Receding horizon H_∞ control for nonlinear discrete-time systems

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Abstract: The H_∞ control problem for discrete-time nonlinear systems with bounded controllers is considered. Applying the receding horizon method for a game problem, a state feedback control law is proposed which imposes a prescribed level of disturbance attenuation with internal stability in a larger domain of the state space than by other methods existing in the literature. Under some further conditions a global solution for the problem is also given. The proposed method cannot only be applied to cases amenable to solution by other methods but also to cases for which the linearisation technique does not work. The application of the method is illustrated by an example.

1 Introduction

The study of the H_∞ control problem for continuous-time nonlinear systems has attracted considerable attention in the last decade and many new results in this area have been published. A majority of these works is based on the combination of game theory and the theory of dissipative systems. Somewhat less attention has been focused on the problem of local disturbance attenuation with internal stability for discrete-time nonlinear systems although fundamental results have also been achieved in this field. ([1–3] and the references therein). One important feature of these results is their local character: the existence of a (possibly very small) neighbourhood of origin is stated where the H_∞ control problem is solved. In [3], it is pointed out that future research on nonlinear H_∞ control should focus either on the case where a global issue is concerned or the case where the analysis of the nonlinear H_∞ control cannot rely solely on 'first-approximation arguments'. This paper intends to make a step in this direction. Another feature of the above mentioned results is that no control constraints are taken into account. A wide-spread and well established method for the stabilisation of nonlinear systems capable of dealing with constraints is the receding horizon control (surveys of recent results are given in [4–6] see also [7, 8] and the references therein). Recently, several papers have been devoted to the investigation of robust and/or H_∞ receding horizon controllers: [6] gives an excellent survey of selected results in the field. Here we particularly want to mention the following papers: [9] which deals with linear, time-varying, unconstrained, continuous-time systems, while [10] is devoted to the design of a H_∞ controller for linear discrete-time systems with control constraints. A method based on the receding horizon approach is proposed in [7] for the stabilisation of an uncertain continuous-time nonlinear system. In [11] and in [12] H_∞ receding horizon algorithms are proposed for continuous-time, nonlinear systems. Whilst [13] and [14] investigate the robustness of the method for discrete-time nonlinear systems. A common feature of these latter papers is the assumption that the size of the perturbation can be estimated by a constant multiplier of the size of the penalty output with a small enough constant. [15] and the present paper presents a generalisation of the discrete-time results mentioned above.

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2 Receding horizon control for a discrete-time nonlinear zero-sum game for two players

2.1 Problem statement and some properties of the value function

Consider a nonlinear discrete-time system described by:

$$x(t+1) = f(x(t), u(t), w(t)), \quad t \in \boldsymbol{Z}$$
$$x(t_0) = x_0 \tag{1}$$

where $x(t) \in \boldsymbol{R}^n$ is the state, $u(t) \in \boldsymbol{R}^{m_1}$ and $w(t) \in \boldsymbol{R}^{m_2}$ are the input variables of the two players, f: $\boldsymbol{R}^n \times \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2} \to \boldsymbol{R}^n$ is a sufficiently smooth function, and $f(0, 0, 0) = 0$. The input variables are constrained to vary in $U \subset \boldsymbol{R}^{m_1}$ and $W \subset \boldsymbol{R}^{m_2}$, respectively, where U, W are closed and contain the origin in their interior.

Let, the system described by (1) be subject to the cost function:

$$J(t_0, t_1, x_0, u(\cdot), w(\cdot)) = \sum_{t=t_0}^{t_1-1} L(\xi(t), u(t), w(t)) + g(\xi(t_1)) \tag{2}$$

where $\xi(\cdot)$ denotes the solution of (1) due to inputs $u(\cdot)$ and $w(\cdot)$ and satisfying the initial condition $\xi(t_0) = x_0$. This cost function is to be minimised by the first player with input variable u and maximised by the second player with input variable w.

For the functions L: $\boldsymbol{R}^n \times \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2} \to \boldsymbol{R}$, g: $\boldsymbol{R}^n \to \boldsymbol{R}$ the following assumption is required to be held.

