

Strong approximation of stochastic processes
using random walks

Outline of the PhD thesis

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This document is the outline of the PhD thesis with the same title. The theorems, lemmas etc. are marked with the same labels and numbers as in the thesis. The statements marked with arabic numbers are the work of the author.

1 Introduction

The main concept of this thesis is to draw the attention to the fact that both for theoretical and practical reasons, it is useful to search for strong (i.e. pathwise, almost sure) approximations of stochastic processes by simple random walks (RWs). The prototype of such efforts was the construction of Brownian motion (BM) as an almost sure limit of simple RW paths, given by Frank Knight in 1962 [8]. Later this construction was simplified and somewhat improved by Pál Révész [11] and then by Tamás Szabados [15]. This sort of results states that one can find a sequence of time and space scaled random walks (B_m) such that it converges to a Brownian motion B almost surely uniformly on every compact intervals as m goes to infinity:

$$\sup_{t \in [0, T]} |B_m(t) - B(t)| \rightarrow 0.$$

Besides the theoretical value of discrete approximations, in some applications the discrete model could be more natural than the continuous one. It provides a general tool for proving statements for continuous time stochastic processes. First, one can prove the discrete version of the statement and then take limit for obtaining the continuous version. Of course, we cannot predict which part of this procedure will be easier.

In this thesis we present both theoretical results, e.g. approximation of continuous martingales and also show examples in which the discrete approximation method can be carried out naturally. During our research some additional statements turned out to be true which are not closely related with our present subject. However, we present some of them because we think that the reader may find them interesting.

2 Approximation of continuous martingales

2.1 Random walks and the Wiener process

A main tool of this thesis is an elementary construction of the Wiener process which is based on a nested sequence of simple random walks that uniformly converges to the Wiener process on bounded intervals with probability 1.

This will be called *RW construction* in the sequel. One of our intentions in this chapter is to extend the underlying “*twist and shrink*” algorithm to continuous local martingales.

We summarize the major steps of the RW construction here. We start with a sequence of independent symmetric random walk $\{\{S_m(k)\}_k\}_m$.

From the independent random walks we want to create dependent ones so that after shrinking temporal and spatial step sizes, each consecutive RW becomes a refinement of the previous one. Since the spatial unit will be halved at each consecutive row, we define stopping times by $T_m(0) = 0$, and for $k \geq 0$,

$$T_m(k+1) = \min\{n : n > T_m(k), |S_m(n) - S_m(T_m(k))| = 2\} \quad (m \geq 1). \quad (1)$$

Using this sequence of stopping times, we define *twisted* RWs \tilde{S}_m recursively for $k = 1, 2, \dots$ using \tilde{S}_{m-1} , starting with $\tilde{S}_0(n) = S_0(n)$ ($n \geq 0$). With each fixed m we proceed for $k =$

0, 1, 2, ... successively, and for every n in the corresponding bridge, $T_m(k) < n \leq T_m(k+1)$. Any bridge is flipped if its sign differs from the desired:

$$\tilde{X}_m(n) = \begin{cases} X_m(n) & \text{if } S_m(T_m(k+1)) - S_m(T_m(k)) = 2\tilde{X}_{m-1}(k+1), \\ -X_m(n) & \text{otherwise,} \end{cases}$$

and then $\tilde{S}_m(n) = \tilde{S}_m(n-1) + \tilde{X}_m(n)$. Then $\tilde{S}_m(n)$ ($n \geq 0$) is still a simple symmetric random walk. The twisted RWs have the desired refinement property:

$$\frac{1}{2}\tilde{S}_m(T_m(k)) = \tilde{S}_{m-1}(k) \quad (m \geq 1, k \geq 0).$$

The last step of the RW construction is *shrinking*. The sample paths of $\tilde{S}_m(n)$ ($n \geq 0$) can be extended to continuous functions by linear interpolation, this way one gets $\tilde{S}_m(t)$ ($t \geq 0$) for real t . Then we define the *m*th approximating RW by

$$\tilde{B}_m(t) = 2^{-m}\tilde{S}_m(t2^{2m}).$$

For the difference of two consecutive approximations one can prove the following

Lemma C. *For any $K > 0$, $C > 1$, and for any $m \geq m_1(C)$, we have*

$$\mathbf{P} \left\{ \sup_{0 \leq k2^{-2m} \leq K} |\tilde{B}_{m+1}(k2^{-2m}) - \tilde{B}_m(k2^{-2m})| \geq K_*^{\frac{1}{4}}(\log_* K)^{\frac{3}{4}}m2^{-\frac{m}{2}} \right\} \leq 3(K2^{2m})^{1-C},$$

where $K_* = \max\{1, K\}$.

Based on this lemma, it is not difficult to show the following convergence result.

Theorem A. *The shrunken RWs $\tilde{B}_m(t)$ ($t \geq 0, m = 0, 1, 2, \dots$) almost surely uniformly converge to a Wiener process $W(t)$ ($t \geq 0$) on any compact interval $[0, K]$, $K > 0$. For any $K > 0$, $C \geq 3/2$, and for any $m \geq m_2(C)$, we have*

$$\mathbf{P} \left\{ \sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| \geq K_*^{\frac{1}{4}}(\log_* K)^{\frac{3}{4}}m2^{-\frac{m}{2}} \right\} \leq 6(K2^{2m})^{1-C}.$$

Now taking $C = 3$ in Theorem A and using the Borel–Cantelli lemma, we get

$$\sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| < O(1)m2^{-\frac{m}{2}} \quad \text{a.s.} \quad (m \rightarrow \infty)$$

and

$$\sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| < K^{\frac{1}{4}}(\log K)^{\frac{3}{4}} \quad \text{a.s.} \quad (K \rightarrow \infty)$$

for any m large enough, $m \geq m_2(3)$.

Next we are going to study the properties of another nested sequence of random walks, obtained by Skorohod embedding. This sequence is not identical, though asymptotically equivalent to the above RW construction. Given a Wiener process W , first we define the stopping times which yield the Skorohod embedded process $B_m(k2^{-2m})$ into W . For every $m \geq 0$ let $s_m(0) = 0$ and

$$s_m(k+1) = \inf \{s : s > s_m(k), |W(s) - W(s_m(k))| = 2^{-m}\} \quad (k \geq 0). \quad (2)$$

With these stopping times the embedded process by definition is

$$B_m(k2^{-2m}) = W(s_m(k)) \quad (m \geq 0, k \geq 0). \quad (3)$$

This definition of B_m can be extended to any real $t \geq 0$ by pathwise linear interpolation. The next lemma describes some useful facts about the relationship between \tilde{B}_m and B_m .

