THESES OF PH.D. DISSERTATION
GRÖBNER BASES IN COMBINATORICS

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1 Introduction

Let $\mathbb{F}$ be a field. In this work we study polynomial functions on finite point sets $\mathcal{V} \subset \mathbb{F}^n$. The polynomial functions mapping $\mathcal{V}$ to $\mathbb{F}$ carry a wealth of information about the combinatorial and geometric properties of $\mathcal{V}$. To obtain information about these functions, it is useful to consider the ideal $I(\mathcal{V})$:

$$I(\mathcal{V}) := \{f \in \mathbb{F}[x_1, \ldots, x_n] : \ f(v) = 0, \text{ whenever } v \in \mathcal{V}\}.$$

The ideals of the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ admit some useful special generating sets, called Gröbner bases. Our principal aim was to describe Gröbner bases and related structures (primarily standard monomials and Hilbert functions) for some combinatorially significant sets $\mathcal{V}$. We give applications to combinatorics to demonstrate the effectiveness of the approach.

1.1 Basic facts related to Gröbner bases

Here we collected some definitions and results about Gröbner bases and standard monomials.

Throughout this abstract $\mathbb{F}$ denotes a fixed (otherwise arbitrary) field. As usual, $\mathbb{F}[x_1, \ldots, x_n]$ denotes the ring of polynomials in variables $x_1, \ldots, x_n$ over $\mathbb{F}$. Occasionally, we use the shorter notation $S = \mathbb{F}[x_1, \ldots, x_n]$.

Let $\mathrm{Mon}(n, d)$ denote the set of monomials $x^u \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d. We write $\mathrm{Mon}(n, \leq d) := \cup_{i=0}^{d} \mathrm{Mon}(n, i)$.

For a subset $F \subseteq [n]$ we write $x_F = \prod_{j \in F} x_j$, and $x^F = \prod_{j \in F}(x_j - 1)$. In particular, $x_\emptyset = x^\emptyset = 1$.

A total order $\prec$ on the monomials composed from variables $x_1, x_2, \ldots, x_m$ is a *term order*, if 1 is the minimal element of $\prec$, and if $u \prec v$, then $u \cdot w \prec v \cdot w$ holds for any monomials u, v, w. We define two important term orders: the lexicographic order $\prec_{lex}$ and the deglex order $\prec_{deg}$. Let $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$ and $x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}$ be two monomials. Then

$$x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \prec_{lex} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}$$

iff $i_k < j_k$ holds for the smallest index k such that $i_k \neq j_k$.

As for deglex, we have $u \prec_{deg} v$ iff either $\deg u < \deg v$, or $\deg u = \deg v$ and $u \prec_{lex} v$.

The *leading monomial* $\mathrm{lm}(f)$ of a nonzero polynomial $f \in S$ is the largest (with respect to $\prec$) monomial which appears with nonzero coefficient in f when expressed as an $\mathbb{F}$-linear combination of monomials.

Let I be a nonzero ideal of S. The *initial ideal* $\mathrm{in}(I)$ of an ideal I is the ideal in S generated by the set of leading monomials $\{\mathrm{lm}(f) : f \in I\}$.

A finite subset $\mathcal{G} = \{g_1, \dots, g_t\} \subseteq I$ is a *Gröbner basis* of I iff for every $f \in I$ there exists a $g_i \in \mathcal{G}$ such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(f)$, i.e., the leading monomials $\mathrm{lm}(g_1), \dots, \mathrm{lm}(g_t)$ generate the initial ideal $\mathrm{in}(I)$. It is easy to verify that a Gröbner basis $\mathcal{G}$ also generates the ideal I.

A Gröbner basis $\{g_1, \dots, g_m\}$ of I is *reduced* if the coefficient of $\mathrm{lm}(g_i)$ is 1, and no nonzero monomial in g_i is divisible by any $\mathrm{lm}(g_j)$, $j \neq i$. By a theorem of Buchberger ([1, Theorem 1.8.7]), for a fixed term order $\prec$, any nonzero ideal of S has a unique reduced Gröbner basis.

