STABILIZING RECEDING HORIZON CONTROL OF SAMPLED-DATA NONLINEAR SYSTEMS VIA THEIR APPROXIMATE DISCRETE-TIME MODELS

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Abstract: Results on stabilizing receding horizon control of sampled-data nonlinear systems via their approximate discrete-time models are presented. The proposed receding horizon control is based on the solution of Bolza-type optimal control problems for the parametrized family of approximate discrete-time models. The sampling period is considered to be fixed, and the discretization parameter is allowed to vary. Sufficient conditions are established which guarantee that the controller that renders the origin to be asymptotically stable for the approximate model also stabilizes the exact discrete-time model for sufficiently small discretization parameters.

Keywords: Predictive control, Feedback stabilization, Sampled-data systems, Optimal control, Numerical methods

1. INTRODUCTION

The stabilization problem of nonlinear systems has received considerable attention in the last decades. The use of computers in the implementation of the controllers necessitated the investigation of sampled-data systems. One way to design a sampled-data control is to implement a continuous-time algorithm with sufficiently small sampling intervals. This approach is proposed in connection with the receding horizon method among the others in (Chen, *et al.*, 2000; Fontes, 2001; Jadababaie and Hauser, 2001). However, some difficulty may arise during the application of this method: 1) the exact solution of the nonlinear continuous-time model is typically unknown, therefore an approximation procedure is unavoidable; 2) because of hardware constraints, it may be impossible to reduce the sampling period to a sufficiently small value that ensures the desired performance of the system; 3) it may be difficult to implement an arbitrarily time-varying control function. An overview and analysis of existing approaches for the stabilization of sampled data systems can be found in the recent papers (Nešić, *et al.*, 1999) and (Nešić and Teel) (see also the references therein). In these papers a systematic investigation of the connection between the exact and approximate models are carried out and conditions are presented which guarantee that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant both for the cases of fixed sampling period and varying integration parameter (Nešić and

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Teel) and for the case when these two parameters coincide (Nešić, *et al.*, 1999) and (Nešić and Teel).

There are several ways to design controllers satisfying the conditions given in (Nešić, *et al.*, 1999) and (Nešić and Teel). In (Grüne and Nešić 2002), optimization based methods are studied; the design is carried out either via an infinite horizon optimization problem or via an optimization problem over a finite horizon with varying length. Another possibility is the application of the widespread receding horizon or model predictive control method. This method obtains the feedback control by solving a finite horizon optimal control problem at each time instant using the current state of the plant as the initial state for the optimization and applying "the first part" of the optimal control. The study of stabilizing property of such schemes has been the subject of intensive research in recent years. From the vast literature we mention here but a few (Mayne and Michalska, 1990; Chen and Allgöwer, 1998; Gyurkovics, 1996; Gyurkovics, 1998; Jadababaie and Hauser, 2001; Fontes, 2001; Hu and Linnemann, 2002) and we refer the reader to the excellent overview papers (Allgöwer, *et al.*, 1999; Mayne, *et al.*, 2000) and to the references therein. A great majority of works deals either with continuous-time systems with or without taking into account any sampling or with discrete-time systems considering the model given directly in discrete-time. To the best of our knowledge, the only exception is the very recent work of (Ito and Kunisch, 2002), where the effect of the sampling and zero-order hold is considered assuming the sampling and discretization parameter to be equal. Relying on the results of (Nešić, *et al.*, 1999) and (Nešić and Teel), the present work studies the conditions under which the stabilizing receding horizon control computed for the approximate discrete-time model also stabilizes the exact discrete-time system.

The paper is organized as follows. In Section 2 the problem is formulated and some results from (Nešić, *et al.*, 1999) and (Nešić and Teel) are presented that are used to prove our main result described in Section 3. An illustrative example is given in Section 4.