In general, roughly saying, \tilde{B}_m is more useful when someone wants to generate stochastic processes from scratch, while B_m is more advantageous when someone needs a discrete approximation of given processes, like in the case of stochastic integration.

Lemma D. *For any $C \geq 3/2$, $K > 0$, take the following subset of the sample space:*

$$A_m = \left\{ \sup_{n > m} \sup_{0 \leq k2^{-2m} \leq K} |2^{-2n}T_{m,n}(k) - k2^{-2m}| < 6(CK_* \log_* K)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m} \right\},$$

where $T_{m,n}(k) = T_n \circ T_{n-1} \circ \dots \circ T_m(k)$ for $n > m \geq 0$ and $k \geq 0$. Then for any $m \geq m_3(C)$,

$$\mathbf{P} \{A_m^c\} \leq 4(K2^{2m})^{1-C}.$$

Moreover, $\lim_{n \rightarrow \infty} 2^{-2n}T_{m,n}(k) = t_m(k)$ exists almost surely and on the set A_m we have

$$\tilde{B}_m(k2^{-2m}) = W(t_m(k)) \quad (0 \leq k2^{-2m} \leq K),$$

cf. (3). Further, on A_m except for a zero probability subset, $s_m(k) = t_m(k)$ and

$$\sup_{0 \leq k2^{-2m} \leq K} |s_m(k) - k2^{-2m}| \leq 6(CK_* \log_* K)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m} \quad (m \geq m_3(C)).$$

If the Wiener process is built by the RW construction described above using a sequence \tilde{B}_m ($m \geq 0$) of nested RWs and then one constructs the Skorohod embedded RWs B_m ($m \geq 0$), it is natural to ask what the approximating properties of the latter are. The answer described by the next theorem is that they are essentially the same as the ones of \tilde{B}_m , cf. Theorem A.

Lemma 1. *For every $K > 0$, $C \geq 3/2$ and $m \geq m_3(C)$ we have*

$$\mathbf{P} \left\{ \sup_{0 \leq t \leq K} |W(t) - B_m(t)| \geq K_*^{\frac{1}{4}} (\log_* K)^{\frac{3}{4}} m 2^{-\frac{m}{2}} \right\} \leq 10(K2^{2m})^{1-C}.$$

2.2 Approximation of continuous martingales

Beside the RW construction of standard Brownian motion, the other main tool applied in this section is a theorem of Dambis (1965) and Dubins–Schwarz (1965). Briefly saying, these theorems state that any continuous local martingale $(M(t), t \geq 0)$ can be transformed into a standard Brownian motion by time-change.

Below it is supposed that an increasing family of sub- σ -algebras $(\mathcal{F}_t, t \geq 0)$ is given in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the given continuous local martingale M is adapted to it.

In the case of a continuous local martingale $M(t)$ vanishing at 0 its quadratic variation $\langle M, M \rangle_t$ is a process with almost surely continuous and non-decreasing sample paths vanishing at 0. This will be one of the two time-changes we are going to use in the sequel. The other one is a quasi-inverse of the quadratic variation:

$$T_s = \inf\{t : \langle M, M \rangle_t > s\}.$$

Theorem B. [12, V (1.6), p.181] *If M is a continuous (\mathcal{F}_t) -local martingale vanishing at 0 and such that $\langle M, M \rangle_\infty = \infty$ a.s., then $W(s) = M(T_s)$ is an (\mathcal{F}_{T_s}) -Brownian motion and $M(t) = W(\langle M, M \rangle_t)$.*

Similar statement is true when $\langle M, M \rangle_\infty < \infty$ is possible.

From now on, W will always refer to the Wiener process obtained from M by the above time-change, the so-called *DDS Wiener process (or DDS Brownian motion)* of M .

Now Skorohod-type stopping times can be defined for M , similarly as for W in (2). For $m \geq 0$, let $\tau_m(0) = 0$ and

$$\tau_m(k+1) = \inf \{ t : t > \tau_m(k), |M(t) - M(\tau_m(k))| = 2^{-m} \} \quad (k \geq 0). \quad (4)$$

The $(m+1)$ st stopping time sequence is a refinement of the m th in the sense that $(\tau_m(k))_{k=0}^\infty$ is a subsequence of $(\tau_{m+1}(j))_{j=0}^\infty$ so that for any $k \geq 0$ there exist j_1 and j_2 , $\tau_{m+1}(j_1) = \tau_m(k)$ and $\tau_{m+1}(j_2) = \tau_m(k+1)$, where the difference $j_2 - j_1 \geq 2$, even.

Lemma 2. *With the stopping times defined by (4) from a continuous local martingale M one can directly obtain the sequence of shrunken RWs $\{B_m\}_m$ that almost surely converges to the DDS Wiener process W of M , cf. (3):*

$$B_m(k2^{-2m}) = W(s_m(k)) = M(\tau_m(k)), \quad s_m(k) = \langle M, M \rangle_{\tau_m(k)}$$

[but $\tau_m(k) \leq T_{s_m(k)}$], where for $m \geq 0$, the non-negative integer k is taking values (depending on ω) until $s_m(k) \leq \langle M, M \rangle_\infty$.

In the sequel B_m will always denote the sequence of shrunken RWs defined by Lemma 2.

2.2.1 Approximation of the quadratic variation process

Our next objective is to show that the quadratic variation of M can be obtained as an almost sure limit of a point process related to the above stopping times that we will call a *discrete quadratic variation process*:

$$\begin{aligned} N_m(t) &= 2^{-2m} \#\{r : r > 0, \tau_m(r) \leq t\} \\ &= 2^{-2m} \#\{r : r > 0, s_m(r) \leq \langle M, M \rangle_t\} \quad (t \geq 0). \end{aligned}$$

Clearly, the paths of N_m are non-decreasing pure jump functions, the jumping times being exactly the stopping times $\tau_m(k)$. Moreover, $N_m(\tau_m(k)) = k2^{-2m}$ and the magnitudes of jumps are constant 2^{-2m} when m is fixed. Proper application of Lemma D yields the following approximation result.

