STABILIZATION OF SAMPLED-DATA NONLINEAR SYSTEMS BY RECEDING HORIZON CONTROL VIA DISCRETE-TIME APPROXIMATIONS

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Abstract: Results on stabilizing receding horizon control of sampled-data nonlinear systems via their approximate discrete-time models are presented. The proposed receding horizon control is based on the solution of Bolza-type optimal control problems for the parametrized family of approximate discrete-time models. This paper investigates the situation when the sampling period and the integration parameter used in obtaining approximate model coincide and can be chosen arbitrarily small. Sufficient conditions are established which guarantee that the controller that renders the origin to be asymptotically stable for the approximate model also stabilizes the exact discrete-time model for sufficiently small sampling parameters.

Keywords: Controller design, Predictive control, Feedback stabilization, Sampled-data systems, Numerical methods

1. INTRODUCTION

The stabilization problem of nonlinear systems has received considerable attention in the last decades. The use of computers in the implementation of the controllers necessitated the investigation of sampled-data systems. One way to design a sampled-data control is to implement a continuous-time algorithm with sufficiently small sampling intervals. This approach is proposed in connection with the receding horizon method among the others in (Chen, *et al.*, 2000; Fontes, 2001; Jadababaie and Hauser, 2001). However, some difficulty may arise during the application of this method: 1) the exact solution of the nonlinear continuous-time model is typically unknown, therefore an approximation procedure is unavoidable; 2) it may be difficult to implement an arbitrarily time-varying control function. An overview and analysis of existing approaches for the stabilization of sampled data systems can be found in the recent papers (Nešić, *et al.*, 1999) and (Nešić and Teel) (see also the references therein). In these papers a systematic investigation of the connection between the exact and approximate models are carried out, numerous examples and counter-examples illustrating the effect of the sampling procedure are reported, and conditions are presented which guarantee that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant both for the cases of fixed sampling period and varying integration parameter (Nešić and Teel) and for the

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case when these two parameters coincide (Nešić, *et al.*, 1999) and (Nešić and Teel).

There are several ways to design controllers satisfying the conditions given in (Nešić, *et al.*, 1999) and (Nešić and Teel). In (Grüne and Nešić 2002), optimization based methods are studied; the design is carried out either via an infinite horizon optimization problem or via an optimization problem over a finite horizon with varying length. Another possibility is the application of the widespread receding horizon or model predictive control method. This method obtains the feedback control by solving a finite horizon optimal control problem at each time instant using the current state of the plant as the initial state for the optimization and applying "the first part" of the optimal control. The study of stabilizing property of such schemes has been the subject of intensive research in recent years. From the vast literature we mention here but a few (Mayne and Michalska, 1990; Gyurkovics, 1996; Chen and Allgöwer, 1998; De Nicolao, *et al.*, 1998a; Gyurkovics, 1998; Jadababaie and Hauser, 2001; Fontes, 2001; Hu and Linnemann, 2002) and we refer the reader to the excellent overview papers (De Nicolao, *et al.*, 1998b; Allgöwer, *et al.*, 1999; Mayne, *et al.*, 2000) and to the references therein. A great majority of works deals either with continuous-time systems with or without taking into account any sampling or with discrete-time systems considering the model given directly in discrete-time. To the best of our knowledge, the only exception is the very recent work of (Ito and Kunisch, 2002), where the effect of the sampling and zero-order hold is considered assuming the existence of a global control Lyapunov function. Relying on the results of (Nešić, *et al.*, 1999) and (Nešić and Teel), the present work studies the conditions under which the stabilizing receding horizon control computed for the approximate discrete-time model also stabilizes the exact discrete-time system in the case when sampling period and the integration parameter coincide and can be chosen arbitrarily small. It should emphasized that these conditions concern directly the data of the problem and the design parameters of the method, but not the result of the design procedure. From practical point of view, it is important to know whether the basin of attraction is sufficiently large when some stabilizing controller is applied. This set is frequently compared with that of the infinite horizon regulator. Here we shall show that the basin of attraction contains any compact subset of the set of initial point which are practically asymptotically controllable to the origin with piecewise constant sampled controllers. In what follows, the notation $\mathcal{B}_\Delta = \{z \in \mathbb{R}^p : \|z\| \leq \Delta\}$ will be used both in $\mathbb{R}^n$ and $\mathbb{R}^m$ and $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ denote the usual class-$\mathcal{K}$, class-$\mathcal{K}_\infty$ and class-$\mathcal{KL}$ functions (see e.g. Nešić, *et al.*, 1999).

