

**CREATING SURFACES BY COMPUTER
AND SYMMETRY GROUPS FOR THE
COMPACT SURFACE OF GENUS 3⁻**

Thesis of PhD dissertation

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2004

1 Introduction

This thesis analyses the symmetry groups of compact surfaces. Especially, we shall present all possible symmetry groups of the 3^- surface, i.e. the connected sum of 3 projective planes. First we give a short survey on plane discontinuous groups and crystallographic groups, in general.

There are 17 crystallographic groups in the Euclidean plane E^2 . The first list of them were published by Fedorov (1890) together with the 219+11 crystallographic groups in space, and these were discovered by Schoenflies (1891) at the same time. Fricke and Klein dealt with plane crystallographic groups in 1897 and György Pólya in 1924. Nowacki described crystallographic groups in an abstract way.

There are 230 crystallographic groups in the Euclidean space E^3 , among them 219 are non-isomorphic, there are 11 reflection pairs. Fedorov in 1890, Schoenflies in 1891, Barlow in 1894 reached the same results independently of each other in different places and by different method. Analysing real crystals by X-ray spectroscopy we can conclude that the crystallographic space is euclidean, indeed „in our size”.

Among the famous problems formulated by Hilbert there are questions about crystallographic groups as well. One of them: Is it true that there are finitely many crystallographic groups in every dimension? Bieberbach answered this question in 1912: it is true, but the number of them is rapidly increasing with the dimension. There are 4783 (non-isomorphic) crystallographic groups in E^4 (there are 111 reflection pairs). This was determined by computer. The number of crystallographic groups in the 5 and 6 dimensional Euclidean space are known: 222018 and 28927922, respectively, but making a list of them is a hopeless task.

The tilings of the hyperbolic plane can still be surveyed [L-M 90, 91], [L-M-V 98]. There arise infinite series, which are characterised later by the signature. Now we have no hope to classify crystallographic groups of 3-dimensional hyperbolic space.

A Riemann surface of genus g^+ ($g \geq 2$) may have an orientation preserving isometry group N/G of finite order at most $84(g-1)$, as it is well-known [Z-V-C 80]. Here $G=O^g$ is the fundamental group of the connected sum of g tori and N is a normalizer group of G in $Isom^+ H^2$, i.e. in the orientation preserving isometry group of the hyperbolic plane. This estimate is sharp for infinitely many g 's, e.g. for $g=3$ first (see e. g. [K-W 99]).

By our knowledge, an analogous estimate is not proved for a non-orientable compact surface, of genus $g \geq 3$ whose universal covering space, as above, may have a hyperbolic metric of constant negative curvature, fixed to $K = -1$ in the following.

Discontinuous isometry groups of sphere of genus 0, projective plane of genus 1^- , Klein-bottle of genus 2^- can be surveyed, if we know discontinuous isometry groups of the 2-sphere and the Euclidean plane.

The 3^- surface – what is connected sum of 3 projective planes – is non-orientable, and it is the first among them, whose covering plane can have hyperbolic metric. As we shall see, the 3^- surface may have finite isometry group, as every compact surface with hyperbolic metric.

2 The applied new methods

The new method of our analysis is unfolding any surface by computer [S 1998]. It means that we determine substantially different side pairings of a $2n$ -gon, so that gluing together the side-pairs (identifying their corresponding points logically) this $2n$ -gon forms a compact surface. Vice versa, the surface can be unfolded to a paired $2n$ -gon. I have solved the problem in algorithmic way on the basis of [L-M 90], [L-M 91] and [M 1992]. Because of exponentially increasing complexity I could solve the problem by Commodore 64 computer only upto $2n=10$. Later on our implementation was running on a computer with Intel Pentium processor of 133 MHz and of 16 Mbyte RAM upto $2n=14$. (In the latter case the running time was about 6,5 hours. I enclosed the source-code of my computer implementation to the end of my dissertation.)

We determined upto $2n=14$ all the fixed-point-free side pairings. We obtained every combinatorially different case by computer. Then we gave estimates for the number of necessary steps depending upon the side-number of the polygon.

We can realize from the table of results, that we have obtained every combinatorially different unfolding of the polygon for connected sum of 3 projective planes, because we must consider the cases $6 \leq 2n \leq 12$ for 3^- surface (Table 1).

The obtained constructions allow us to analyse symmetry groups of 3^- surface unknown so far. Our arguments provide a general method to determine possible symmetry groups of any surface, as 2-dimensional spaceform of constant curvature. The estimates, detailed in the dissertation, show exponential increase by genus g of the surface.

3 New results, theorems

As it has already been mentioned in [S 02], the general construction of universal covering allows us to consider any compact non-orientable surface as an orbit structure Π^2/\mathbf{G} . Here Π^2 is a complete simply connected plane, one of S^2 , E^2 , H^2 , i.e. the sphere, Euclidean and hyperbolic plane, respectively, and \mathbf{G} is an isometry group acting on Π^2 freely and with a compact fundamental domain $\mathcal{F}_{\mathbf{G}}$ (a topological polygon), endowed with consecutive side pairings (Fig.1.a-b)

$$(1.1) \quad a_i: s_{ai-1} \rightarrow s_{ai}, \quad a_i^{-1}: s_{ai} \rightarrow s_{ai-1}, \quad 1 \leq i \leq g$$

of orientation reversing isometries (glide reflections). This leads to the canonical presentation of the fundamental group \mathbf{G} as described in (1.1).

$$(1.2) \quad \mathbf{G} = (a_1, a_2, \dots, a_g \mid a_1 a_1 a_2 a_2 \dots a_g a_g = 1) =: \otimes^g$$

S^2 , $g=1$ leads to the *projective plane*
 E^2 , $g=2$ leads to the *Klein bottle*
 H^2 , $g \geq 3$ leads to the other non-orientable compact surfaces,
 e.g. to our 3^- surface, being discussed.