Lemma 3. *Let M be a continuous local martingale vanishing at 0. Fix $K > 0$ and take a sequence $a_m = O(m^{-2-\epsilon}2^{2m})K$ with some $\epsilon > 0$, where $a_m \geq K \vee 1$ for any $m \geq 1$ ($x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$).*

Then for any $C \geq 3/2$ and $m \geq m_4(C)$ we have

$$\begin{aligned} \mathbf{P} \left\{ \sup_{0 \leq t \leq K} |\langle M, M \rangle_t \wedge a_m - N_m(t \wedge T_{a_m})| \geq 12(Ca_m \log_* a_m)^{\frac{1}{2}} m^{\frac{1}{2}} 2^{-m} \right\} \\ \leq 3(a_m 2^{2m})^{1-C}. \end{aligned}$$

The statement of the next theorem corresponds to the main result in Karandikar [5], though the method applied is different and here we give a rate of convergence as well.

Theorem 1. *Using the same notations as in Lemma 3, we have*

$$\sup_{0 \leq t \leq K} |\langle M, M \rangle_t - N_m(t)| < O(1)m^{\frac{1}{2}}2^{-m} \quad a.s. \quad (m \rightarrow \infty)$$

and

$$\sup_{0 \leq t \leq K} |\langle M, M \rangle_t - N_m(t)| < K^{\frac{1}{2}}(\log K)^{\frac{1}{2}} \quad a.s. \quad (K \rightarrow \infty)$$

for any m large enough, $m \geq m_4(3)$.

2.2.2 Strong approximation of continuous martingales

Now we are ready to discuss the strong approximation of continuous local martingales by time-changed random walks. Applying Lemma 3 and DDS-construction one obtains

Lemma 5. *Fix $K > 0$ and take a sequence $a_m = O(m^{-7-\epsilon}2^{2m})K$ with some $\epsilon > 0$, where $a_m \geq K \vee 1$ for any $m \geq 1$.*

Then for any $C \geq 3/2$ and $m \geq m_5(C)$ we have

$$\mathbf{P} \left\{ \sup_{0 \leq t \leq K} |M(t \wedge T_{a_m}) - B_m(N_m(t \wedge T_{a_m}))| \geq 2a_m^{\frac{1}{4}}(\log_* a_m)^{\frac{3}{4}}m2^{-\frac{m}{2}} \right\} \leq 14(a_m2^{2m})^{1-C}.$$

Theorem 3. *With the same notations as in Lemma 5, we have*

$$\sup_{0 \leq t \leq K} |M(t) - B_m(N_m(t))| < O(1)m2^{-\frac{m}{2}} \quad a.s. \quad (m \rightarrow \infty)$$

and

$$\sup_{0 \leq t \leq K} |M(t) - B_m(N_m(t))| < K^{\frac{1}{4}}(\log K)^{\frac{3}{4}} \quad a.s. \quad (K \rightarrow \infty)$$

for any m large enough, $m \geq m_5(3)$.

Kiefer [7] proved in the Brownian case $M = W$ that using Skorohod embedding one cannot embed a standardized RW into W with convergence rate better than $O(1)n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{4}}$, where n is the number of points used in the approximation. Since this theorem gives a rate of convergence $O(1)n^{-\frac{1}{4}} \log n$ (the number of points used is $n = K2^{2m}$), this rate is close to the best we can have with a Skorohod-type embedding.

2.3 Symmetrically evolving martingales

It is important both from theoretical and practical (e.g. simulation) points of view that the shrunken RW B_m and the corresponding discrete quadratic variation process N_m be independent when approximating M as in Theorem 3. This leads to the question of independence of the DDS Brownian motion W and quadratic variation $\langle M, M \rangle$ in the case of a continuous local martingale M . For, by Lemma 2, B_m depends only on W and, by (5), N_m is determined by $\langle M, M \rangle$ alone. Conversely, if the processes B_m and N_m are independent for any m large enough, then so are W and $\langle M, M \rangle$ too by Lemma 1 and 1. It will turn out from the next theorem that the basic notion in this respect is the symmetry of the increments of M given the past. Thus we will say that a stochastic process $M(t)$ ($t \geq 0$) is *symmetrically evolving* (or has symmetric increments given the past) if for any positive integer n , reals $0 \leq s < t_1 < \dots < t_n$ and Borel sets of the line U_1, \dots, U_n we have

$$\mathbf{P} \{ \Gamma | \mathcal{F}_s^0 \} = \mathbf{P} \{ \Gamma^- | \mathcal{F}_s^0 \},$$

where $\Gamma = \{M(t_1) - M(s) \in U_1, \dots, M(t_n) - M(s) \in U_n\}$, Γ^- is the same, but each U_j replaced by $-U_j$, and $\mathcal{F}_s^0 = \sigma(M(u), 0 \leq u \leq s)$ is the filtration generated by the past of M . If $M(t)$ has finite expectation for any $t \geq 0$, then this condition expresses a very strong martingale property.

Our Theorem 4 below is basically a reformulation of Dubins-Émery-Yor's Theorem of [2]. Their theorem is strongly built on Ocone's Theorem A of [10]. In Ocone's paper it is shown that a continuous local martingale M is conditionally (w.r.t. to the sigma algebra generated by $\langle M, M \rangle$) Gaussian martingale if and only if it is J -invariant. Here J -invariance means that M and $\int_0^t \alpha \, dM$ have the same law for any predictable process α with range in $\{-1, 1\}$. In fact, it is proved there too that J -invariance is equivalent to H -invariance which means that it is enough to consider deterministic integrands of the form $\alpha^{(r)}(t) = \mathbf{I}_{[0,r]}(t) - \mathbf{I}_{(r,\infty)}(t)$.

Dubins, Émery and Yor in [2] proved that these conditions are equivalent to the independence of the DDS Brownian motion and the quadratic variation. Further, in this paper and the paper of Vostrikova and Yor in [20] shorter proofs with additional equivalent conditions were given in the case when M is a continuous martingale. In these references the equivalent condition of the independence of the DDS BM and $\langle M, M \rangle$ explicitly appears. Besides, in [2], the conjecture that a continuous martingale M has the same law as its Lévy transform $\hat{M} = \int \text{sgn}(M) \, dM$ if and only if its DDS BM and $\langle M, M \rangle$ are independent is proved to be equivalent to the conjecture that the Lévy transform is ergodic. Below we give a new, long, but elementary proof for any continuous local martingale M that the DDS BM and $\langle M, M \rangle$ are independent if and only if M is symmetrically evolving i.e. Ocone. Then, in Subsection 2.3.2 we present some remarkable properties of Ocone martingales and some of our recent results. Here, we also give a couple of examples for martingales being Ocone or non-Ocone.

