Underactuated Mechanical Systems with Constant Linear Constraints

László Lemmer and Bálint Kiss
Department of Control Engineering and Information Technology
Budapest University of Technology and Economics
Magyar Tudósok krt. 2, H-1117, Budapest
Email: lemmer@iit.bme.hu, bkiss@iit.bme.hu

Abstract

This paper presents methods to build the equations of motion of constrained mechanical systems with special emphasis on linear mechanical systems with linear kinematic constraints. On the one hand, an explicit closed form solution is given for the constraint forces, on the other hand, the minimal set of equations of motion is derived with the elimination of the constraint forces. Afterwards, the number of maximal possible degree of underactuation and minimal required number of position outputs are analyzed to have a controllable and an observable system, respectively.

Keywords: constrained mechanical system, underactuated mechanical system, maximal degree of underactuation, controllability, observability

1 Introduction

The importance of the modeling of mechanical systems is growing steadily with the wide usage of robotic applications and intelligent actuators. Today, the usage of a minimal number of actuators and sensors is often required in favor of cost reduction. To introduce the notion of *underactuation*, which is an important notion in this field, we consider the following linear second-order (multi-variable) differential equation:

$$M\ddot{q} + D\dot{q} + Sq = Fu. \tag{1}$$

The dimension of the vector of coordinates q is n and of inputs u is m. Accordingly, the constant matrices M, D and S have dimension $n \times n$. The input mapping matrix F is of dimension $n \times m$. The most common examples of processes to be modeled with such differential equations are representatives of a

Notation	Mechanical interpretation	Electrical interpretation
M	inertia	inductivity
D	damping	resistance
S	stiffness	elastance
q	displacement	quantity of charge
u	force/torque	voltage

Table 1: Analogy between mechanical an electrical systems

special class of mechanical systems (gear systems) and electrical networks. The analogy between them is given in Table 1. If $\operatorname{rank} F = n$, we say the system to be *fully actuated* otherwise it is *underactuated*.

This paper focuses on the study of some properties of underactuated mechanical systems, which are useful for control purposes. Section 2 addresses the issue of the modeling of constrained mechanical systems and pay special attention to the case of constant linear constraints. It gives a closed form for constraint forces in the linear case. The paper also presents the mathematical model with a minimal number of generalized coordinates, shows that in case of constant linear constraints the transformation of the coefficient matrices is a congruent transformation, and gives some properties of the resulting matrices. Section 3 discusses the controllability and observability properties in a special way: it determines the minimal number of independent force inputs and position measurements required to make such a system controllable and observable, respectively. In Section 4 the results of the paper are presented on two application examples.

2 Constrained Mechanical Systems

Holonomic constraints can be introduced to the unconstrained mathematical model with help of constraint forces. In special cases, from the equations of motion with constraint forces, a minimal set of equations can be obtained for a reduced set of generalized coordinates. In this section, we describe both approaches and pay particular attention to a special class of mechanical systems with constant linear constraints.

2.1 Mathematical Model with Constraint Forces

The equations of motion of a constrained system read [8]

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L(q,\dot{q})}{\partial q} = \left[\frac{\partial c(q)}{\partial q}\right]^T \lambda + F(q)\tau, \tag{2a}$$

subject to

$$c(q) = 0, \tag{2b}$$

where q, λ and τ stand for the n-dimensional vector of the generalized coordinates, the m-dimensional ($m < n$) vector of Lagrange multipliers and the p-dimensional ($p \leq n$) vector of the generalized forces, respectively. The $n \times p$ dimensional matrix $F(q)$ is referred to as the input mapping matrix. The coefficient matrix of the vector of Lagrange multipliers is the transpose of the Jacobian matrix of the m-dimensional vector of (independent) holonomic constraints (2b) with respect to the generalized coordinates (henceforth referred to as $J(q)$). The product $J(q)\lambda$ gives the vector of the constraint forces. The Lagrangian L is defined as the difference between the kinetic and the potential energy:

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q). \tag{3}$$

Usually, (2a) is written in the following form:

$$M(q)\ddot{q} + h(q, \dot{q}) = J^T\lambda + F(q)\tau, \tag{4}$$

where $M(q)$ is the n-by-n symmetric and positive definite inertia matrix (recall Table 1) and $h(q, \dot{q})$ is the n-dimensional vector of the Coriolis and centrifugal force terms that are quadratic in velocity, the potential (gravity, spring forces) and possibly frictional force terms.

2.2 Mathematical Model with a Reduced Set of Generalized Coordinates

Let us consider the time derivative of (2b):

$$\frac{d}{dt}c(q) = \frac{\partial c(q)}{\partial q}\dot{q} = J(q)\dot{q} = 0. \tag{5}$$

Note that the vector of generalized velocities must lie in the null space of $J(q)$ to fulfill (5). Let $N(q)$ be an n-by-$(n-m)$ matrix with linearly independent columns that span this (right) null space. Now we give two identities to be reused in the sequel:

$$J(q)N(q) = 0, \tag{6}$$

$$\dot{q} = N(q)v. \tag{7}$$

Equation (7) expresses that the generalized velocities can be determined with a reduced set of variables i.e. the $(n-m)$ coordinates for a basis in the null space of $J(q)$. For further use, let us derive its time derivative:

$$\ddot{q} = N(q)\dot{v} + \dot{N}(q)v. \tag{8}$$

The transpose of (6) implies that the pre-multiplication of (4) by $N^T(q)$ eliminates λ and also reduces the number of equations by the number of the constraints (i.e. by $m = \dim\lambda$). After substitution of the expressions for $\dot{q}$ and $\ddot{q}$ from (7) and (8) into (4) and pre-multiplication by N^T, we have the reduced set of equations

$$H_r(q)\dot{v} + h_r(q,v) = F_r(q)\tau, \tag{9}$$

where

$$H_r(q) = N^T(q)H(q)N(q), \tag{10a}$$

$$h_r(q,v) = N^T(q)H(q)\dot{N}^T(q)v + N^T(q)h(q,N(q)v), \tag{10b}$$

$$F_r(q) = N^T(q)F(q). \tag{10c}$$

Equations (9) and (7) are a set of first order ordinary differential equations appropriate to describe the dynamical behavior of the constrained system without constraint forces such that the initial conditions satisfy (2b). Nevertheless, the factors $H_r(q)$, $F_r(q)$ and the term $h_r(q,v)$ do not always depend on the whole set of the generalized coordinates q and if so, we do not need to take the whole set of (7).