At first we analyse symmetry groups, what we obtain for 8 fundamental hexagons of 3^- surface. We formulate the following theorem (Fig 1.a-b):

Theorem 1.1. *The hexagon domains together with their vertex figures generate the following (combinatorially not extendable) symmetry groups N/G of our 3^- surface, where the fundamental group of 3^- is denoted by G , and N is its normalizer in the isometry group of the hyperbolic plane H^2 .*

1. *The polygon symbol is **aabbcc**: $|N/G|=12$. The normalizer N is generated by four reflections, which reflect the domain \mathcal{F}_N in their sides, as Fig 1.a shows. The Conway-Macbeath signature of N is ***2223**, which has a boundary component with its enumerated dihedral corners.*

2. aabcbC: $|N/G|=2$. *The normalizer N maps the centre figure of \mathcal{F}_G onto its vertex figure by \mathcal{F}_N (with signature **2* \otimes** , where the crosscap means the projective plane) with the following presentation*

$$N=(m, h, t, g \text{ --- } m^2, h^2, mtmt^{-1}, hggt)$$

(Fig1. a) .

3. aabcBC: $|N/G|=4$. *We can see in Fig 1.a the fundamental domain \mathcal{F}_N by the normalizer N with signature **2*222** (This has a half turn center of order two and a boundary component with the indicated dihedral corners).*

4. aabccb: $|N/G|=8$. \mathcal{F}_N *is the reflection domain that can be seen in Fig 1.b with signature ***2224**.*

5. abacbc, 8. abcaBC: $|N/G|=4$. *The fundamental domain \mathcal{F}_N is generated by the normalizer*

$$N=(m_1, m_2, h(2), t \text{ --- } m_1^2, m_2^2, h^2, m_1tm_1t^{-1}, m_2thm_2ht^{-1}),$$

Fig 1.b with signature **2*222**.

6. abacbC, 7. abacBC: $|N/G|=2$. *With the side pairing of \mathcal{F}_N we can see in Fig 1.b the following geometric presentation*

$$N=(m_1, m_2, h, t \text{ --- } m_1^2, m_2^2, h^2, m_1tm_1t^{-1}, m_2thm_2ht^{-1})$$

*with the signature **2**** (with the two boundary component).*

In cases 3, 5 and 8 the normalizers are isomorphic and the symmetry groups are equivariant. \square

Then we analyse the symmetry groups of surface 3^- and their (fundamental) tilings considered to the 65 combinatorially different fundamental domains given by computer. Especially important are the maximal symmetry groups, which are combinatorially not extendable. Our results are seen in Table 2 as well:

Theorem 1.2. *In Table 2 you see that the surface 3^- has 2 maximal, i.e. not extendable, symmetry groups: ***2223/G** of order 12 and ***2224/G** of order 8. The other groups N/G ($G = \otimes^3$) are their subgroups, having a lattice structure. \square*

This is in (a rough) analogy to the 17 classes of the Euclidean (E^2) plane crystallographic groups \mathbf{N}/\mathbf{T} , where

$$\mathbf{N}_1 = \mathbf{p}6\mathbf{m}\mathbf{m} = *236 \quad \text{and} \quad \mathbf{N}_2 = \mathbf{p}4\mathbf{m}\mathbf{m} = *244$$

are the maximal normalizers (without additional translation) of the torus group

$$\mathbf{T} = \mathbf{p}1 = \mathbf{O} = (a_1, b_1 \mid a_1 b_1 a_1^{-1} b_1^{-1} (=1)).$$

Therefore, our classification can be considered as an extension of the 17 discontinuous E^2 groups to those of the other compact surfaces of hyperbolic metric. The computer implementation of [S 98] has listed the 65 combinatorial fundamental domains (Table 1) for the 3^- surface H^2/\otimes^3 . The general algorithm, for finding all the fundamental domains for g^- surface in [L-M 90, 91], [L-M-V 98] is based on the fixed-point-free pairings of a $2g$ -gon, with one vertex class, at least one side pairing is orientation reversing. Then comes a tree graph construction with additional vertices. Along this graph the surface is cut and unfolded onto a topological polygon at most of $6(g-1)$ sides, at most of $2(g-1)$ vertex classes, in each class 3 vertices at least (this process is indicated in Fig.4.a-b).

In such a way we obtain not only the possible groups \mathbf{N}/\mathbf{G} but also the possible normalizer tilings of the 3^- surface H^2/\mathbf{G} up to a combinatorial (topological) equivalence of domains $\mathcal{F}_\mathbf{N}$. Of course, different fundamental domains for $\mathbf{G} = \otimes^3$ may induce the same domain for a normalizer \mathbf{N} , i.e. equivariant tilings for the 3^- surface H^2/\mathbf{G} (see e.g. Fig. 1. c-e, 4. a-b). But combinatorially different $\mathcal{F}_\mathbf{N}$'s for fixed \mathbf{N} will be distinguished as providing different tilings for H^2/\mathbf{G} . Our Table 1 lists the typical maximal normalizer(s) \mathbf{N} for each $\mathcal{F}_\mathbf{G}$, sometimes not uniquely, that can be tiled by an appropriate $\mathcal{F}_\mathbf{N}$. By Table 2 we can turn to other fundamental tilings by symmetry breakings of subgroup actions.

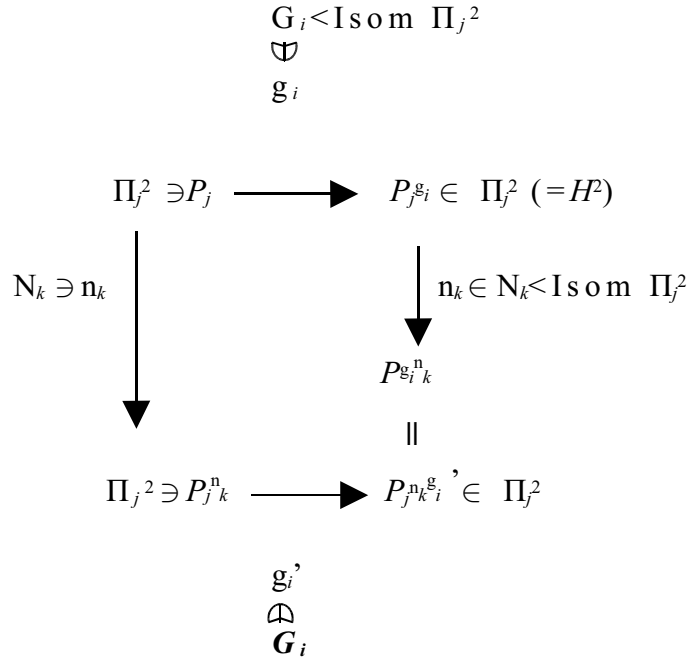
Then the complete classification of fundamental tilings with $\mathcal{F}_\mathbf{N}$'s for the 3^- surface by [L-M 90] is relatively easy but it would be too lengthy to list here. As an information we list all the combinatorially different polygon symbols $\mathcal{F}_\mathbf{N}$ in Table 3 for occurring normalizers \mathbf{N} by [L-M-V 98].