2.3.1 Distributional characterization of Ocone martingales

Theorem 4. (a) *If the Wiener process $W(t)$ ($t \geq 0$) and the non-decreasing, vanishing at 0, continuous stochastic process $C(t)$ ($t \geq 0$) are independent, then $M(t) = W(C(t))$ is a symmetrically evolving continuous local martingale vanishing at 0, with quadratic variation C .*

(b) *Conversely, if M is a symmetrically evolving continuous local martingale, then its DDS Brownian motion W and its quadratic variation $\langle M, M \rangle$ are independent processes.*

2.3.2 Properties of Ocone martingales

In this subsection we give some equivalent conditions for martingales being Ocone martingale. Originally, Ocone proved the equivalence of the conditions (ii-iv). The the equivalence of the parts (i-iii-v-vi) are due to Émery-Dubins-Yor.

Theorem D. *Let M be a continuous martingale with natural filtration $\mathcal{F} = (\mathcal{F}_t)$. The following five statement are equivalent:*

- (i) *the DDS-Brownian motion β^M of M and $\langle M \rangle$ are independent;*
- (ii) *conditionally on $\langle M \rangle$ M is Gaussian martingale;*
- (iii) *for every \mathcal{F} -predictable process H taking values in $\{-1, 1\}$, the two pairs of processes have the same law*

$$\left(\int H \, dM, \langle M \rangle \right) \stackrel{d}{=} (M, \langle M \rangle);$$

- (iv) *for every deterministic function h of the form $\mathbf{I}_{[0,a]} - \mathbf{I}_{(a,\infty)}$, the martingale $\int h \, dM$ has the same law as M ;*

(v) for every \mathcal{F} -predictable process H measurable for the σ -field $\mathcal{B}(\mathbb{R}_+) \otimes \sigma(\langle M \rangle)$ and such that $\int_0^\infty H_s^2 d\langle M \rangle_s < \infty$ a.s.,

$$\mathbf{E} \left[\exp \left(i \int_0^\infty H_s dM_s \right) \middle| \langle M, M \rangle \right] = \exp \left(-\frac{1}{2} \int_0^\infty H_s^2 d\langle M \rangle_s \right);$$

(vi) for every deterministic function h of the form $\sum_{j=1}^n c_j \mathbf{I}_{[0, a_j]}$,

$$\mathbf{E} \left[\exp \left(i \int_0^\infty h(s) dM_s \right) \right] = \mathbf{E} \left[\exp \left(-\frac{1}{2} \int_0^\infty h^2(s) d\langle M \rangle_s \right) \right].$$

Rather interestingly, in the theory of Ocone martingales there is a ten years old conjecture originally introduced by Yor [2]. It says that a divergent martingale M is Ocone martingale if and only if the Lévy transform $\widehat{M} = \int \text{sign}(M) dM$ of the martingale M has the same law as M . If it was true it would mean that one could formally compress the conditions of Theorem D into only one condition.

Dubins, Émery and Yor proved that this conjecture is equivalent to an other conjecture known since the late 70's. This conjecture says that the transformation $\mathcal{T} : B \rightarrow \int \text{sign}(B) dB$ is ergodic on the above introduced measure space.

A result by Dubins and Smorodinsky [3, 1992] increases the plausibility of this conjecture. They established that the discrete version of the Lévy transformation taking effect on the standard symmetric random walk is ergodic on the corresponding measure space.

At the end of this subsection we prove that the processes $\mathcal{T}^n B$ $n \geq 0$ are pairwise weakly orthogonal in the following sense [12]:

Two local martingales M and N are said to be weakly orthogonal if $\mathbf{E} M_s N_t = 0$ for every s and $t \geq 0$. This condition is equivalent to that of for all $s \geq 0$ $\mathbf{E} \langle M, N \rangle = 0$. In our present case this means that the martingales $\mathcal{T}^n M_t$ and $\mathcal{T}^k M_t$ are not jointly Gaussian.

Proposition 1. *The martingales $\mathcal{T}^n W$, ($n \geq 0$) are pairwise weakly orthogonal. More precisely, for all $s, t \geq 0$ and $n \neq k$ non-negative integer we have*

$$\mathbf{E} \mathcal{T}^n W_s \mathcal{T}^k W_t = 0.$$

2.3.3 Some examples of Ocone and non-Ocone martingales

In this subsection we present some remarkable properties of Ocone martingales and some of our recent results. In the detailed study we also give a couple of examples for martingales being Ocone or non-Ocone.

Proposition 2. *Let M be continuous Ocone martingale. Suppose $\langle M \rangle_\infty = \infty$ and M is of the form*

$$M_t = x + \int_0^t \sigma(M_s) d\beta_s,$$

where σ is a nowhere vanishing function and β is a Brownian motion. Then, M is Gaussian martingale.

Example B. ([20]). Let (B_t, C_t) , $t \geq 0$ a planar Brownian motion. The process

$$\mathcal{A}_t = \frac{1}{2} \int_0^\infty (C_s dB_s - B_s dC_s)$$

is an example of Ocone martingale. This assertion is a consequence of the following general theorem which was inspired by Marc Yor.

Theorem 5. Let $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a regular function. Denote the adjoint of its derivative by $\Psi(\mathbf{x}) = (\Phi')^T(\mathbf{x})$ and suppose that the following conditions hold:

$$\Psi(\mathbf{x})\Phi(\mathbf{x}) = \mathbf{x} \quad \text{and} \quad \mathbf{x} \cdot \Phi(\mathbf{x}) = 0 \quad \text{for any } \mathbf{x} \in \mathbb{R}^d$$

where \cdot stands for the usual scalar product in \mathbb{R}^d . If B is a standard d -dimensional Brownian motion then the martingale

$$M_t = \int_0^t \Phi(B_s) \cdot \bar{d}B_s$$

is an Ocone martingale.

Corollary 1. If $\Phi(\mathbf{x}) = A\mathbf{x}$ where A is a regular matrix then the conditions above are equivalent with the conditions that A is orthogonal and anti-symmetric.

Applying this statement one can easily prove that the martingale in Example B is Ocone martingale.

2.4 Approximation of local time, excursion, meander and bridge

The Brownian motion construction in our focus is especially suitable for obtaining results on the approximation of local time of the Brownian motion. Indeed, if two points of an approximating random walk are at the same altitude, they will remain at the same altitude after refinements forever. However, this may be disadvantageous for approximating the other three processes, the excursion, bridge and meander because the first 0 hit changes randomly from one approximation to the next one if we look into the excursion.