2. PROBLEM STATEMENT AND BACKGROUNDS

Consider a nonlinear control system described by

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $f(0,0) = 0$, U is closed and $0 \in U$. The system is to be controlled digitally using piecewise constant control functions $u(t) = u(kT) =: u_k$, if $t \in [kT, (k+1)T)$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period. The function f is assumed to have enough regularity to guarantee the existence and uniqueness of solutions of (1) with $x(0) = x_0 \in \mathcal{X}_\infty$ and $u(t) \equiv \overline{u} \in U$ over the whole sampling interval $[0, T]$ at least for $T \leq T^*$. Let $t \in [0, T] \longmapsto \phi(t, x_0, \overline{u})$ denote this solution. Then, the exact discrete-time model of the system can be defined as

$$x_{k+1} = F_T^e(x_k, u_k), \qquad (2)$$

where $F_T^e(x, u) := \phi(T, x, u)$.

We emphasize that F_T^e in (2) is not known in most cases, therefore the controller design can be carried out by means of an approximate discrete-time model

$$x_{k+1} = F_{T,h}^a\left(x_k, u_k\right), \qquad (3)$$

where $T \in (0, T^*]$ is again the sampling parameter, while parameter h is used to obtain the required accuracy between the exact and the approximate models.

We write F if we refer for a general discrete-time parametrized system

$$x_{k+1} = F\left(x_k, u_k\right), \qquad (4)$$

where F may depend on some parameters. In particular, F may stand for both $F_{T,h}^a$ and F_T^e. In what follows, the notation $\mathcal{B}_\Delta = \{z \in \mathbb{R}^p : \|z\| \leq \Delta\}$ will be used both in $\mathbb{R}^n$ and $\mathbb{R}^m$ and $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ denote the usual class-$\mathcal{K}$, class-$\mathcal{K}_\infty$ and class-$\mathcal{KL}$ functions (see e.g. Nešić, *et al.*, 1999)

Assumption A1. We assume that both $F_{T,h}^a$ and F_T^e are continuous in x and u and satisfy a local Lipschitz condition of the following type: for each $\Delta > 0$ there exist $T^* > 0$, $L > 0$ and $h^* > 0$ such that

$$\|F(x,u) - F(y,u)\| \leq e^{LT}\|x - y\|,$$

holds for all $u \in \mathcal{B}_\Delta$, all $T \in (0, T^*]$, all $h \in (0, h^*]$ and all $x, y \in \mathcal{B}_\Delta$, where F may stand for both $F_{T,h}^a$ and F_T^e.

The problem is to define a state-feedback controller

$$v_{T,h}^a : \overline{\mathcal{X}} \subset \mathcal{X}_\infty \to U \qquad (5)$$

using the approximate model (3) which stabilizes the origin for the exact model (2) in an appropriate sense.

Some recent results of (Nešić, *et al.*, 1999) and (Nešić and Teel) give sufficient conditions for a controller designed by means of the approximate model to be stabilizing for the exact model, as well. Since these results of Nesic, Teel and Kokotović play a central role in our development, for the sake of completeness, we formulate some definitions and theorems given in (Nešić, *et al.*, 1999), (Nešić and Teel) and (Grüne and Nešić 2002).

Definition 1. Let strictly positive real numbers (T, Δ_1, Δ_2) be given. If there exists $h^* > 0$ such that

$$\sup_{\{x \in \mathcal{B}_{\Delta_1},\, h \in (0,h^*]\}} \|v^a_{T,h}\| \leq \Delta_2 \qquad (6)$$

then the family of controllers (5) is said to be (T, Δ_1, Δ_2)-uniformly bounded.

Definition 2. Let a triple of strictly positive numbers (T, Δ_1, Δ_2) be given and suppose that there exist $\gamma \in \mathcal{K}$ and $h^* > 0$ such that

$$(x,u) \in \mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}, \quad h \in (0,h^*] \Longrightarrow \\ \| F^a_{T,h}(x,u) - F^e_T(x,u) \| \leq T\gamma(h), \quad (7)$$

then we say that the family $F^a_{T,h}$ is (T, Δ_1, Δ_2)-consistent with F^e_T.

