is constant when restricted to any stable manifold, so it can be regarded as a function defined on $\bar{\Delta}$.

Lemma 20. *If $f: X \to \mathbb{R}^d$ is piecewise η-Hölder, and β satisfies $1 > \beta \geq \alpha^{\eta/2}$, then the associated function $\bar{f}: \bar{\Delta} \to \mathbb{R}^d$ is bounded and Lipschitz-continuous wrt the metric β^s:*

$$\left|\bar{f}(\bar{x}) - \bar{f}(\bar{y})\right| \leq C\beta^{s(\bar{x},\bar{y})}.$$

Proof. Let $\bar{x}, \bar{y} \in \bar{\Delta}$ such that $s(x, y) \geq 2n$ then **(P5)** ensures

$$\left|\tilde{f}F^k x - \tilde{f}F^k y\right|, \left|\tilde{f}F^k \Xi x - \tilde{f}F^k \Xi y\right| \leq C_f (C\alpha^{2n-k})^\eta, \quad 0 \leq k \leq n.$$

For all $k > 0$ **(P4)** gives

$$\left|\tilde{f}F^n x - \tilde{f}F^n \Xi x\right|, \left|\tilde{f}F^n y - \tilde{f}F^n \Xi y\right| \leq C_f (C\alpha^n)^\eta.$$

Hence $|h(x) - h(y)| \leq 2C_f \sum_{k=0}^{n} (C\alpha^{2n-k})^\eta + 2C_f \sum_{k=n+1}^{\infty} (C\alpha^n)^\eta \leq \text{const}C_f \alpha^{n\eta}$ given $1 > \beta \geq \alpha^{\eta/2}$ the latter estimate $\leq \bar{C}_f \beta^s$. □

Actually our function κ is locally constant on the phase space, hence it is Hölder continuous. Moreover since it is constant on local stable manifolds the above cohomology is not needed to consider its pullback $\tilde{\kappa}$ as a function on $\bar{\Delta}$. For that reason we will use the same notation κ for the function on $\bar{\Delta}$ as well.

The other free flight function ψ is only Hölder when the horizon is finite. Even in that case only the presence of secondary singularities make ψ to be $\frac{3}{5}$-Hölder. In the infinite horizon case ψ is only Hölder when we exclude certain part of the phase-space. Actually later we will exclude this part, but the reason for that is completely different. However in the proofs we do not need any assumption on ψ, only on κ.

Chapter 4

Analysis of the Fourier-transform operator

In our case the Birkhoff sum $S_n = \sum_k \kappa \circ T^k$ is not an independent sum. Since Nagaev's 1957 work, [Nag 57], for concluding a limit theorem, one traditionally uses the Fourier-transform of the transfer operator.

$$P_t(h) \stackrel{\text{def}}{=} P(e^{i\langle t,\kappa\rangle} h) \qquad (h \in \mathcal{L}, t \in \mathbb{R}^2)$$

It has the following simple connection with the characteristic function of the Birkhoff-sum:

$$\int \exp(i \langle t, S_n\rangle) d\mu = \int P_t^n(\rho) d\bar{m}.$$

4.1 Quasicompactness

The purpose of this section is to prove the Doeblin-Fortet inequality for the Fourier transform of the Perron-Frobenius operator: Simplifying the notations for a fixed t denote $\zeta = e^{i\left\langle t,\bar{f}\right\rangle}$, so $P_t(\bar{\varphi}) = P(\zeta\bar{\varphi})$. For to prove the inequality we need the assumption of Hölder continuity for the measurable f.

Lemma 21. *If f and β satisfies the conditions of lemma 20, then the operator P_t satisfies the Doeblin-Fortet inequality $\forall t \in \mathbb{R}^d$.*

Proof.

$$\|P_t^n\bar{\varphi}\|_{\mathcal{L}} = \|P_t^n\bar{\varphi}\|_{\mathcal{C}} + \|P_t^n\bar{\varphi}\|_h \leq \|P_t^n\|_{\mathcal{C}}\|\bar{\varphi}\|_{\mathcal{C}} + \|P_t^n\bar{\varphi}\|_h.$$

By (ϵii) $\|P_t^n\|_{\mathcal{C}} \leq 2$, so we only have to bound the continuity modulus.

$$P_t^n(\bar{\varphi}) = P^n(\zeta_n\bar{\varphi}) \text{ where } \quad \zeta_n(\bar{x}) := \prod_{k=0}^{n-1} \zeta(\bar{F}^k\bar{x}).$$

It follows that

$$P_t^n(\bar{\varphi})(\bar{x}) = \sum_{\bar{x}^{-n}:T^n\bar{x}^{-n}=\bar{x}} \frac{\zeta_n(\bar{x}^{-n})\bar{\varphi}(\bar{x}^{-n})}{J\bar{F}^n(\bar{x}^{-n})}$$

If $\bar{x}$ and $\bar{y}$ lie in the same element $\bar{\Delta}_{\omega,j}$, then the inverse images can be coupled: $\bar{x}_i^{-n}$ and $\bar{y}_j^{-n}$ form a pair if $\forall k, 0 \leq k \leq n \quad \bar{F}^k(\bar{x}_i^{-n})$ and $\bar{F}^k(\bar{y}_j^{-n})$ belong to the same element of the Markov partition $\{\bar{\Delta}_{\omega,j}\}$. That this is really a coupling is ensured by (e) in the definition of $\mathcal{D}$. For notational simplicity suppose that the inverse images are numbered according to the coupling. We have then the following expression for the continuity modulus:

$$|P_t^n(\bar{\varphi})(\bar{x}) - P_t^n(\bar{\varphi})(\bar{y})| = \left| \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \frac{\zeta_n(\bar{x}_i^{-n})\bar{\varphi}(\bar{x}_i^{-n})}{J\bar{F}^n(\bar{x}_i^{-n})} - \frac{\zeta_n(\bar{y}_i^{-n})\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \right|.$$

The right hand side can be written as $|I + II|$ where

$$I = \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \zeta_n(\bar{x}_i^{-n}) \left(\frac{\bar{\varphi}(\bar{x}_i^{-n})}{J\bar{F}^n(\bar{x}_i^{-n})} - \frac{\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \right),$$

and

$$II = \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \frac{\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \left(\zeta_n(\bar{x}_i^{-n}) - \zeta_n(\bar{y}_i^{-n}) \right).$$

The first quantity can be estimated as follows:

$$|I| \leq \sum_{\bar{x}_i^{-n}:\bar{F}^n\bar{x}_i^{-n}=\bar{x}} \left| \frac{\bar{\varphi}(\bar{x}_i^{-n})}{J\bar{F}^n(\bar{x}_i^{-n})} - \frac{\bar{\varphi}(\bar{y}_i^{-n})}{J\bar{F}^n(\bar{y}_i^{-n})} \right|$$