In the thesis we present a different, rather combinatorial, construction for excursion, bridge, meander which is due to Phillippe Marchal. In his paper he also gave a local time approximating algorithm but this one is not so natural as in the first three cases.

2.4.1 Approximation of local time

Let \mathcal{L} denote the Brownian local time at level 0. Let $\{B_m\}$ be our usual sequence of scaled random walks that converges to B almost surely uniformly on every compact interval. Let $\ell_m(k) := \#\{0 \leq l < k \mid \tilde{S}_m(l) = 0\}$ and $\mathcal{L}_m(t) := \frac{1}{2^m} \ell_m(\lfloor t2^{2m} \rfloor)$ the local time of the m th approximation at point 0 up to time t . We prove that the sequence of the local time of the discrete approximations is almost surely uniformly converges to the local time of the Brownian motion. The main idea of the proof is the following. When the random walk, say the m th, hits 0 at time k the finer random walk, the $(m+1)$ th, has $\frac{1}{2}(T_{m+1}(k+1) - T_{m+1}(k))$ 0 hits between $T_{m+1}(k)$ and $T_{m+1}(k+1)$ (for the definition T_{m+1} see (1)) which is geometrically distributed random variable with parameter $\frac{1}{2}$ so we can evaluate the time the Brownian motion spends at 0.

Theorem 6. As $n \rightarrow \infty$

$$\mathbf{P} \left(\sup_{[0, K]} |\mathcal{L}_n(t) - \mathcal{L}(t)| \geq C K^{1/4} (\log_* K)^2 n^2 2^{-n/2} \right) \leq \lambda(C) \left(K^{1/2} 2^n \right)^{2-C},$$

with appropriate C dependent positive constant $\lambda(C)$ after an appropriately large n and fixed C .

Proper choice of the constant C and proper usage of the Borel-Cantelli lemma gives the following corollary:

Corollary 2.

$$\sup_{[0,K]} |\mathcal{L}_n(t) - \mathcal{L}(t)| < O(1)n^2 2^{-\frac{n}{2}} \quad a.s. \quad (n \rightarrow \infty)$$

$$\sup_{[0,K]} |\mathcal{L}_n(t) - \mathcal{L}(t)| < K^{\frac{1}{4}}(\log K)^2 \quad a.s. \quad (K \rightarrow \infty)$$

for all m large enough.

The following corollary is necessary for proving an stochastic integral approximation result Theorem 9 in Section 2.5.

Before this global convergence theorem we introduce some notation. Let $\mathcal{L}^a(t)$ and $\mathcal{L}_n^a(t)$ respectively be the local time of the standard Brownian motion and its n th approximation respectively at level a until time t . For arbitrary real number a let us define $\mathcal{L}_n^a(t) = \mathcal{L}_n^{[a]_n}(t)$ where $[a]_n = \frac{[a2^n]}{2^n}$. More, $\mathcal{L}_n^{[a]_n}(t) = \frac{1}{2^m} \ell_m^{[a2^n]}(\lfloor t2^{2m} \rfloor) = \frac{1}{2^m} \#\{0 \leq l < \lfloor t2^{2m} \rfloor \mid \tilde{S}_m(l) = [a2^n]\}$

Corollary 3.

$$\sup_{a \in \mathbb{R}} \sup_{[0,K]} |\mathcal{L}_n^a(t) - \mathcal{L}^a(t)| < O(1)2^{-(\frac{1}{2}-\varepsilon)n} \quad a.s. \quad (n \rightarrow \infty)$$

for an arbitrary small positive ε .

2.4.2 Excursion, bridge, meander – Marchal’s construction

We present Phillipe Marchal’s algorithm for approximating the Brownian excursion, bridge and meander. His algorithms allow to generate directly the excursion, the bridge and the meander. In his paper he also gave a local time approximating algorithm but this one is not so natural as in the first three cases.

2.5 Pathwise stochastic integration

Our objective in this section is to define a sequence of stochastic sums converging to $\int_0^t Y_s dM(s)$ with probability 1 uniformly on every compact interval, where M is a continuous local martingale and Y is an integrable stochastic process with respect to M . It will turn out that one can carry out this procedure for two types of processes. The first is of the form $f'_-(M)$ where f'_- denotes the left-hand side derivative of the difference of two convex functions. The other family of processes is that of with right continuous paths and left limit.

The main idea is that one can take first a dyadic partition on the “spatial” axis that gives the random stopping times of the Skorohod embedding on the time axis. So we get an approach of stochastic integrals which is basically different from the usual definition of stochastic integration.

Similar approach can be found in Karandikar’s papers. Because of this similarity we think that showing some of his results could make our presentation more complete. His results can be found in Subsection 2.5.1.

The main difference of the two approaches is for which process the discretisation is applied. Karandikar applied the discretisation to the integrand while we apply it to the integrator martingale.

2.5.2 Discretization applied to the integrator

Theorem 7. Let $(W_t, (\mathcal{F}_t))$ be a Brownian motion with the standard filtration. Let f be an r.c.l.l. adapted process and for $n \geq 1$ let $\{s_n(i) : i \geq 0\}$ be the stopping times defined in (2).

Let (I_t^n) be defined as follows. For $s_n(k) \leq t < s_n(k+1), k \geq 0$,

$$I_t^n = \sum_{i=0}^{k-1} f_{s_n(i)}(W_{s_n(i+1)} - W_{s_n(i)}) + f_{s_n(k)}(W_t - W_{s_n(k)}).$$

Then, for all $T < \infty$ we have

$$\sup_{0 \leq t \leq T} \left| I_t^n - \int_0^t f dW \right| \rightarrow 0 \quad a.s. \quad (n \rightarrow \infty)$$

Essentially, the proof is based on the fact that the sequence $\{s_m(k)\}_k$ is getting dense on every compact interval as m goes to infinity.

Based on this theorem and the DDS construction one can easily prove the following generalization.

Theorem 8. Let M be continuous (\mathcal{F}_t) -martingale with quadratic variation such that $\langle M \rangle_\infty = \infty$. Let Y be an r.c.l.l. (\mathcal{F}_t) -adapted process and for $n \geq 1$ let $\{\tau_n(i) : i \geq 0\}$ be the stopping times defined in (4).