Definition 3. Let a pair of strictly positive real numbers (T, D), a family of functions $V_{T,h} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ and a positive definite function $\sigma_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given. Suppose that for any pair of strictly positive real numbers (δ_1, δ_2) with $\delta_2 < D$ there exist $h^* > 0$ and $c > 0$ such that for all $x \in \mathcal{B}_D$, $h \in (0, h^*]$, we have

$$\sigma_1(\|x\|) \leq V_{T,h}(x) \leq \sigma_2(\|x\|), \qquad (8)$$

$$V_{T,h}(F^a_{T,h}(x, v^a_{T,h}(x))) - V_{T,h}(x) \leq -T\sigma_3(\|x\|) \\ +T\delta_1, \qquad (9)$$

and, for all $x_1, x_2 \in \mathcal{B}_D - \mathcal{B}_{\delta_2}$, with $\|x_1 - x_2\| \leq c$ we have

$$|V_{T,h}(x_1) - V_{T,h}(x_2)| \leq \delta_1, \qquad (10)$$

then we say that the family (4), (5) is (T, D)-stable with a continuous Lyapunov function.

Theorem 1 (Grüne and Nešić, 2002; Nešić and Teel) Suppose that there exists a triple of strictly positive numbers (T, D, M) such that

(i) the family of the closed-loop systems $(F^a_{T,h}, v^a_{T,h})$ is (T, D)-stable with a continuous Lyapunov function;
(ii) the family of controllers $v^a_{T,h}$ is (T, D, M)-uniformly bounded;
(iii) the family $F^a_{T,h}$ is (T, D, M)-consistent with F^e_T.

Then, there exists $\beta \in \mathcal{KL}$, $D_1 \in (0, D)$ and for any $\delta > 0$ there exists $h^* > 0$ such that for all $x_0 \in \mathcal{B}_{D_1}$ and $h \in (0, h^*]$, the solutions of the family $(F^e_T, v^a_{T,h})$ satisfy:

$$\|\phi^e_T(k, x_0)\| \leq \beta(\|x_0\|, kT) + \delta, \;\; k \in \mathbb{N}_0. \, (11)$$

The question is, how to find an approximate model, a family of controllers and a suitable Lyapunov function so that conditions (i), (ii) and (iii) of Theorem 1 be satisfied. In (Grüne and Nešić, 2002), an optimization based approach is proposed considering both infinite horizon problems and problems over finite interval with varying length. To relax the computational burden needed in the case of infinite horizon optimization and in the case of optimization over a varying time interval, the application of the receding horizon method offers good vistas. In (Grüne and Nešić 2002), it was pointed out that the results presented in that paper was not directly applicable for RHC. In what follows we briefly outline a version of the receding horizon control method which can be considered as the discrete time counterpart of the method investigated in (Gyurkovics, 1998) see also (Chen and Allgöwer, 1998; Jadababaie and Hauser, 2001; De Nicolao, *et al.*, 1998; Parisini *et al.*, 1998; Ito and Kunisch, 2002; Hu and Linnemann 2002), the review papers (Allgöwer, *et al.*, 1999; Mayne, *et al.*, 2000) and the references therein.

In order to define a feedback controller, let $\mathcal{X}^e_f$, $\mathcal{X}^a_f$ denote given compact subsets of $\mathbb{R}^n$ containing the origin and let equations (2), (3) be subject to the cost functions

$$J^e_T(N, x, \mathbf{u}) = \sum_{k=0}^{N-1} Tl(x^e_T(k), u(k)) + \\ g(x^e_T(N)), \qquad (12)$$

$$J^a_{T,h}(N, x, \mathbf{u}) = \sum_{k=0}^{N-1} Tl_h(x^a_{T,h}(k), u(k)) + \\ g(x^a_{T,h}(N)), \qquad (13)$$

respectively, under the constraints

$$u(k) \in U, \;\; x^e_T(N) \in \mathcal{X}^e_f, \;\; x^a_{T,h}(N) \in \mathcal{X}^a_f. \, (14)$$

Here $x^e_T(.) = \phi^e_T(.; x, \mathbf{u})$ and $x^a_{T,h}(.) = \phi^a_{T,h}(.; x, \mathbf{u})$ denote the solution of (2) and (3), respectively, with the control function $\mathbf{u} = \{u_0, u_1, ..., u_{N-1}\}$ and satisfying the initial conditions $x^e_T(0) = x$ and $x^a_{T,h}(0) = x$. The final state constraint given in (14) is implicit in $\mathbf{u}$, since both x^e_T and $x^a_{T,h}$ depend on $\mathbf{u}$. A control sequence $\mathbf{u}$ is said to be admissible at the initial point x with respect to (2), (3), respectively, if the constraints (14) are satisfied. The sets of admissible controllers are denoted by $\mathcal{U}^e_T(x)$ and $\mathcal{U}^a_{T,h}(x)$. Consider the optimization problem

$$P^a_{T,h}(N, x) : \min_{\mathbf{u} \in \mathcal{U}^a_{T,h}(x)} J^a_{T,h}(N, x, \mathbf{u}).$$