If the time derivatives of the generalized coordinates, which are the arguments in (9) and (7) can be chosen as a subset of coordinates in the basis of the null space of $J(q)$, the following notation can be used:

$$\dot{q} = N(q_r)\dot{q}_r, \tag{11}$$

where q_r stands for the coordinates according to the basis in the null space. Then the compact set of equations of motion reads

$$H_c(q_r)\ddot{q}_r + h_c(q_r,\dot{q}_r) = F_c(q_r)\tau. \tag{12}$$

2.3 Mathematical Model of Mechanical Systems with Constant Linear Constraints

Now we consider a special class of mechanical systems where the kinetic energy and the potential energy are expressed in a quadratic form in generalized velocities and positions with constant symmetric positive definite and respectively positive semidefinite coefficient matrices:

$$K(\dot{q}) = \frac{1}{2}\dot{q}^T M\dot{q}, \quad M^T = M, \quad M > 0, \tag{13}$$

$$P(q) = \frac{1}{2}q^T Sq, \quad S^T = S, \quad S \geq 0. \tag{14}$$

The constraints are supposed to have a linear form with constant coefficients:

$$c(q) = Cq = 0, \tag{15}$$

where C has full row rank. We also assume a damping (viscous friction) linear in velocities that will be derived with help of Rayleigh's dissipative function with a symmetric positive semidefinite and constant coefficient matrix:

$$R(\dot{q}) = \frac{1}{2}\dot{q}^T D\dot{q}, \quad D^T = D, \quad D \geq 0. \tag{16}$$

For (4) we obtain

$$M\ddot{q} + D\dot{q} + Sq = C^T\lambda + F\tau. \tag{17}$$

where F is a constant and full column rank input mapping matrix and τ is the vector of the external generalized forces.

2.3.1 Constraint Forces in Mechanical Systems with Constant Linear Constraints

Let us express $\ddot{q}$ from (17) and substitute it into the second derivative of (15):

$$CM^{-1}\left(C^T\lambda + F\tau - D\dot{q} - Sq\right) = 0. \tag{18}$$

Lemma 1. *The product $CM^{-1}C^T$ is positive definite.*

Proof. M and accordingly also M^{-1} are positive definite: $x^T M^{-1}x > 0$ for all $x \neq 0$. Let us consider the linear transformation $x = C^T y$. As C^T has full column rank, $x = 0$ if and only if $y = 0$. Consequently $y^T CM^{-1}C^T y > 0$ for all $y \neq 0$. □

Now we can express the Lagrange multipliers from (18):

$$\lambda = \left(CM^{-1}C^T\right)^{-1} CM^{-1}\left(F\tau - D\dot{q} - Sq\right). \tag{19}$$

Remark 1. *The coefficient $\left(CM^{-1}C^T\right)^{-1} CM^{-1}$ is a reflexive generalized inverse of C^T providing the least-squares solution of $C^T\lambda = F\tau - D\dot{q} - Sq$ with the norm $\|x\|_{M^{-1}} = \sqrt{x^T M^{-1}x}$. (The notion of the generalized inverse can be found for example in [7].)*

Theorem 1. *The constraint forces in the mechanical system* (17) *can be expressed with help of a projection in the form*

$$\tau_c = P\left(F\tau - D\dot{q} - Sq\right), \tag{20}$$

where

$$P = C^T\left(CM^{-1}C^T\right)^{-1} CM^{-1}. \tag{21}$$

Proof. With the substitution of (19) for λ into the expression of the constraint forces ($\tau_c = C^T\lambda$) we obtain the form given in the theorem. Now we show that P is idempotent:

$$P^2 = C^T \underbrace{\left(CM^{-1}C^T\right)^{-1} CM^{-1}C^T}_{I} \left(CM^{-1}C^T\right)^{-1} CM^{-1} = C^T \left(CM^{-1}C^T\right)^{-1} CM^{-1} = P. \tag{22}$$

□

The equations of motion of a mechanical system with constant linear constraints can be described with the differential equations: $M\ddot{q} + D\dot{q} + Sq = \tau_c + F\tau$ where τ_c is as in (20). Sometimes it is convenient to calculate the constraint forces and to have the values of all generalized coordinates, however, it is possible to give equations with a less number of coordinates. In the succeeding subsection we discuss the later approach.

2.3.2 Mathematical Model of Mechanical Systems with Constant Linear Constraints with a Reduced Set of Generalized Coordinates

As C in (15) is a constant matrix, its null space can also be spanned by the linearly independent constant basis vectors $b_1, b_2, \ldots, b_{n-m}$. Let us build the matrix

$$N = [b_1, b_2, \ldots, b_{n-m}], \tag{23}$$

and so we have

$$q^{(i)} = N q_r^{(i)}, \quad i \in \mathbb{N}. \tag{24}$$

Theorem 2. *The motion of the linear mechanical system* (17) *subject to constant linear constraints* (15) *can be described with a reduced number $(n-m)$ of differential equations*

$$M_r \ddot{q}_r + D_r \dot{q}_r + S_r q_r = F_r \tau, \tag{25}$$

where

$$F_r = N^T F, \tag{26}$$

and the coefficient matrices on the left-hand side are obtained with the congruent transformation of N:

$$M_r = N^T M N, \quad D_r = N^T D N, \quad S_r = N^T S N. \tag{27}$$

The original coordinates and its derivatives can be determined with help of the transformation given in (24)*.*

Proof. Let us use (24) for substitution into (17). We premultiply the obtained equation by N^T and considering $CN = 0$ we obtain (25) with the coefficient matrices as in (26) and (27). (Notice that the same results are obtained considering (9) and (10) from Subsection 2.2.) □

Prior to the analysis of the matrices obtained with the congruent transformation of N, let us introduce some technical definitions.

