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Bounds on the Rubbling and Optimal Rubbling Numbers of Graphs

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Abstract

A pebbling move on a graph removes two pebbles at a vertex and adds one pebble at an adjacent vertex. Rubbling is a version of pebbling where an additional move is allowed. In this new move, one pebble each is removed at vertices v and w adjacent to a vertex u, and an extra pebble is added at vertex u. A vertex is reachable from a pebble distribution if it is possible to move a pebble to that vertex using rubbling moves. The rubbling number is the smallest number m needed to guarantee that any vertex is reachable from any pebble distribution of m pebbles. The optimal rubbling number is the smallest number m needed to guarantee a pebble distribution of m pebbles from which any vertex is reachable. We give bounds for rubbling and optimal rubbling numbers.

Keywords: pebbling, rubbling, bounded diameter

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1 Introduction

Graph pebbling has its origin in number theory. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. How many pebbles will guarantee that every vertex is reachable (*pebbling number*), or that all vertices are reachable at the same time (*cover pebbling number*)? How can we place the smallest number of pebbles such that every vertex is reachable (*optimal pebbling number*)? For a comprehensive list of references for the extensive literature see the survey papers [8,9].

Graph rubbling is an extension of graph pebbling. In this version, we also allow a move that removes a pebble each from the vertices v and w that are adjacent to a vertex u, and adds a pebble at vertex u. The basic theory of rubbling and optimal rubbling is developed in [1]. The rubbling number of complete m-ary trees are studied in [6], while the rubbling number of caterpillars are determined in [13].

The current paper extends the theory of graph rubbling by providing bounds for the rubbling numbers of graphs. In Section 3, we give an upper bound for the rubbling number in terms of the number of vertices and the diameter of the graph. In Sections 4–5, we investigate how big the rubbling number of diameter 2 graphs can be. Let $f(n, d)$ be the maximum rubbling number of diameter d graphs with n vertices. We construct a family of graphs whose rubbling numbers match all known values of $f(n, 2)$. We also prove an upper bound for $f(n, 2)$. Similar questions for pebbling are studied in [2,5], more details are given in Section 5. In Section 6, we give a sharp upper bound for the optimal rubbling number of a graph in terms of the number of vertices. We also give sharp upper and lower bounds in terms of the diameter. Similar results for the optimal pebbling number are presented in [3,11]. Our results are extensions of these.

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2 Preliminaries

Throughout the paper, let G be a simple connected graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \to \mathbb{Z}$ where $p(v)$ is the number of pebbles placed at v. A *pebble distribution* is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles $\sum_{v\in V(G)} p(v)$. We are going to use the notation $p(v_1, \ldots, v_n, *) = (a_1, \ldots, a_n, q(*))$ to indicate that $p(v_i) = a_i$ for $i \in \{1, \ldots, n\}$ and $p(w) = q(w)$ for all $w \in V(G) \setminus \{v_1, \ldots, v_n\}$.

Consider a pebble function p on the graph G. If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v\to u)$ removes two pebbles at vertex v, and adds one pebble at vertex u to create a new pebble function

$$p_{(v,v\to u)}(v, u, *) = (p(v) - 2, p(u) + 1, p(*)).$$

If $\{w, u\} \in E(G)$ and $v \neq w$, then the *strict rubbling move* $(v, w\to u)$ removes one pebble each at vertices v and w, and adds one pebble at vertex u to create a new pebble function

$$p_{(v,w\to u)}(v, w, u, *) = (p(v) - 1, p(w) - 1, p(u) + 1, p(*)).$$

A *rubbling move is* either a pebbling move or a strict rubbling move. A *rubbling sequence* is a finite sequence $s = (s_1, \ldots, s_k)$ of rubbling moves. The pebble function gotten from the pebble function p after applying the moves in s is denoted by p_s. The pebble function gotten after applying the moves in a multiset S of rubbling moves in any order is denoted by p_S.

A rubbling sequence s is *executable* from the pebble distribution p if $p_{(s_1,\ldots,s_i)}$ is nonnegative for all i. A vertex v of G is *reachable* from the pebble distribution p if there is an executable rubbling sequence s such that $p_s(v) \geq 1$. The *rubbling number* $\rho(G)$ of a graph G is the minimum number m with the property that every vertex of G is reachable from any pebble distribution of size m. The *optimal rubbling number* $\rho_{\text{opt}}(G)$ of a graph G is the size of a distribution with the least number of pebbles from which every vertex is reachable.