Let $\prec$ be a fixed term order on the monomials. A monomial $w \in S$ is called a *standard monomial for I* if $w \notin \mathrm{in}(I)$, i.e., it is not a leading monomial of any $f \in I$. Let $\mathcal{V} \subseteq \mathbb{F}^n$ be an arbitrary finite subset. Then let $\mathrm{Sm}(\prec, I)$ stand for the set of all standard monomials of $I := I(\mathcal{V})$ with respect to the term order $\prec$ over $\mathbb{F}$.

Finally we introduce the notion of affine Hilbert function of an ideal. Let I be an ideal of $S = \mathbb{F}[x_1, \dots, x_n]$. The *Hilbert function* of the algebra S/I is the sequence $h_{S/I}(0), h_{S/I}(1), \dots$. Here $h_{S/I}(m)$ is the dimension over $\mathbb{F}$ of the quotient $\mathbb{F}[x_1, \dots, x_n]_{\le m}/(I \cap \mathbb{F}[x_1, \dots, x_n]_{\le m})$ (see [5, Section 9.3]).

In the case when $I = I(\mathcal{V})$ for some finite set of points $\mathcal{V} \subseteq \mathbb{F}^n$, the number $h_I(m) := h_{S/I(\mathcal{V})}(m)$ is the dimension of the space of functions from $\mathcal{V}$ to $\mathbb{F}$ which can be represented as polynomials of degree at most m.

In the combinatorial literature $h_I(m)$ is usually given in terms of inclusion matrices. For families $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ the *inclusion matrix* $I(\mathcal{F}, \mathcal{G})$ is a (0,1) matrix of size $|\mathcal{F}| \times |\mathcal{G}|$ whose rows and columns are indexed by the elements of $\mathcal{F}$ and $\mathcal{G}$, respectively. The entry at position (F, G) is 1 if $G \subseteq F$ and 0 otherwise ($F \in \mathcal{F}, G \in \mathcal{G}$).

1.2 The background of this work

The Gröbner basis theory has an interesting history. An early version of the notion appeared already in Macaulay's work on polynomial rings. Gröbner bases were first introduced in the 1960s by H. Hironaka. He used a division algorithm in his landmark paper on the resolution of singularities. Later B. Buchberger discovered them again, and he treated them in detail in his

Ph.D. thesis. Buchberger gave the name "Gröbner bases" to honor his thesis adviser W. Gröbner. His thesis contains also the Buchberger's criterion and the famous Buchberger's algorithm, now bearing his name, in an implicit form. He dealt also with the programming of the resulting algorithm.

The linear algebra bound method is the second source of our results. This method yields upper bounds for the size of various combinatorial structures. The general description of this method is the following. Let $\mathcal{F}$ denote a finite set of points in the affine space $\mathbb{F}^n$. Suppose that we can define for each $v \in \mathcal{F}$ a corresponding polynomial $p_v(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ such that the polynomials $\{p_v : v \in \mathcal{F}\}$ will be linearly independent over the field $\mathbb{F}$. We denote by $\mathcal{T}$ the set of monomials, which occur in the polynomials p_v. Then the linear algebra bound method gives the upper bound $|\mathcal{F}| \leq |\mathcal{T}|$.

We combined this simple method with a reduction argument via Gröbner bases. Suppose that $\mathcal{F} \subseteq \mathcal{G}$ for some fixed finite set of points $\mathcal{G} \subseteq \mathbb{F}^n$. In some cases we may choose the set $\mathcal{T}$ to be a subset of $\mathrm{Sm}(I(\mathcal{G}), \prec)$, where $\mathcal{G}$ is a well–behaved set of points in the sense that we have enough information about the standard monomials of the ideal $I(\mathcal{G})$. Then we obtain a better upper bound for $|\mathcal{F}|$. One example appears in our solution of a conjecture of Babai and Frankl.

Concerning the background of our results, it is appropriate to recall Smolensky's work on Boolean function. Smolensky has observed in [16] that there is a fruitful connection between the complexity theory of Boolean functions and algebraic geometry. His approach was the following.

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ in n variables can be represented as a real multivariate polynomial $g(x_1, \ldots, x_n)$, whose zero set is the same as the set of zeroes of f, together with the polynomials $x_i^2 - x_i$, $i = 1, \ldots, n$. Consider the finite set of points defined by $g = 0$ and $x_i^2 - x_i = 0$, $i = 1, \ldots, n$. This set of points can be then investigated in terms of the Hilbert function of its ideal, and Smolensky showed in [16], that the values of the Hilbert function are related to certain lower bounds in complexity theory.