If this optimization problem has a solution denoted by $\mathbf{u}^{a,*} = \{u_0^{a,*}, u_1^{a,*}, ..., u_{N-1}^{a,*}\}$, then the first element of $\mathbf{u}^{a,*}$, i.e. $u_0^{a,*}$ is applied at the state x. Since the optimal solution of $P_{T,h}^a(N,x)$ naturally depends on x, in this way a feedback has been defined on the basis of the approximate discrete-time model, i.e. $v_{T,h}^a(x) := u_0^{a,*}$.

Conditions, under which $v_{T,h}^a$ asymptotically stabilizes the origin for the approximate model (3), are well-established. In order to ensure the stabilizing property of $v_{T,h}^a$ for the exact model (2), one needs somewhat stronger conditions: in fact, it has to be shown that the estimations in (8)-(10) and the boundedness of $v_{T,h}^a$ are uniform in h.

3. STABILIZATION WITH RECEDING HORIZON CONTROL

The stabilizing property of the receding horizon control $v_{T,h}^a$ will be investigated under the following assumptions.

Assumption A2. The running and the terminal cost functions satisfy the following:

(i) l and l_h are continuous with respect to x and u, uniformly in small h in the case of l_h.

(ii) There exist $h^* > 0$ and two class-$\mathcal{K}_\infty$ functions φ_1 and φ_2 such that the inequality

$$\varphi_1(\|x\|) + \varphi_1(\|u\|) \leq l(x,u),\, l_h(x,u) \leq \varphi_2(\|x\|) + \varphi_2(\|u\|),$$

holds for all x, u and $h \in (0, h^*]$.

(iii) g is continuous, strictly positive definite radially unbounded function.

Assumption A3. There exist an $\eta > 0$ and $h^* > 0$ such that for all $x \in \mathcal{G}_\eta = \{x : g(x) \leq \eta\}$ there exists a $k(x) \in U$ such that inequality

$$Tl_h(x, k(x)) + g\left(F_{T,h}^a(x, k(x))\right) \leq g(x) \quad (15)$$

holds true for all $h \in (0, h^*]$.

Definition 4.(Practical asymptotic controllability of (2)) The exact discrete-time model (2) is said to be practically asymptotically controllable from $\mathcal{B}_\Delta \cap \mathcal{X}_\infty$ to the origin, if there is a $\beta \in \mathcal{KL}$ such that for every $x \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty$ and for every $r_0 > 0$ there exists a control sequence $\mathbf{u} = \mathbf{u}^{r_0}(x)$ such that

$$\|\phi_T^e(t; x, \mathbf{u}^{r_0}(x))\| + \|u_t^{r_0}(x)\| \leq \max\{\beta(\|x\|, t), r_0\}, t \in \mathbb{N}.$$

Assumption A4. Let $\Delta > 0$, $0 < T \leq T^*$ be fixed. The exact discrete-time model is practically asymptotically controllable from $\mathcal{B}_\Delta \cap \mathcal{X}_\infty$ to the origin.

Let $r_0 > 0$ be such that $\overline{\mathcal{B}}_{r_0} \subset \mathcal{G}_{\eta/2} = \{x : g(x) \leq \eta/2\}$. Since $\beta \in \mathcal{KL}$, there is a $\tau^* \in \mathbb{N}$ such that $\beta(\Delta, \tau) \leq r_0$, if $\tau \geq \tau^*$, therefore, under the Assumption A4 $\|\phi_T^e(k; x, \mathbf{u}^{r_0}(x))\| + \|u_k^{r_0}(x)\| \leq \max\{\beta(\Delta, 0), r_0\} =: \Delta_1$, $k = 0, ..., N-1$, $\phi_T^e(N; x, \mathbf{u}^{r_0}(x)) \in \mathcal{G}_{\eta/2}$, $N \geq \tau^*$. In what follows, N is assumed to be fixed satisfying the inequality $N \geq \tau^*$, and $\mathcal{X}_f^e = \mathcal{G}_{\eta/2}$.