Let (Y_t^n) and (I_t^n) be defined as follows. For $\tau_n(k) \leq t < \tau_n(k+1), k \geq 0$,

$$Y_t^n = Y_{\tau_n(i)}$$

$$I_t^n = \int_0^t Y_s^n dM_s = \sum_{i=0}^{k-1} Y_{\tau_n(i)}(M_{\tau_n(i+1)} - M_{\tau_n(i)}) + Y_{\tau_n(k)}(M_t - M_{\tau_n(k)}).$$

Then, for all $T < \infty$ we have

$$\sup_{0 \leq t \leq T} \left| I_t^n - \int_0^t Y dM \right| \rightarrow 0 \quad a.s. \quad (n \rightarrow \infty)$$

The next important corollary is a simple consequence of this theorem.

Corollary 4. Under the assumption of Theorem 8 one can obtain the following convergence result:

$$\sup_{0 \leq t \leq K} \left| \sum_{\tau_n(i+1) \leq t} Y_{\tau_n(i)}(M_{\tau_n(i+1)} - M_{\tau_n(i)}) - \int_0^t Y dM \right| \rightarrow 0 \quad a.s. \quad (n \rightarrow \infty).$$

This corollary implies that I_t^n can be thought of as a stochastic sum of stochastically weighted, independent, identically distributed, symmetrical and $\pm \frac{1}{2^n}$ valued random variables.

2.5.3 The non cadlag-case

Karandikar's approach, where the integrand is the base of the discretization, provides a method for approximating stochastic integrals where the integrand is cadlag process. Therefore, we cannot apply his method for integrals in which the integrand is the sign function of the Brownian motion $\text{sign}(B_t)$ which is often used as a basis of examples in stochastic analysis. In this case we can carry out our approximating method which is applicable on a much smaller class

of integrands, namely for $f'_-(B)$ where f'_- is the derivative of the difference of two convex functions.

In [15] Tamás Szabados gave an approximation theorem using discrete Itô formula. This result was valid for integrals like $\int f(B) dB$ for $f \in C^2$, i.e. for two times continuously differentiable real functions. Using Theorem 3 and Itô-Tanaka formula [12] one could prove the following more general statement of this kind.

Theorem 9. *Let f be a difference of two convex functions and let M be a continuous local martingale such that $\langle M \rangle_\infty = \infty$ almost surely. Then for arbitrary $T > 0$*

$$\sup_{t \in [0, T]} \left| \int_0^t f'_-(M_m(s)) dM_m(s) - \int_0^t f'_-(M(s)) dM(s) \right| \rightarrow 0$$

almost surely as m tends to infinity.

3 Approximation of the exponential functional of Brownian motion

This study is devoted to a special application of our discrete approximation method.

Here, we consider the exponential functional of Brownian motion

$$\mathcal{I}_\nu = \int_0^\infty \exp(B(t) - \nu t) dt$$

and mainly its discrete version the exponential functional of random walk. This leads us to the investigation of the $Y = 1 + \xi_1 + \xi_1 \xi_2 + \dots$ type random variables which can also be written in difference equation form

$$Y \stackrel{d}{=} 1 + \xi Y. \quad (5)$$

3.1 Introduction to the exp functional of Brownian motion

The *geometric Brownian motion* plays a fundamental role in the Black–Scholes theory of option pricing, modeling the price process of a stock. It can be explicitly given in terms of Brownian motion (BM) B as

$$S(t) = S_0 \exp(\sigma B(t) + (\mu - \sigma^2/2)t), \quad t \geq 0.$$

In the case of Asian options one is interested in the average price process

$$A(t) = \frac{1}{t} \int_0^t S(u) du, \quad t \geq 0.$$

The following interesting result is true for the distribution of a closely related, widely investigated exponential functional of BM:

$$\mathcal{I} = \int_0^\infty \exp(B(t) - \nu t) dt \stackrel{d}{=} \frac{2}{Z_{2\nu}} \quad (\nu > 0).$$

Here $Z_{2\nu}$ is a gamma distributed random variable with index 2ν and parameter 1, while $\stackrel{d}{=}$ denotes equality in distribution. This result was proved by [4] using discrete approximations

with gamma distributed random variables and also by [19], using rather ingenious stochastic analysis tools.

As a consequence, the p th integer moment of \mathcal{I} is finite iff $p < 2\nu$ and

$$\mathbf{E}(\mathcal{I}^p) = 2^p \frac{\Gamma(2\nu - p)}{\Gamma(2\nu)}. \quad (6)$$

On the other hand, all negative integer moments, also given by (6), are finite and they characterize the distribution of \mathcal{I} .

The situation is much nicer when BM with negative drift is replaced in the model by the negative of a subordinator $(\alpha_t, t \geq 0)$, that is, by the negative of a non-decreasing process with stationary and independent increments, starting from the origin. Then, as was shown by [1], all positive integer moments of $\mathcal{J} = \int_0^\infty \exp(-\alpha_t) dt$ are finite:

$$\mathbf{E}(\mathcal{J}^p) = \frac{p!}{\Phi(1) \cdots \Phi(p)}, \quad \Phi(\lambda) = -\frac{1}{t} \log \mathbf{E}(\exp(-\lambda\alpha_t)), \quad (7)$$

and in this case the positive integer moments characterize the distribution of \mathcal{J} .

To achieve a similar favorable situation in the BM case, at least in an approximate sense, it is a natural idea to use a simple, symmetric random walk (RW) as an approximation, with a large enough negative drift. Besides, in some applications a discrete model could be more natural than a continuous one. It seems important that, as we shall see below, the discrete case is rather different from the continuous case in many respects.

So let $(X_j)_{j=1}^\infty$ be an i.i.d. sequence with $\mathbf{P}(X_1 = \pm 1) = \frac{1}{2}$ and $S_0 = 0$, $S_k = \sum_{j=1}^k X_j$ ($k \geq 1$). Introduce the following approximation of \mathcal{I} :

$$Y = \sum_{k=0}^{\infty} \exp(S_k - k\nu) = 1 + \xi_1 + \xi_1 \xi_2 + \cdots, \quad \xi_j = \exp(X_j - \nu),$$

where $\nu > 0$. Later, we apply proper scaling to get a real approximation of \mathcal{I} . In this paper we investigate the properties of Y type random variables which, in this simple case, will be called the discrete exponential functional of the given RW, or shortly, the discrete exponential functional.

In Section 3.2, below it turns out that the distribution of Y is singular w.r.t. Lebesgue measure if $\nu > 1$. Here, we prove a more general result according to the singularity of the distribution of Y when ξ_j above has more general distribution.