Now let $0 \leq d_1 < \ldots < d_t \leq n$ be a strictly increasing sequence of nonnegative integers. Let

$$\mathcal{F} := \binom{[n]}{d_1} \cup \ldots \cup \binom{[n]}{d_t}$$

denote a symmetric subset of the Boolean cube $\{0,1\}^n$, i.e., the union of the complete uniform families $\binom{[n]}{d_i}$. Clearly $V(\mathcal{F})$, the set of characteristic vectors of $\mathcal{F}$, is a zero set of a corresponding symmetric Boolean function.

In [3] Egidi and Bernasconi determined completely the Hilbert function over $\mathbb{Q}$ of the ideals associated to the zero set of symmetric Boolean functions. They applied this characterisation to analyse the behaviour of the majority function on symmetric subsets of the Boolean cube.

2 The results of the thesis

2.1 The complete ℓ-wide and complete d-uniform set families

First we define the notion of ℓ-wide families.

Let $v_F \in \{0,1\}^n$ denote the characteristic vector of a set $F \subseteq [n]$. For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, let $V(\mathcal{F}) := \{v_F : F \in \mathcal{F}\} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$.

Let $\mathcal{F}^{k,\ell}$ denote the *complete ℓ-wide* family

$$\mathcal{F}^{k,\ell} = \{F \subseteq [n] : k - \ell < |F| \leq k\}.$$

A set family $\mathcal{F} \subseteq 2^{[n]}$ is *ℓ-wide* if $\mathcal{F} \subseteq \mathcal{F}^{k,\ell}$ for a suitable k.

In Chapter 3 and 4 of the thesis we describe (reduced) Gröbner bases of the ideal of polynomials, over an arbitrary field $\mathbb{F}$, which vanish on the characteristic vectors of elements of the set system $\mathcal{F}^{k,\ell}$.

As an application, we obtain results on certain inclusion matrices related to $\mathcal{F}^{k,\ell}$. We prove also a special case of a conjecture of Frankl related to the determination of the maximum number of subsets of $[n]$ with no shattered set of size t and with no chain of size $\ell + 1$.

Now we introduce polynomials, which appear in the reduced Gröbner bases of the ideal of the characteristic vectors of complete ℓ-wide families.

Let ℓ, t be positive integers. We define $\mathcal{H}(t,\ell)$ as the set of those subsets $H = \{s_1 < \ldots < s_t\}$ of $[n]$ for which t is the smallest index j with $s_j < 2j - \ell + 1$. In the special case $\ell = 1$ we let $\mathcal{H}(t) := \mathcal{H}(t,1)$.

We remark that $\mathcal{H}(t,\ell) = \emptyset$ for $t < \ell$. Also if $t > (n+\ell)/2$, then $\mathcal{H}(t,\ell) = \emptyset$ again.

The elements of $\mathcal{H}(t,\ell)$ are t-subsets of $[n]$, and we have $H \in \mathcal{H}(t,\ell)$ iff $s_1 \geq 3 - \ell, s_2 \geq 5 - \ell, \ldots, s_{t-1} \geq 2t - \ell - 1$ and $s_t < 2t - \ell + 1$. It follows that $s_t = 2t - \ell$ (in the case $t = 1$ we have $\ell = 1$ as well). For $t > 1$ we have also $s_{t-1} = 2t - \ell - 1$.

For a subset $J \subseteq [n]$ and an integer $0 \le i \le |J|$ we denote by $\sigma_{J,i}$ the i-th elementary symmetric polynomial of the variables x_j, $j \in J$:

$$\sigma_{J,i} := \sum_{T \subseteq J, |T|=i} x_T \ \in \mathbb{Z}[x_1, \dots, x_n].$$

Now let $0 < \ell \le t < (n+\ell)/2$, $0 \le k \le n$ and $H \in \mathcal{H}(t,\ell)$. Then put $H' = H \cup \{2t-\ell+1, 2t-\ell+2, \dots, n\}$.