Lemma 1. If assumptions A1, A2 and A4 hold true, then

$$V_N^e(x_0) := \inf\{J_T^e(N, x_0, \mathbf{u}) : \mathbf{u} \in \mathcal{U}_T^e(x_0)\}$$

is lower-semicontinuous, bounded, the optimal control $\mathbf{u}^* \in \mathcal{U}_T^e(x_0)$ exists and it is bounded, (uniformly in x_0) for all $x_0 \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty$.

Proof. Under the assumptions of the lemma, the lower-semicontinuity of V_N^e is known (see e.g. Vasil'ev, 1988). Let $x_0 \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty$ be arbitrary and let $\mathbf{u}(x_0) = \left\{u_0^{r_0}(x_0), ..., u_{N-1}^{r_0}(x_0)\right\}$ be the fore-part of the sequence $\mathbf{u}^{r_0}(x)$ produced by Assumption A4. Then, by Assumption A2,

$$V_N^e(x_0) \leq 2NT\varphi_2(\Delta_1) + \eta/2 =: V_{Max}^e.$$

On the other hand, for the optimal trajectory-control pair $(\phi_T^{e,*}, \mathbf{u}^*)$, we have $T\left(\varphi_1\left(\left\|\phi_T^{e,*}(k, x_0, \mathbf{u}^*)\right\|\right) + \varphi_1\left(\|u_k^*\|\right)\right) \leq V_{Max}^e$, therefore $\phi_T^{e,*}(k, x_0, \mathbf{u}^*) \in \overline{\mathcal{B}}_{K^e}$, $u^*(k) \in \overline{\mathcal{B}}_{K^e}$ where $K^e := \varphi_1^{-1}\left(V_{Max}^e/T\right)$.□

Assume that F_T^e and $F_{T,h}^a$ are (T, K^e+1, K^e+1)-consistent, and let h_1^* and $\gamma(.)$ are produced by Definition 2. As an immediate consequence of the (T, K^e+1, K^e+1)-consistency, we get that

$$\left\|\phi_{T,h}^a(k, x_0, \mathbf{u}^*) - \phi_T^e(k, x_0, \mathbf{u}^*)\right\| \leq T\gamma(h)\frac{e^{(N+1)LT} - 1}{e^{LT} - 1}, \quad (16)$$

$k = 1, ..., N$ for all $x_0 \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty$. From (16) and from (iii) of Assumption A2 it follows that there exist $0 < \varepsilon_0 \leq 1$ and $0 < h_2^* \leq h_1^*$ such that $\mathcal{G}_{\eta/2} + \mathcal{B}_{\varepsilon_0} \subset \mathcal{G}_\eta$,

$$\left\|\phi_{T,h}^a(k, x_0, \mathbf{u}^*) - \phi_T^e(k, x_0, \mathbf{u}^*)\right\| \leq \varepsilon_0, \quad (17)$$

$$g\left(\phi_{T,h}^a(N, x_0, \mathbf{u}^*)\right) \leq \frac{3\eta}{4} < \eta \quad (18)$$

$k = 1, ..., N$, $h \in (0, h_2^*]$. Thus $\mathcal{U}_{T,h}^a(x_0) \neq \emptyset$ if $x_0 \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty$ and $h \in (0, h_2^*]$.

Lemma 2. Let $\mathcal{X}_f^a = \mathcal{G}_\eta$. If Assumptions A1, A2 and A4 hold true and h_2^* is defined above so that (17) and (18) are satisfied, then function $V_N^a : \mathcal{B}_\Delta \cap \mathcal{X}_\infty \to \mathbb{R}_{\geq 0}$, $V_N^a(x_0) := \inf\left\{J_{T,h}^a(N, x_0, \mathbf{u}) : \mathbf{u} \in \mathcal{U}_{T,h}^a(x_0)\right\}$ is lower semicontinuous, uniformly bounded, the optimal control $\mathbf{u}^{a,*}(x_0) \in \mathcal{U}_{T,h}^a(x_0)$ exists and (T, Δ, M)-uniformly bounded for all $h \in (0, h_2^*]$.