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 The model

Consider the nonlinear control system described by

$$\dot{x}(t) = f\left(x(t), u(t)\right), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $f(0,0) = 0$, U is closed and $0 \in U$. The system is to be controlled digitally using piecewise constant control functions $u(t) = u(kT) =: u_k$, if $t \in [kT, (k+1)T)$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period.

Assumption A1 (i) Function f is continuous, two times continuously differentiable at least in a neighborhood of the origin in both variables, and for any pair of positive numbers (Δ_1, Δ_2) there exist a $T^* > 0$ such that for any $x_0 \in \mathcal{B}_{\Delta_1}$, $\overline{u} \in U \cap \mathcal{B}_{\Delta_2}$ and $T \in (0, T^*]$ equation (1) with $\overline{u} \equiv u(t)$,and $x(0) = x_0$ has a unique solution on $[0, T]$ denoted by $\phi(., x_0, \overline{u})$.

(ii) For any $\Delta > 0$ there exists an $L_f = L_f(\Delta)$ such that

$$\|f(x,u) - f(y,u)\| \leq L_f \|x - y\|,$$

for all $x, y \in \mathcal{B}_\Delta$ and $u \in \mathcal{B}_\Delta$.

(iii) f is sufficiently smooth. □

Then, the exact discrete-time model of the system can be defined as

$$x_{k+1} = F_T^e(x_k, u_k), \tag{2}$$

where $F_T^e(x,u) := \phi(T, x, u)$. (A discussion about the case of finite escapes can be found e.g. in (Nešić, *et al.*, 1999). It has to be emphasized that F_T^e in (2) is generally not known.

Remark 1 If Assumption A1 is valid, then F_T^e is continuous in x and u and it satisfies a local Lipschitz condition of the following type: for each $\Delta > 0$ there exist $T^* > 0$, and $L > 0$ such that

$$\|F_T^e(x,u) - F_T^e(y,u)\| \leq e^{LT} \|x - y\|, \tag{3}$$

holds for all $u \in \mathcal{B}_\Delta$, all $T \in (0, T^*]$, and all $x, y \in \mathcal{B}_\Delta$.

Suppose that a parametrized family of approximate discrete-time plant models is given

$$x_{k+1} = F_T^a\left(x_k, u_k\right), \tag{4}$$

where T is the sampling parameter. We shall assume that the numerical approximation scheme preserves the properties of f in the following sense.

Assumption A2 (i) $F_T^a(0,0)=0$, F_T^a is continuous in both variables and it satisfies the same kind of local Lipschitz condition as F_T^e (see Remark 1);

(ii) there exist a $r_0>0$, a $T^*>0$ and a $L_{r_0}^a>0$ such that for any $T\in(0,T^*]$ we have

$$\|F_T^a(x,u)-x\|\le TL_{r_0}^a(\|x\|+\|u\|) \qquad (5)$$

if $\|x\|+\|u\|\le r_0$.□

Remark 2 Observe that, if (i) Assumption A1 hold true then for many one-step numerical methods, the assertions of Assumption A2 can be proven.

2.2 Practical asymptotic controllability and stabilizability

Let $\Gamma\subset\mathbb{R}^n$ be a given compact set with a bound Δ_0, containing a neighborhood of the origin.