We formulate the main results in our

Theorem 1.3. *The 3^- surface, as a connected sum of 3 projective planes, allows hyperbolic (H^2) metric structures such that 12 isometry groups \mathbf{N}/\mathbf{G} can act on the 3^- surface, induced by normalizers \mathbf{N} of the fundamental group $\mathbf{G} = \otimes^3$ in the isometry group of H^2 , up to homeomorphism equivariance. These 12 normalizers \mathbf{N} provide 65+58 fundamental tilings for our 3^- surface H^2/\mathbf{G} (Tables 1-4). \square*

The completeness proof of our classification: The basic tool is the algorithmic enumeration of fundamental domains for any compact plane group of given signature [L-M 90], [L-M 91], [L-M-V 98], namely, for the fundamental group \mathbf{G} of a compact surface and for its normalizer \mathbf{N} (see Tables 1-3).

The diagram



symbolizes how the fundamental group $\mathbf{G}_i = \{ \mathbf{g}_i \}$ acts on the universal covering plane $\Pi_j^2 = \{ P_j \}$ to form the orbit plane Π_j^2 / \mathbf{G}_i as a surface, and how a \mathbf{G}_i -normalizer $\mathbf{N}_k < \text{Isom } \Pi_j^2$, mapping any \mathbf{G}_i -orbit $P_j^{\mathbf{g}_i}$ onto another one $P_j^{\mathbf{n}_k \mathbf{g}_i} = P_j^{\mathbf{g}_i \mathbf{n}_k}$ for any $\mathbf{n}_k \in \mathbf{N}_k$, induces an isometry group $\mathbf{G}_i / \mathbf{N}_k$ of the surface:

$$\mathbf{G}_i < \mathbf{N}_k < \text{Isom } \Pi_j, \text{ thus } \mathbf{n}_k \mathbf{G}_i = \mathbf{G}_i \mathbf{n}_k \in \mathbf{G}_i / \mathbf{N}_k$$

as usual. Here Π_j^2 is either S^2 or E^2 or H^2 . \mathbf{G}_i and \mathbf{N}_k will be determined up to a homeomorphism equivariance by the signature, described in the Introduction.

Definition The action of \mathbf{G}_1 on Π_1^2 is φ -equivariant to that of \mathbf{G}_2 on Π_2^2 if there is a homeomorphism.

$$\varphi: \Pi_1^2 \rightarrow \Pi_2^2: P_1 \rightarrow P_2 := P_1^\varphi \text{ such that } \mathbf{G}_2 = \varphi^{-1} \mathbf{G}_1 \varphi.$$

If φ above yields $\mathbf{N}_2 = \varphi^{-1} \mathbf{N}_1 \varphi$, then $\mathbf{N}_1 / \mathbf{G}_1$ and $\mathbf{N}_2 / \mathbf{G}_2$ are also called equivariant. If $\mathbf{N}_2 > \varphi^{-1} \mathbf{N}_1 \varphi$ then $\mathbf{N}_2 / \mathbf{G}_2 > \mathbf{N}_1 / \mathbf{G}_1$, i.e. \mathbf{N}_2 provides a richer symmetry group of Π_2^2 / \mathbf{G}_2 than \mathbf{N}_1 provides that for Π_1^2 / \mathbf{G}_1 .

Isomorphic, i.e. equivariant normalizers \mathbf{N} 's of \mathbf{G} form an equivalence class, and we are interested in determining the different classes and their subgroup relations. Here the relations of groups and maximal (proper) subgroups are satisfactory.

Any \mathbf{G} (and \mathbf{N}) is defined (will be determined) by a *fundamental* (topological) polygon \mathcal{F}_G (\mathcal{F}_N) with their side pairing isometries as generators, first in a combinatorial way, then metrically in a plane Π^2 by its signature. Hence the vertex classes with their stabilizers and the corresponding defining relations have been determined by a polygon symbol up to a combinatorial equivalence as indicated and illustrated above. \square

Although we may have many combinatorially different domains \mathcal{F}_G (\mathcal{F}_N) — our algorithm [L-M 90], [L-M 91], [L-M-V 98] enumerates all of them. Any \mathcal{F}_G by its barycentric subdivision and its \mathbf{G} -images, at the neighbourhoods of non \mathbf{G} -equivalent sides and vertices, by defining relations, gives us — in a finite algorithmic procedure — complete information on the systems of locally minimal closed geodesics, as on the orientation preserving ones as on the orientation reversing ones, and on their \mathbf{G} -images as well. Any element n of a normalizer \mathbf{N} maps these systems onto itself, now metrically if the domain \mathcal{F}_G is well deformed by a homeomorphism ϕ . Then we determine \mathcal{F}_N step by step.

Of course, any \mathcal{F}_G can be deformed in such a way that any possible normalizer \mathbf{N} occurs, since any combinatorial \mathcal{F}_G can be cut and glue onto any other one by the usual topological procedure. But now *we can concentrate on the cases where the \mathbf{N} -images of \mathcal{F}_N tile \mathcal{F}_G by the representatives of \mathbf{N}/\mathbf{G} , and this is a finite procedure.*

Theorem 1.4. *For $\mathbf{G} = \otimes^3$ we have 65 types of fundamental (topological) polygons as listed in Table 1 by computer. \square*

We examined each of them with the above respects of view. From the combinatorial structure of \mathcal{F}_G we selected a normalizer element and cut \mathcal{F}_G into a smaller domain with induced side pairing step by step, first by combinatorial line reflection, then by rotations especially by halfturn, glide reflection and translation, preserving the \mathbf{G} -equivalence of sides. We always check the homomorphism criterion for any candidate \mathbf{N} . Thus we obtain an \mathcal{F}_N and so \mathbf{N} by its presentation, then \mathbf{N}/\mathbf{G} , moreover, the smallest \mathcal{F}_N for \mathcal{F}_G , so the richest \mathbf{N} and \mathbf{N}/\mathbf{G} with tiling \mathcal{F}_G by the images of \mathcal{F}_N under representatives of \mathbf{N}/\mathbf{G} as required.