Proof. Because of the choice of h_2^*, $\mathcal{U}_{T,h}^a(x) \neq \emptyset$ for all $h \in (0, h_2^*]$, therefore the existence of optimal control and the lower semicontinuity of V_N^a is well known under the assumptions of the lemma (see e.g. Vasil'ev). From (ii) of Assumption A2 it follows that

$$V_N^a(x_0) \leq NT\left(\varphi_2(K^e+1) + \varphi_2(K^e)\right) + \eta \\ =: V_{Max}^a$$

where V_{Max}^a is independent of h and x_0. On the other hand,

$$V_N^a(x_0) \geq Tl_h(\phi_{T,h}^a(k, x_0, \mathbf{u}^{a,*}(x_0)), u_k^{a,*}(x_0))$$

therefore $\left\|\phi_{T,h}^a(k, x_0, \mathbf{u}^{a,*}(x_0))\right\| \leq K^a$, $\|u_k^{a,*}(x_0)\| \leq K^a$, where $K^a := \varphi_1^{-1}\left(V_{Max}^a/T\right)$ is also independent of h and x_0, thus $M = K^a$ can be taken. □

Lemma 3. Suppose that Assumption A1-A4 hold. Then there exist two class $\mathcal{K}_\infty$ functions σ_1, σ_2, a positive definite function $\sigma_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a $h^* > 0$ such that for all $x \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty$, $h \in (0, h^*]$ inequalities (8) and (9) are satisfied.

Proof. The lower estimation in (8) immediately follows from (ii) of Assumption A2 by taking $\sigma_1 = T\varphi_1$. It can be proven in a standard way that $V_N^a(F_{T,h}^a(x, v_{T,h}^a(x))) - V_N^a(x) \leq -Tl_h(x, v_{T,h}^a(x))$, therefore σ_3 in (9) can be chosen as $\sigma_3 = \varphi_1$. Also under the Assumption A3 it can be shown that $V_N^a(x) \leq V_{N-1}^a(x) \leq ... \leq V_0^a(x) = g(x)$, if $x \in \mathcal{G}_\eta$. Let $h^* = h_2^*$, where h_2^* is produced by lemma 2, and let ρ_1 be such that $\mathcal{B}_{\rho_1} \subset \mathcal{G}_\eta$. We define a function ζ:$\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as $\zeta(r) = \max_{\|x\| \leq r} g(x) + r^2/2$ and σ_2 as $\sigma_2(r) = \zeta(r)$ if $0 \leq r \leq \frac{\rho_1}{2}$, and $\sigma_2(r) = \max\{\zeta(r), \zeta(\frac{\rho_1}{2}) + \frac{2}{\rho_1}V_{Max}^a\left(r - \frac{\rho_1}{2}\right)\}$ if $r \geq \frac{\rho_1}{2}$. Obviously, $\sigma_2 \in \mathcal{K}_\infty$, $g(x) \leq \sigma_2(\|x\|)$ and $\sigma_2(\|x\|) \geq V_{Max}^a$, if $x \notin \mathcal{G}_\eta$, therefore σ_2 gives the upper bound estimation in (8).□

For the application of Theorem 1, one has to show that V_N^a is uniformly continuous (both in x_0 and h). To this end we introduce a subset $\mathcal{X}_N^{\varepsilon_1} \subset \mathcal{B}_\Delta \cap \mathcal{X}_\infty$ in the following way: for r_0 we know that $\mathcal{B}_{r_0} \subset \mathcal{G}_{\eta/2}$ and let $\varepsilon_1 > 0$ be chosen so that $\varepsilon_1 < \min\{\eta/2, T\varphi_1(r_0)\}$. Let $x_1 = \phi_{T,h}^a(1; x_0, \mathbf{u}^{a,*}(x_0))$ and let $\mathbf{u}^{a,*}(x_1)$ denote the optimal controller for the initial state x_1. Then

$$\mathcal{X}_N^{\varepsilon_1} = \{x_0 \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty : \\ g(\phi_{T,h}^a(N, x_0, \mathbf{u}^{a,*}(x_0))) \leq \eta - \varepsilon_1, \quad (19) \\ g(\phi_{T,h}^a(N, x_1, \mathbf{u}^{a,*}(x_1))) \leq \eta - \varepsilon_1\}.$$

Lemma 4 Suppose that Assumptions A1-A4 hold. Then function V_N^a is uniformly continuous in $\mathcal{X}_N^{\varepsilon_1}$, $\mathcal{X}_N^{\varepsilon_1} \supset \mathcal{G}_\eta$ and $\mathcal{X}_N^{\varepsilon_1}$ is invariant with respect to the receding horizon controller $v_{T,h}^a$.