Young [You 98] gets her D-F inequality by estimating the same quantity in the case where $n = N$. For the estimate of the second term we have to say something about the continuity modulus of ζ:

$$|\zeta(a) - \zeta(b)| = \left| e^{i\langle t,\bar{f}(a)\rangle} - e^{i\langle t,\bar{f}(b)\rangle} \right| \leq |t| \left| \bar{f}(a) - \bar{f}(b) \right|.$$

By lemma 20 this latter is

$$\leq |t| \, C\beta^{s(a,b)}.$$

Then the continuity modulus of ζ_N:

$$\begin{aligned}
\left|\zeta_N(\bar{x}_i^{-N}) - \zeta_N(\bar{y}_i^{-N})\right| &= \sum_{k=0}^{N-1} \left|\zeta(\bar{F}^k(\bar{x}_i^{-N})) - \zeta(\bar{F}^k(\bar{y}_i^{-N}))\right| \\
&\leq \sum_{k=0}^{N-1} |t| \, C\beta^{s(\bar{F}^k(\bar{x}_i^{-N}),\bar{F}^k(\bar{y}_i^{-N}))} \\
&= \sum_{k=0}^{N-1} |t| \, C\beta^{s(\bar{x},\bar{y})+N-k} \\
&\leq \frac{\beta \, |t| \, C\beta^{s(\bar{x},\bar{y})}}{1-\beta}
\end{aligned}$$

II can be estimated by taking absolute value term by term. Then the continuity modulus is multiplied by $P^N|\varphi|y \le e^{\epsilon l} \left\| P^N |\varphi| \right\|_{\mathcal{C}} \le e^{l\epsilon} 2^N \left\|\varphi\right\|_{\mathcal{C}}$. From these it is easy to see, that in the D-F inequality this estimate of II contributes to the coefficient of $\|\varphi\|_{\mathcal{C}}$ by $\frac{2^N \beta C|t|}{1-\beta}$, so it doesn't bother Young's estimate of I. □

Later we will use this statement for $f = \kappa$, where the inequality is much stronger. Since κ is locally constant the terms in II vanish. Therefore we have a uniform inequality for all the t values.

4.2 Minimality

Next we have to investigate the t values, for which P_t has an eigenvalue on the unit circle. Otherwise P_t is strictly contracting by quasicompactness. As we will see, this is the question of minimality. Unfortunately this question has to be investigated on Young's symbolic system $(\bar{\Delta}, \bar{F}, \bar{\mu}_\Delta)$, since the operators are defined on that space. Nevertheless we are able to prove the minimality of κ on the symbolic system as well.

Definition 8. *We say that f is* cohomologous *to g (notation: $f \sim g$) if $\exists h$ measurable such that $f - g = h - h \circ T$. Under the* minimal support *of a function f (notation: $S(f)$) we mean the minimal translated closed subgroup of $\mathbb{R}^d$, which supports its values. We call a translated closed subgroup the* minimal lattice *of f if it is the intersection of minimal supports in the cohomology class of f ($M(f) = \cap_{g:g\sim f} S(g)$). We call f* minimal *if $S(f) = M(f)$. We call f* degenerate *if $M(f)$ is contained in a smaller dimensional affine subspace of $\mathbb{R}^d$.*

Lemma 22. *Fix the function f. Then $P_t^\rho \bar{g} = \lambda \bar{g}$ with $|\lambda| = 1 \iff e^{it\bar{f}}\bar{g} = \lambda \bar{g} \circ \bar{F}$. Moreover $\bar{g}$ can be supposed to take values on the unit circle.*

Proof. $\Longrightarrow$ If $P_t^\rho \bar{g} = \lambda \bar{g}$ then by (ϵi) $\bar{g} \in \mathcal{L} \Longrightarrow \bar{g} \in L_2(\bar{m})$, and also $\bar{g} \in L_2(\bar{\mu}_\Delta)$ we can take:

$$\left\langle e^{it\bar{f}}\bar{g}, \bar{g}\circ\bar{F} \right\rangle_{\bar{\mu}_\Delta} = \left\langle P\left(e^{it\bar{f}}\bar{g}\right), \bar{g} \right\rangle_{\bar{\mu}_\Delta} = \langle \lambda\bar{g}, \bar{g}\rangle_{\bar{\mu}_\Delta} = \lambda \left\|\bar{g}\right\|^2_{L_2(\bar{\mu}_\Delta)}.$$

From Cauchy-Schwartz inequality it follows that $e^{it\bar{f}}\bar{g} = \lambda\bar{g}\circ\bar{F}$. By ergodicity we can suppose $|\bar{g}| \equiv 1$.

$\Longleftarrow$ If $e^{it\bar{f}}\bar{g} = \lambda\bar{g}\circ\bar{F}$ then $P_t^\rho(\bar{g}) = \frac{1}{\rho}P(\rho e^{it\bar{f}}\bar{g}) = \frac{\lambda}{\rho}P(\bar{g}\circ\bar{F}\rho) = \lambda\bar{g}\frac{P(\rho)}{\rho} = \lambda\bar{g}$. Since $|\bar{g}| = 1 \Longrightarrow \bar{g} \in \mathcal{C}$, then it follows that $\bar{g} \in \mathcal{L}$ [I-TM 50].

□

This lemma shows that the t values for which the above-mentioned property holds form a closed subgroup of $\mathbb{R}^d$, moreover the eigenvalues and -functions preserve the group structure. If $P_{t_1}^\rho \bar{g}_1 = \lambda_1\bar{g}_1\circ\bar{F}$ and $P_{t_2}^\rho\bar{g}_2 = \lambda_2\bar{g}_2\circ\bar{F}$, then $P_{t_1+t_2}^\rho \bar{g}_1\bar{g}_2 = \lambda_1\lambda_2(\bar{g}_1\bar{g}_2)\circ\bar{F}$. Also, for $t \in G$, $t \mapsto \bar{g}_t$ and $t \mapsto \lambda_t$ are uniquely determined by ergodicity. (Here G denotes the subgroup of $\mathbb{R}^d$ formed by these t values.) This uniqueness can be easily derived from the multiplicative structure, and the already known spectral picture for $P = P_0$. Since λ_t is a multiplicative functional of t, so the logarithm is a linear one, and therefore $-i\log\lambda_t = tr$ for some r real vector. (Taking the adequate branch of the logarithm.)

Theorem 23. *$M(\bar f) = \widehat{\mathbb{R}^d/G} + r$. There exist minimal functions in each cohomology class. The minimal function is unique iff it is constant.*

Proof. $\subset$ We are going to prove that $\forall t \in G, \forall x \in M(\bar f)$ one has $e^{itx} = e^{itr}$. Since $t \in G$ we have $e^{it\bar f}\bar g = \lambda \bar g \circ \bar F$. Taking the logarithm

$$t\bar f \equiv -i\log\lambda + i\log\bar g - i\log\bar g\circ\bar F \pmod{2\pi}. \tag{4.1}$$

Remember that the first term on the right hand side is tr. By denoting $h = i\log\bar g$ we get that $t\bar f - (Z + tr) = h - h\circ\bar F$ for some Z, which takes values in $2\pi\mathbb{Z}$. To lift it to vector valued equation let us denote $\vec h = \frac{th}{|t|^2}$, $\vec Z = \frac{tZ}{|t|^2} + \bar f^{t^\perp} - r^{t^\perp}$, we get that $\bar f \sim \vec Z + r$, and the right hand side takes values in $H = t^\perp \oplus \frac{2\pi t}{|t|^2}\mathbb{Z} + r$. By definition $H \supset M(f)$, and since $\forall x \in H$ $e^{itx} = e^{itr}$ this is true for $\forall x \in M(\bar f)$.