Assumption 1:

(a) There exists a function Φ: $[0, \infty) \rightarrow [0, \infty)$ such that $L(x, u, 0) \geq \Phi(\|u\|) > 0$, if $\|u\| \neq 0$ and $\Phi(0) = 0$.
(b) The function $x \rightarrow L(x, 0, 0)$ can be written as $L(x, 0, 0) = Q(h_1(x))$, where $Q(s) > 0$ if $s \neq 0$, $Q(0) = 0$, $h_1(0) = 0$, $h_1(x) \neq 0$ for all nonzero x satisfying $f(x, 0, 0) = x$ and the system:

$$x(t+1) = f(x(t), 0, 0)$$
$$y(t) = h_1(x(t)) \quad t \geq t_0$$

is zero-state detectable, i.e. from $h_1(x(t)) = 0$ for $\forall t = t_0, t_0 + 1, \ldots$, it follows that

$$\lim_{t \rightarrow \infty} x(t) = 0$$

(c) The function $w \rightarrow L(0, 0, w)$ is negative definite.
(d) The function g is positive definite ($g(0) = 0$, $g(x) > 0$, $x \neq 0$).
(e) Both L and g are sufficiently smooth.

Let $\eta > 0$, and let $\mathcal{G}_\eta \subset \boldsymbol{R}^n$ be defined as $\mathcal{G}_\eta = \{x \in \boldsymbol{R}^n$: $g(x) \leq \eta\}$.

Let $y_{a|b}$ denote a function y: $t \in [a, b] \cap \boldsymbol{Z} \rightarrow y(t)$.

Definition 1: The input pair $(u_{t_0|t_1-1}, w_{t_0|t_1-1})$ is said to be admissible on $[t_0, t_1]$ for $x_0 \in \boldsymbol{R}^n$, if: (i) $u(t) \in U$, $w(t) \in W$ for $t = t_0, \ldots, t_1 - 1$; and (ii) the trajectory $\xi_{t_0|t_1}$ corresponding to $(u_{t_0|t_1-1}, w_{t_0|t_1-1})$ initiated at $\xi(t_0) = x_0$ satisfies the end point condition $\xi(t_1) \in \mathcal{G}_\eta$. Furthermore, the corresponding trajectory $\xi_{t_0|t_1}$ is said to be an admissable trajectory, and the triplet $(\xi_{t_0|t_1}, u_{t_0|t_1-1}, w_{t_0|t_1-1})$ is said to be an admissible process.

Remark 1: The definition of admissible processes is not quite usual in game problems because of the presence of the terminal constraint set $\mathcal{G}_\eta$, although the possibility of such a formulation of the problem is mentioned in [16], and a similar set plays an important role in [6, 11, 13, 14]. In the latter papers, admissible inputs for the two players are defined separately but, inevitably, referring to each other. Therefore it seems to be useful to make this mutual dependence clear in the definition. It remains a very important question if the sets of admissible inputs are sufficiently ample: it may be required e.g. that there exist input functions of the first player such that they form admissible input pairs with any input function of the second player with certain property. In Section 3 we shall apply the results of the present Section for H_∞ control problems, where such considerations are crucial.

The game problem is the following: for any admissible process the cost function (2) is to be minimised by the first player (the control input) and maximised by the second player (the disturbance input), respectively.

Remark 2: In view of this problem formulation, the conditions (a), (c)–(e) in assumption 1 seem to be quite natural. Condition (b) is required to ensure the applicability of LaSalle's invariance principle; a similar condition appears in most of the papers dealing with receding horizon stabilisation or with H_∞-problems (see e.g. [1–8, 11–14]).

Let $\Omega_T \subset \boldsymbol{R}^n$ denote the set of all initial points x_0, for which there exits an admissible process on $[0, T]$. (By definition, let $\Omega_0 = \mathcal{G}_\eta$.)

Definition 2: An admissible input $(u^*_{0|T-1}, w^*_{0|T-1})$ is said to be optimal if:

$$J(0, T, x_0, u^*_{0|T-1}, w_{0|T-1}) \leq J(0, T, x_0, u^*_{0|T-1}, w^*_{0|T-1}) \leq J(0, T, x_0, u_{0|T-1}, w^*_{0|T-1})$$

for any admissible $(u^*_{0|T-1}, w_{0|T-1})$ and $(u_{0|T-1}, w^*_{0|T-1})$

$$\hat{V}_T(x_0) = J(0, T, x_0, u^*_{0|T-1}, w^*_{0|T-1})$$

is called the value of the game

Assumption 2: For any $T \geq 0$ and for any $x_0 \in \Omega_T$, there exists an optimal input, $\hat{V}_T(\cdot)$ is continuous, and $(u^*_{0|T-1}, w^0_{0|T-1})$ is admissible with $w^0(t) = 0$, $t = 0, \ldots, T-1$.

(Note that the admissibility of the identically zero disturbance is usually considered self-evident and it is used without any extra postulate.)

Lemma 1: If assumptions 1 and 2 hold, then for any $x_0 \in \Omega_T$, $\hat{V}_T(x_0) \geq 0$ and $\hat{V}_T(x_0) > 0$, if $x_0 \neq 0$.

