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Asymptotics of random unitaries

Outline of PhD thesis

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1 Introduction to random matrices

Random matrices are matrix valued random variables or in other words matrices whose entries are random variables. There are different kind of random matrices depending on the size, the distribution of the elements, and the correlation between the elements.

Wishart was the first who studied random matrices in 1928 ([16]), and he was motivated by multivariate statistics. He considered n pieces of m dimensional independent identically distributed random vectors. The covariance matrix of these random variables is the expectation of an $m \times m$ positive random matrix, what we call Wishart matrix if the components of the random vectors are normally distributed random variables.

Another point of view was given by physics. Wigner obtained some properties of the eigenvalues of complex, selfadjoint or real, symmetric random matrices in the papers [14, 15]. He used large symmetric random matrices in order to have a model of the energy levels of nuclei.

The main question was the behaviour of the eigenvalues of the random matrices. If we have the joint eigenvalue density, then we have all the information about the eigenvalues, but for this we need to know the joint density of the entries, and the invariance of the distribution of the random matrix under unitary conjugation. Therefore, though Wigner in [15] gave the joint eigenvalue density of the selfadjoint random matrices if the entries are Gaussian, but in the general case he studied the empirical distribution of the eigenvalues. This means that he defined the random function for an $n \times n$ random matrix A_n

$$F_n(x) := \frac{\#\{i : \lambda_i(A_n) < x\}}{n}.$$

Wigner in [14] proved the convergence of the expectation of F_n as $n \to \infty$, and later it turned out that the almost sure convergence also holds ([4, 10, 12]) even in the case of covariance type matrices. Moreover in Gaussian case the convergence rate of the empirical eigenvalue distribution is exponential with some lower semicontinuous rate function.

The question of non-selfadjoint matrices is also interesting. For example if all the entries are independent, identically distributed random variables, then we get a random matrix whose eigenvalues are not real. This random matrix defines a whole family of random matrices, if we take any linear combination of the matrix and its adjoint. In the Gaussian case the linear combination is also Gaussian, so it is possible to obtain the joint eigenvalue density, and the rate function for the exponential rate of convergence can proven ([13]), but the same universal theorem holds as in the case of selfadjoint random matrices, i.e. the empirical eigenvalue distribution measure of the matrix we only need the finiteness of some moments of the entries (see [7]).

The other very important type of random matrices is the set of unitary random matrices. The construction of a random unitary matrix is different from the above random matrices, since the entries are correlated. The set of $n \times n$ unitary matrices is not a subspace of the $n \times n$ matrices, as in the previous examples, but it is a group with respect to the matrix multiplication. Therefore the matrix density is considered with respect to the translation invariant measure, the so-called Haar measure of this group, not with respect to the Lebesgue measure. The matrix which is distributed according to this measure, i.e. has uniform distribution on the set of $n \times n$ unitary matrices, is called Haar unitary random matrix. Here the eigenvalues are not real, but they are on the unit circle. By the definition of the Haar unitary, since it is invariant under multiplication by a unitary matrix, clearly it is invariant under unitary conjugation. Therefore it is possible to obtain the joint eigenvalue density, and the convergence of the empirical eigenvalue distribution, and since the joint density of the eigenvalues is known, we can prove the exponential convergence with some rate function.

2 Overview of the dissertation

In the dissertation I will study most of the above topics in the following order.

In Section 1 I give an overview of different kind of random matrices. In the case of independent normally distributed entries, it is easy to determine the joint distribution of the entries. This joint distribution can be described by the eigenvalues, so if we find the Jacobian of the transformation which transforms the entries into the eigenvalues and some independent parameters, we get the joint density of the eigenvalues. I put down a more detailed version of this calculations, which was first given by Wigner [15] and Mehta [11] in the case of selfadjoint and non-selfadjoint random matrices. Since these matrices are invariant under unitary conjugation, the joint density of the eigenvalues contains all the information about the random matrices. The other important question concerning the random matrices is the limit distribution of the sequence of the empirical eigenvalue distribution as the matrix size goes to infinity. First I deal with random matrices with independent normally distributed entries, and note that some of the methods work for not normally distributed entries. I recall the result from [8] about the exponential rate of convergence of the empirical eigenvalue distribution.

In Section 2 I give an introduction into the large deviation theory. This theory is related to the sequence of random variables with non-random limits, for example in the case of law of large numbers. After recalling the first large deviation theorem of Cramèr the large deviation principle for random matrices is defined. The large deviation theorem for the different kind of Gaussian random matrices mentioned in the Section 1 are also here, as the theorem of Ben Arous and Guionnet [5], and the theorems of Hiai and Petz. Since the rate function in the case of random matrices is some weighted energy, and the limit distribution is the so-called equilibrium measure of this functional, I recall some basic notions of potential theory, and some theorems in order to obtain the equilibrium measures of the logarithmic energy with different rate functions.