$\supset$ We are going to prove that if for $t \in \mathbb{R}^d$ and $\forall x \in M(\bar f)$ we have $e^{itx} = e^{itr}$, then $t \in G$. The condition means that $\exists Z,\ Z \sim \bar f,\ S(Z) \subset t^\perp \oplus \frac{2\pi t}{|t|^2}\mathbb{Z} + r$. Combining the condition with the cohomological equation we get $e^{itZ} = e^{itr} = e^{it(\bar f - h + h\circ\bar F)}$. After rearranging one obtains $e^{it\bar f}e^{-ith} = e^{itr}e^{-ith\circ\bar F}$, and by the previous lemma $t \in G$.

$\exists$ Let us revisit the congruence (4.1). Observe that $i\log\bar g$ is also a linear functional of t, so $i\log\bar g = ts$ for some $s : \bar\Delta \to \mathbb{R}^d$. The function Z derived from this congruence is also linear in t, so $Z = tz$. Denote by H the orthocomplement of the linear subspace generated by G. Recalling the definition of r, s and z we can see, that r^H, s^H and z^H can be arbitrary, so let the latter one agree with $\bar f^H$, and the others be 0. We get $\bar f - (z+r) = s - s\circ\bar F$. Consider now $S(z+r)$. In the definition of Z we said that it takes values in $2\pi\mathbb{Z}$, but $Z = tz$ gives $\forall t \in G$ $e^{it(z+r)} = e^{itr}$, so from the already proven part of the theorem it follows that $S(z+r) = M(\bar f)$. Uniqueness is obvious: if $M(\bar f)$ is not a single point, then taking any $h : X \to M(\bar f)$ nonconstant $\bar f - h + h\circ\bar F$ is also a minimal function, and by ergodicity is not equal to $\bar f$.

□

Let us remark, that $M(S_n) = M(\bar f) + (n-1)r$. One of the inclusions ($\subset$) is trivial, the other ($\supset$) follows from ergodicity of iterates.

4.2.1 Minimality of the free flight function

Start with a simple observation

Lemma 24.

$$\kappa \sim \psi$$

Proof. Fix an arbitrary point $w \in D$. For $x = (q, v) \in X$ define $h(x) = w - q$. if $\psi(x) \in D + z$ for some $z \in \mathbb{Z}^2$, then $\kappa(x) = z$, and, of course,

$$\psi(x) = \kappa(x) + h(x) - h(Tx)$$

□

Theorem 25. *κ is minimal in the class of ψ.*

Proof. Suppose the contrary and denote the minimal function by κ'! Apply the factorisation by the minimal lattice: $\kappa_f : X \to \mathbb{Z}^2/M(\kappa)$! Then $\kappa_f \sim \kappa'_f$, and κ'_f is the constant function. Denote by n the cardinality of this Abelian group $\mathbb{Z}^d/M(\kappa)$! (We can suppose $n < \infty$.) In this case $\forall x$ periodic, such that $n|\text{per}(x) = p$ the Birkhoff sum $S_p(\kappa_f)x = 0$. The proof of the theorem is based on our forthcoming lemma 26. It is a variant of a statement which was originally applied in [BChS 91] to establish the non-singularity of the limiting covariance in the CLT. To contradict the non-minimality we are going to find a periodic point for each sublattice of finite index, not satisfying the above equation.

Lemma 26. *For any finite index sublattice $Z \subset \mathbb{Z}^2$ there exists a periodic point x such that the period p is a multiple of $|\mathbb{Z}^2 : Z|$ and $\sum_{i=0}^{p-1} \kappa(T^i x) \not\equiv 0 \pmod{Z}$*

Proof of lemma. The idea is a suitably adapted, simplified and generalised version of an argument of [BChS 91]. The original idea is well explained in [B 00]. Fix the lattice Z, denote the index by i, and fix $\Lambda \subset \mathbb{T}_0^d$, the basic product set of the Young system of our billiard ($\mu(\Lambda) > 0$). Take a billiard in the elongated torus $\mathbb{T}(Z) = \mathbb{R}^d/Z$, which is an appropriate projection of our Lorentz process. Consider the images of Λ on the elongated torus. Take two of them Λ_0 and Λ_1. By using the ergodicity of powers of the billiard in $\mathbb{T}(Z)$ we see that there exists an $n \in \mathbb{Z}_+$ such that $\Lambda_0 \cap T(Z)^{-ni}\Lambda_1$ contains a Markov intersection Λ^* of positive measure where $T(Z)$ denotes the Poincaré section map of the billiard on $\mathbb{T}(Z)$. The fact that $\Lambda_0 \cup T(Z)^{-ni}\Lambda_1$ contains a Markov intersection Λ^* of positive measure requires a proof. This is the only part in our paper where we have to go beyond properties (P1-8) of Young systems formulated in section 3.1 and to use some more detailed arguments from her construction.

Lemma 27 (Sublemma)**.** *For the billiard on $\mathbb{T}(Z)$ there exists an $n \in \mathbb{Z}_+$ such that $\Lambda_0 \cap T(Z)^{-ni}\Lambda_1$ contains a Markov intersection Λ^* of positive measure.*

By identifying Λ with Λ_0, $\cap_{l=-\infty}^{\infty} T^{lni}\Lambda^*$ consists of exactly one point x^*. Clearly $T^{ni}x^* = x^*$ and, moreover, the claim of the lemma is also evident. □

To conclude the proof it is sufficient to observe that the relation $\kappa \sim \kappa'$ and the periodicity of x also imply that $\sum_{i=0}^{p-1} \kappa(T^i x) \not\equiv 0 \pmod{Z}$. Hence the theorem. □

Proof of sublemma. In order that our ideas be clear with a minimal knowledge of sections 7 and 8 of [You 98] we summarise some facts from this reference. First, let us note that often it is convenient to use the semi-metric p determined by the density $\cos\phi dr$. We will write $p(.)$ for the p-length of a curve, while $l(.)$ denotes its Euclidean length. Finally, as before, $d(.,.)$ denotes Euclidean distance. In particular, $\gamma_\delta^u(x)$ will denote that piece of a γ_{loc}^u-curve whose endpoints have p-distance δ from its 'centre' x.

Facts:

(i) $\delta_1 > 0$ is a suitably small number, $\delta = \delta_1^4$ and $\alpha_1 = \alpha^{\frac{1}{4}}$.