Proof: According to assumption 2, $\hat{V}_T(x_0)$ is well defined for $x_0 \in \Omega_T$ and $(\xi^0_{0|T}, u^*_{0|T-1}, w^0_{0|T-1}$ is admissible. By the definition of optimality, we have:

$$\hat{V}_T(x_0) \geq J(0, T, x_0, u^*_{0|T-1}, w^0_{0|T-1}) = g(\xi^0(T)) + \sum_{t=0}^{T-1} L(\xi^0(t), u^*(t), 0)$$

Now, from (a) and (d) of assumption 1 it is clear that $\hat{V}_T(x_0) \geq 0$, and $\hat{V}_T(x_0) = 0$ only if $\xi^0(T) = 0$, $u^*(t) = 0$, and $L(\xi^0(t), 0, 0) = 0$, $t = 0, \ldots, T-1$. Then, from assumption 1(b) it follows that $\xi^0(t) = 0$ for each $t = 0, \ldots, T-1$, i.e. $x_0 = 0$. □

Let $U(x) \subseteq U$, $W(x) \subseteq W$ be such that for any $x \in \mathcal{G}_\eta$, $f(x, u, w) \in \mathcal{G}_\eta$ for all $u \in U(x)$, $w \in W(x)$.

Assumption 3: For all $x \in \mathcal{G}_\eta$, $U(x) \neq \emptyset$, $W(x) \neq \emptyset$.

From the assumption that $U(x) \neq \emptyset$ and $W(x) \neq (\emptyset)$, if $x \in \mathcal{G}_\eta$, it follows immediately that $\Omega_T \subseteq \Omega_{T+1}$ for any $T \geq 0$. In fact, if $x_0 \in \Omega_T$ and $(\bar{\xi}_{0|T}, \bar{u}_{0|T-1}, \bar{w}_{0|T-1})$ is an admissible process, then concatenating it with $(f(\bar{\xi}(T), u, w), u, w)$, where $u \in U(\bar{\xi}(T))$, $w \in W(\bar{\xi}(T))$, we obtain an admissible process on $[0, T+1]$, thus $x_0 \in \Omega_{T+1}$.

Assumption 4: For all $x \in \mathcal{G}_\eta$:

$$g(x) \geq \min_{u \in U(x)} \max_{w \in W(x)} \{L(x, u, w) + g(f(x, u, w))\}.$$

Remark 3: Assumption 4 is a straightforward generalisation of the discrete-time counterpart of assumption A3 of [8]. We shall show later that this assumption holds true under the conditions discussed in the literature when the H_∞ control problem is locally solvable.

Assumption 5: If $x_0 \in \Omega_T$, and $(\tilde{u}^*_{0|T}, \tilde{w}^*_{0|T})$ and $(u^*_{0|T-1}, w^*_{0|T-1})$ are optimal inputs on $[0, T+1]$ and $[0, T]$, respectively, then $(u^*_{0|T-1}, \bar{w}_{0|T-1})$ is admissible on $[0, T]$, where $\bar{w}_{0|T-1}$ is the restriction of $\tilde{w}^*_{0|T}$ to $[0, T-1] \cap \boldsymbol{Z}$. Moreover, $\tilde{w}^*(T) \in W(\bar{\xi}(T))$, where $\bar{\xi}_{0|T}$ is the trajectory of (1) corresponding to $(u^*_{0|T-1}, \bar{w}_{0|T-1})$.

Remark 4: Assumption 5 expresses the certain dominance of the first player over the second player: the assumption requires, roughly speaking, the restriction of the optimal input of the maximising player on a longer time interval to a shorter one to be admissible with the optimal input of the minimising player on the shorter time interval, and the possibility that the system can be kept in the terminal constraint set if the last element of this maximising input is in effect. This assumption is not very restrictive, but it plays a crucial role in the proof of the monotonicity property of the value of the game, which is formulated in the following theorem.

Theorem 1: Under assumptions 2, 3 and 5 the inequality:

$$\hat{V}_{T+1}(x_0) \leq \hat{V}_T(x_0) \quad \text{for all } T \geq 0 \text{ and for all } x_0 \in \Omega_T \tag{3}$$

holds true if and only if assumption 4 is valid.

Proof: (a) *Sufficiency.* Suppose that assumption 4 is valid and let $T \geq 0$ and $x_0 \in \Omega_T$ be fixed.