In this way we obtained Table 2 from Table 1 by Table 3 and by a careful analysis.

4 Formulation of conjectures and further tasks

Finally we can formulate two conjectures on the basis of given results for any surface g^- , $g \geq 3$. We intend to deal with these conjectures in forthcoming papers.

Remark 1.1. In Fig.1.b we have indicated the general construction scheme for any g^- surface, $g \geq 3$. This shows *our natural general conjecture* that

$\mathbf{N}_{g^-} = *222g$ with $|\mathbf{N}/\mathbf{G}| = 4g$, as reflection group in the $(\pi/2, \pi/2, \pi/2, \pi/g)$ quadrangle, is the maximal normalizer of $\mathbf{G} = \otimes^g$ in the isometry group $\text{Isom } H^2$ of the hyperbolic plane.

Remark 1.2 Our construction scheme can be generalized again for regular $4(g-1)$ -gon with side pairing glide reflection and translation each of number $g-1$, with $g-1$ vertex classes, 4 vertices in each with $\pi/2$ angles. *Then $\mathbf{N}_{g^-} = *222[2(g-1)]$ is conjectured as second richest normalizer, $|\mathbf{N}/\mathbf{G}| = 4(g-1)$.*

2n	p1	p2	p3	n2	n3	n4	n5	n6	n7
4	1			2					
6	1			2	8				
8		4			22	47			
10		18			24	279	473		
12		34	82		11	682	4928	7192	
14		38	1022			838	20979	110266	144906

Table 1

Relations of (maximal) subgroups N/G by normalizers N:

— invariant ones - - - - - non-invariant ones

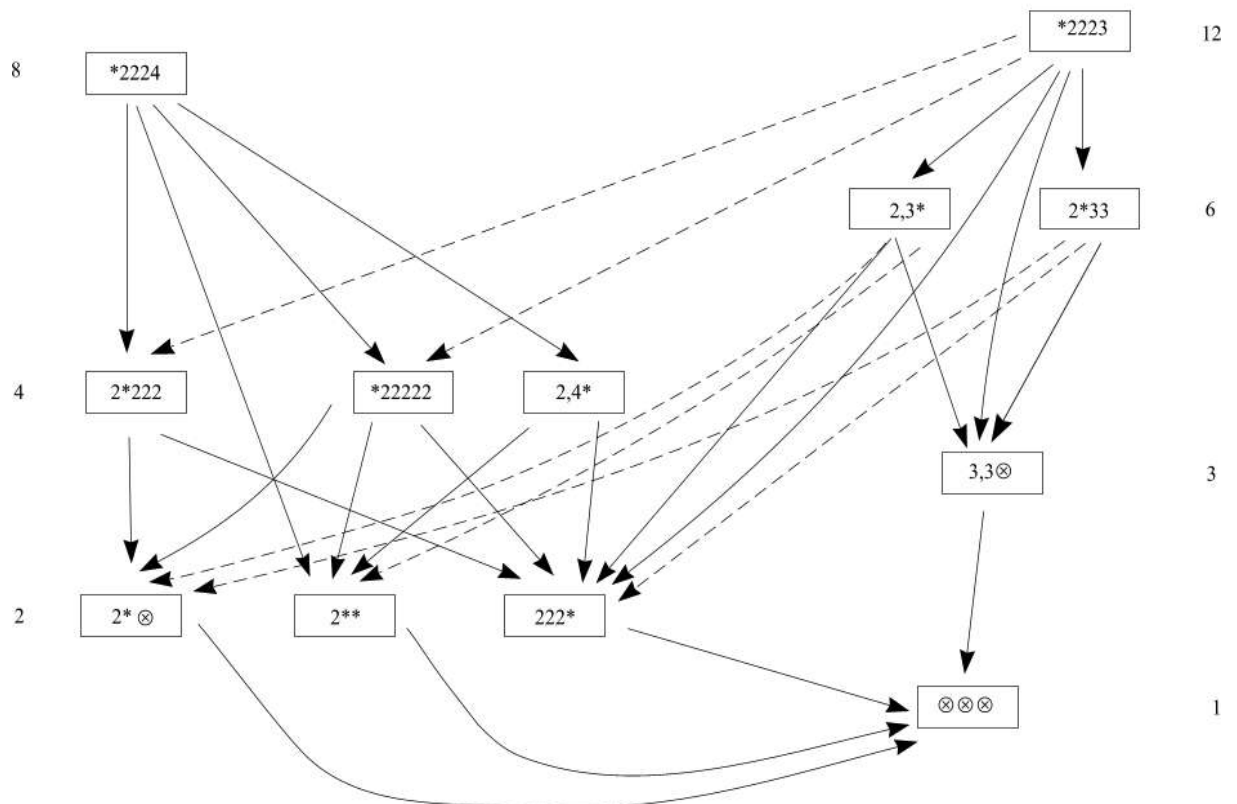


Table 2

Table 3. The list of fundamental domains for 3^- surface with their typical maximal tiling normalizers with factors and indices $|N/G|$