We write

$$f_{H,k} = f_{H,k}(x_1, \dots, x_n) := \sum_{j=0}^{t} (-1)^{t-j} \binom{k-j}{t-j} \sigma_{H',j}.$$

Note that $f_{H,k}$ depends on t and ℓ through H. Moreover, H uniquely determines t and ℓ.

For $0 \le \ell - 1 \le k \le n$ we write

$$D(k,\ell) = \{\{g_1 < \ldots < g_t\} \subseteq [n] : \ t \le k \text{ and } g_j \ge 2j - \ell + 1 \text{ if } 1 \le j \le t\}.$$

Following R. P. Anstee and A. Sali (see [2]) we say that a set

$$S = \{s_1 < s_2 < \ldots < s_d\} \subseteq [n]$$

is *order shattered* by the family $\mathcal{F} \subseteq 2^{[n]}$ if the following holds: in the case $S = \emptyset$ the family $\mathcal{F}$ has to contain a set; when $|S| > 0$, then there are 2^d sets in $\mathcal{F}$ that can be divided into two families $\mathcal{F}_0$ and $\mathcal{F}_1$ such that $s_d \notin F$ for all $F \in \mathcal{F}_0$, $s_d \in F$ for all $F \in \mathcal{F}_1$, and both $\mathcal{F}_0, \mathcal{F}_1$ order shatter the set $S \setminus \{s_d\}$, furthermore $T \cap F_0 = T \cap F_1$ holds for $T = \{s_d+1, s_d+2, \dots, n\}$ and all $F_0 \in \mathcal{F}_0, F_1 \in \mathcal{F}_1$.

Let

$$\operatorname{osh}(\mathcal{F}) := \{S \subseteq [n] : \mathcal{F} \text{ order shatters } S\}.$$

In [2] R. P. Anstee, Rónyai L. and Sali A. described with the aid of ballot sequences the order-shattering of the complete uniform families $\binom{[n]}{d}$, where $d \le n/2$:

$$\operatorname{osh}\binom{[n]}{d} = \{\{s_1 < \ldots < s_j\} \subset [n] : \ j \le d \text{ and } s_i \ge 2i \text{ for } 1 \le i \le j\}. \quad (1)$$

It is immediate that for a nonempty family $\mathcal{F} \subseteq 2^{[n]}$ we have

$$\operatorname{osh}(\operatorname{co}(\mathcal{F})) = \operatorname{osh}(\mathcal{F}), \quad (2)$$

where

$$\mathrm{co}(\mathcal{F}) = \{[n] \setminus F : \; F \in \mathcal{F}\}.$$

In particular,

$$\mathrm{osh}\binom{[n]}{d} = \mathrm{osh}\binom{[n]}{n-d}.$$

The following Theorem generalises the description (1) to ℓ-wide families.

Theorem 2.1 *(Thm. 4.6 in the thesis)*
(a) Let $0 \leq k < (n+\ell)/2$. *Then*

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) = D(k,\ell)$$

(b) If $k \geq (n+\ell)/2$, *then*

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) = D(n-k+\ell-1,\ell).$$

Consider a family $\mathcal{F}$ of subsets of $[n]$. We say that $\mathcal{F}$ *shatters* S if

$$\{E \cap S : \; E \in \mathcal{F}\} = 2^S. \tag{3}$$

Then define

$$\mathrm{sh}(\mathcal{F}) = \{S \subseteq [n] : \; \mathcal{F} \text{ shatters } S\}. \tag{4}$$

Recall that a *chain* of size p in $2^{[n]}$ is a sequence $A_1, \ldots, A_p$ of subsets of $[n]$ with $A_1 \subset \ldots \subset A_p$.

Let $g(n,t,d)$ denote the maximum number of subsets of $[n]$ with no shattered set of size t and no chain of size $d+1$. In [6] Frankl proposed the following conjecture.

Conjecture *Assume that* $2t \leq n+d$. *Then*

$$g(n,t,d) \leq \sum_{i=\max(0,t-d)}^{t-1} \binom{n}{i}. \tag{5}$$

We prove in the thesis the following special case of Frankl's conjecture.