Definition 1. System (2) is *practically asymptotically controllable* (PAC) *from* Γ *to the origin*, if there exist a $\beta(.,.)\in\mathcal{KL}$ and a continuous positive and increasing function $\sigma(.)$ such that for all $x\in\Gamma$ and for all $r>0$ there exists a control sequence $\mathbf{u}^r(x)=\{u_0^r(x),u_1^r(x),...\}$, $u_t^r(x)\in U$, $\|u_t^r(x)\|\le\sigma(\|x\|)$ such that the corresponding solution ϕ of (2) satisfies

$$\|\phi_t(x,\mathbf{u}^r(x))\|\le\max\{\beta(\|x\|,tT),r\}, \qquad (6)$$

for all $t\ge0$. Moreover, system (2) is *semiglobally practically asymptotically controllable to the origin* if it is PAC from any compact set $\Gamma\subset\mathbb{R}^n$.

Definition 2. System (2) is *practically asymptotically stabilizable* (PAS) *in* Γ *about the origin*, if there exist a $\beta(.,.)\in\mathcal{KL}$ and a continuous positive and increasing function $\sigma(.)$ such that for any $r>0$ there exists a feedback $k^r:\Gamma\to U$, $\|k^r(x)\|\le\sigma(\|x\|)$ such that for any $x\in\Gamma$ the solution ϕ^c of $x_{t+1}=F_T^e(x_t,k^r(x_t))$, $x_0=x$ inequality

$$\|\phi_t^c(x)\|\le\max\{\beta(\|x\|,tT),r\} \qquad (7)$$

holds true for all $t\ge0$. Moreover, system (2) is *semiglobally practically asymptotically stabilizable about the origin* if it is PAS in Γ for any compact set $\Gamma\subset\mathbb{R}^n$.

Theorem 1 System (2) is practically asymptotically stabilizable in Γ about the origin if and only if it is practically asymptotically controllable from Γ to the origin.

Proof: Because of lack of space, the proof is omitted.

Remark 3 In view of Theorem 1, it is reasonable to consider the problem of PAS for system (2) over a compact set Γ from which it is PAC. If this latter property is semiglobal, then the same will also be true with respect to stabilization, otherwise semiglobal stabilization is impossible.

Remark 4 Theorem 1 remains valid if, similarly to (Kreisselmeier and Birkhölzer, 1994), the properties PAC and PAS are required with vanishing controllers, i.e. the left hand sides of (6) and (7) are substituted by $\|\phi_t(x,\mathbf{u}^r(x))\|+\|u_t^r(x)\|$ and $\|\phi_t^c(x)\|+\|k^r(\phi_t^c(x))\|$, respectively.

2.3 Basic definitions and assumptions

In what follows, a stabilizing feedback will be constructed for the *approximate model* and conclusion about the stability of the closed-loop exact model is drawn on the basis of the closeness of solutions of the two models. This closeness is characterized by the following definition:

Definition 3 Let a pair of strictly positive numbers (Δ_1,Δ_2) be given and suppose that there exist $\gamma\in\mathcal{K}$ and $T^*>0$ such that

$$(x,u)\in\mathcal{B}_{\Delta_1}\times\mathcal{B}_{\Delta_2},\quad T\in(0,T^*]\Longrightarrow \\ \|F_T^a(x,u)-F_T^e(x,u)\|\le T\gamma(T), \qquad (8)$$

then the family F_T^a is said to be (Δ_1,Δ_2)-*consistent* with F_T^e. Moreover, if for any pair of strictly positive numbers (Δ_1,Δ_2) there exist $\gamma\in\mathcal{K}$ and $T^*>0$ such that (8) holds true, then F_T^a is said to be *semiglobally consistent* with F_T^e.

Sufficient checkable conditions for consistency properties can be found in (Nešić, *et al.*, 1999) and (Nešić and Teel).

In order to define a receding horizon feedback controller, let (4) be subject to the cost function

$$J_T(N,x,\mathbf{u})=\sum_{k=0}^{N-1}Tl_T(x_k^a,u_k)+g(x_N^a),$$

where $x_k^a=\phi_k^a(x,\mathbf{u})$, $k=0,1,...,N$, l_T and g are given functions.