			10/1	aabcbdeeDc	→ 10/16
6/1	aabbcc	$2^*33 / 3m, 6$	2	aabccBdeeD	$2^*\otimes / m, 2$
2	aabcbC	$\otimes^3 / 1, 1$	3	aabcdbeCed	$\otimes^3 / 1, 1$
3	aabcBC	$2^*\otimes / m, 2$	4	aabcdBeCDE	$2^*\otimes / m, 2$
4	aabccb	$2^*222 / mm, 4$	5	aabcdBedcE	$222^* / m, 2$
5	abacbc	$*2224 / mm \text{ o } m, 8$	6	aabcdceDeB	→ 8/16
6	abacbC	$2^{**} / m, 2$	7	aabcdCedEB	→ 8/19
7	abacBC	$2^{**} / m, 2$	8	aabcdCeDEB	→ 8/18
8	abcaBC	$2^*222 / mm, 4$	9	aabcdecDeB	→ 8/18
			10	aabcdeCDEB	→ 8/16
8/1	aabbcdC	→ 6/4	11	aabcdeCedB	→ 8/15
2	aabcbdCd	$2^*\otimes / m, 2$	12	aabcbdeeDcB	$*2223 / m \text{ o } 3m, 12$
3	aabcbddc	$2^*\otimes / m, 2$	13	abacdbeCDE	$\otimes^3 / 1, 1$
4	aabcBdcD	$222^* / m, 2$	14	abacdbedcE	$\otimes^3 / 1, 1$
5	aabcBdCD	$2^{**} / m, 2$	15	abacdBceDe	$2^*\otimes / m, 2$
6	aabcdbCd	$2^{**} / m, 2$	16	abacdBeCed	$2^*\otimes / m, 2$
7	aabcdBCD	$2^*\otimes / m, 2$	17	abacdCbedE	$222^* / m, 2$
8	aabcdBdc	$\otimes^3 / 1, 1$	18	abacdCbeDE	$2^{**} / m, 2$
9	aabcdcDB	→ 6/6	19	abacdeBcDe	$2^{**} / m, 2$
10	aabcdCDB	→ 6/5	20	abcadBeCDe	$2^*222 / m \text{ o } 2, 4$
11	aabcedcB	→ 6/8	21	abcadcebDE	$\otimes^3 / 1, 1$
12	abacbdcd	$\otimes^3 / 1, 1$	22	abcadcedBE	$2^*\otimes / m, 2$
13	abacbdCD	$\otimes^3 / 1, 1$	23	abcAdeBCEd	$2^*222 / mm, 4$
14	abacBdCd	$2^*\otimes / m, 2$	24	abcAdecBEd	$2^*222 / mm, 4$
15	abacdbCD	$2^*\otimes / m, 2$			
16	abacdbdc	$*2223 / mm \text{ o } 3, 12$	12/1	aabcdceffEdB	$2^*\otimes / m, 2$
17	abacdBcD	$\otimes^3 / 1, 1$	2	aabcbddCeffEB	$*2223 / 3m \text{ o } m, 12$
18	abAcdbDc	$*22222 / mm, 4$	3	aabcdecfDfeB	→ 12/11
19	abAcdBDC	$*22222 / mm, 4$	4	aabcdeCfDEFB	→ 10/23
20	abcadBCD	$2^*\otimes / m, 2$	5	aabcdeCfedFB	→ 10/23
21	abcadcbD	$2^*\otimes / m, 2$	6	abacdeBcfDfe	$2^*\otimes / m, 2$
22	abcdAbeD	$*2224 / 4m, 8$	7	abacdeCbfDEF	$2^*\otimes / m, 2$
	and	$*2223 / 2 \text{ o } 3m, 12$	8	abacdeCbfedF	$222^* / m, 2$
			9	abcadeBdfCEf	$*2223 / 3m \text{ o } m, 12$
			10	abcadecfDbEF	$2^*\otimes / m, 2$
			11	abcadecfeBdF	$2^*\otimes / m, 2$

Table 4. The list of polygon symbols \mathcal{F}_N by [L-M-V 98] for non-trivial normalizers N of $G = \otimes^3$. In the symbols ...a...a... refers to side pairing by glide reflection, ...b...B... refers to hyperbolic translation, — refers to line reflection; ...anb... means rotation or dihedral centre of order n at joint of a and b , ...c2C... refers to half-turn about the midpoint of a side, ...dnD... refers to rotation of order n at joint of d and D .

- 222*** (13 domains): —a2Ab2Bc2C, —a2Ab2c2C2B, —a2b2B2c2CA, —a2b2c2C2B2A,
—a2A—b2Bc2C, —a2A—b2c2C2B, —a2A—b2B—c2C,
—a2Abc2Cd2DB, —ab2Bc2Cd2DA, —ab2Bc2d2D2CA,
—a2bc2Cd2DB2A, —a2A—bc2cd2DB, —ab2Bcd2De2ECA
- 2** (4):** —a2Ab—B, —a—A—b2B, —a2b—B2A, —ab—Bc2CA
- 2*⊗ (16):** — ab2Ba, —a2ba2b, —a2b2b2A, —a—ab2B, —a2b—a2b,
—a—a—b2B, —abac2Cb, —abbAc2C, —abbc2CA, —abc2CbA,
— ab2cb2cA, —a2bccBA, —a2A—bccB, —ab—ac2Cb, —ab2BcddCA,
— abc2CdbdA
- 3,3⊗ (8):** a3a3b3B3, a3b3a3b3, aab3Bc3C, aab3c3C3B, a3Ab3cb3c, a3Abc3Cb,
aab3Cd3DB, a3ABc3CdBd
- 24* (5):** —a2Ab4B, —a4b2B4A, —a2b4B2A, —a2A—b4B, —ab2Bc4CA
- *22222 (1):** —2—2—2—2—2
- 2*222 (2):** —2—2—2a2A2, —2—2—2—a2A
- 2*33 (2):** —3—3a2A3, —3—3—a2A
- 23* (5):** —a2Ab3B, —a3b2B3A, —a2b3B2A, —a2A—b3B, —ab2Bc3CA
- *2224 (1):** —2—2—2—4
- *2223 (1):** —2—2—2—3

65+
58 tilings

Hexagonal domains with generating closed geodesics and some typical normalizers for the 3 surface from [S 02]