Definition 1 (M-norm). *The norm* $\|x\|_M = \sqrt{\langle x, x\rangle_M}$ *associated with the (elliptical) inner product* $\langle x, y\rangle_M = x^T M y$, *(*$x, y \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$, $M > 0$, $M = M^T$*) will be called M-norm.*

Definition 2 (M-orthogonal vector system). *The vectors* $x_1, x_2, \dots, x_l \in \mathbb{R}^n$, *(*$l \le n$*) build an M-orthogonal vector system if for every* (x_i, x_j), $i \neq j$ *pairs* $\langle x_i, x_j\rangle_M = x_i^T M x_j = 0$, *(*$M \in \mathbb{R}^{n\times n}$, $M > 0$, $M = M^T$*) holds.*

In the following we will consider the inertia matrix M_r in (25) but the same properties hold true for the matrices D_r and S_r if they are nonsingular.

Theorem 3. *If the basis* $b_1, b_2, \dots, b_{n-m}$ *for the null space of C in* (15) *builds an M-orthogonal vector system, the congruent transformation with N as defined in* (23) *transforms M into a diagonal form with diagonal elements (i.e. eigenvalues)* $\lambda_i = \|b_i\|_M^2$.

Even if the basis $b_1, b_2, \dots, b_{n-m}$ *does not build an M-orthogonal vector system, the difference between the eigenvalue λ_i of the transformed matrix* $M_r = N^T M N$ *and* $\|b_i\|_M^2$ *is bounded:*

$$\left|\lambda_i - \|b_i\|_M^2\right| \le \sum_{j\neq i} \left|\langle b_i, b_j\rangle_M\right|. \tag{28}$$

Proof. To prove the theorem we calculate the elements of M_r:

$$M_r = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_{n-m}^T \end{bmatrix} M \begin{bmatrix} b_1 & b_2 & \cdots & b_{n-m} \end{bmatrix} = \begin{bmatrix} \langle b_1, b_1\rangle_M & \langle b_1, b_2\rangle_M & \cdots & \langle b_1, b_{n-m}\rangle_M \\ \langle b_2, b_1\rangle_M & \langle b_2, b_2\rangle_M & \cdots & \langle b_2, b_{n-m}\rangle_M \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_{n-m}, b_1\rangle_M & \langle b_{n-m}, b_2\rangle_M & \cdots & \langle b_{n-m}, b_{n-m}\rangle_M \end{bmatrix} \tag{29}$$

The first statement is a consequence resulting from Definitions 1 and 2. The second statement is obtained with the direct use of Geršgorin's theorem that gives bounds for eigenvalues of arbitrary quadratic matrices (see for example in [2,3]). □

Remark 2. *For singular stiffness (S) or damping(D) matrices it is not possible to define the elliptical inner product and its induced norm such as in Def. 1. Nonetheless the eigenvalue localization theorem for the transformed matrices can be formulated in the following form*

$$\left|\lambda_i - b_i^T X b_i\right| \le \sum_{j\neq i} \left|b_i^T X b_j\right|, \tag{30}$$

where S and D has to be substituted for X. This is a direct consequence of Geršgorin's theorem again.

Sometimes it can be useful to have a diagonal inertia matrix in the reduced equations of motion of the constrained system. According to Theorem 3, it can be achieved with an M-orthogonal basis for the null space of the constraint matrix C. An M-orthonormal basis can be obtained with the Gram–Schmidt procedure using the elliptical inner product defined in Def. 1. This generalized Gram–Schmidt sequence is presented in Algorithm 1 where the projection operator P_{M,u_i} is defined as

$$P_{M,u_i} b_k = \frac{\langle u_i, b_k \rangle_M}{\|u_i\|_M^2} u_i. \tag{31}$$

Algorithm 1 Generalized Gram–Schmidt Process

$u_1 \leftarrow b_1$
$e_1 \leftarrow \frac{u_1}{\|u_1\|_M}$
for $k = 2$ to $n - m$ **do**
  $u_k \leftarrow b_k - \sum_{i=1}^{k-1} P_{M,u_i} b_k$
  $e_k \leftarrow \frac{u_k}{\|u_k\|_M}$
end for

The Gram–Schmidt procedure requires any linearly independent set of basis vectors that span the subspace. The vectors $u_1, u_2, \ldots, u_{n-m}$ and $e_1, e_2, \ldots, e_{n-m}$ obtained from the Gram–Schmidt procedure build an M-orthogonal vector system and an orthonormal basis in the null space of C, respectively. Wirh use of these vectors for the congruent transformation ($N = [u_1, u_2, \ldots, u_{n-m}]$ or $N = [e_1, e_2, \ldots, e_{n-m}]$) we obtain diagonal or identity inertia matrix, respectively.