(ii) The product set Λ has a sort of centre $x_0 \in A_{\delta_0} = \{x \in X | \ \gamma_{3\delta_0}^u(x) exists\} \neq \emptyset$. Denote $\Omega = \gamma_{3\delta_0}^u(x_0)$. Moreover, let us fix a small, rectangular shaped neighbourhood U of x_0 such that $\Lambda \cap U$ itself is a product set with $\mu(\Lambda \cap U) > 0$.

(iii) For the product set Λ one has a simply connected, rectangular-shaped region $Q(x_0)$ such that $\partial Q(x_0)$ is made up of two u-curves and two s-curves. The two u-curves are roughly $2\delta_0$ in length and they are either from $\Gamma^u(x_0)$ or do not meet any element of $\Gamma^u(x_0)$. The two s-curves are approximately 2δ long and have the same properties wrt $\Gamma^s(x_0)$. $\hat{Q}(x_0)$ is a proper u-subrectangle of $Q(x_0)$, i. e. it shares the s-boundaries of $Q(x_0)$ and its u-boundaries, which must have the same properties as those of $Q(x_0)$, are strictly inside $Q(x_0)$.

(iv) Denote $\Omega_\infty = \{y \in \Omega | \text{ for } \forall\ n \geq 0\ \ d(T^n y, S) > \delta_1 \alpha^n\}$. There are unions of a finite number of closed connected curves ω such that $\Omega_n \supset \Omega_{n+1}$ and $\Omega = \cap_n \Omega_n$. In addition, if ω is a component of Ω_n, then $T^n\omega$ is a connected smooth curve with $d(T^n\omega, S) \geq \frac{1}{2}\delta_1\alpha^n$, and, in particular, $T^{n+1}\omega$ is also a connected smooth curve.

(v) If for a point x one has $R(x) = n$, then x belongs to an s-subrectangle Q_ω of $Q(x_0)$ (where ω is some component figuring in (iv)) such that $T^j Q_\omega \cap S = \emptyset$ for every $0 \leq j \leq n$. Also, $10\delta_0 \leq p(T^n\omega) \leq 20\delta_0$ and $T^n\omega$ u-crosses $\hat{Q}(x_0)$ with segments $2\delta_0$ in length sticking out on both sides.

(vi) Finally, for some $R_1 \geq R_0$ large enough it is true that if, for some $n \geq R_1$, a component ω of Ω_n u-crosses the middle half of Q under T^n , then the entire s-subrectangle of Q associated with ω u-crosses Q under T^n.

When now turning to the billiard on $\mathbb{T}(Z)$ we will extend our previous usage of notations: for instance, $x_0^{(0)}, \ldots, x_0^{(Z)}$ will denote the different copies of x_0, and similarly $U^{(0)}, \ldots, U^{(Z)}$ the different copies of U. $\mu(Z, .)$ will denote the invariant probability measure for our 'elongated' billiard system. We note that Young's construction uses powers of T which are multiple of some given natural number. Here, for simplicity, we take this number to be equal to one and use the ergodicity of T. However, for our billiard it is known that any power of T is also ergodic so our simplification is by no means a restriction.

In fact, claim (vi) is the main fact necessary for our purposes. Introduce the function

$$w(x) = \chi_{\{p(\gamma^u(x)) \geq 10\delta_0\}}(x)\chi_{\{x \in \Lambda^{(Z)} \cap U^{(Z)}\}}(x).$$

By ergodicity,

$$\frac{1}{n}\sum_{k=0}^{n-1}\int \chi_{\{x \in \Lambda^{(0)} \cap U^{(0)}\}}(x)w(T^k x)d\mu_1(Z, x) \to \mu(\Lambda^{(0)} \cap U^{(0)})\bar{w}$$

where $\bar{w} = \int w(x)d\mu_1(Z, x) > 0$. Therefore, for some $x \in \Lambda^{(0)} \cap U^{(0)}$ there exist arbitrarily large indices k such that $T^k x \in \Lambda^{(Z)} \cap U^{(Z)}$ and $p(\gamma^u(T^k x)) \geq 10\delta_0$. Since $x \in \Omega_\infty^{(0)} \subset \Omega_k^{(0)}$, by property (vi) we are done.

□

We note here, after the proof, that the minimality is proven, indeed, on the tower $\bar{\Delta}$. The periodic point which led to the contradiction in the indirect proof was constructed inside a Markov-intersection of the basic hyperbolic set. Hence the period of this point on X is also the period of this point on the tower $\bar{\Delta}$.

4.3 Nagaev type theorems

4.3.1 Finite horizon case

Though we are interested in the Birkhoff-sum of κ we are going to use a general f during the proof emphasising that the only requirements for these results to hold are Hölder continuity and boundedness.

Expand now P_t in a Taylor series around $t = 0$! $P_t(\bar{\varphi}) = P(e^{i\langle t,\bar{f}\rangle}\bar{\varphi}) = P(\bar{\varphi}) + itP\left(\bar{f}\bar{\varphi}\right) - \frac{t^2}{2}P\left(\bar{f}^2\bar{\varphi}\right) + o(t^2)\left\|\bar{f}^2\bar{\varphi}\right\|_{\mathcal{L}}$. From lemma 20 it follows that the norm exists, so the second order Taylor-expansion at zero makes sense. Let us denote the operator $\bar{\varphi} \mapsto P(\bar{f}\bar{\varphi})$ by M (mean) and $\bar{\varphi} \mapsto P(\bar{f}^2\bar{\varphi})$ by Σ (covariance).

Denote by λ_t the leading -also simple- eigenvalue of P_t, (we know that $\lambda_0 = 1$) and by τ_t the projection operator corresponding to λ_t. The invariant density ρ is known to be bounded away from zero and infinity, and is Hölder. We know that $\tau_0 = \rho\bar{m}$, since ρ is the invariant density. Consider the second order Taylor polynomial of these two objects:

$$\lambda_t = 1 + iat - b\frac{t^2}{2} + o(t^2)$$
$$\tau_t = \rho\bar{m} + \eta t + \chi t^2 + o(t^2)$$

By definition $\tau_t P_t = \lambda_t \tau_t$. Expressing the terms by the above equations and considering the coefficients of t and t^2 we get the following:

$$i\rho\bar{m}M + \eta P = \eta + ia\rho\bar{m}$$
$$-\frac{1}{2}\rho\bar{m}\Sigma + i\eta M + \chi P = \chi + ia\eta - \frac{b\rho\bar{m}}{2}$$

evaluating these on ρ we get from the first that $a = \bar{m}M(\rho)$. We are allowed to suppose that $M(\rho)$ is a constant. This is because if we change $\bar{f}$ to a cohomologous $\bar{f}'$ the maximal eigenvalue does not change. Just like in the case of P_t^ρ we will study a conjugated operator with the same spectrum. Let us solve the equation: $P(\bar{f}\rho) - \int f d\nu = Pu - u$. This is solvable since the left hand side $\in \ker \bar{m}$. Let us consider $\bar{f}' = \bar{f} - \frac{u}{\rho} + \frac{u\circ\bar{F}}{\rho\circ\bar{F}}$! This is clearly cohomologous to $\bar{f}$. Let us consider $M'(\rho) = P(\bar{f}'\rho) = P(\bar{f}\rho) - Pu + P(\frac{u\circ\bar{F}}{\rho\circ\bar{F}}\rho)$. This latter term is $\frac{u}{\rho}P\rho = u$. So by the definition of u $M'(\rho) = \int f d\nu$ constant. Evaluating the second equation on ρ we get $b = \bar{m}\Sigma'(\rho) = \int \bar{f}'^2 d\bar{\nu}$, remember, that a was the average of the function, now b is some second moment, and we can define covariance by $\sigma^2 = b - a^2$. It is also remarkable, that σ is the second central moment of a function cohomologous to $\bar{f}$. If f is nondegenerate, each such quadratic form (and consequently σ) is nondegenerate also. We have proved the following theorem:

Theorem 28. *There are constants $\epsilon > 0$, $K > 0$ and $\theta < 1$ and a function $\rho : (-\epsilon, \epsilon)^d \to \mathcal{L}$ such that*

$$\left\| P_t^n h - \lambda_t^n \rho_t \int_{\bar{\Delta}} h d\bar{m} \right\|_{\mathcal{L}} \le K\theta^n \|h\|_{\mathcal{L}} \quad \forall |t| < \epsilon, \;\; n \ge 1, \;\; h \in \mathcal{L},$$

and $\|\rho_t - \rho\|_{\mathcal{L}} = O(t)$, $\lambda_t = 1 + ait - (\sigma^2 + a^2)\frac{t^2}{2} + o(t^2)$.

4.3.2 Infinite horizon case, proof of theorem 5

In the forthcoming section we are going to establish the global limit law in the infinite horizon case of Theorem 5 via an asymptotic expression at $t \to 0$ of λ_t, the leading eigenvalue of the Fourier-transform of the Perron-Frobenius operator. We will rely upon ideas of [BG 06] and of our study in chapter 2.

For the latter we expect the same behaviour what we have seen in the single term characteristic function. This will mean that the effect of dependence is negligible: both the independent and the dependent sums have the same asymptotic expansion for the Fourier transform, namely $1 + \mathbf{c}n|t|^2 \log|t|$. (In this formula $\mathbf{c}$ describes the direction-dependence, and $|t|$ the length, remember the definition of $\mathbf{c}$) in 2.1!

The following theorem of [BG 06] provides a condition for a limit law in the non-standard domain of attraction of the Gaussian law for models possessing a Young tower with an exponential tail bound for the height function, in general.

Theorem 29. *([BG 06], Theorem 3.5) Assume that the distribution of $g : \bar{\Delta} \to \mathbb{R}$ is in the nonstandard domain of attraction of the normal law. Remember that ω was the level function of the tower. Denote the "tower-sum" by $G = \sum_{k=1}^{\omega} g \circ \bar{F}^{-k}$. Let L, l be as in subsection 2.2. Assume, moreover, that $l(x \log x)/l(x) \to 1$, $L(x \log x)/L(x) \to 1$ when $x \to \infty$. Finally, assume that there exists a real number $a \neq -1/2$ such that*

$$\int g(e^{itG} - 1) = (a + o(1))itL(1/|t|). \tag{4.2}$$

Write $L_1(x) = (2a+1)L(x)$, and choose a sequence $B_n \to \infty$ such that $\frac{n}{B_n^2} L_1(B_n) \to 1$. Then for λ_t the leading eigenvalue of the $P_t(h) = P(e^{itg}h)$ Fourier-transform-operator:

$$\lambda_t = 1 - \frac{t^2}{2} L_1(1/|t|)(1 + o(1))$$

and consequently for the $S_n = \sum_{k=0}^{n-1} g \circ \bar{F}^k$ Birkhoff-sum:

$$\frac{S_n - n\int g}{B_n} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1).$$

We will indeed show that in our case the integral condition (4.2) of the above theorem holds for $g = \kappa$ with $a = 0$. (This is actually the case discussed by [AD 01b].) We note that this case will allow a simpler treatment than that of the stadium because here the expansion rate is much larger (though the logarithm of the free flight function in a corridor does not form a random walk as it is the case in the stadium).

Theorem 30 (Nagaev-type theorem). *There are constants $\epsilon > 0$, $K > 0$ and $\theta < 1$ and a function $\rho : (-\epsilon, \epsilon) \to \mathcal{L}$ such that*

$$\left\| P_t^n h - \lambda_t^n \rho_t \int_{\bar{\Delta}} h d\bar{m} \right\|_{\mathcal{L}} \leq K\theta^n \|h\|_{\mathcal{L}} \quad \forall |t| < \epsilon, n \geq 1, h \in \mathcal{L} \tag{4.3}$$

and

$$\|\rho_t - \rho\|_{L^3} = O(|t|^{1/10}) \ (t \to 0), \ \ \lambda_t = 1 + (1 + o(1))\mathbf{c}|t|^2 \log|t|$$

The proof of Theorem 30 can also be derived from considerations in [BG 06], namely in the proof of their theorem 3.5 they check the conditions of [KL 99] and use Corollary 1 of that paper. Corollary 2 of [KL 99] is this theorem when translated to the concrete situation.

4.3.3 Proof of the integral condition (4.2)

In our case the function κ is locally constant, so we can pass the integral immediately to Δ. Denote the "tower-sum" by $K = \sum_{k=1}^{\omega} \kappa \circ F^{-k}$. It is easy to see that the dominating terms of the integral in (4.2) are those corresponding to parts of the phase space when the process is close to a singular point say x_0 (discussed in 2.1).

The estimate of this integral is based on the following fact. We have already observed that high values of κ are typically reached rapidly: $\kappa \circ T^{-1}$ is in the order of $\sqrt{|\kappa|}$. During this fast trajectory segment the tower-sum can be estimated with the last term, hence it is also of order $\sqrt{|\kappa|}$. Since $|e^{it\bar{K}} - 1| \le t\bar{K}$, the integral can be estimated by $t\sum_n n\sqrt{n} \cdot \mu\{|\kappa| = n\} = O(t)$. The trajectories which do not provide fast reach have polynomially small (in $|\kappa|$) relative measure in the level-sets. These domains can be discarded due to the following lemma:

Lemma 31. *Any part of the integration domain A with measure $\mu(A \cap \{|\kappa| = N\}) = O(N^{-3-\alpha})$ with any $\alpha > 0$ can be thrown away:*

$$\int_{\Delta} \kappa(e^{i\langle t,K\rangle} - 1) = \int_{\Delta \setminus A} \kappa(e^{i\langle t,K\rangle} - 1) + O(|t|)$$

This is proved in proposition (4.17) of [BG 06]. Though the integration domains are in the hyperbolic Young-tower, if we identify the sets to be disregarded on the original phase-space X, then by measure preservation their pullbacks will satisfy the lemma.

First we are going to discard that part A_1 of the neighbourhood of x_0 where the last step was not fast enough, i. e. $A_1 = \{x \in U_0 \mid |\kappa \circ T^{-1}| > |\kappa|^{\frac{3}{4}}\} \subset X$. We also discard the set A_2 of those points where $\kappa \circ T^{-1} = w_0$, i. e. $A_2 = U_0 \setminus U_0'$.