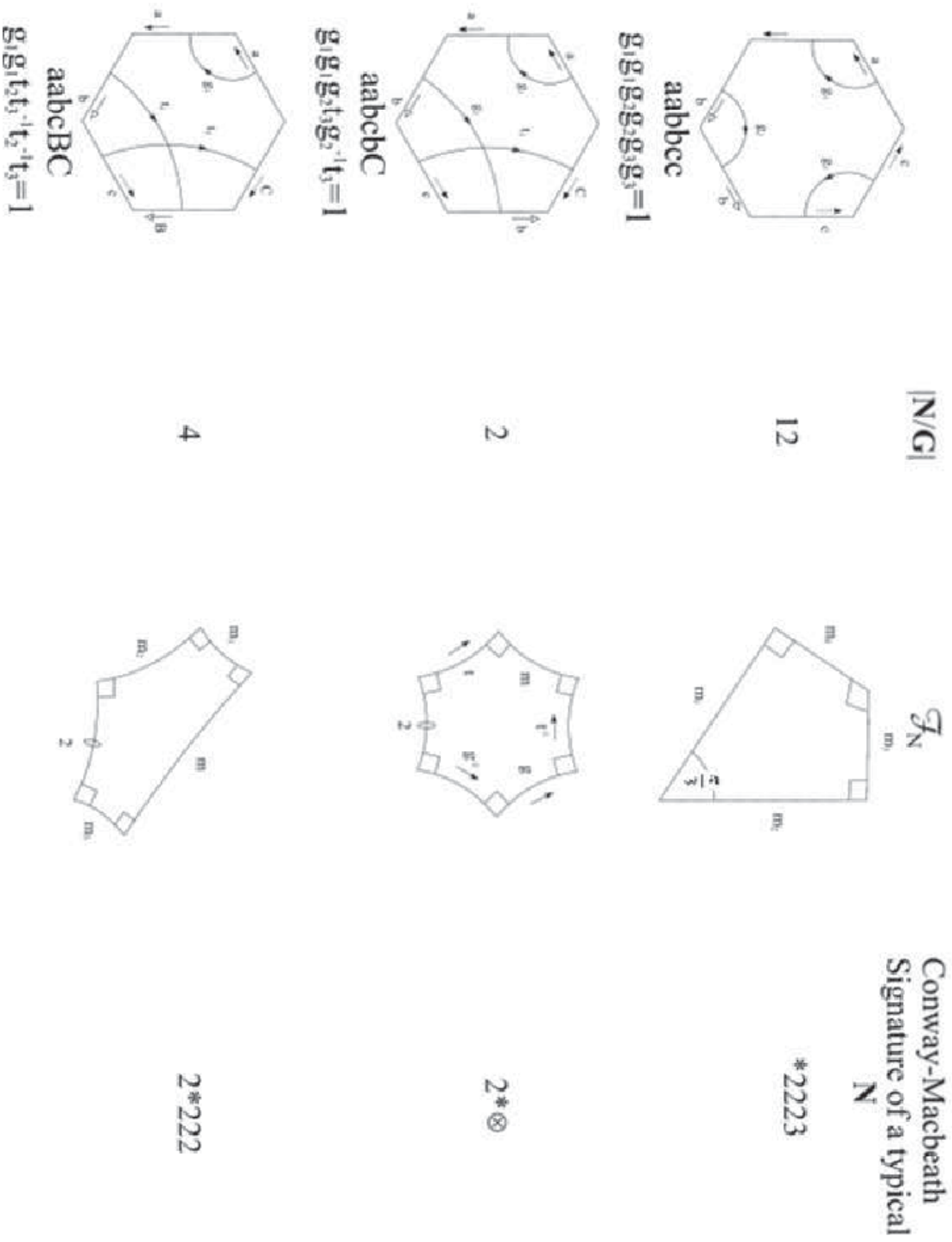


Fig 1.a

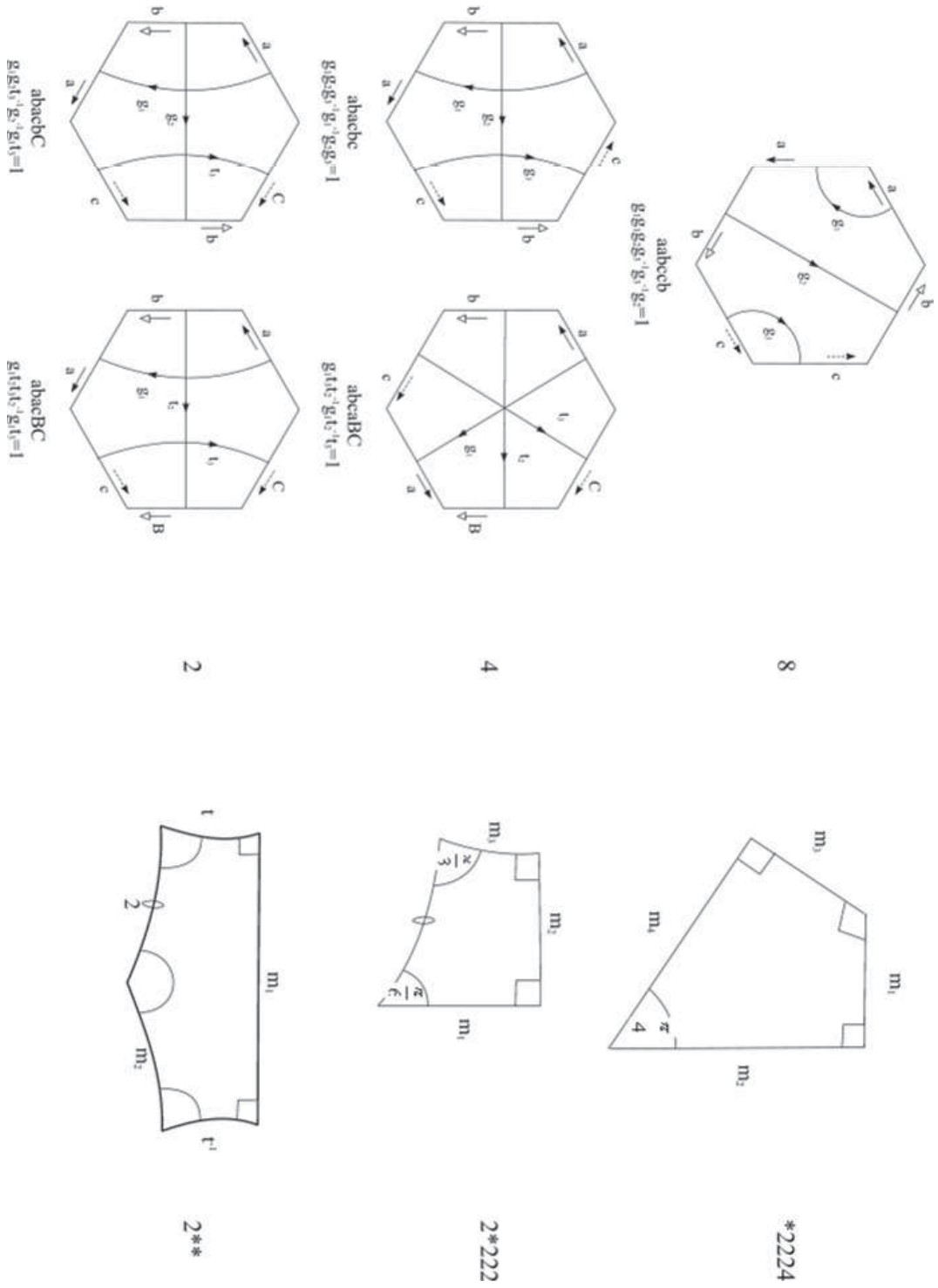


Fig 1. b

a) The 6/1 tiling of polygon symbol **aabbcc**, its barycentric subdivision; b) maximal normalizer for g surface, $g=3$, its fundamental domain $\mathcal{F}_N = (1,2)$; c) - e) Some domains for 3^- surface with tilings by \mathcal{F}_N