Theorem 4. *The basis of the null space of the constraint matrix always can be chosen so that the inertia matrix (M_r) in the equations of motion of the constrained system* (25) *is the identity matrix.*

Proof. Eq. (29) shows that the inertia matrix after the congruent transformation is a Gramian matrix with the M-norm. As the M-norm Gramian matrix of any M-orthonormal basis is the identity matrix, we only have to show that the null space of the constraint matrix C contains an M-orthonormal basis. The Gram–Schmidt procedure started with the basis vectors $b_1, b_2, \ldots, b_{n-m}$ will terminate with an M-orthonormal set of vectors in this null space. This M-orthonormal set of vectors is then a basis because its members are linearly independent. □

3 Controllability and Observability Properties of Mechanical Systems with Constant Linear Constraints

For the controllability and observability analysis of mechanical systems with constant linear constraints we take the model (25) where the constraint forces have already been eliminated. To have a compact form we premultiply it by M_r^{-1} and after rearrangement we obtain

$$\ddot{q}_r = \tilde{D}\dot{q}_r + \tilde{S}q_r + \tilde{F}\tau, \tag{32}$$

where $\tilde{D} = -M_r^{-1}D_r$, $\tilde{S} = -M_r^{-1}S_r$, and $\tilde{F} = M_r^{-1}F_r$. The outputs are defined as independent linear combinations of the generalized coordinates:

$$y = Gq_r, \tag{33}$$

where G has full row rank. Accordingly the state equations read

$$\dot{x} = \begin{bmatrix} 0 & I \\ \tilde{S} & \tilde{D} \end{bmatrix} x + \begin{bmatrix} 0 \\ \tilde{F} \end{bmatrix} \tau, \tag{34a}$$

$$y = \begin{bmatrix} G & 0 \end{bmatrix} x, \tag{34b}$$

with the state vector: $x = (q_r^T, \dot{q}_r^T)^T$.

In the following subsections we study the controllability and observability properties of this system and present conditions on the minimal number of actuators and senors that make the model controllable and observable, respectively.

3.1 Minimal Number of Inputs for Controllability

In the sequel we will use the following rank theorem of controllability where n_{ss} is the dimension of the state space:

Theorem 5. *The pair (A, B) is controllable if and only if*

$$\text{rank} \begin{bmatrix} \lambda_i I - A, B \end{bmatrix} = n_{ss} \tag{35}$$

for all n_{ss} eigenvalues λ_i of A.

The eigenvalue λ_i is an uncontrollable eigenvalue of A if and only if

$$\text{rank} \begin{bmatrix} \lambda_i I - A, B \end{bmatrix} < n_{ss}. \tag{36}$$

Proof. See [1]. □

The nullity of $\lambda_i I - A$ is the number of the linearly independent eigenvectors of A belonging to the eigenvalue λ_i. Accordingly, it follows the theorem below.

Theorem 6. *The minimal number of linearly independent inputs in the linear system $\dot{x} = Ax + Bu$ to make it controllable is the maximal number of the linearly independent eigenvectors of A belonging to one of its eigenvalues. The directions of the inputs have to be such that their projection to the subspace spanned by the left eigenvectors belonging to the same eigenvalue also span it.*

Note that we did not make use of a special form of the state equations and so this is a general result for the minimal number of independent inputs for controllability. In the following subsections we will make use of special assumptions that make sense for second-order (mechanical) systems.

3.1.1 Second-Order Systems with Nonzero First- and Zero-Order Terms

Now we study the controllability of state equations of the form such as in (34) where $\tilde{D} \leq 0$ and $\tilde{S} \leq 0$. To determine the minimal required number of independent inputs to make it controllable, we use Theorem 6.

Let λ be an eigenvalue of the system matrix in (34) and the two n-dimensional partitions of the corresponding left eigenvector be v_1 and v_2. With these notations we can write:

$$\begin{pmatrix} v_1^T & v_2^T \end{pmatrix} \begin{bmatrix} 0 & I \\ \tilde{S} & \tilde{D} \end{bmatrix} = \begin{pmatrix} v_1^T & v_2^T \end{pmatrix} \lambda, \tag{37}$$

or accordingly

$$v_2^T \tilde{S} = v_1^T \lambda, \tag{38a}$$

$$v_1^T = v_2^T \left(\lambda I - \tilde{D} \right). \tag{38b}$$

Eq. (38b) implies that the number of independent eigenvectors belonging to one eigenvalue is limited up to n (the degrees of freedom), i.e. independent (force) inputs with a number equal the degrees of freedom always make this type of systems controllable. After substitution of v_1 from (38b) into (38a) we have

$$v_2^T \left(\lambda^2 I - \lambda \tilde{D} - \tilde{S} \right) = 0, \tag{39}$$

or after premultiplication by the inertia matrix

$$v_2^T \left(\lambda^2 M + \lambda D + S \right) = 0. \tag{40}$$

The relationship between (37) and (40) delivers a special formulation of Theorem 6 for this system class:

Theorem 7. *The maximal degree of underactuation of the controllable second-order system* (25) *is* $\max_{\lambda} \operatorname{rank} \left(\lambda^2 M + \lambda D + S \right)$ *where the maximum is taken over all eigenvalues of the system matrix in* (34) *but obviously at least 1 input is needed. The projections of the inputs to the null space of* $\left(\lambda^2 M + \lambda D + S \right)$

have to span this null space for all eigenvalues of A.

3.1.2 Second-Order Systems with Pure Damping Term

Let us analyze the case when $\tilde{S} = 0$ in the state equations (34). With the same notation as in the previous subsection (37) gives

$$0 = v_1^T \lambda, \tag{41a}$$

$$v_1^T = v_2^T (\lambda M + D). \tag{41b}$$

Equations. (41a) and (41b) imply that $\lambda = 0$ is eigenvalue of the system matrix in (34) with multiplicity of at least n. We will use this fact to prove the following theorem.