Assumption A3 (i) g is continuous, there exists a class-$\mathcal{K}_\infty$ function γ_1 such that $\gamma_1(\|x\|)\le g(x)$ and for any $\Delta>0$ there exists a $L_g=L_g(\Delta)>0$ constant such that $|g(x)-g(y)|\le L_g\|x-y\|$ for all $x\in\mathcal{B}_\Delta$.

(ii) l_T is continuous with respect to x and u, uniformly in small T, and for any $\Delta_1>0$, $\Delta_2>0$

there exist $T^* > 0$ and $L_l = L_l(\Delta_1, \Delta_2) > 0$ such that

$$|l_T(x,u) - l_T(y,u)| \le L_l \|x - y\|$$

for all $T \in (0, T^*]$, $x, y \in \mathcal{B}_{\Delta_1}$ and $u \in \mathcal{B}_{\Delta_2}$

(iii) There exist a $T^* > 0$ and two class-$\mathcal{K}_\infty$ functions φ_1 and φ_2 such that the inequality

$$\begin{aligned}\varphi_1(\|x\|) + \varphi_1(\|u\|) &\le l_T(x,u)\\ &\le \varphi_2(\|x\|) + \varphi_2(\|u\|),\end{aligned}$$

holds for all $x \in \mathbb{R}^n$, $u \in U$ and $T \in (0, T^*]$.

Assumption A4 There exists a $T^* > 0$ such that the exact discrete-time model (2) is PAC from Γ to the origin for all $T \in (0, T^*]$.

Let $\beta(.,.)$ and $\sigma(.)$ be functions generated by Assumption A4 and let Δ_1 be such that $\Delta_1 \ge 1 + \max_{x\in\Gamma} \beta(\|x\|, 0)$. Moreover, for $0 < r \le \Delta_1$, we introduce the notation

$$\mathcal{U}_r = U \cap \mathcal{B}_{\sigma(r)} \tag{9}$$

Assumption A5 The family F_T^a is semiglobally consistent with F_T^e.

Assumption A6 There exist $T^* > 0$ and $\eta > 0$ such that for all $x \in \mathcal{G}_\eta = \{x : g(x) \le \eta\}$ there exists a $k_T(x) \in \mathcal{U}_{\rho_0}$ such that inequality

$$T l_T(x, k_T(x)) + g\left(F_T^a(x, k_T(x))\right) \le g(x) \tag{10}$$

holds true for all $T \in (0, T^*]$, where ρ_0 is such that $\mathcal{G}_\eta \subset \mathcal{B}_{\rho_0}$.

In what follows, T_0^* denotes the minimum of T^* generated by Assumptions A1-A6 and Remark 1.

Proposition 1. If Assumptions A4-A6 hold true, then there exists $T^* > 0$ such that, for any $T \in (0, T^*]$ system (4) is asymptotically controllable from Γ to the origin.

Proof. The proof is an immediate consequence of Theorem 1 and A6.

Suppose that Assumption A4 holds true. Consider the optimization problem

$$P_T^a(N,x) : \min\{J_T(N,x,\mathbf{u}) : u_k \in \mathcal{U}_{\Delta_1}\}$$

where $\mathbf{u} = \{u_0, u_1, ..., u_{N-1}\}$. (We note that in problem $P_T^a(N,x)$ no terminal constraint is given.)

If this optimization problem has a solution denoted by $\mathbf{u}^*$ then its first element, i.e. u_0^* is applied at the state x. Since the optimal solution of $P_T^a(N,x)$ naturally depends on x, in this way a feedback has been defined on the basis of the approximate discrete-time model, i.e. $v_T^a(x) := u_0^*$.

Conditions, under which v_T^a asymptotically stabilizes the origin for the approximate model (4) with a fixed $T > 0$, are well-established (see e.g. (Gyurkovics, 1998, Chen and Allgöwer, 1998; De Nicolao, *et al.*, 1998a,b; Jadababaie and Hauser, 2001; Hu and Linnemann, 2002), the review papers (Allgöwer, *et al.*, 1999; Mayne, *et al.*, 2000) and the references therein).

