Hypergrpah Extensions of the Erdős-Gallai Theorem

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Abstract: The Erdős-Gallai Theorem gives the maximum number of edges in a graph without a path of length k. We extend this result for Berge paths in r-uniform hypergraphs. We also find the extremal hypergraphs avoiding t-tight paths of a given length and consider this extremal problem for other definitions of paths in hypergraphs.

Keywords: extremal graph, hypergraph, Berge path

1 Introduction

For reference we state the classical result:

Theorem 1 (Erdős-Gallai [2]) *Let G be a graph on n vertices containing no path of length k. Then $e(G) \leq \frac{1}{2}(k-1)n$. Equality holds iff G is the disjoint union of complete graphs on k vertices.*

We consider several generalizations of this theorem for hypergraphs. This is due to the fact that there are several possible ways to define paths in hypergraphs. One such definition of paths in hypergraphs is due to Berge.

Definition 2 *A Berge path of length k in a hypergraph is a collection of k hyperedges $h_1, \ldots, h_k$ and $k+1$ vertices $v_1, \ldots, v_{k+1}$ such that for each $1 \leq i \leq k$ we have $v_i, v_{i+1} \in h_i$.*

We find the extremal sizes of r-uniform hypergraphs avoiding Berge cycles of length k. Interestingly, the size of the extremal hypergraphs depend on the relationship between r and k. Specifically, we distinguish between the cases when $k \leq r$ and when $k > r$.

Theorem 3 *Fix $r > 2$, let $k > r$ and let $\mathcal{H}$ be a hypergraph containing no Berge path of length k. Then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$.*

On the other hand, if $k \leq r$, we have the following theorem.

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Theorem 4 *Fix $2 < k \leq r$. If $\mathcal{H}$ is an r-uniform hypergraph with no path of length k, then $e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}$.*

Remark 5 *Both of the above theorems are sharp as the following two examples show. In the first case, if $k > r$, partition the n vertices into sets of size k. In each k set, take all possible subsets of size r to be in the hypergraph. Such a hypergraph has exactly $\frac{n}{k}\binom{k}{r}$ hyperedges and clearly contains no k-path as every component is composed of exactly k vertices.*

In the second case, $k \leq r$. Here we partition the vertices into sets of size $r+1$ and then on each $r+1$ set, we select exactly $k-1$ of its subsets of size r to be in the hypergraph. This hypergraph has exactly $\frac{k-1}{r+1}n$ hyperedges and as each component is encompasses exactly $k-1$ edges, it is clear there is no path of length k. In this paper we will not deal with the case when $k = 2$, as it is trivial, but it is interesting to note that the above construction is not best possible when $k = 2$.

We further generalize the idea of a Berge path:

Definition 6 *Fix $r \geq 2$ and $t, 1 \leq t \leq r-1$. A t-tight path of length k in a r-uniform hypergraph is a Berge-path on $k+1$ vertices $\{v_1, v_2, \ldots, v_{k+1}\}$ and k hyperedges $\{h_1, h_2, \ldots h_k\}$ such that consecutive hyperedges intersect in at least t points.*

Of course a 1-tight path is the same as a Berge path. A slight variation of our t-tight paths have been studied before in other settings. (See for example [1].) As in the case of Berge paths, we can get quite exact results regarding hypergraphs avoiding t-tight paths.

Theorem 7 *Fix $r \geq 2$ and $t, 1 \leq t \leq r-1$. Fix k large. Let $\mathcal{H}$ be an extremal r-uniform hypergraph on n vertices containing no t-tight path of length k. Then*

$$(1+o(1))\frac{\binom{n}{i}\binom{k}{r}}{\binom{k}{t}} \leq e(\mathcal{H}) \leq \frac{\binom{n}{t}\binom{k}{r}}{\binom{k}{t}}.$$

The lower bound follows directly from a theorem of Rödl [4].