Theorem 8. *Pure damping cannot increase the degree of underactuation of a controllable second-order system. In other words, $S = 0$ (i.e. no springs in the system) implies that the number of independent actuators equals the degree of freedom of the mechanical system to make it controllable.*

Proof. After substitution of $\lambda = 0$ into (35) in Theorem 5 we obtain

$$\operatorname{rank} \begin{bmatrix} 0 & -I & 0 \\ 0 & -\tilde{D} & \tilde{F} \end{bmatrix} = 2n \tag{42}$$

as a necessary condition of the controllability and consequently rank $\tilde{F}$ has to be n. Notice that rank $\tilde{F} = n$ is a sufficient condition, as well. □

3.1.3 Second-Order Systems with No Damping Term

Now we substitute $\tilde{D} = 0$ into (37) and obtain

$$v_2^T \tilde{S} = v_1^T \lambda, \tag{43a}$$

$$v_1^T = v_2^T \lambda. \tag{43b}$$

Again v_2 determines v_1 according to (43b) and its substitution into (43a) delivers

$$v_2^T \tilde{S} = v_2^T \lambda^2, \tag{44}$$

that is, v_2 is the left eigenvector of $\tilde{S}$ and all eigenvalues of the system matrix in (34) are square roots of the eigenvalues of $\tilde{S}$. (Recall that $\tilde{S}$ is negative semi-definite and so the square roots are conjugate pairs on the imaginary axis.) The theorem below summarizes these results:

Theorem 9. *The minimal required number of independent (force) inputs to make the second-order (mechanical) system* (34) *with* $D = 0$ *controllable is the highest multiplicity in the eigenvalues of* $\tilde{S}$*. The projections of the inputs to the subspace spanned by the left eigenvectors of* $\tilde{S}$ *(belonging to the same eigenvalue) have to span this subspace.*

Proof. The number of inputs must be at least the maximal number of linearly independent eigenvectors belonging to the same eigenvalue of $\tilde{S}$. The matrix $\tilde{S}$ is a symmetric matrix and accordingly also a normal matrix that is similar to a diagonal matrix (see for example in [3, 6]). Consequently, $\tilde{S}$ has a complete orthonormal set of eigenvectors and so all the eigenvectors belonging to one eigenvalue are linearly independent. □

3.2 Minimal Number of Outputs for Observability

The duality between controllability and observability allows us to give properties of observability similar to the results for controllability from Subsection 3.1. Here we use the dual equivalent of Theorem 5:

Theorem 10. *The pair* (A, C) *is observable if and only if*

$$\operatorname{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n_{ss} \tag{45}$$

for all n *eigenvalues* λ_i *of* A.

The eigenvalue λ_i *is an unobservable eigenvalue of* A *if and only if*

$$\operatorname{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} < n_{ss}. \tag{46}$$

Proof. See [1]. □

Because of similar reasons as in Subsection 3.1 we can give the following theorem about the minimal number of independent outputs to make it observable:

Theorem 11. *The minimal number of independent outputs in the linear system* $\dot{x} = Ax + Bu$, $y = Cx$ *to make it observable is the largest number in the Weyr characteristic of* A*. The output directions have to be such that their projections to the subspace spanned by the right eigenvectors belonging to the same eigenvalue also span it.*

Observe that the right eigenvalues play a role now. This is because in (45) the rows of C can increase the rank. In the next parts we will apply Theorem 11 to the system class defined above.

3.2.1 Second-Order Systems with Nonzero First- and Zero-Order Terms

Now we assume the general case when $\tilde{D} \leq 0$ and $\tilde{S} \leq 0$. Again λ stands for an eigenvalue of the system matrix in (34) and the two n-dimensional partitions of a corresponding right eigenvector will be referred to as u_1 and u_2:

$$\begin{bmatrix} 0 & I \\ \tilde{S} & \tilde{D} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{47}$$

or accordingly

$$u_2 = \lambda u_1, \tag{48a}$$

$$\tilde{S} u_1 + \tilde{D} u_2 = \lambda u_2. \tag{48b}$$

Eq. (48a) implies that the number of independent eigenvectors belonging to one eigenvalue is limited up to n (the degrees of freedom), i.e. n independent (position) measurements always make this type of systems observable. With the substitution of u_2 from (38a) into (38b) and premultiplication by the inertia matrix we obtain

$$\left(\lambda^2 M + \lambda D + S\right) u_1 = 0, \tag{49}$$

and the specialization of Theorem 11 for this system class reads:

Theorem 12. *Second-order systems with state equations as given in* (34) *can be made observable with independent (position) measurements with a minimal number of* $\max_{\lambda} \left\{n - \operatorname{rank}\left(\lambda^2 M + \lambda D + S\right)\right\}$ *where the maximum is taken over all eigenvalues of the system matrix in* (34) *but at least 1 output is required. The projections of the output directions to the null space of* $\left(\lambda^2 M + \lambda D + S\right)$ *have to span this null space for all eigenvalues of* A.

3.2.2 Second-Order Systems with Pure Damping Term

Let us analyze the case when $\tilde{S} = 0$ in the state equations (34). Recall that it was implied by (41) that $\lambda = 0$ is eigenvalue of the system matrix. We can use this property also here.

Theorem 13. *The mathematical system* (34) *with* $\tilde{S} = 0$ *(i.e. no springs in the system) is observable if and only if the rank of* G *equals* n *(the degree of freedom), i.e. there are* n *independent position measurements.*

Proof. With the substitution of $\lambda = 0$ into (45) in Theorem 10 we obtain

$$\operatorname{rank} \begin{bmatrix} 0 & -I \\ 0 & -\tilde{D} \\ G & 0 \end{bmatrix} = 2n \tag{50}$$

as a necessary condition of the controllability that can be fulfilled only with rank G. Because of the identity block this is a sufficient condition, as well. □

3.2.3 Second-Order Systems with No Damping Term

Let us substitute $\tilde{D} = 0$ into (47). So we have

$$u_2 = \lambda u_1, \tag{51a}$$

$$\tilde{S} u_1 = \lambda u_2. \tag{51b}$$

Again u_1 determines u_2 according to (51a) and its substitution into (51b) delivers

$$\tilde{S} u_1 = \lambda^2 u_1. \tag{52}$$

As we already know, the eigenvalues of the system matrix are square roots of the eigenvalues of $\tilde{S}$. Now the u_1's are right eigenvectors of $\tilde{S}$. Let us summarize this:

Theorem 14. *The minimal required number of independent (position) outputs to make the second-order (mechanical) system* (34) *with $D = 0$ observable equals the highest multiplicity in the eigenvalues of $\tilde{S}$. The projections of the inputs to the subspace spanned by the right eigenvectors (belonging to the same eigenvalue) of $\tilde{S}$ have to span this subspace.*

Proof. See the proof of the Theorem 9. □

Remark 3. *Notice the interesting consequence of the duality of the controllability and observability properties that the minimal required number of outputs to make the system observable equals the minimal required number of inputs to make it controllable.*

4 Application Examples

In this section we present two example applications. In the first example we give the equations of motion of a mechanical system where four rotating bodies are connected with a harmonic drive and a spring. We also model damping on the bodies and on the spring. Both modeling approaches (the modeling with constraint torques and the reduction of the number of generalize coordinates) will be applied with symbolic calculations. In the second example, it will be shown on a simple setup that changes in system parameters can destroy the controllability property of an underactuated system.

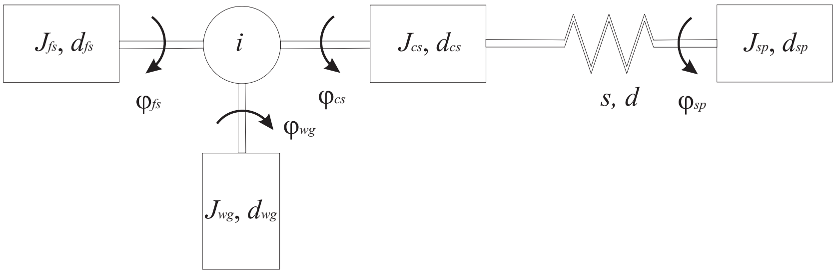

Figure 1: Constrained system with harmonic drive and spring

4.1 Mathematical Modelling of a Harmonic Drive System

Fig. 4.1 shows a mechanical system where the moment of inertia and the (viscous) damping factor of the corresponding body is J_x and d_x with the corresponding subscript ($x \in \{fs, wg, cs, sp\}$), respectively. The bodies are connected with a harmonic drive with gear transmission ratio i and a spring with stiffness s and damping d. The Lagrangian of this system reads

$$L = \frac{1}{2}\left\{J_{fs}\dot{\varphi}_{fs}^2 + J_{wg}\dot{\varphi}_{wg}^2 + J_{cs}\dot{\varphi}_{cs}^2 + J_{sp}\dot{\varphi}_{sp}^2 - s\left(\varphi_{cs} - \varphi_{sp}\right)^2\right\} \tag{53}$$

and Rayleigh's dissipative function is

$$R = \frac{1}{2}\left\{d_{fs}\dot{\varphi}_{fs}^2 + d_{wg}\dot{\varphi}_{wg}^2 + d_{cs}\dot{\varphi}_{cs}^2 + d_{sp}\dot{\varphi}_{sp}^2 + d\left(\dot{\varphi}_{cs} - \dot{\varphi}_{sp}\right)^2\right\} \tag{54}$$

where φ_x is the position coordinate of the corresponding body. It is easy to show that with the Lagrangian and the dissipative function above we obtain the following coefficient matrices on the left side of (17):

$$M = \begin{bmatrix} J_{fs} & 0 & 0 & 0 \\ 0 & J_{wg} & 0 & 0 \\ 0 & 0 & J_{cs} & 0 \\ 0 & 0 & 0 & J_{sp} \end{bmatrix}, \quad D = \begin{bmatrix} d_{fs} & 0 & 0 & 0 \\ 0 & d_{wg} & 0 & 0 \\ 0 & 0 & d_{cs} + d & -d \\ 0 & 0 & -d & d_{sp} + d \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s & -s \\ 0 & 0 & -s & s \end{bmatrix}. \tag{55}$$

The harmonic drive introduces the following linear constraint (see [4, 5]):

$$i\varphi_{fs} + \varphi_{wg} - (i + 1)\varphi_{cs} = 0. \tag{56}$$

Accordingly, on the right-hand side of (17) the factor for the Lagrange multiplier λ is

$$C = \begin{bmatrix} i & 1 & -(i+1) & 0 \end{bmatrix}. \tag{57}$$

The input mapping matrix F depends on the application (where the actuators are mounted) but usually it is built of the identity vectors $(e_1, e_2, \ldots, e_n)$.

Let us determine the projector P in (20). As we have only 1 constraint equation and so C has only 1 row, P will be a matrix of rank 1 according to (21). The middle inverse factor is a scalar that can be brought to the front and so we have

$$P = \frac{J_{fs}J_{wg}J_{cs}}{i^2 J_{wg}J_{cs} + J_{fs}J_{cs} - (i+1)^2 J_{fs}J_{wg}} \begin{bmatrix} \frac{i^2}{J_{fs}} & \frac{i}{J_{wg}} & -\frac{i(i+1)}{J_{cs}} & 0 \\ \frac{i}{J_{fs}} & \frac{1}{J_{wg}} & -\frac{i+1}{J_{cs}} & 0 \\ -\frac{i(i+1)}{J_{fs}} & -\frac{i+1}{J_{wg}} & \frac{(i+1)^2}{J_{cs}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{58}$$

With this projection one can determine the constraint torque such as given in (20). Observe that the last body (with subscript ‘sp’) does not play any role in the constraint (as we expected).