Proof. Let $x_0 \in \mathcal{X}_N^{\varepsilon_1}$ and $\varepsilon > 0$ be arbitrary and let $\mathbf{u}^{a,*}(x_0)$ denote the optimal controller for x_0. Let $x_0' \in \mathcal{B}_\Delta \cap \mathcal{X}_\infty$. Because of the uniform continuity of the composite function on $\mathcal{B}_\Delta \cap \mathcal{X}_\infty$ there exists a $\delta_1 = \delta_1(\varepsilon_1)$ such that $g(\phi_{T,h}^a(N, x_0', \mathbf{u}^{a,*}(x_0))) \leq \eta$, thus $\mathbf{u}^{a,*}(x_0)$ is admissible with respect to x_0'. Therefore $V_N^a(x_0') - V_N^a(x_0) \leq \sum_{k=0}^{N-1} T[l_h(\phi_{T,h}^a(k, x_0', \mathbf{u}^{a,*}(x_0)), u_k^{a,*}) - l_h(\phi_{T,h}^a(k, x_0, \mathbf{u}^{a,*}(x_0)), u_k^{a,*})] + g(\phi_{T,h}^a(N, x_0', \mathbf{u}^{a,*}(x_0))) - g(\phi_{T,h}^a(N, x_0, \mathbf{u}^{a,*}(x_0)))$. Using again the uniform continuity of the functions $F_{T,h}^a$, l_h and g, we can state that for a sufficiently small $\delta > 0$ we have $V_N^a(x_0') - V_N^a(x_0) < \varepsilon$, if $\|x_0 - x_0'\| < \delta$ with $\delta \leq \delta_1$. This shows the upper semi-continuity of V_N^a on $\mathcal{X}_N^{\varepsilon_1}$. If $x_0' \in \mathcal{X}_N^{\varepsilon_1}$, then the role of x_0 and x_0' can be changed to obtain the inequality $V_N^a(x_0) - V_N^a(x_0') < \varepsilon$. Thus the uniform continuity of V_N^a on $\mathcal{X}_N^{\varepsilon_1}$ is proven. To show that $\mathcal{G}_\eta \subset \mathcal{X}_N^{\varepsilon_1}$ we remember that, under the Assumption A3, $V_N^a(x_0) \leq g(x_0)$ if $x_0 \in \mathcal{G}_\eta$. Let $x_0 \in \mathcal{G}_\eta$. Then $Tl_h(x_0, u_0^{a,*}(x_0)) + g(\phi_{T,h}^a(N, x_0, \mathbf{u}^{a,*}(x_0))) \leq V_N^a(x_0)$,therefore

$$g(\phi_{T,h}^a(N, x_0, \mathbf{u}^{a,*}(x_0))) \leq g(x_0) - \\ Tl_h(x_0, u_0^{a,*}(x_0)) \leq g(x_0) - T\varphi_1(\|x_0\|). \quad (20)$$

If $\|x_0\| < r_0$, then $g(x_0) \leq \eta/2$ and from (20) it follows that $\phi_{T,h}^a(N, x_0, \mathbf{u}^{a,*}(x_0)) \in \mathcal{G}_{\eta/2} \subset \mathcal{G}_{\eta-\varepsilon_1}$. On the other hand, if $\|x_0\| \geq r_0$, then it follows that again from (20) that

$$g(\phi_{T,h}^a(N, x_0, \mathbf{u}^{a,*}(x_0))) \leq g(x_0) - T\varphi_1(r_0) \\ \leq \eta - \varepsilon_1.$$

The invariance of $\mathcal{X}_N^{\varepsilon_1}$ with respect to $v_{T,h}^a$ is included in the definition of $\mathcal{X}_N^{\varepsilon_1}$.□