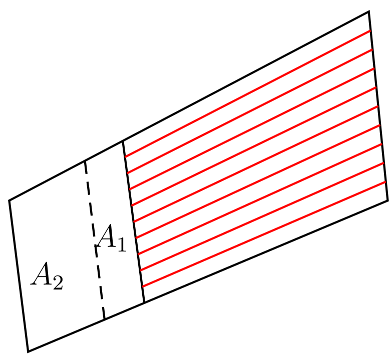


Figure 4.1: The discarded sets A_1 and A_2 in the level set of κ and the foliation with nearly unstable curves

We already know that the relative measure of $(A_1 \cup A_2)$ inside $\{|\kappa| = N\}$ is $O(N^{-1/2})$ (cf. propositions 16 and 17).

Let us foliate the remaining part with curves whose direction is nearly unstable (the derivative is in the unstable cone for T^{-1}).

The third discarded set A_3 will consist of points, for which the backward $Z \log N$-step trajectory meets the $\{|\kappa| > N^{\frac{4}{5}}\}$ set, where Z is a large number to be chosen later. To estimate the relative measure of A_3 we prove the following lemma:

Lemma 32. *There exists a constant C such that, for any large enough integer Z it is true: for any large enough N, given any unstable curve (for the mapping T^{-1}) in the set $\{|\kappa| = N\} \setminus \bigcup_{i=1,2} A_i$, the points for which $|\kappa|$ increases above $N^{4/5}$ within $Z \log N$ iterations of T^{-1} occupy a subset whose relative measure is less than $CN^{-1/11}$ in that curve.*

Proof. The proof is a suitable modification of Lemma 4.14 of [BG 06]. Note first that in the set $\{|\kappa| = N\} \setminus A_1$ one has $|\kappa \circ T^{-1}| < N^{\frac{3}{4}}$. The points which reach $|\kappa| > N^{4/5}$ get closer to the singularity than const $\cdot N^{-8/5}$. We are going to measure the set of points of the latter type.

The map T^{-1} satisfies Chernov's axioms [Ch 99] as this was checked for our infinite horizon case by Chernov in section 8 of the same article. Using exactly the same ideas as in [BG 06] we choose $\delta = (Z[\mathcal{D}, \mathcal{D}, 0]n)^{-1/\sigma}$ and apply Theorem 3.1 in [Ch 99].

We obtain a decreasing sequence $W_0^1 \supset W_1^1 \supset \cdots \supset W_{[Z \log N]}^1$ of subsets of the given LUM such that

$$\forall c > 0, \forall 0 \leq p \leq Z \log N \qquad \mathrm{Leb}\{x \in W_p^1 \mid \mathrm{dist}(T^{-p}x, S_0) \leq cN^{-8/5}\} \leq CcN^{-8/5}$$

(by equation (3.3) in [Ch 99]), and

$$\forall 0 \leq p \leq Z \log N, \quad \mathrm{Leb}(W_p^1 \setminus W_{p+1}^1) \leq \frac{C}{N} \mathrm{Leb(LUM)}$$

(By (iv), (3.5) in [Ch 99] and our choice of δ).

The second estimate is itself a relative measure estimate. For the first one we observe that $N^{-8/5} = N^{-1/10}N^{-6/4} \leq N^{-1/10}$const $\cdot$ Leb(LUM). This gives a measure at most $C \log N \;\; N^{-1/10}$, hence the lemma. □

The last discarded set will be defined on the tower: We are going to throw away that part A_4 of the integration domain, which is too high on the tower $\omega > Z \log |\kappa|$. Since the tower has exponentially small tails, if Z was chosen large enough, then the discarded set has measure $\mu_\Delta(A_4 \cap \{|\kappa| = N\}) = O(N^{-4})$, and thus by lemma 31 the integral can be restricted to its complement.

Now it remained to estimate the integral on the non-discarded set.

Lemma 33.
$$\int_{\pi_X^{-1}(X \setminus \bigcup_{i=1}^3 A_i) \setminus A_4} \kappa(e^{i\langle t,K \rangle} - 1) = O(|t|)$$

Proof. Consider the integral on the left. By the definition of the discarded sets we have:

$$\left| \int_{\pi_X^{-1}(X \setminus \bigcup_{i=1}^3 A_i) \setminus A_4} \kappa(e^{i\langle t,K \rangle} - 1) \right| \leq$$

$$\leq |t| \int_{\pi_X^{-1}(X \setminus \bigcup_{i=1}^3 A_i) \setminus A_4} |\kappa||K| \leq C|t| \sum_n \mu\{|\kappa| = n\} n \log n \; n^{4/5} \leq C|t|$$

□

Chapter 5

Proof of the results

5.1 Local limit theorems

First we are going to deal with κ. Since it is an integer valued vector function $P_t = P_u$ if $t - u \in 2\pi\mathbb{Z}^2$. Hence we can consider $t \in 2\pi\mathbb{T}^2$. We are going to apply a Nagaev type theorem (28 or 30) in a neighbourhood of zero. On the complement of the zero neighbourhood, which is compact by the above factorisation, we are going to apply the following theorem:

Lemma 34 ([AD 01]). *Suppose that $\mathcal{K}$ is a compact set of $\mathcal{L}$ operators such that each element of $\mathcal{K}$ is a Doeblin-Fortet operator, and none of them has an $\mathcal{L}$-eigenvalue on the unit circle. Then $\exists K > 0$ and $\theta < 1$ such that*

$$\|Q^n\|_{\mathcal{L}} \leq K\theta^n \quad \forall n \geq 1, \quad Q \in \mathcal{K}.$$

However to ensure compactness of the operator family one needs continuity in the parameter, which is not the case if the horizon is infinite. By using results in [KL 99] it is enough to prove uniform continuity in a weaker norm. We do not want to give details, the conditions of [KL 99] were checked in [BG 06] for a good choice of norms, and for a wide class of models including infinite horizon Lorentz-process. One of the conclusions of [KL 99] -if applied to this situation- is the above uniform exponential bound.

Let B_n denote the normalisation i. e. $B_n = \sqrt{n}$ in the finite horizon case, and $B_n = \sqrt{n \log n}$ in the infinite horizon case.

Theorem 35. *Let $k_n \in \mathbb{Z}^2$ be such that $\frac{k_n}{B_n} \to k \in \mathbb{R}^2$. Let the joint distribution of $(x, T^n x, S_n(x) - k_n)$ be denoted by Υ_n, where $x \in X$ is μ distributed. Then*

$$\lim_{n\to\infty} B_n^2 \Upsilon_n = \frac{e^{-\frac{1}{2}k\Sigma^{-1}k^T}}{\det \Sigma 2\pi} \; \mu^2 \times \sharp.$$

where $\sharp$ is the counting measure on $\mathbb{Z}^2$. In the infinite horizon case Σ is the matrix in (1.1).