In order to ensure the stabilizing property of v_T^a for the exact model (2) in the given set Γ, one needs somewhat stronger conditions than it is usual in receding horizon investigations: in fact, one has to derive certain estimations for the value of $P_T^a(N,x)$ which are *uniform* in small T.

3. MAIN RESULT

For any $x \in \mathbb{R}^n$, let

$$V_{N,T}^a(x) = \inf\{J_T(N,x,\mathbf{u}) : u_k \in \mathcal{U}_{\Delta_1}\}$$

where $\mathbf{u} = \{u_0, u_1, ..., u_{N-1}\}$, if the right hand side is finite and let $V_{N,T}^a(x) = \infty$ otherwise.

From assumptions A1-A4, it follows immediately that for any $T \in (0, T_0^*]$, any $N > 0$ and any $x \in \mathcal{B}_{\Delta_1}$, $(\Gamma \subset \mathcal{B}_{\Delta_1})$, $P_T^a(N,x)$ has a solution $\mathbf{u}^*(x)$, $V_{N,T}^a(.)$ is positive definite and continuous. With argumentations standard in receding horizon literature one can prove the following lemma.

Lemma 1 Suppose that Assumptions A1-A3 and A6 hold true. Then for any $T \in (0, T_0^*]$ and $N \ge 1$, the following statements are valid:

1.) For any $x \in \mathcal{G}_\eta$, $\phi_N^a(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$ and $V_{N,T}^a(x) \le g(x)$.

2.) If $\phi_N^a(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$ for some $x \in \Gamma$, then $V_{\overline{N},T}^a(x) \le V_{N,T}^a(x)$ for all $\overline{N} \ge N$, and

$$V_{N,T}^a(F_T^a(x, v_T^a(x))) - V_{N,T}^a(x) \le -T l_T(x, v_T^a(x))$$

3.) If for $x \in \Gamma$ and for some t , $0 \le t < N$, $\phi_t^a(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$, then $\phi_N^a(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$.□

Lemma 2 If Assumptions A1-A6 hold true, then there exist a T_1^*, $0 < T_1^* \le T_0^*$ and a constant $V_{\max}^a$ such that $V_{N,T}^a(x) \le V_{\max}^a$ for all $x \in \Gamma$ and $T \in (0, T_1^*]$

Proof. Because of lack of space, the proof is omitted.

Lemma 3 Suppose that Assumptions A1-A6 hold true. If $T \in (0, T_1^*]$ and $N \in \mathbb{N}$ are chosen so that

$$TN > \frac{V_{\max}^a - \eta}{c} =: \tau_2^*$$

where $c > 0$ is such a positive constant that $\varphi_1(\|x\|) \ge c$ for all $x \notin \mathcal{G}_\eta$, then

$$\phi_N^a(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta \qquad \text{for all } x \in \Gamma. \tag{11}$$

Proof. Because of lack of space, the proof is omitted.

In what follows, we shall assume that

$$\tau_2^* \leq NT \leq \tau_2^* + 1 \quad (12)$$

Assumption A7 There exist two positive constants c_1, c_2 such that

$\|f(x,u)\| \leq c_1 + c_2 l_T(x,u)$, for all $x \in \mathbb{R}^n$ and $u \in U$.

Let

$$\Gamma_{\max} = \left\{x \in \mathbb{R}^n : V_{N,T}^a(x) \leq V_{\max}^a\right\}. \quad (13)$$

Clearly, $\Gamma \subset \Gamma_{\max}$.

Lemma 4 Suppose that Assumptions A1-A7 hold true. For any $r > 0$ let

$$K_r = 2\left(c_1/\varphi_1(r) + c_2\right) + 1$$

and let $R_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as

$$R_r(s) = \min\left\{\gamma_1\left(s/2\right), s/(4K_r)\right\}. \quad (14)$$

Then there exists a $T_2^* > 0$ such that

$$\|F_T^a(x,u) - x\| \leq K_r T l_T(x,u)$$

for all $x \in \mathcal{B}_{\Delta_r}$, $u \in \mathcal{U}_{\Delta_1}$, where $\Delta_r = R_r^{-1}\left(V_{\max}^a\right)$.