Theorem 8 *The packing number $m(n, k, l) = (1+o(1))\frac{\binom{n}{l}}{\binom{k}{l}}$. (This is the size of the largest k-uniform family of subsets of an n-set such that every l-set is contained in at most 1 member of the family.)*

Definition 9 *([5]) A tight path of length k in a r-uniform hypergraph is a collection of $k+r-1$ vertices $\{v_1, v_2, \ldots, v_{k+r-1}\}$ and k hyperedges $\{h_1, h_2, \ldots h_k\}$ such that for each $1 \leq i \leq k, h_i = \{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$.*

Note that tight cycles in hypergraphs have also been studied before regarding the extremal number for Hamiltonian cycles. See [6] and [3].

Theorem 10 *Let $\mathcal{H}$ be an extremal r-uniform hypergraph containing no tight path of length k. Then*

$$(1+o(1))\frac{k-1}{r}\binom{n}{r-1} \leq |e(\mathcal{H})| \leq (k-1)\binom{n}{r-1}.$$

Finally, note that there is an important difference between $(r-1)$-tight paths and tight paths in r-uniform hypergraphs. We can investigate this difference by considering the following.

Definition 11 *Let $k > r \geq 2$ and $1 \leq J < k$. Then a Berge path of length k in a r-uniform hypergraph $\mathcal{H}$ on hyperedges $e_1, \ldots, e_k$ satisfies interseciton conditions (J) if*

$$\text{for } 1 \leq l \leq J \text{ and } i > l, \ \ |e_i \cap e_{i-l}| = \max\{r-l, 0\}.$$

Of course a Berge path satisfying intersection conditions (1) is the same as an $r-1$ tight path. Furthermore, a Berge path satisfying intersection conditions (k-1) is exactly a tight path. It is interesting that as proved above, an extremal hypergraph excluding $(r-1)$-tight paths of length k contains assymptotically $\frac{k-r+1}{r}\binom{n}{r-1}$ hyperedges. On the other hand, our best construction for a hypergraph excluding a tight path of lenght k contains $\frac{k-1}{r}\binom{n}{r-1}$ hyperedges. Supposedly, each Berge path satisfying intersection conditions (J) falls somewhere between these two. However, while there $k-1$ different intersection conditions, there are only $r-1$ possible block sizes in the construction of the conjectured extremal hypergraphs.

The rest of the paper is organized as follows. In section 2 we prove theorem 3, the proof of 4 is omitted in this extended abstract. In section 3 we look at t-tight paths. In section 4 we consider tight paths and $(r-1)$-tight paths satisfying a certain number of intersection conditions as in Definition 11. Some of the proofs are omitted here.

2 Berge Paths

PROOF: (*Proof of Theorem* 3) We prove this by induction on n. Clearly, for small values of n, the theorem trivially holds. Now, fix n such that the theorem holds for all $n' < n$ Then let $\mathcal{H} = (\mathcal{E}, V)$ be a r-uniform hypergraph on n vertices with $e(\mathcal{H}) > \frac{n}{k}\binom{k}{r}$. We can assume that the following holds for the minimal degree, δ, in $\mathcal{H}$.

$$\delta = e(\mathcal{H})\frac{r}{n} > \frac{1}{r}\binom{k-1}{r-1} \tag{1}$$

Otherwise, if there is a vertex x in $\mathcal{H}$ with degree no more than $\frac{1}{r}\binom{k-1}{r-1}$, then we may delete this vertex (and all the edges incident with it) from $\mathcal{H}$. The result will be a hypergraph on $n-1$ vertices with more than $\frac{n-1}{k}\binom{k}{r}$ edges. Thus by the induction hypothesis, this hypergraph will have a path of length k.

Let P be the longest path in $\mathcal{H}$. Let $v_1, v_2, \ldots, v_{l+1}$ be the vertices of P, and $h_1, h_2, \ldots, h_l$ the hyperedges such that for each $i, v_i, v_{i+1} \in h_i$. Suppose that $l < k$.