Theorem 2.2 *(Thm. 4.1 in the thesis) Suppose that* $2t \leq n+\ell$ *and let* $\mathcal{F} \subseteq 2^{[n]}$ *be an* ℓ*-wide family with no shattered set of size* t. *Then*

$$|\mathcal{F}| \leq \sum_{i=\max(0,t-\ell)}^{t-1} \binom{n}{i}.$$

We denote by $I(k,\ell)$ the ideal $I(V(\mathcal{F}^{k,\ell}))$ of the characteristic vectors of the complete ℓ-wide family $\mathcal{F}^{k,\ell}$. If $\ell = 1$, then $I(d) := I(d,1)$ denotes the ideal of the characteristic vectors of the complete d-uniform family $\binom{[n]}{d}$.

In the next two Theorems we describe the reduced Gröbner bases and the minimal generating system of the ideals $I(k,\ell)$.

For $0 < \ell \leq k+1$ let $\mathcal{B}(k,\ell)$ denote the collection of subsets $U \subseteq [n]$, where $U = \{u_1 < \ldots < u_{k+1}\}$ and $u_j \geq 2j - \ell + 1$ holds for $j = 1,\ldots,k$. In the special case $\ell = 1$ let $\mathcal{B}(d) := \mathcal{B}(d,1)$.

Theorem 2.3 *(Corollary 3.4 and Corollary 3.5 in the thesis) Let d,n be integers such that $n > 0$ and $0 \leq d \leq n/2$. Let $\mathbb{F}$ be a field, and $\prec$ be an arbitrary term order on the monomials of $S = \mathbb{F}[x_1,\ldots,x_n]$ for which $x_n \prec x_{n-1} \prec \ldots \prec x_1$. Then the following set of monomials is the minimal generating system of the initial ideal $in(I(d)) = in(I(n-d))$:*

$$\cup_{t=1}^{d}\{x_H :\ H \in \mathcal{H}(t)\} \cup \{x_U :\ U \in \mathcal{B}(d)\} \cup \{x_i^2 :\ i = 2,\ldots,n\}.$$

The following set of polynomials is the reduced Gröbner basis with respect to $\prec$ of the ideal $I(d)$:

$$\{x_2^2 - x_2,\ldots,x_n^2 - x_n\} \cup \{x_J :\ J \in \mathcal{B}(d)\} \cup$$

$$\{f_{H,d} :\ H \in \mathcal{H}(t) \textit{ for some } 0 < t \leq d\}.$$

Similarly, the following set is the reduced Gröbner basis of $I(n-d)$:

$$\{x_2^2 - x_2,\ldots,x_n^2 - x_n\} \cup \{x^J :\ J \in \mathcal{B}(d)\} \cup$$

$$\{f_{H,n-d} :\ H \in \mathcal{H}(t) \textit{ for some } 0 < t \leq d\}.$$

Theorem 2.4 *(Corollary 4.3 and Corollary 4.4 in the thesis) Let $n > 0$, k and ℓ be integers such that $0 < \ell - 1 \leq k \leq n$. Let $\mathbb{F}$ be a field, and $\prec$ be an arbitrary term order on the monomials of $S = \mathbb{F}[x_1,\ldots,x_n]$ for which $x_n \prec x_{n-1} \prec \ldots \prec x_1$. If $k < (n+\ell)/2$, then*

$$\{x_i^2 :\ i = 1,\ldots,n\} \cup \{x_U :\ U \in \mathcal{B}(k,\ell)\} \cup$$

$$\{x_H :\ H \in \mathcal{H}(t,\ell) \textit{ for some } t,\ \ell \leq t \leq k\}$$

generates minimally $\mathrm{in}(I(k,\ell))$. The following set is the reduced Gröbner basis with respect to $\prec$ of the ideal $I(k,\ell)$:

$$\{x_1^2 - x_1,\ldots,x_n^2 - x_n\} \cup \{x_J :\ J \in \mathcal{B}(k,\ell)\} \cup$$

$$\{f_{H,k}:\ H \in \mathcal{H}(t,\ell) \text{ for some } \ell \le t \le k\}.$$

Next assume that $k \ge (n+\ell)/2$. *Then*

$$\{x_i^2:\ i=1,\dots,n\} \cup \{x_U:\ U \in \mathcal{B}(n-k+\ell-1,\ell)\} \cup$$

$$\{x_H:\ H \in \mathcal{H}(t,\ell) \text{ for some } \ t,\ \ell \le t \le n-k+\ell-1\}$$

minimally generates $\mathrm{in}(I(k,\ell))$. *In the case* $k \ge (n+\ell)/2$ *the following set is the reduced Gröbner basis with respect to* $\prec$ *of the ideal* $I(k,\ell)$*:*

$$\{x_1^2-x_1,\dots,x_n^2-x_n\} \cup \{x^J:\ J \in \mathcal{B}(n-k+\ell-1,\ell)\} \cup$$

$$\{f_{H,k}:\ H \in \mathcal{H}(t,\ell) \text{ for some } \ell \le t \le n-k+\ell-1\}.$$