Proof. We are going to prove a similar result on the tower $\bar{\Delta}$. Before that, we have to make it clear how the $\bar{\mu}_\Delta^2 \times \sharp$ limit on the tower implies the $\mu^2 \times \sharp$ limit in the original $X^2 \times \mathbb{Z}^2$ space. The Markov-extension is an extension, so if the limit is proved for that, it is also valid for the original system. What we have to deal is the factorisation. As detailed before in section 3.2.2 κ is locally constant, so it can be considered as a function on $\bar{\Delta}$. What remained to change are the two μ_Δ distributed variables to their $\bar{\mu}_\Delta$ distributed versions. The σ-algebra $\bar{\mathcal{S}}$, generated by factorised functions, is the multiplication of the σ-algebra generated by the rectangles in Δ in the stable direction, and the Borel-algebra in the unstable direction (mod 0). The forthcoming limit theorem for $\bar{\Upsilon}_n$ proves the measure limit not only for the σ-algebra $\bar{\mathcal{S}}$, but also for $F\bar{\mathcal{S}}$, because the application of F means the application of $(x, y, \xi) \mapsto (Fx, Fy, \xi - \kappa(x) + \kappa(y))$, and the limit is invariant under this action. Since $\bigvee_{n>0} F^n\bar{\mathcal{S}} = \mathcal{S}$ (mod 0) it is enough to prove the limit theorem for $\bar{\Upsilon}_n$.

For to do this we are going to integrate test functions: $w(\bar{x}, \bar{y}, \xi)$. We will restrict ourselves to product functions $w(\bar{x}, \bar{y}, \xi) = w_1(\bar{x})w_2(\bar{y})w_3(\xi)$, where w_1 and w_2 are in $\mathcal{L}$, and w_3 is summable (integrable with respect to $\sharp$). For simplicity we are going to use the inverse transform: $w(\bar{x}, \bar{y}, \xi) = \int w_1(\bar{x})w_2(\bar{y})\hat{w}_3(t)e^{it\xi}dt$.

$$
\begin{aligned}
B_n^2 \int_{\bar{\Delta}^2\times\mathbb{Z}^2} w d\bar{\Upsilon}_n &= B_n^2 \int w(\bar{x}, \bar{F}^n\bar{x}, S_n(\bar{x}) - k_n) d\bar{\mu}_\Delta \\
&= B_n^2 \int \int_{2\pi\mathbb{T}^2} \hat{w}(\bar{x}, \bar{F}^n\bar{x}, t) e^{it(S_n(\bar{x})-k_n)} dt \;\, d\bar{\mu}_\Delta \\
&= B_n^2 \int \rho^{-1}(\bar{x}) P^n \rho(\bar{x}) \left(\int_{2\pi\mathbb{T}^2} \hat{w}(\bar{x}, \bar{F}^n\bar{x}, t) e^{it(S_n(\bar{x})-k_n)} dt \right) d\bar{\mu}_\Delta \\
&= B_n^2 \int \int \rho^{-1}(\bar{x}) e^{-itk_n} w_2(\bar{x}) \hat{w}_3(t) P_t^n \left(\rho(\bar{x}) w_1(\bar{x})\right) dt \;\, d\bar{\mu}_\Delta
\end{aligned}
$$

Using lemma 34 and the Nagaev type theorems (28 and 30) we can substitute $P_t^n \rho w_1$ by $\lambda_t^n \rho_t \int_{\bar{\Delta}} \rho w_1 d\bar{m}$ in the domain $|t| < \delta$ and we get an error term $O(B_n^2\theta^n)$ inside the integration wrt $\bar{\mu}_\Delta$. This involves the error terms both from lemma 34 and from the Nagaev-type theorems (28 and 30).

$$
\begin{aligned}
& B_n^2 \int_{\bar{\Delta}^2\times\mathbb{Z}^2} w d\bar{\Upsilon}_n = \\
& \int \left(\rho^{-1}(\bar{x}) w_2(\bar{x}) \left(\int w_1 d\bar{\mu}_\Delta \right) \left(\int_{|t|<\delta B_n} \hat{w}_3\left(\frac{t}{B_n}\right) e^{-it\frac{k_n}{B_n}} \lambda^n_{\frac{t}{B_n}} \rho_{\frac{t}{B_n}}(\bar{x}) dt \right) + o(1) \right) d\bar{\mu}_\Delta \\
& \qquad \to \int_{\mathbb{R}^d} \left(\int w_1 d\bar{\mu}_\Delta \right) \left(\int w_2 d\bar{\mu}_\Delta \right) \hat{w}_3(0) e^{-itk} e^{\frac{-t\Sigma t^T}{2}} dt \\
& \qquad\qquad = \frac{1}{(2\pi)^2} \int_{\mathbb{Z}^2} w(\bar{x}, \bar{y}, \xi) d\bar{\mu}_\Delta^2 \times d\sharp \quad \frac{1}{\det \Sigma} 2\pi e^{-\frac{1}{2}k\Sigma^{-1}k^T}
\end{aligned}
$$

In the above limit the order of the error term is meant in $\mathcal{L}$-norm (cf. lemma 34 and theorems 28 and 30), this implies that limiting makes the error term vanish (cf. definition of $\mathcal{L}$-norm). The same applies for the $\bar{x}$ dependence of $\rho_{\frac{t}{\sqrt{n}}}$. Remember that in the infinite horizon case only L^3-norm continuity was stated, but this is also enough. The convergence in t is dominated, since in the finite horizon case $\exists C \quad \forall |t| \le \delta B_n \quad \left|\lambda^n_{\frac{t}{B_n}}\right| \le e^{-C|t|^2}$. In the infinite horizon case $\exists C \quad \forall |t| \le \delta\sqrt[3]{n} \quad \forall n > n_0(\delta) \quad \left|\lambda^n_{\frac{t}{B_n}}\right| \le e^{-C|t|^2}$, and $\exists C \quad \forall \delta\sqrt[3]{n} \le |t| \le B_n \quad \forall n > n_0(\delta) \quad \left|\lambda^n_{\frac{t}{B_n}}\right| \le e^{-C|t|^2/\log|t|}$. □

Proof of theorem 2. For the non-discrete free flight function ψ we can deduce the local limit theorem easily. We know theorem 35 for κ, and we know the cohomology relation from lemma 24. Applying the mapping $(x, y, \xi) \mapsto (x, y, \xi + h(x) - h(y))$ to the triple $(x, T^n x, S_n(x) - k_n)$ we get the same with ψ in the place of κ. Applying the same mapping to the limit measure, it transforms the following way: the third component will be convoluted with the distribution of h and that of $-h$. The reason for convolution is that the coordinate functions were independent in the limit in theorem 35. □