In Section 3.3 we determine the moments of the discrete exponential functional in order to work out a (6) and a (7) type equations for the discrete case. Dealing with the moments, beyond these results we have found a recursion of certain moments in the expansion of the moments of the discrete approximation. We will describe this recursion in Subsection 3.3.1.

Finally, in Section 3.4 section we use a nested sequence of RWs to obtain a.s. converging approximation of \mathcal{I} , and this way an elementary proof of result (3.1) of Dufresne and Yor as well.

3.2 The distribution of the discrete exponential functional

In Subsection 3.2.1 we review some fractal notions which are essential in the later subsections. In Subsection 3.2.2 we deal with the case when the distribution function of Y is ‘‘overlapping’’. The generalization of this simple case can be found in Subsection 3.2.3. In this outline we also recall some fractal properties of Y .

In (5) let ξ_j take the positive values $\gamma_1 < \dots < \gamma_N < 1$, and let $p_i = \mathbf{P}(\xi = \gamma_i)$. (In our basic case $N = 2$, $\gamma_1 = e^{-1-\nu}$, $\gamma_2 = e^{1-\nu}$, $p_1 = p_2 = \frac{1}{2}$.) The similarity transformations $T_i(x) = \gamma_i x + 1$ ($1 \leq i \leq N$) form an iterated function scheme. If

$$\mathbf{E}(\log \xi) = \sum_{i=1}^N p_i \log \gamma_i < 0$$

then simple consideration proves that the following functional equation holds for the distribution function of Y :

$$F(y) = \sum_{i=1}^N p_i F(T_i^{-1}(y)). \quad (8)$$

3.2.3 The general case

Next we are going to show that the distribution of Y is singular w.r.t. Lebesgue measure even in the overlapping case if $\nu > 1$. Again, we consider the slight generalization introduced above. The proof below is based on [13] and on personal communication with K. Simon.

Theorem 10. *Let ξ take the values γ_i ($i = 1, \dots, N$), $0 < \gamma_1 < \dots < \gamma_N < 1$, and let $p_i = \mathbf{P}(\xi = \gamma_i)$. Take an i.i.d. sequence $(\xi_j)_{j=1}^{\infty}$, $\xi_j \stackrel{d}{=} \xi$. Then the distribution of $Y = 1 + \xi_1 + \xi_1 \xi_2 + \dots$ is singular w.r.t. Lebesgue measure, if*

$$-\chi_{\mathbf{P}} = \mathbf{E}(\log \xi) = \sum_{i=1}^N p_i \log \gamma_i < \sum_{i=1}^N p_i \log p_i = -H_{\mathbf{P}}.$$

Here $\chi_{\mathbf{P}}$ is the Lyapunov exponent of the iterated function scheme (T_1, \dots, T_N) corresponding to the Bernoulli measure \mathbf{P} .

Returning to our basic case, this condition holds iff $\nu > \log 2 \approx 0.693$. Combining this with the condition $\gamma_2 < 1$, this means that the distribution of Y is singular w.r.t. Lebesgue measure for any $\nu > 1$.

3.3 The moments of the discrete exponential functional

Using the binomial theorem, the following recursion can be obtained for the p th moment $e_p = \mathbf{E}(Y^p)$ of Y for any positive integer p :

$$e_p = \frac{1}{1 - \mu_p} \sum_{k=0}^{p-1} \binom{p}{k} \mu_k e_k, \quad (9)$$

supposing $\mu_p < 1$, where $\mu_k = \mathbf{E}(\xi^k)$, $e_k = \mathbf{E}(Y^k)$, $k \geq 0$.

3.3.1 Permutations with given descent set

For integer $p \geq 1$ it follows from (9) by induction that $\mathbf{E}(Y^p)$ is a rational function of the moments μ_1, \dots, μ_p :

$$\mathbf{E}(Y^p) = \frac{1}{(1 - \mu_1) \cdots (1 - \mu_p)} \sum_{(j_1, \dots, j_{p-1}) \in \{0,1\}^{p-1}} a_{j_1, \dots, j_{p-1}}^{(p)} \mu_1^{j_1} \cdots \mu_{p-1}^{j_{p-1}}, \quad (10)$$

where the coefficients of the numerator are *universal* constants, independent of the distribution of ξ_j .

These universal coefficients $a_{j_1 \dots j_{p-1}}^{(p)}$ make a symmetrical, Pascal's triangle-like table if each row is listed in the increasing order of the binary numbers $j_{p-1}2^{p-2} + \dots + j_12^0$, defined by the multiindices (j_1, \dots, j_{p-1}) .

A natural question may arise at this point, independently of any probability theory background mentioned above. Suppose that one defines a recursive sequence $(e_p)_{p=1}^\infty$ by (9) with coefficients $a_{j_1 \dots j_{p-1}}^{(p)}$ given by (10). Can one attach any direct mathematical meaning to this coefficients $a_{j_1 \dots j_{p-1}}^{(p)}$ then? The answer is yes, and rather surprisingly the coefficient $a_{j_1 \dots j_{p-1}}^{(p)}$ is equal to the number of permutations $\pi \in S_p$ having descent $\pi(i) > \pi(i+1)$ exactly where $j_i = 1$, $1 \leq i \leq p-1$, cf. Theorem 11 below. Beyond this identification, we give a new recursion for the coefficients $a_{j_1 \dots j_{p-1}}^{(p)}$ and for the number of permutation with given descent set.

We recall that the descent set of a permutation $\pi \in S_p$ is defined as $D(\pi) = \{i : \pi(i) > \pi(i+1), 1 \leq i \leq p-1\}$

Theorem 11. *The coefficient $a_{j_1 \dots j_{p-1}}^{(p)}$ is equal to the number of permutations $b^{(p)}(S)$ with descent set S given by*

$$S = \{s_1, \dots, s_m\} = \{k : j_k = 1, 1 \leq k \leq p-1\}, \quad m = \sum_{k=1}^{p-1} j_k. \quad (11)$$

We give an algebraic and a simple combinatorial proof as well which basically translates the well-known method by which permutations in S_p can be obtained from permutations in S_{p-1} by adjoining the number p . Finally, we present a new recursion for the coefficients $a_{j_1 \dots j_{p-1}}^{(p)}$ and for the number of permutation with given descent set as well.