As a simple application we obtain a new proof for the following inequality by Frankl:

Let p be an arbitrary prime and k an integer. Let $\mathcal{F}(k,p)$ denote the following set system:

$$\mathcal{F}(k,p) = \{K \subseteq [n]:\ |K| \equiv k \pmod{p}\}.$$

Corollary 2.5 *(Corollary 3.10 in the thesis) Let* p *be an arbitrary prime and* k *an integer. Assume that* $0 \le \ell < p$ *and* $2\ell \le n$. *Then*

$$\mathit{rank}_{\mathbb{F}_p}\ I(\mathcal{F}(k,p), \binom{[n]}{\le \ell}) \le \binom{n}{\ell}.$$

2.2 A conjecture of Babai and Frankl

Let p be a prime and $d \in \mathbb{Z}$. Let $q = p^\alpha$, $\alpha \ge 1$ be an arbitrary prime power. Then let

$$\mathcal{F}(d,q) := \{F \subseteq [n]:\ |F| \equiv d \pmod{q}\}.$$

The following Theorem was conjectured by L. Babai and P. Frankl. In Chapter 5 we prove this result with reduction via Gröbner bases of the ideal $I(V(\mathcal{F}(k,q)))$.

Theorem 2.6 *(Thm. 5.7 in the thesis) Let* k *be an integer and* $q = p^\alpha$, $\alpha \ge 1$, *a prime power. Suppose that* $2(q-1) \le n$. *Assume that* $\mathcal{F} = \{A_1,\dots,A_m\}$ *is a family of subsets of* $[n]$ *such that*

$$(a)\ |A_i| \equiv k \ (mod\ q) \text{ for } i = 1,\dots,m$$

(b) $|A_i \cap A_j| \not\equiv k \pmod q$ *for* $1 \le i, j \le m,\ i \ne j$.

Then

$$m \le \binom{n}{q-1}.$$

The following Theorem is a generalisation of Frankl's well–known rank estimate (Corollary 2.5) for prime powers:

Theorem 2.7 *(Thm. 5.9 in the thesis) Let* p *be a prime and* k *an integer. Let* $q = p^\alpha$, $\alpha \ge 1$. *If* $\ell \le q-1$ *and* $2\ell \le n$, *then*

$$rank_{\mathbb{F}_p}\ I(\mathcal{F}(k,q), \binom{[n]}{\le \ell}) \le \binom{n}{\ell}.$$

We prove this by studying the deglex standard monomials of the ideal $I(V(\mathcal{F}(k,q)))$.

2.3 Gröbner bases and standard monomials for permutations

In Chapter 6 we describe the reduced Gröbner bases and the standard monomials of permutations.

Let S_n denote the symmetric group (acting on $[n]$). Let $\mathbb{F}$ be a fixed, but arbitrary field. Let $\alpha_1, \dots, \alpha_n$ be n different elements of $\mathbb{F}$ and put

$$V_{(1^n)} := V_{(1^n)}(\alpha_1, \dots, \alpha_n) := \{(\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)}) :\ \pi \in S_n\}.$$

$V_{(1^n)}$ is the set of all permutations of the α_i, viewed as a subset of $\mathbb{F}^n$.

Let i be a nonnegative integer and write

$$h_i(x_1, \dots, x_n) = \sum_{\substack{a_1 + \cdots + a_n = i \\ a_1 \ge 0, \dots, a_n \ge 0}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

for the i-th complete symmetric polynomial.