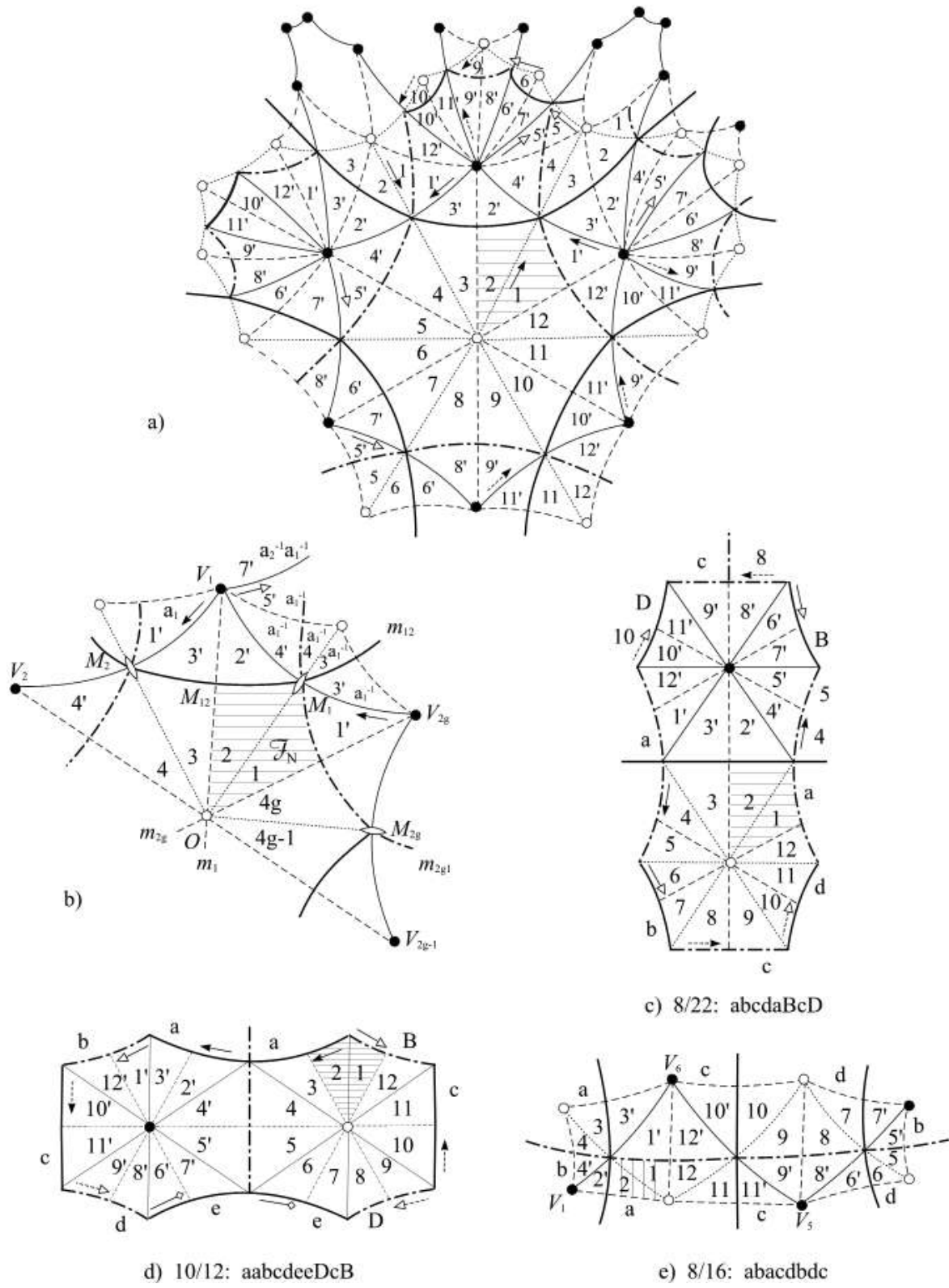


Fig 2

Two 12-gonal fundamental domains for $G=\otimes^3$ with maximal normalizer $N=^*2223$ leading to equivariant tilings a) $12/2$ **aabccddCeffEB** b) $12/9$ **abcadeBdfCEf**