Remark. (Hu and Linnemann, 2002) investigated a similar version of the receding horizon control for a fixed discrete-time model. The main idea in (Hu and Linnemann, 2002) is to define the receding horizon control by solving an analogous optimal control problem without an explicit terminal constraint set. The stabilizing property of the proposed control has been proved in the set $\mathcal{X}_N^{(H-L)}$ which consists of such initial states that the corresponding optimal trajectory terminates in a suitable level set of the terminal cost function, moreover, the set $\mathcal{X}_N^{(H-L)}$ approaches the set of all controllable states as N increases. It can easily be shown that, if the terminal set $\mathcal{X}_f^{(H-L)}$ of Hu and Linnemann is chosen equal to $\mathcal{G}_{\eta-\varepsilon_1}$ then $\mathcal{X}_N^{(H-L)} \subset \mathcal{X}_N^{\varepsilon_1}$, thus $\mathcal{X}_N^{\varepsilon_1}$ is "not too small" if N is sufficiently large.

Theorem 2 Suppose that Assumptions A1-A4 hold true. Let $\Delta^* = \max\{K^e+1, K^a\}$. If the family $F_{T,h}^a$ is (T, Δ^*, Δ^*)-consistent with F_T^e, then there exists $\beta \in \mathcal{KL}$, $D_1 \in (0, \Delta)$ and for any $\delta > 0$ there exists $h^* > 0$ such that for all $x_0 \in \mathcal{B}_{D_1} \cap \mathcal{X}_N^{\varepsilon_1}$ and $h \in (0, h^*]$, the solutions of the family $(F^e_T, v^a_{T,h})$ satisfy: $\|\phi^e_T(k,x_0)\| \leq \beta(\|x_0\|, kT) + \delta, \quad k \in \mathbb{N}_0$.

Proof. The statement of the theorem is an immediate consequence of Theorem 1 and Lemma 1 - Lemma 4.□

4. ILLUSTRATIVE EXAMPLE

Consider continuous-time system (Mayne and Michalska, 1990)

$$\dot{x}_1 = -x_2 + (\mu + (1-\mu)x_1)u,$$
$$\dot{x}_2 = x_1 + (\mu - 4(1-\mu)x_2)u.$$

Let the approximate discrete-time model is defined by Euler method as follows: let $\overline{x} = x^a(k), \quad \overline{u} = u(k)$. With $x_0 = \overline{x}$ let

$$x_{1,k+1} = x_{1,k} + h\left[-x_{2,k} + (\mu + (1-\mu)x_{1,k})\overline{u}\right],$$
$$x_{2,k+1} = x_{2,k} + h\left[x_{1,k} + (\mu - 4(1-\mu)x_{2,k})\overline{u}\right],$$

$k = 0, 1, ..., m-1$, and let $x^a(k+1) = x_m$, where $h = T/m$. The running and the terminal costs are given by $l_h(x,u) = \frac{1}{2}\|x\|^4 + u^2, \quad g(x) = x^{'}Px$, where P is given below. Simulations for the continuous-time system were carried out using ODE45 program in MATLAB when $T = 0.1$, $m = 20$, $\mu = 0.5$ and

$$P = \begin{pmatrix} 2.8438 & -0.1589 \\ -0.1589 & 4.1541 \end{pmatrix}$$

The trajectory of the continuous-time system is shown in Figure 1.

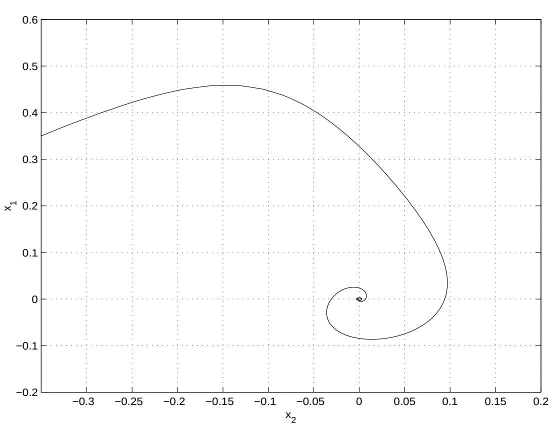

Fig. 1

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