In the following we give the equations of motion with the reduced set of equations obtained after the elimination of the constraint torques. We will use the congruent transformation given in Theorem 2 for this purpose. To do this we have to give a basis for the null space of C. It is easy to prove that the vectors $b_1 = (0,0,0,1)^T$, $b_2 = (1,-i,0,0)^T$, $b_3 = (0,i+1,1,0)^T$ lie in the null space of C and they are linearly independent and so they build a basis for that null space. The congruent transformation with use of $N = [b_1, b_2, b_3]$ would deliver the required equations, however, in order to show the application of the results of Theorem 4, we apply step by step the generalized Gram–Schmidt orthogonalization procedure (Algorithm 1) to this set of vectors:

1. Using $\|b_1\|_M = \sqrt{J_{sp}}$ we obtain for the first basis vector $u_1 = (0,0,0,1)^T$, $e_1 = (0,0,0,1/\sqrt{J_{sp}})^T$.
2. The second vector is M-orthogonal to these vectors therefore $P_{M,u_1} b_2 = 0$ and because $\|b_2\|_M = \sqrt{J_{fs} + i^2 J_{wg}}$ we have $u_2 = (1,-i,0,0)^T$, $e_2 = 1/\sqrt{J_{fs} + i^2 J_{wg}}(1,-i,0,0)^T$.
3. Also the third vector b_3 is M-orthogonal to u_1 ($P_{M,u_1} b_3 = 0$) but its projection onto u_2 is nonzero

$$P_{M,u_2} b_3 = -\frac{i(i+1)J_{wg}}{J_{fs} + i^2 J_{wg}} \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \tag{59}$$

and so for u_3 and its M-norm we obtain

$$u_3 = \frac{1}{J_{fs} + i^2 J_{wg}} \begin{pmatrix} i(i+1)J_{wg} \\ (i+1)J_{fs} \\ J_{fs} + i^2 J_{wg} \\ 0 \end{pmatrix}, \quad \|u_3\|_M = \sqrt{J_{cs} + (i+1)^2 \frac{J_{fs}J_{wg}}{J_{fs} + i^2 J_{wg}}}. \tag{60}$$

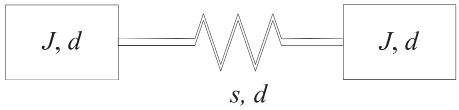

Figure 2: Bodies connected with a spring

Consequently the last normalized basis vector reads

$$e_3 = \frac{1}{\sqrt{J_{cs}\left(J_{fs} + i^2 J_{wg}\right)^2 + (i+1)^2 J_{fs} J_{wg}\left(J_{fs} + i^2 J_{wg}\right)}} \begin{pmatrix} i(i+1)J_{wg} \\ (i+1)J_{fs} \\ J_{fs} + i^2 J_{wg} \\ 0 \end{pmatrix}. \tag{61}$$

Now we could use the M-orthonormal basis (e_1, e_2, e_3) for the congruent transformation and so we would obtain for the inertia matrix the identity matrix of dimension 3: $M_r = I$. But, for sake of simplicity we will apply the elements of the M-orthogonal basis: $N = [u_2, u_3, u_1]$. So we will have the inertia matrix in diagonal form.

$$M_r = N^T M N = \operatorname{diag}\left(\|u_2\|_M^2, \|u_3\|_M^2, \|u_1\|_M^2\right) = \operatorname{diag}\left(J_{fs} + i^2 J_{wg}, J_{cs} + (i+1)^2 \frac{J_{fs} J_{wg}}{J_{fs} + i^2 J_{wg}}, J_{sp}\right) \tag{62a}$$

$$D_r = N^T D N = \begin{bmatrix} d_{fs} + i^2 d_{wg} & i(i+1)\frac{J_{wg} d_{fs} - J_{fs} d_{wg}}{J_{fs} + i^2 J_{wg}} & 0 \\ i(i+1)\frac{J_{wg} d_{fs} - J_{fs} d_{wg}}{J_{fs} + i^2 J_{wg}} & d_{cs} + i^2 (i+1)^2 \frac{J_{wg}^2 d_{fs} + J_{fs}^2 d_{wg}}{(J_{fs} + i^2 J_{wg})^2} + d & -d \\ 0 & -d & d_{sp} + d \end{bmatrix} \tag{62b}$$

$$S_r = N^T S N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & s & -s \\ 0 & -s & s \end{bmatrix} \tag{62c}$$

Substituting these matrices and the transformed input mapping matrix $N^T F$ into (25) we obtain a minimal set of equations that describe the motion of the example system.

4.2 Change of Controllability in Dependence of System Parameters

Now we consider the mechanical system which is shown in Fig. 4.2. Here two bodies with the same moment of inertia (J) are connected with a spring that has a stiffness of s. The viscous damping parameters are assumed to be equal on all parts (d). With similar Lagrangian and dissipative function as in the previous example we obtain the following inertia, damping and stiffness matrices, respectively:

$$M = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \quad D = \begin{bmatrix} 2d & -d \\ -d & 2d \end{bmatrix}, \quad S = \begin{bmatrix} s & -s \\ -s & s \end{bmatrix}. \tag{63}$$

There are no constraints in the system therefore $C = 0$ and so at least two differential equations are required to describe the motion of this second example system. We give the system matrix of (34):

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{s}{J} & \frac{s}{J} & -2\frac{d}{J} & \frac{d}{J} \\ \frac{s}{J} & -\frac{s}{J} & \frac{d}{J} & -2\frac{d}{J} \end{bmatrix}. \tag{64}$$

Its eigenvalues are:

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{d}{J}, \quad \lambda_{3,4} = -\frac{3d \pm \sqrt{9d^2 - 8sJ}}{2J}. \tag{65}$$

We substitute these results into the expression in Theorem 7:

1. With the substitution of $\lambda_1 = 0$ we obtain

$$\lambda_1^2 M + \lambda_1 D + S = s \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{66}$$

and its null space[1] is

$$\ker\left(\lambda_1^2 M + \lambda_1 D + S\right) = \begin{cases} \operatorname{span}\left\{(1,1)^T\right\} & \text{for } s \neq 0, \\ \operatorname{span}\left\{(1,0)^T, (0,1)^T\right\} & \text{for } s = 0. \end{cases} \tag{67}$$

It implies that with $s = 0$ we have to actuate both degrees of freedom for controllability. Recall that it also follows from Theorem 8. When $s \neq 0$, for the controllability with 1 input, this only input must not actuate both bodies with the same torque in the opposite direction because in that case the only column of the input mapping matrix F would be orthogonal to the corresponding eigenvector of A.