Theorem 36. *Let $j_n \in \mathbb{Z}^2$ be such that $\frac{j_n}{B_n} \to j \in \mathbb{R}^d$, and $k_n \in \mathbb{Z}^2$ be such that $\frac{k_n}{B_n} \to k \in \mathbb{R}^d$. Denote the joint distribution of $S_m - j_m, S_{m+n} - j_m - k_n$ by $\upsilon_{m,n}$! Then*

$$\lim_{m,n\to\infty} B_m^2 B_n^2 \upsilon_{n,m} \to \frac{e^{-\frac{1}{2}j\Sigma^{-1}j^T} e^{-\frac{1}{2}k\Sigma^{-1}k^T}}{\det^2 \Sigma (2\pi)^2} \sharp^2.$$

Proof. Again as in the previous proof if we consider the joint distribution $\Upsilon_{m,n}$ of the 5-tuple $(x, T^m x, T^{m+n} x, S_m(x) - j_m, S_{m+n}(x) - j_m - k_n)$, then it is enough to prove, that

$$\lim_{m,n\to\infty} B_m^2 B_n^2 \bar{\Upsilon}_{m,n} \to \frac{e^{-\frac{1}{2}j\Sigma^{-1}j^T} e^{-\frac{1}{2}k\Sigma^{-1}k^T}}{\det^2 \Sigma (2\pi)^2} \bar{\mu}_\Delta^3 \times \sharp^2.$$

To prove convergence we are going to integrate test functions: $w(\bar{x}, \bar{y}, \bar{z}, \xi, \zeta)$. Again as in the previous proof we restrict ourselves to product functions $w(\bar{x}, \bar{y}, \bar{z}, \xi, \zeta) = w_1(\bar{x}) w_2(\bar{y}) w_3(\bar{z}) w_4(\xi) w_5(\zeta)$, such that w_1, w_2 and w_3 are in $\mathcal{L}$, w_4 and w_5 are summable (integrable wrt $\sharp$). We are going to use the inverse transform:

$$\begin{aligned} w(\bar{x}, \bar{y}, \bar{z}, \xi, \zeta) &= \int \hat{w}(\bar{x}, \bar{y}, \bar{z}, t, u) e^{i(t\xi + u\zeta)} dt \ du \\ &= \int w_1(\bar{x}) w_2(\bar{y}) w_3(\bar{z}) \hat{w}_4(t) \hat{w}_5(u) e^{i(t\xi + u\zeta)} dt \ du \end{aligned}$$

$$B_m^2 B_n^2 \int\limits_{\bar{\Delta}^3 \times \mathbb{Z}^4} w \, d\bar{\Upsilon}_{m,n} =$$

$$B_m^2 B_n^2 \int \int\limits_{2\pi\mathbb{T}^4} \rho^{-1} e^{-i(t+u)j_m - iuk_n} P_{t+u}^m \left(\rho e^{iu(S_{m+n} - S_m)} \hat{w}(\bar{x}, \bar{F}^m \bar{x}, \bar{F}^{m+n} \bar{x}, t, u) \right) dt \ du \ d\bar{\mu}_\Delta$$

$$\begin{aligned} P_{t+u}^m \left(\rho e^{iu(S_{m+n} - S_m)} \hat{w}(\bar{x}, \bar{F}^m \bar{x}, \bar{F}^{m+n} \bar{x}, t, u) \right) &= \\ = e^{iuS_n} w_2(\bar{x}) w_3(\bar{F}^n \bar{x}) \hat{w}_4(t) \hat{w}_5(u) P_{t+u}^m (\rho w_1(\bar{x})) &= \\ = e^{iuS_n} w_2(\bar{x}) w_3(\bar{F}^n \bar{x}) \hat{w}_4(t) \hat{w}_5(u) \left(\rho_{t+u} \lambda_{t+u}^m \left(\int w_1 d\bar{\mu}_\Delta \right) + O(B_m^2 \theta^m) \right) \end{aligned}$$

Collecting the terms for the outer integration wrt $\bar{\mu}_\Delta$ and using again P invariance

$$\begin{aligned} \int_{\bar{\Delta}} e^{iuS_n} w_2(\bar{x}) w_3(\bar{F}^n \bar{x}) \rho_{t+u} d\bar{m} &= \int_{\bar{\Delta}} P^n \left(e^{iuS_n} w_2(\bar{x}) w_3(\bar{F}^n \bar{x}) \rho_{t+u} \right) d\bar{m} \\ &= \int_{\bar{\Delta}} w_3(\bar{x}) P_u^n \left(w_2(\bar{x}) \rho_{t+u} \right) d\bar{m} \\ &= \int_{\bar{\Delta}} w_3(\bar{x}) \left(\lambda_u^n \rho_u \int_{\bar{\Delta}} w_2(\bar{x}) \rho_{t+u} d\bar{m} + O(\theta^m) \right) d\bar{m} \end{aligned}$$

From this point the variables can be handled separately and the argument of the previous proof should be repeated twice to get the statement of this theorem. □

5.2 Recurrence

In this subsection we want to apply the local limit theorem in order to get the recurrence for the planar Lorentz-process, a result already proved in [Sch 98] and in [Conze 99] for the finite horizon case. Theorem 35 ensures that $\mu(S_n \in D) > \frac{C}{B_n^2}$ for some $C > 0$. It immediately extends to any fixed domain.

Theorem 37. *The planar Lorentz process is almost surely recurrent.*

Proof. The proof follows the ideas used in [KSz 85]. The sequence of events

$$A_n = \{S_n \in D\}$$

fulfils the condition of Lamperti's Borel-Cantelli [Spi 64]:

$$\sum_{k=1}^{\infty} \mu\{A_k\} = \infty$$

is clear by the main theorem

$$\liminf_{n\to\infty} \frac{\sum\limits_{j,k=1}^{n} \mu(A_j A_k)}{\left(\sum\limits_{k=1}^{n} \mu(A_k)\right)^2} < c$$

in the finite horizon case the denominator is of order $\log^2 n$, the numerator will be decomposed as follows:

$$\sum_{j,k=1}^{n} \mu(A_j A_k) \leq \sum_{\min(j,k)<\log n} \mu(A_j A_k) + \sum_{|j-k|<\log n} \mu(A_j A_k) + \sum_{j,k,|j-k|\geq \log n} \mu(A_j A_k).$$

The first sum can be estimated by $2\log n \sum_{k=1}^{n} m(A_k)$ which is of order $\log^2 n$. The same is true for the second term as well. Concerning the third one, by theorem 36 we know that the asymptotics of this term is proportional to $\frac{1}{jk}$, so the sum is of order $\log^2 n$.

In the infinite horizon case the denominator is of order $(\log\log n)^2$. We apply the same decomposition rule for the numerator as in the finite horizon case, but with $\log\log n$ in the summation limits instead of $\log n$. In this way all the terms will be of order $(\log\log n)^2$.

Consequently, by Lamperti's lemma

$$\mu\{A_k \text{ i. o.}\} > \frac{1}{c}.$$

Since this event is invariant under the ergodic dynamics, it happens almost surely. □

Finally we note that, as observed by Simányi [Sim 89] the recurrence of the planar Lorentz process is equivalent to saying that the corresponding billiard in the whole plane (with an infinite invariant measure) is ergodic (see also [Pen 00]).

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