3 Upper bound on the rubbling number

All the known upper bounds for the pebbling number π are also upper bounds for the rubbling number since $\rho \leq \pi$. The following result is the rubbling version of the upper bound $\pi(G) \leq (n - d)(2^d - 1) + 1$ [4, Theorem 1]. The difference between the pebbling upper bound and the rubbling upper bound is $2^{d-1}(n - d - 1) \geq 0$. The improvement is 0 for P_n (the path on n vertices)

n	3	4	5	6	7	8	9	10
$f(n,2)$	4	4	5	5	5	5	6	?
$\rho(G_n)$	4	4	5	5	5	5	6	6

Table 1
Rubbling numbers of G_n and all known maximum rubbling numbers for diameter 2 graphs with n vertices.

since then $d = n - 1$.

Theorem 3.1 *If G is a graph with n vertices and diameter d, then*

$$\rho(G) \leq (n - d + 1)(2^{d-1} - 1) + 2.$$

The upper bound is sharp for $d = 0$ since $\rho(K_1) = 1$ and for $d = 1$ since $\rho(K_n) = 2$ for $n > 1$. It is also sharp for $d = n - 1$ since $\rho(P_n) = 2^{n-1}$. If the diameter of G is 2, then the upper bound becomes $\rho(G) \leq n + 1$. This is no surprise since $\rho(G) \leq \pi(G)$ and we know [5] that $\pi(G)$ is either n or $n + 1$. However, this upper bound is not sharp.

4 Lower bound for $f(n, 2)$

There is no lower bound that forces ρ to grow with the number of vertices of the graph. In fact $\rho(K_n) = 2$ for all n. The only known lower bound $\rho(G) \geq 2^d$ for the rubbling number ρ is coming from the diameter d of the graph G. So we could ask whether $\{\rho(G) \mid \operatorname{diam}(G) = d\}$ is a finite set for all $d \geq 2$. The family of star shaped graphs constructed in [2] can be used to show that this is not the case for $d \geq 3$. For $d = 2$ we need a more elaborate construction.

We give a graph G_n for all $n \geq 3$ and prove the following:

Theorem 4.1 *For $n \geq 3$ we have* $\operatorname{diam}(G_n) = 2$ *and* $\rho(G_n) = \lfloor \sqrt{2n - 1} \rfloor + 2$.

5 Upper bound for $f(n, 2)$

Table 1 shows the maximum rubbling numbers

$$f(n, 2) = \max\{\rho(G) \mid n = |V(G)| \text{ and } 2 = \operatorname{diam}(G)\}$$

of diameter 2 graphs with n vertices. The values were calculated by a computer program [14]. The program checked all diameter 2 graphs with a given number of vertices. These graphs were generated by Nauty [10]. We have $f(n, 2) = \rho(G_n)$ for $n \in \{3, \ldots, 9\}$. It is not clear whether this is true for all n.

Problem 5.1 *Is it true that* $f(n, 2) = \rho(G_n)$ *for all* $n \geq 3$*?*

We can show that the difference cannot be large:

Theorem 5.2 *We have* $f(n, 2) \leq \sqrt{2n-1} + 5$.

Corollary 5.3 $\lfloor \sqrt{2n-1} \rfloor + 2 \leq f(n, 2) \leq \sqrt{2n-1} + 5$.

There are more existing results for similar questions about pebbling. It is known [12] that $f(n, 2) = n + 1$ since the pebbling number of a diameter 2 graph is either n or $n+1$. A classification of diameter 2 graphs with pebbling number $n + 1$ is also known from [5]. Diameter 3 graphs are also studied. In [2], it is shown that $f(n, 3) = \frac{3}{2}n + O(1)$.

6 Bounds on the optimal rubbling number

We saw in [1] that $\rho_{\text{opt}}(P_n) = \lceil \frac{n+1}{2} \rceil$. We show that the path requires the most pebbles for optimal rubbling amongst the graphs with a given number of vertices. The proof follows the ideas of [3].

Proposition 6.1 *If* G *is a tree with* n *vertices, then* $\rho_{opt}(G) \leq \lceil \frac{n+1}{2} \rceil$.

Corollary 6.2 *If* G *is a connected graph with* n *vertices, then* $\rho_{opt}(G) \leq \lceil \frac{n+1}{2} \rceil$.

Since $\rho_{\text{opt}}(K_n) = 2$ for $n \geq 2$, there is no useful lower bound for the optimal rubbling number in terms of the number of vertices. We can still find bounds in terms of the diameter. Similar results are presented in [11] for the optimal pebbling number.

Proposition 6.3 *If* G *is a connected graph with diameter* d*, then* $\lceil \frac{d+2}{2} \rceil \leq \rho_{opt}(G)$.

If the diameter of G is d, then every vertex is reachable from the distribution that has 2^d pebbles on a single vertex. Hence $\rho_{\text{opt}}(G) \leq 2^d$.

Proposition 6.4 *For all nonnegative integer* d *there is a graph* G *with diameter* d *and* $\rho_{opt}(G) = 2^d$.

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