Lemma 8. *The following recursion holds for any $p \geq 2$ and multiindex $(j_1, \dots, j_{p-1}) \in \{0, 1\}^{p-1}$:*

$$a_{j_1 \dots j_{p-1}}^{(p)} = \sum_{i=1}^{p-1} \delta_i a_{j_1^{(i)} \dots j_{p-2}^{(i)}}^{(p-1)} = \sum_{(i_1, \dots, i_{p-2}) \in L(j_1, \dots, j_{p-1})} a_{i_1 \dots i_{p-2}}^{(p-1)},$$

where $a^{(1)} = 1$, $\delta_i = |j_i - j_{i-1}|$ for $i \geq 2$, $\delta_1 = 1$, $j_k^{(i)} = j_k$ for $1 \leq k \leq i-1$, $j_k^{(i)} = j_{k+1}$ for $i \leq k \leq p-2$, and $L(j_1, \dots, j_{p-1})$ is the set of all distinct binary sequences obtained from (j_1, \dots, j_{p-1}) by deleting exactly one digit. For example, $a_{0110}^{(5)} = 11 = a_{110}^{(4)} + a_{010}^{(4)} + a_{011}^{(4)}$.

3.4 Approximation of the exponential functional of Brownian motion

In this section we are going to show that taking our usual sequence of RWs the resulting sequence of discrete exponential functionals (3.1) converges almost surely to the corresponding exponential functional \mathcal{I} of BM. Based on this, using convergence of moments, we will give an elementary proof of theorem (3.1) of Dufresne and Yor.

We need a more general result about this approximation. This is a version of Lemma C originally introduced in [15, Lemma 4]. The proof can be read easily from the proof there. Namely, for almost every ω there exists an $m_0(\omega)$ such that for any $m \geq m_0(\omega)$ and for any $K \geq e$, one has

$$\sup_{j \geq 1} \sup_{0 \leq t \leq K} |B_{m+j}(t) - B_m(t)| \leq K^{\frac{1}{4}} (\log K)^{\frac{3}{4}} m 2^{-\frac{m}{2}}.$$

Using this result one gets that the discrete approximation converges to the exp. func. of Brownian motion:

Lemma 9. Let $B_m(t)$, $t \geq 0$, $m \geq 0$ be our usual a sequence of shrunken simple symmetric RWs that a.s. converges to BM $(B(t), t \geq 0)$, uniformly on bounded intervals. Then for any $\nu > 0$, as $m \rightarrow \infty$,

$$Y_m = 2^{-2m} \sum_{k=0}^{\infty} \exp\left(2^{-m} \tilde{S}_m(k) - \nu k 2^{-2m}\right) \\ \rightarrow \mathcal{I} = \int_0^{\infty} \exp(B(t) - \nu t) dt < \infty \quad a.s.$$

Applying the results of the previous sections to Y_m one can determine the limes $\lim_{m \rightarrow \infty} \mathbf{E}(Y_m^p)$ for various p .

3.4.1 The moments of the exponential functional of BM

Lemma 10. If p is a positive integer such that $\frac{p}{2} < \nu$, then

$$\lim_{m \rightarrow \infty} \mathbf{E}(Y_m^p) = \frac{1}{\prod_{k=1}^p \left(\nu - \frac{k}{2}\right)} < \infty.$$

The result for negative moments can be carried out by simple combinatorial and analytical consideration using the asymptotic of the exp and cosh functions.

Lemma 11. For all integer $p \geq 1$, we have

$$\lim_{m \rightarrow \infty} \mathbf{E}(Y_m^{-p}) = \lim_{m \rightarrow \infty} \mathbf{E}(Y_m^{-1}) \prod_{k=1}^{p-1} \left(\nu + \frac{k}{2}\right),$$

where $\lim_{m \rightarrow \infty} \mathbf{E}(Y_m^{-1}) < \infty$.

Finally, it turns out that Y_m^{-1} converges to \mathcal{I}^{-1} in any L^p . This makes it possible to recover the result of Yor and Dufresne ([4] and [19]) the distributional characterization of the exponential functional of Brownian motion.

Theorem 12. Then the following statements hold true:

- (a) Y_m^{-1} converges to \mathcal{I}^{-1} in L^p for any $p \geq 1$ real and $\lim_{m \rightarrow \infty} \mathbf{E}(Y_m^{-p}) = \mathbf{E}(\mathcal{I}^{-p}) < \infty$;
- (b) $\mathcal{I} \stackrel{d}{=} 2/Z_{2\nu}$, where $Z_{2\nu}$ is a gamma distributed random variable with index 2ν and parameter 1;
- (c) Y_m converges to \mathcal{I} in L^p for any integer p such that $1 \leq p < 2\nu$ (supposing $\nu > \frac{1}{2}$) and then $\lim_{m \rightarrow \infty} \mathbf{E}(Y_m^p) = \mathbf{E}(\mathcal{I}^p) < \infty$. The same is true for any real q , $1 \leq q < p$.

3.4.2 Properties of the exponential functional process

Theorem 12 says that the one dimensional marginal distributions of the process $X(\nu) = \frac{1}{\mathcal{I}(\nu)}$, $0 < \nu$ are gamma distributed with index 2ν as that of a gamma process. In spite of the equality of the one dimensional marginals X is completely different from the gamma process as the next proposition shows.

Proposition 4. The process

$$X(\nu) := \frac{1}{\int_0^{\infty} e^{B_s - \nu s} ds}$$

is almost surely continuous so it is not a gamma process.

An interesting consequence of the technique used in the proof of Lemma 11 is the following lemma. It says that every multi indices moment of the process $(\mathcal{I}^{-1}(\nu))_{\nu>0}$ can be derived from some other multi indices moments with bigger individual moments. We remark that this statement seems to be insufficient to determine all multi indices moments.

For simplicity, we introduce some notations.

Fix a positive integer d and vectors $\underline{k} \in \mathbb{Z}_+^d$, $\underline{\nu} \in \mathbb{R}_+^d$. Let

$$M((\underline{\nu}, \underline{k})) = \mathbf{E} (\mathcal{I}(\nu_1)^{-k_1} \dots \mathcal{I}(\nu_d)^{-k_d}).$$

Finally, let $(\underline{\nu}, \underline{k})_i = (\underline{\nu}, (k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_d))$.

For the multi indices moment $M((\underline{\nu}, \underline{k}))$ we have the following

Lemma 12.

$$M((\underline{\nu}, \underline{k})) = \left(\frac{k_1 + \dots + k_d}{2} + \frac{k_1 \nu_1 + \dots + k_d \nu_d}{k_1 + \dots + k_d} \right)^{-1} \sum_{i=1}^d \frac{k_i}{k_1 + \dots + k_d} M((\underline{\nu}, \underline{k})_i).$$

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