In Section 3 I give the construction of the so called Haar unitary random matrix, which is a unitary matrix valued random variable with the distribution according to the Haar measure on the set of $n \times n$ unitary matrices. There is a collection of the main properties of this random matrix, as the distribution of the entries, the correlation between any two entries, and the joint eigenvalue density function. There is also an elementary proof of the theorem of Diaconis and Shashahani, which claims that the trace of different powers of the Haar unitary random matrices are asymptotically independent and normally distributed as the matrix size goes to infinity. From this one can deduce that the empirical eigenvalue distribution tends to the uniform distribution on the unit circle. I also prove this for the Haar distributed orthogonal random matrices with the same method. Finally I recall the theorem of Hiai and Petz [9], which proves the large deviation theorem for Haar unitary random matrices.

In Section 4 the starting point is a new kind of random matrix, the $n \times n$ truncation of an $m \times m$ random matrix. I give a more detailed proof of the theorem of Życzkowski and Sommers which gives the joint eigenvalue density of these random matrices, and then I calculate the normalization constant [1]. The joint eigenvalue density then helps to prove the large deviation theorem for the empirical eigenvalue distribution of the truncation, as the matrix size goes to infinity, and m/n converges to a constant λ, which is the main result of the thesis. After minimizing the rate function of this large deviation we get the limit distribution of the empirical eigenvalue distribution.

Finally in Section 5 I point to the connection of the free probability and the random matrix theory. I define the noncommutative probability space, the noncommutative random variables, and random matrix models of different noncommutative random variables, using the random matrices mentioned in the previous sections. I put down the definition of the Brown measure of a noncommutative random variable, and study he relationship between the Brown measures of the random variables and the empirical eigenvalue distribution of their random matrix model.

3 Main results

Section 3 deals with Haar unitary random matrices. I proved that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of an $n \times n$ Haar distributed unitary random matrix U_n, then $\lambda_1^m, \dots, \lambda_n^m$ are independent if $m \geq n$ ([1]).

Next method of moments is applied in order to obtain the limit distribution of the powers of the Haar unitary random matrix. I give a proof for the theorem of Diaconis and Shahshahani from [6] with elementary method. Theorem 3.4 states that

$$\operatorname{Tr} U_n \stackrel{n\to\infty}{\longrightarrow} \xi$$

in distribution, where ξ is a standard complex normal variable. Theorem 3.5 gives similar result to the higher powers of U_n, i.e.

$$\operatorname{Tr} U_n^l \stackrel{n\to\infty}{\longrightarrow} \sqrt{l}\xi,$$

and finally in Theorem 3.6 the asymptotic independence of the traces of different powers is shown, so we arrive to the theorem of Diaconis and Shahshahani. The proof is based on the fact that

$$\lim_{n\to\infty} \mathbb{E}\left(\prod_{i=1}^{l} \left(\operatorname{Tr} U_n^i\right)^{a_i} \left(\overline{\operatorname{Tr} U_n^i}\right)^{b_i}\right) = \prod_{i=1}^{l} \delta_{a_i b_i} a_i! i^{a_i},$$

i.e. the joint moments of the traces converge to the joint moments of the l independent complex normal variable.

Section 3.5 contains similar results for $n \times n$ Haar distributed orthogonal random matrices. The construction of O_n implies that the density of the entries

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} x^{-\frac{1}{2}}(1-x)^{\frac{n-3}{2}},$$

on the interval $[0,1]$, and Theorem 3.8 concludes that the limit distribution of $\sqrt{n}O_{ij}$ is standard normal, as the matrix size n goes to infinity. Similarly to the case of unitary matrices method of moments shows that the distribution of $\operatorname{Tr} O_n$ is asymptotically normal.

Section 4 studies the $m \times m$ truncation $U_{[n,m]}$ of an $n \times n$ Haar unitary random matrix. The eigenvalues lie in the unit disc $\mathcal{D}$. Subsection 4.1 gives the detailed calculation of the joint eigenvalue density of $U_{[n,m]}$ which is given by

$$C_{[n,m]} \prod_{i<j} |\zeta_i - \zeta_j|^2 \prod_{i=1}^{m} (1 - |\zeta_i|^2)^{n-m-1}$$

and was determined by Życzkowski and Sommers in [17]. In [1] we gave the normalization constant

$$C_{[n,m]}^{-1} = \pi^m m! \prod_{k=0}^{m-1} \binom{n-m+k-1}{k}^{-1} \frac{1}{n-m+k}.$$

by using complex contour integral.