For $0 \le i \le n$ we write σ_i for the i-th elementary symmetric polynomial:

$$\sigma_i(x_1, \dots, x_n) = \sum_{S \subset [n],\ |S| = i} x_S.$$

For $1 \leq k \leq n$ we introduce the polynomials $f_k \in S$ as follows:

$$f_k := \sum_{i=0}^{k} (-1)^i h_{k-i}(x_k, x_{k+1}, \ldots, x_n)\sigma_i(\alpha_1, \ldots, \alpha_n).$$

We remark, that $f_k \in \mathbb{F}[x_k, x_{k+1}, \ldots, x_n]$. Moreover, $\deg f_k = k$ and the leading monomial of f_k is x_k^k with respect to any term order $\prec$ for which $x_1 \succ x_2 \succ \ldots \succ x_n$.

Theorem 2.8 *(Thm. 6.2 in the thesis) Let $\mathbb{F}$ be an arbitrary field and let $\prec$ be an arbitrary term order on the monomials of $\mathbb{F}[x_1, \ldots, x_n]$ such that $x_n \prec \ldots \prec x_1$. Then the reduced Gröbner basis of $I(V_{(1^n)})$ is*

$$\{f_i : \ 1 \leq i \leq n\}.$$

Moreover the set of standard monomials is

$$\mathrm{Sm}(\prec, I(V_{(1^n)})) = \{x_1^{\alpha_1} \ldots x_n^{\alpha_n} : \ 0 \leq \alpha_i \leq i-1 \text{ for } 1 \leq i \leq n\}. \qquad (6)$$

The Fundamental Theorem of Symmetric Polynomials asserts that every symmetric polynomial $f \in \mathbb{F}[y_1, \ldots, y_n]$ admits a unique expression of the form

$$f = \sum_{p \geq 0} a_p \sigma_1^{p_1} \sigma_2^{p_2} \cdots \sigma_n^{p_n},$$

where $p = (p_1, p_2, \ldots, p_n)$, $a_p \in \mathbb{F}$, and the σ_i are the elementary symmetric polynomials in the y_i. Here we prove the following generalisation, which appeared first in Garsia's paper [8]:

Corollary 2.9 *Every polynomial $f \in \mathbb{F}[y_1, \ldots, y_n]$ has a unique expansion of the form*

$$f(y_1, \ldots, y_n) = \sum_{w \in \mathcal{N}} \sum_{p \geq 0} a_{w,p} w \sigma_1^{p_1} \sigma_2^{p_2} \cdots \sigma_n^{p_n},$$

where $a_{w,p} \in \mathbb{F}$ and $\mathcal{N} = \{y_1^{\alpha_1} \ldots y_n^{\alpha_n} : \ 0 \leq \alpha_i \leq i-1 \text{ for } 1 \leq i \leq n\}$.

2.4 The lex and the deglex standard monomials of partitions

In Chapter 7 we give a combinatorial description for the lexicographic standard monomials and the Hilbert function of the ideal $I(V_\lambda)$ (see the definition of V_λ below). As applications, a basis of the orthogonal complement $(S^\lambda)^\perp$ (with respect to the James scalar product) of the Specht module S^λ is exhibited.

A sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of natural numbers is a *partition* of n, if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$. If λ is a partition of n, then we denote this fact by $\lambda \vdash n$.

Suppose that $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n. The Ferrers diagram of λ is an array of n squares (boxes) having k left-justified rows, with row i containing λ_i squares for $1 \leq i \leq k$. A λ-*tableau* t is obtained by filling the squares of the Ferrers diagram of λ with the numbers $1, 2, \dots, n$ bijectively.

For example, if $n = 7$, $\lambda = (4, 2, 1)$ then

2	5	3	6
4	1		
7			

and

1	2	4	5
3	7		
6			

(7)

are two of the 7! λ-tableaux.

A tableau t is *standard* if the rows and the columns of t are increasing sequences.

Let $\alpha_0, \dots, \alpha_{k-1}$ be k different elements of $\mathbb{F}$, $\lambda \vdash n$ and V_λ be the set of all vectors $v = (v_1, \dots, v_n) \in \mathbb{F}^n$ such that

$$|\{j \in [n] : \ v_j = \alpha_i\}| = \lambda_{i+1}$$

for $0 \leq i \leq k-1$.

We work with a partial order $\leq$ on $\mathbb{N}^n$: if $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{N}^n$, then set $u \leq v$ iff $u_i \leq v_i$ for $1 \leq i \leq n$.