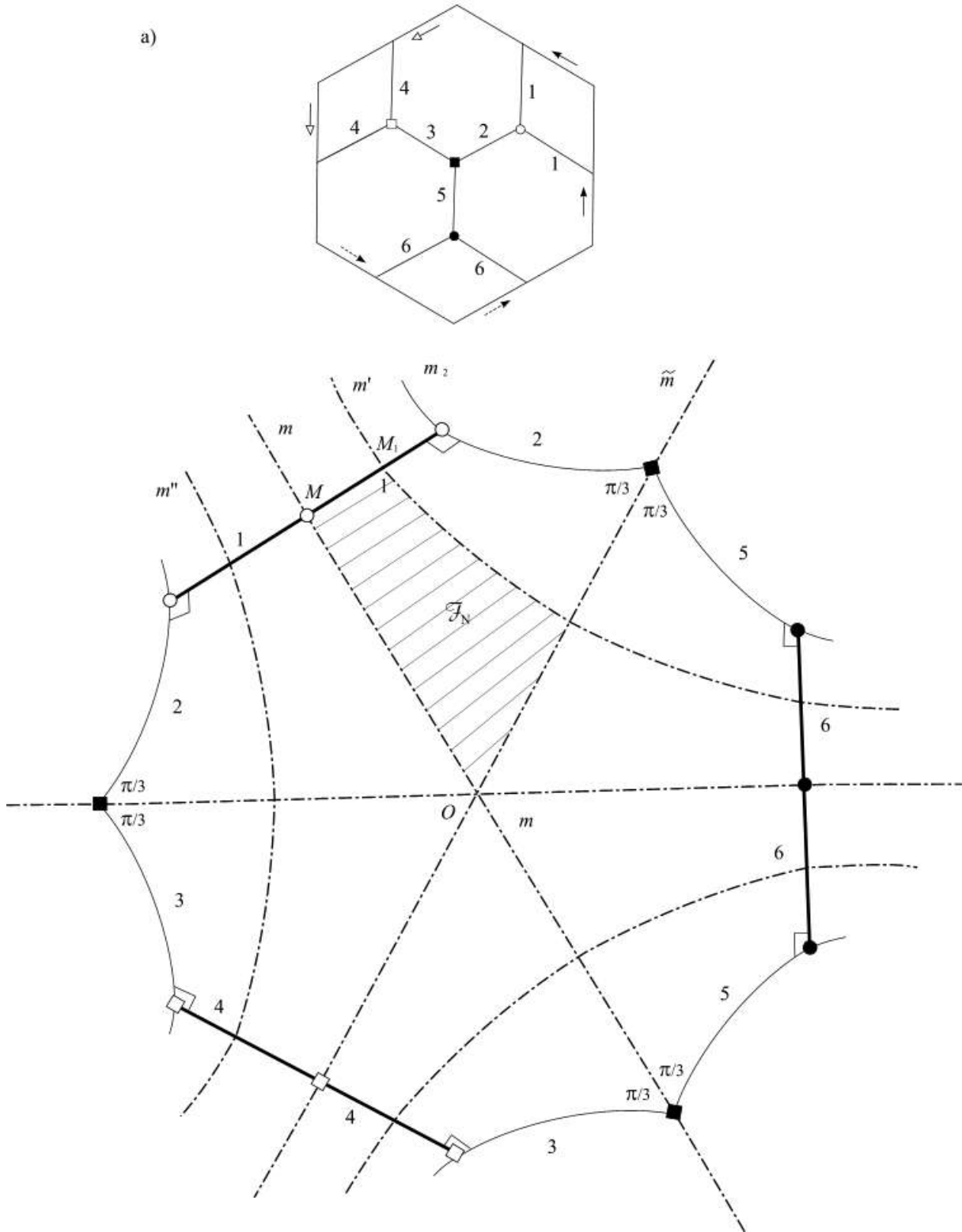


Fig 3. a

b)

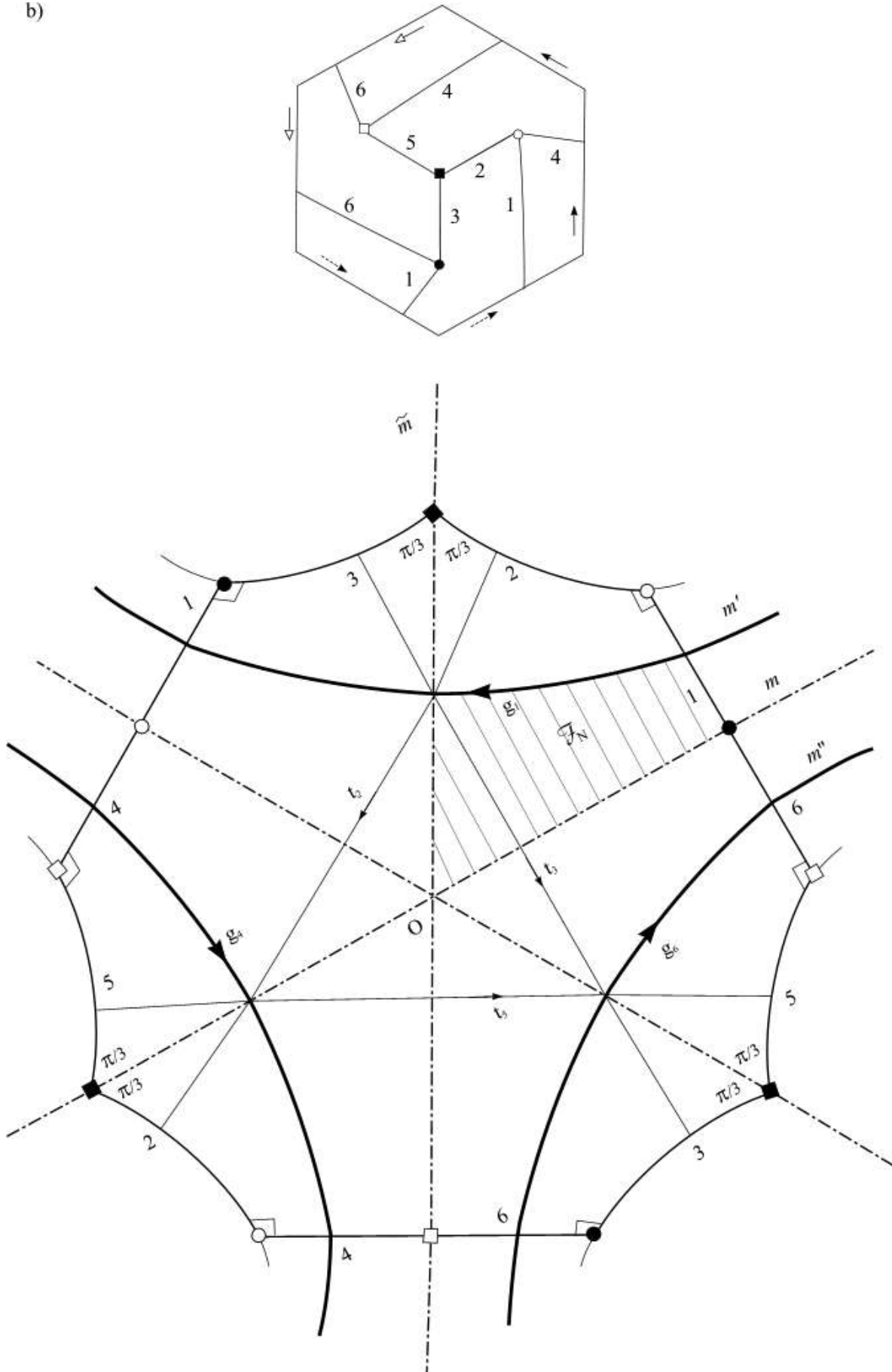


Fig 3. b

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