Theorem 4.3 is the main result of the dissertation. The theorem claims that the following large deviation theorem holds for the empirical eigenvalue distribution of $U_{[n,m]}$ ([2, 3]). Let $1 < \lambda < \infty$. If $m/n \to \lambda$ as $n \to \infty$, then the sequence of the distributions of the empirical eigenvalue distribution

$$P_n = \frac{1}{n} \sum_{i=1}^{n} \delta(\lambda_i),$$

(where $\lambda_1, \ldots \lambda_n$ are the eigenvalues of $U_{[n,m]}$) satisfies the large deviation principle in the scale $1/n^2$ with rate function

$$I(\mu) := -\iint_{\mathcal{D}^2} \log|z-w|\, d\mu(z)\, d\mu(w)$$
$$-(\lambda-1) \int_{\mathcal{D}} \log(1-|z|^2)\, d\mu(z) + B,$$

for $\mu \in \mathcal{M}(\mathcal{D})$, where

$$B := -\frac{\lambda^2 \log \lambda}{2} + \frac{\lambda^2 \log(\lambda-1)}{2} - \frac{\log(\lambda-1)}{2} + \frac{\lambda-1}{2},$$

which means

$$\liminf_{n\to\infty} \frac{1}{n^2} \log P_n(G) \geq -\inf_{\mu\in G} I(\mu),$$

for all G open subset of the set $\mathcal{M}(\mathcal{D})$ of the probability measures supported on the unit disc $\mathcal{D}$, and

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P_n(F) \leq -\inf_{\mu\in F} I(\mu),$$

for all $F \subset \mathcal{M}(\mathcal{D})$ closed set.

Lemma 4.3 gives the equilibrium measure associated to a radially symmetric weight function on the unit disc $\mathcal{D}$ as follows. Suppose that we have a radially symmetric function $Q : \mathcal{D} \to (-\infty, \infty]$, i. e., $Q(z) = Q(|z|)$ such that Q is differentiable on $(0,1)$ with absolute continuous derivative bounded below, moreover $rQ'(r)$ increasing on $(0,1)$ and

$$\lim_{r\to 1} rQ'(r) = \infty.$$

Let $r_0 \geq 0$ be the smallest number for which $Q'(r) > 0$ for all $r > r_0$, and let R_0 be the smallest solution of $RQ'(R) = 1$. Clearly $0 \leq r_0 < R_0 < 1$. The functional

$$I_Q(\mu) := \iint_{\mathcal{D}^2} \log \frac{1}{|z-w|}\, d\mu(z)\, d\mu(w) + 2\int_{\mathcal{D}} Q(z)\, d\mu(z)$$

attains its minimum at a measure μ_Q supported on the annulus

$$S_Q = \{z : r_0 \leq |z| \leq R_0\},$$

and the density of μ_Q is given by

$$d\mu_Q(z) = \frac{1}{2\pi}(rQ'(r))'\, dr\, d\varphi, \qquad z = re^{\mathrm{i}\varphi}.$$

By Lemma 4.3 there exists a unique $\mu_0 \in \mathcal{M}(\mathcal{D})$ given by the density

$$d\mu_0(z) = \frac{(\lambda - 1)r}{\pi\,(1 - r^2)^2}\, dr\, d\varphi, \quad z = re^{\mathrm{i}\varphi}$$

on $\{z : |z| \le 1/\sqrt{\lambda}\}$ such that $I(\mu_0) = 0$, and this is the limit of the empirical eigenvalue distribution.

The statement of Theorem 4.4 is similar to Theorem 4.3. Here the large deviation theorem is proven for the empirical eigenvalue distribution of the random matrix sequence $Q_m U_n Q_m$, where Q_m is an $n \times n$ a non-random projection with rank m, and $m/n \to \lambda$ as $n \to \infty$, where $0 < \lambda < 1$. The scale is again $1/n^2$ and the rate function is

$$\tilde{I}(\widetilde{\mu}) := \begin{cases} I(\mu), & \text{if} \quad \widetilde{\mu} = (1 - \lambda^{-1})\delta(0) + \lambda^{-1}\mu, \\ \\ +\infty, & \text{otherwise} \end{cases}$$

Furthermore, the measure

$$\widetilde{\mu}_0 = (1 - \lambda^{-1})\delta(0) + \lambda^{-1}\mu_0$$

is the unique minimizer of $\tilde{I}$, and $\tilde{I}(\widetilde{\mu}_0) = 0$.

Section 5 shows that (Q_m, U_n, U_n^*) is a random matrix model of the noncommutative random variables (q, u, u^*) where q is a projection with

$$\varphi(q) = \frac{1}{\lambda},$$

and u is a Haar unitary element, and moreover u and q are in free relation. Moreover the Brown measure of quq is the same as the limit of the empirical eigenvalue distribution of $Q_m U_n Q_m$.

Own papers

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