We define the *downward hull* of $W \subseteq \mathbb{N}^n$ by:

$$W^{\leq} := \{w \in \mathbb{N}^n : \ \text{there exists } v \in W \text{ such that } w \leq v\}.$$

A *ballot sequence*, or a *lattice permutation*, is a vector of nonnegative integers $m = (i_1, i_2, \dots, i_n)$ such that, for any prefix $m_k = (i_1, \dots, i_k)$ and any nonnegative integer l, the number of l's in m_k is at least as large as the number of $(l+1)$'s in that prefix.

It is well–known (see [15]) that lattice permutations are in one to one correspondence with standard tableaux. Given a standard tableau t with n elements, form the sequence $m = (i_1, i_2, \ldots, i_n)$, where $i_k = i-1$ if k appears in row i of t. This way we obtain a lattice permutation. It is easy to construct the inverse map.

A lattice permutation m has type λ iff the corresponding t is a standard λ-tableau. For $\lambda \vdash n$ we define

$$\mathrm{st}(\lambda) := \{u \in \mathbb{N}^n : \ u \text{ is a lattice permutation of type } \lambda\}.$$

We write $b_\lambda := \mathrm{st}(\lambda)^{\leq}$ and $B_\lambda = \{x^u : \ u \in b_\lambda\}$. Clearly b_λ is downward closed.

Consider the special cases $\lambda = (n-d, d)$ and $\lambda = (1^n)$. In the former case we have necessarily $d \leq n/2$, and it is not difficult to see that B_λ consists of the monomials $x_{i_1} x_{i_2} \ldots x_{i_j}$, where $j \leq d$, $i_1 < i_2 < \ldots < i_j$ and $i_l \geq 2l$ for $1 \leq l \leq j$. In [2] it is shown that $\mathrm{Sm}(\prec_{lex}, I(V_\lambda)) = B_\lambda$. In [11] this is generalised to an arbitrary term order $\prec$, for which $x_n \prec \ldots \prec x_1$.

Now let $\alpha_1, \ldots, \alpha_n$ be n different elements of $\mathbb{F}$, put $\lambda = (1^n)$ and consider the set $V_\lambda = \{(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}) : \ \pi \in S_n\}$.

Let $\prec$ be an arbitrary term order on the monomials of $\mathbb{F}[x_1, \ldots, x_n]$ such that $x_n \prec \ldots \prec x_1$. We proved in Theorem 2.2 of [10] that

$$\mathrm{Sm}(\prec, I(V_\lambda)) = \{x_1^{\beta_1} \ldots x_n^{\beta_n} : \ 0 \leq \beta_i \leq i-1 \text{ for } 1 \leq i \leq n\}. \qquad (8)$$

Clearly $g = (0, 1, \ldots, n-1)$ is the only lattice permutation of type $\lambda = (1^n)$, that is, $\mathrm{st}(\lambda) = \{g\}$. By (8) we have in this case $\mathrm{Sm}(\prec, I(V_\lambda)) = B_\lambda$.

The main contribution of Chapter 7 is the following result.

Theorem 2.10 *(Thm. 7.8 and Corollary 7.13 in the thesis) Let $\mathbb{F}$ be an arbitrary field and λ be an arbitrary partition of n. Then*

$$\mathrm{Sm}(\prec_{lex}, I(V_\lambda)) = B_\lambda.$$

Moreover

$$h_{I(V_\lambda)}(m) = |B_\lambda \cap \mathrm{Mon}(n, \leq m)|,$$

if $m \geq 0$.

In their study of the q-Kostka polynomials [9] A. M. Garsia and C. Procesi have obtained that $\mathrm{Sm}(\prec_{deg}, V_\lambda) = B_\lambda$ holds for any partition λ (Proposition

3.2 in [9]). They worked over $\mathbb{Q}$, but their argument is valid over an arbitrary field. The associated graded ring gr $S/I(V_\lambda)$ is also described there.

From Theorem 2.10 we obtain a new, perhaps simpler proof of the result of Garsia and Procesi on the deglex standard monomials of V_λ (Theorem 7.12 in the thesis). In particular, we avoid the use of Tanisaki's ideals (see (1.5) in [9]).

Theorem 2.10 can be viewed as an extension of the Garsia, Procesi result to the lexicographic case.

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