2. The eigenvalue $\lambda_2 = -d/J$ gives

$$\lambda_2^2 M + \lambda_2 D + S = \left(s - \frac{d^2}{J}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{68}$$

with the null space

$$\ker\left(\lambda_2^2 M + \lambda_2 D + S\right) = \begin{cases} \operatorname{span}\left\{(1,1)^T\right\} & \text{for } d^2 \neq sJ, \\ \operatorname{span}\left\{(1,0)^T, (0,1)^T\right\} & \text{for } d^2 = sJ. \end{cases} \tag{69}$$

[1] It is a symmetric matrix and so the left and right null spaces are identical. Otherwise we should take care to analyze the left null space.

We obtained the same null spaces as before with $\lambda_1 = 0$ but the necessary parameter condition for the controllability with one input is $d \neq \sqrt{sJ}$. (We assume a passive system therefore $d \geq 0$.)

3. With the eigenvalue $\lambda_3 = -(3d + \sqrt{9d^2 - 8sJ})/(2J)$ we have

$$\lambda_3^2 M + \lambda_3 D + S = \left(\frac{3d^2 + d\sqrt{9d^2 - 8sJ}}{2J} - s \right) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{70}$$

   and assuming $d \geq 0$ its null space reads

$$\ker\left(\lambda_2^2 M + \lambda_2 D + S\right) = \operatorname{span}\left\{(1, -1)^T\right\}. \tag{71}$$

   Accordingly the next necessary condition for the controllability with 1 input is, that this input does not actuate both bodies with the same torque.

4. The eigenvalue $\lambda_3 = -(3d - \sqrt{9d^2 - 8sJ})/(2J)$ gives

$$\lambda_4^2 M + \lambda_4 D + S = \left(\frac{3d^2 - d\sqrt{9d^2 - 8sJ}}{2J} - s \right) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{72}$$

   and

$$\ker\left(\lambda_4^2 M + \lambda_4 D + S\right) = \begin{cases} \operatorname{span}\left\{(1, -1)^T\right\} & \text{for } d \neq \sqrt{sJ}, \\ \operatorname{span}\left\{(1, 0)^T, (0, 1)^T\right\} & \text{for } d = \sqrt{sJ}. \end{cases} \tag{73}$$

   This eigenvalue does not impose additional conditions for the controllability with 1 input. (The conditions are already present in the second and third cases.)

Let use summarize all four cases: The example system in this section is controllable with one actuator mounted to any of the two bodies if and only if both of the conditions $s \neq 0$ and $d \neq \sqrt{sJ}$ are satisfied. Otherwise both of the bodies have to be actuated with an separate motor.

One important aspect of this result is that with change of damping parameters it can happen that we lose the controllability of an underactuated system. The friction phenomena in a mechanical system are not constant they are influenced by several factors. They usually are not linear and the viscous damping is only a first order approximation. Therefore the confidence interval of damping parameters and its effect on the controllability should be analyzed in an underactuated system.

5 Conclusion

In this paper the modeling, the controllability and the observability of linear mechanical systems with constant linear kinematic constraints were addressed. Two approaches were delivered in the modeling part. In the first approach, we determined the constraint forces as function of position, velocity and external force variables. With this result it is possible to determine the accelerating force for every degree of freedom. The second approach gives a transformation to obtain the simplest (with less number of generalized coordinates) equations of motion for the addressed system class. With the results in the second part it is possible to give a lower bound for the required number of actuators (sensors) of a fully state controllable (observable) system.

In future research, the modeling of linear mechanical systems with non-constant (for example depending on q) linear constraints is a possible direction. The maximal degree of underactuation can also be further analyzed because it can be sufficient for certain systems that only a subspace of the state space is controllable, the uncontrollable modes being stable.

Acknowledgements

The authors would like to thank the Bayerische Forschungsstiftung and the Hungarian Science Research Fund (grant OTKA K71762) for supporting this research.

References

[1] Panos J. Antsaklis and Anthony N. Michel. *Linear Systems.* Birkhäuser, Boston, 2nd corrected printing edition, 2006.

[2] Felix R. Gantmacher. *Matrizentheorie.* Springer-Verlag, Berlin, 1986.

[3] Peter Lancaster. *Theory of Matrices.* Academic Press, Inc., New York and London, 1969.

[4] László Lemmer. The decoupling of a harmonic-drive-spring system for position and torque control on two different axes. In *Proceedings, 15th Mediterranean Conference on Control and Automation*, Athens, Greece, June 2007.

[5] László Lemmer and Bálint Kiss. Modeling, identification, and control of harmonic drives for mobile vehicles. In *Proceedings, IEEE International Conference on Mechatronics*, Budapest, Hungary, July 2006.

[6] Carl Meyer. *Matrix Analysis and Applied Linear Algebra.* Society for Industrial and Applied Mathematics, 2000.

[7] Calyampudi Radhakrishna Rao and Sujit Kumar Mitra. *Generalized Inverse of Matrices and Its Applications.* John Wiley & Sons, Inc., 1971.

[8] Ahmed A. Shabana. *Dynamics of Multibody Systems.* Cambridge University Press, The Pitt Building, Cambridge CB2 2RU, United Kingdom, 2nd edition, 1998.