Proof. Because of lack of space, the proof is omitted.

Lemma 5. Suppose that Assumptions A1-A7 hold true. Then there exist a $T_3^* > 0$ and a class-$\mathcal{K}_\infty$ function ψ_2 such that

$$V_{N,T}^a(x) \leq \psi_2(\|x\|) \quad (15)$$

for all $x \in \Gamma_{\max}$ and all $T \in (0, T_3^*]$. Moreover, for any $r_0 > 0$ there exist a class-$\mathcal{K}_\infty$ function $\psi_1^{r_0}$ such that

$$V_{N,T}^a(x) \geq \psi_1^{r_0}(\|x\|) \quad (16)$$

for all $x \in \Gamma_{\max} \backslash \mathcal{B}_{r_0}$ and all $T \in (0, T_3^*]$.

Proof. Let $(r_0, \overline{T} = T^*$and $L_{r_0}^a$ be generated by Assumption A2) and let NT satisfy (12). Let us choose a positive r so that

$$0 < r \leq \min\left\{r_0, r_0/\left(4L_{r_0}^a(\tau_2^* + 1)\right)\right\}$$

and let $T_3^* = \min\left\{\overline{T}, T_1^*, T_2^*\right\}$ where T_2^* is generated by Lemma 4. Let $x_0 \in \Gamma_{\max}$ be arbitrary with $\|x_0\| \geq r_0$ and let the corresponding optimal trajectory be $\xi_k^* = \phi_k^a(x_0, \mathbf{u}^*(x_0))$. Then there are two possibilities:

a.) If $\|\xi_N^* - x_0\| \leq \frac{1}{2}\|x_0\|$, then $\|\xi_N^*\| \geq \frac{1}{2}\|x_0\|$, therefore

$$V_{N,T}^a(x_0) \geq g(\xi_N^*) \geq \gamma_1(\frac{1}{2}\|x_0\|)$$

b.) If $\|\xi_N^* - x_0\| > \frac{1}{2}\|x_0\|$, then we shall introduce the set of integers

$$\begin{aligned}\iota_1 := \{k : 0 \leq k \leq N-1,\\ \|\xi_k^*\| + \|u_k^*(x_0)\| > r\}\\ \iota_2 = \{0, 1, ..., N-1\} \backslash \iota_1.\end{aligned}$$

With this definition we have that

$$\begin{aligned}\frac{1}{2}\|x_0\| &\leq \|\xi_N^* - x_0\| \\ &\leq \sum_{k=0}^{N-1} \|F_T^a(\xi_k^*, u_k^*(x_0)) - \xi_k^*\| \\ &\leq \sum_{k \in \iota_1} T K_r l_T(\xi_k^*, u_k^*(x_0)) \\ &\quad + \sum_{k \in \iota_2} T L_{r_0}^a(\|\xi_k^*\| + \|u_k^*(x_0)\|) \\ &\leq K_r V_{N,T}^a(x_0) + rNTL_{r_0}^a \\ &\leq K_r V_{N,T}^a(x_0) + r(\tau_2^* + 1)L_{r_0}^a\end{aligned}$$

By the choice of r , $r(\tau_2^* + 1)L_{r_0}^a \leq \frac{1}{4}\|x_0\|$, thus $V_{N,T}^a(x_0) \geq \frac{1}{4K_r}\|x_0\|$.

Let $\psi_1^{r_0} : [0, \infty) \to \mathbb{R}_{\geq 0}$ be defined as $\psi_1^{r_0}(s) = R_r(s)$, then $\psi_1^{r_0} \in \mathcal{K}_\infty$ and (16) is valid.