We claim that if there is a cycle of length $l+1$ on the vertices $v_1, v_2, \ldots, v_{l+1}$, then these vertices constitute a component of the hypergraph $\mathcal{H}$. To see this, suppose that C is such a cycle. Then if an edge h in the cycle C does not lie completely within the vertices $v_1, v_2, \ldots, v_{l+1}$, then deleting h from C we have an l- path which can be extended (by the edge h) to a path of length at least $l+1$. Thus every edge h in the cycle C must be contained within the vertices $v_1, v_2, \ldots, v_{l+1}$. In fact, something stronger is true. For each vertex in the cycle, v_i, the neighborhood of v_i lies within $v_1, v_2, \ldots, v_{l+1}$. (The neighborhood of a vertex is the set of vertices in $\mathcal{H}$ which are connected to v_i by an edge.) Indeed, suppose that for some i, the vertex v_i has a neighbor y outside of $\{v_1, v_2, \ldots, v_{l+1}\}$. Then the edge containing both v_i and y is not an edge of C (by the above argument.) Thus, removing an appropriate edge of C so that it is a path of length l with v_i as an endpoint, we can extend this to a path of length $l+1$ with y as an endpoint, a contradiction. This proves our claim.

In the remainder of the proof we show that there must be an $l+1$ cycle on the vertices $v_1, v_2, \ldots, v_{l+1}$. As this constitutes a connected component of $\mathcal{H}$ with at most $\binom{l+1}{r} \le \binom{k}{r}$ hyperedges, we may delete this component from $\mathcal{H}$ and by the induction hypothesis find a path of length k on what remains.

It remains to show that the vertices $v_1, v_2, \ldots, v_{l+1}$ form a cycle of length $l+1$. Let $\mathcal{H}'$ be the hypergraph obtained by deleting the edges of P from $\mathcal{H}$. Specifically, let $\mathcal{H}' = \mathcal{H} \backslash \{h_1, h_2, \ldots, h_l\}$. Note that by the choice of P, the neighborhoods of v_1 and v_{l+1} in $\mathcal{H}'$ must fall within $\{v_1, v_2, \ldots, v_{l+1}\}$

Clearly, if there is an edge of $\mathcal{H}'$ containing both v_1 and v_{l+1}, then the edges of P form a cycle of length $l+1$. On the other hand, if there exist edges $g_1, g_2 \in \mathcal{H}'$ such that for some i, $1 < i < l+$, $v_1, v_{i+1} \in g_1$ and $v_l + 1, v_i \in g_2$ then there is an $l+1$ cycle, on the vertices

$$v_1, v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{l+1}, v_i, v_{i-1}, v_{i-2}, \ldots, v_1.$$

Thus by the pigeon hole principle, if in $\mathcal{H}'$ the degrees of both v_1 and v_{l+1} are greater then $\binom{\frac{k-2}{2}}{r-1}$, then there is a $l+1$ cycle on $v_1, v_2, \ldots, v_{l+1}$ in $\mathcal{H}$.

We will show that this is in fact true; that the minimum degree in $\mathcal{H}'$ is greater than $\binom{\frac{k-2}{2}}{r-1}$. Of course, it is sufficient to show that the minimum degree in $\mathcal{H}$ is greater than $\binom{\frac{k-2}{2}}{r-1}+k-1$. Indeed from Equation (1) it is enough to show that the following holds.

$$\binom{k-1}{r-1} \geq \binom{\frac{k-2}{2}}{r-1} + k - 1 \tag{2}$$

We leave it as an exercise for the reader to verify that Inequality (2) holds for all $k > r \geq 3$.

This completes the proof of the theorem. □

3 t-Tight Paths

Proof: (*Proof of Theorem* 7) First consider the lower bound. By Theorem 8, there is a family $\mathcal{B}$ of k-sets of an initial n set such that no t set is contained in more than 1 element of the family and such that

$$|\mathcal{B}| \geq (1+o(1))\frac{\binom{n}{t}}{\binom{k}{t}}.$$

We claim that the r-uniform hypergraph $\mathcal{H}$ obtained by replacing each member of $\mathcal{B}$ with all its $\binom{k}{r}$ r-sets contains no t-tight path. Any such path would have vertices in at least 2 different members of $\mathcal{B}$. Specifically, such a path would contain vertices u and v with $u \in B_1 \backslash B_2$ and $v \in B_2 \backslash B_1$ where B_1 and B_2 are two distinct members of $\mathcal{B}$. But $|B_1 \cap B_2| < t$ (the same holds for all distinct pairs of members of $\mathcal{B}$.) Thus there can be no t-tight path in $\mathcal{H}$ from u to v.