On the other hand, we define

$$\begin{aligned}\mu(s) &= \frac{s^2}{2} + \max_{\|x\| \leq s} g(x) \\ \nu(s) &= \mu(\frac{\rho_1}{2}) + \frac{2}{\rho_1}(s - \frac{\rho_1}{2})V_{\max}^a\end{aligned}$$

and $\psi_2 : [0, \infty) \to \mathbb{R}_{\geq 0}$ by

$$\psi_2(s) = \begin{cases}\mu(s), & \text{if } 0 \leq s \leq \rho_1/2, \\ \max\{\mu(s), \nu(s)\}, & \text{if } s > \rho_1/2,\end{cases}$$

then $\psi_2 \in \mathcal{K}_\infty$ and (15) holds true.□

Corollary 1 Under the assumptions of Lemma 5 $\Gamma_{\max} \subset \mathcal{B}_\Delta$, where $\Delta = R_{\rho_1}^{-1}(V_{\max}^a)$ and $\rho_1 > 0$ is such that $B_{\rho_1} \subset \mathcal{G}_\eta$ and R_r given by (14)

Lemma 6 Suppose that Assumptions A1-A6 hold true. Then $V_{N,T}^a(.)$ is locally Lipschitz continuous in $\Gamma_{\max}$, uniformly in small T, i.e. there exist $L_v > 0$ and $\delta_v > 0$ such that for all $T \in (0, T_3^*]$ and $N \in \mathbb{N}$ with $TN \leq (\tau_2^* + 1)$, inequality

$$\left|V_{N,T}^a(x) - V_{N,T}^a(y)\right| \leq L_v\|x - y\| \quad (17)$$

holds true for all $x, y \in \Gamma_{\max}$ with $\|x - y\| \leq \delta_v$.

Proof. Because of lack of space, the proof is omitted.

We summarize the basic properties of $V_{N,T}^a$ in the following theorem.

Theorem 2 Suppose that Assumptions A1-A7 are valid. Then there exist such positive numbers

τ^* and T^* that for any $T \in (0, T^*]$ and $N \in \mathbb{N}$ with $\tau^* \leq NT \leq \tau^* + 1$

1.) there exists a function $\psi_2 \in \mathcal{K}_\infty$ such that $V^a_{N,T}(x) \leq \psi_2(\|x\|)$ for any $x \in \Gamma_{\max}$;

2.) for any $r_0 > 0$ there exists a function $\psi_1^{r_0} \in \mathcal{K}_\infty$ such that

$$\psi_1^{r_0}(\|x\|) \leq V^a_{N,T}(x), \qquad (18)$$

for any $x \in \Gamma_{\max} \backslash \mathcal{B}_{r_0}$. Moreover $V^a_{N,T}(0) = 0$ and $V^a_{N,T}(x) > 0$ for any $x \in \mathcal{B}_{r_0} \backslash \{0\}$;

3.) for any $x \in \Gamma_{\max}$,

$$V^a_{N,T}(F^a_T(x, v^a_T(x))) - V^a_{N,T}(x) \leq -T\varphi_1(\|x\|);$$

4.) $V^a_{N,T}(.)$ is locally Lipschitz continuous in $\Gamma_{\max}$ uniformly in small T.

Proof. The assertions of the theorem follow from Lemmas 1-6 by taking $T^* = T_3^*$ and $\tau^* = \tau_2^*$.□

Theorem 3 Suppose that Assumptions A1-A7 hold true. Then there exist such positive numbers τ^* and T^* that for any $T \in (0, T^*]$ and $N \in \mathbb{N}$ with $\tau^* \leq NT \leq \tau^* + 1$, the exact discrete-time model with the receding horizon controller

$$x_{t+1} = F^e_T(x_t, v^a_T(x_t)), \quad x_0 \in \Gamma \qquad (19)$$

is practically asymptotically stable about the origin.

Proof. The proof can follow the same line as that of Theorem 2 in (Nešić, *et al.*, 1999) with slight modifications. In fact, one has only to take care of the domain of validity of (18) and the local character of the Lipschitz continuity of $V^a_{N,T}$. Because of lack of space, the details are omitted.

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