We now look at the upper bound. If $t = 1$ then we are done by Theorem 3. Suppose then that $t \geq 2$. Let $\mathcal{H}$ be a hypergraph on n vertices with more than $\binom{n}{t}\binom{k}{r}/\binom{k}{t}$ hyperedges. Then it is easy to see that there exists a vertex $x_1 \in V$ with

$$\frac{\binom{n}{t}\binom{k}{r}}{\binom{k}{t}} = \frac{\binom{n-1}{t-1}\binom{k-1}{r-1}}{\binom{k-1}{t-1}}$$

Let $\mathcal{H}_1 = \{h \backslash \{x_1\} \,:\, h \in \mathcal{H}\}$ be the link of x_1. Then continuing we can clearly find vertices $x_2, \ldots, x_{t-1}$ such that for $1 < i < t$, $\mathcal{H}_i$ is the link of x_i in $\mathcal{H}_{i-1}$ and such that for $1 < i < t$, the degree of x_i in $\mathcal{H}_{i-1}$ is greater than $\binom{n-i}{t-i}\binom{k-i}{r-i}/\binom{k-i}{t-i}$. But then $\mathcal{H}_{t-1}$ is simply a $(r-t+1)$-graph on $n-t+1$ vertices with more than $\frac{n-t+1}{k-t+1}\binom{k-t+1}{r-t+1}$ edges. But then applying Theorem 3 we can find a path of length $k-t+1$ in $\mathcal{H}_{t-1}$. If the minimal degree in $\mathcal{H}_{t-1}$ is large enough, we can then extend this path using the vertices $x_1, \ldots, x_{t-1}$ to a t-tight path of length k in $\mathcal{H}$. □

4 Even Tighter Paths

In this section we consider the relationship between tight and t-tight paths as mentioned in Definition 11. We prove Theorem 10 and a related theorem concerning $(r-1)$-tight paths satisfying intersection conditions J for fixed $1 \leq J \leq k-1$. First we will need a simple averaging argument.

Lemma 12 *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with $c\binom{n}{r-1}$ edges. Then there exists an nonempty sub-hypergraph, $\mathcal{H}'$, of $\mathcal{H}$ such that*

$$\forall S \in \binom{\mathcal{V}(\mathcal{H})}{r-1}, d_{\mathcal{H}'}(S) \leq c \Rightarrow d_{\mathcal{H}'}(S) = 0 \tag{3}$$

where $d_{\mathcal{H}'}(S)$ refers to the number of hyperedges of $\mathcal{H}'$ containing the set S.

PROOF: (*Proof of Theorem* 10) The lower bound of course follows from our usual construction using Theorem 8. The upper bound follows just as easily from Lemma 12. If an r-uniform hypergraph $\mathcal{H}$ on n vertices satisfies 3 then it is quite clear that we can find a tight path of length k in $\mathcal{H}$: in fact every edge of $\mathcal{H}$ will be contained in such a path. □

It is thus perhaps unsurprising that the upper bound (and trivially the lower bound) of Theorem 7 for $(r-1)$-tight paths also holds for paths satisfying intersection conditions (2) if k is big enough compared to r. On the other hand it is easy to see that our construction for maximal hypergraphs containing no tight path does indeed contain Berge paths satisfying intersection conditions $(k-2)$. We give the following theorem which is clearly best possible up to a factor of r.

Theorem 13 *Let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k which satisfies intersection conditions (J). (Where r, k, J are as in Def.* 11.*) If $k - J > r - 1$, set $a := r - 1$. Otherwise set $a := k - J$. Then $e(\mathcal{H}) < (k-a)\binom{n}{r-1}$.*

Note that for $J = 1$ (i.e. for $(r-1)$-tight paths), the theorem returns a weaker result than Theorem 7.

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