Then also $x_0 \in \Omega_{T+1}$. By assumption 2, there is an optimal input on $[0, T+1]$, which is denoted by $(\tilde{u}^*_{0|T}, \tilde{w}^*_{0|T})$. By definition of optimality, we have:

$$\begin{aligned}\hat{V}_{T+1}(x_0) &= J(0, T+1, x_0, \tilde{u}^*_{0|T}, \tilde{w}^*_{0|T}) \\ &\leq J(0, T+1, x_0, \tilde{u}_{0|T}, \tilde{w}^*_{0|T}) \\ &= g(\tilde{\xi}(T+1)) + \sum_{t=0}^{T-1} L(\tilde{\xi}(t), \tilde{u}(t), \tilde{w}^*(t)) \\ &\quad + L(\tilde{\xi}(T), \tilde{u}(T), \tilde{w}^*(T))\end{aligned}$$

where $(\tilde{u}_{0|T}, \tilde{w}^*_{0|T})$ is any admissible input on $[0, T+1]$ with $\tilde{w}^*_{0|T}$ and $\tilde{\xi}_{0|T+1}$ is the corresponding trajectory. Choose now the first T components of $\tilde{u}_{0|T}$ to be equal to the controller optimal on $[0, T)$: $\tilde{u}(t) = u^*(t)$, $t = 0, \ldots, T-1$. According to assumption 5, $u^*_{0|T-1}$ and the restriction $\tilde{w}^*_{0|T-1}$ of $\tilde{w}^*_{0|T}$ to $[0, T-1]$ gives an admissible pair, and:

$$\begin{aligned}&\sum_{t=0}^{T-1} L(x(t; 0, x_0, \tilde{u}_{0|T}, \tilde{w}^*_{0|T}), \tilde{u}_{0|T}(t), \tilde{w}^*_{0|T}(t)) \\ &= \sum_{t=0}^{T-1} L(x(t; 0, x_0, u^*_{0|T-1}, \tilde{w}^*_{0|T-1}), u^*_{0|T-1}(t), \tilde{w}^*_{0|T-1}(t)) \\ &= J(0, T, x_0, u^*_{0|T-1}, \tilde{w}^*_{0|T-1}) - g(x(T; 0, x_0, u^*_{0|T-1}, \tilde{w}^*_{0|T-1}))\end{aligned}$$

Since

$$\begin{aligned}J(0, T, x_0, u^*_{0|T-1}, \tilde{w}^*_{0|T-1}) &\leq J(0, T, x_0, u^*_{0|T-1}, w^*_{0|T-1}) \\ &= \hat{V}_T(x_0)\end{aligned}$$

we have

$$\begin{aligned}\hat{V}_{T+1}(x_0) \leq \hat{V}_T(x_0) &+ g(\tilde{\xi}(T+1)) + L(\tilde{\xi}(T), \tilde{u}(T), \tilde{w}^*(T)) \\ &- g(x(T; 0, x_0, u^*_{0|T-1}, \tilde{w}^*_{0|T-1}))\end{aligned} \tag{4}$$

By admissibility of $(u^*_{0|T-1}, \tilde{w}^*_{0|T-1})$, $\bar{x} := x(T; 0, x_0, u^*_{0|T-1}, \tilde{w}^*_{0|T-1})$ belongs to $\mathcal{G}_\eta$, therefore:

$$\begin{aligned}g(\bar{x}) &\geq \min_{u \in U(\bar{x})} \max_{w \in W(\bar{x})} \{L(\bar{x}, u, w) + g(f(\bar{x}, u, w))\} \\ &\geq \min_{u \in U(\bar{x})} \{L(\bar{x}, u, \tilde{w}^*(T)) + g(f(\bar{x}, u, \tilde{w}^*(T)))\} \\ &= L(\bar{x}, u_0, \tilde{w}^*(T)) + g(f(\bar{x}, u_0, \tilde{w}^*(T)))\end{aligned} \tag{5}$$

The last element of $\tilde{u}_{0|T}$ has not yet been fixed. Let it be defined by:

$$\tilde{u}_{0|T}(T) = u_0$$

then $\tilde{\xi}(T+1) = x(T+1; 0, x_0, \tilde{u}_{0|T}, \tilde{w}^*_{0|T}) = f(\bar{x}, u_0, \tilde{w}^*(T))$. Hence, (4) and (5) give the required inequality $\hat{V}_T(x_0) \geq \hat{V}_{T+1}(x_0)$.

(b) *Necessity.* Apply (3) with $T = 0$ and $x_0 \in \Omega_0 = \mathcal{G}_\eta$, then:

$$\begin{aligned}g(x_0) = \hat{V}_0(x_0) &\geq \hat{V}_1(x_0) \\ &= \min_{u \in U(x_0)} \max_{w \in W(x_0)} \{L(x_0, u, w) + g(f(x_0, u, w))\}\end{aligned}$$

Since x_0 is arbitrary, this is exactly assumption 4. □

Corollary 1: Under the assumptions 1 to 5, $\hat{V}_T(0) = 0$ for all $T \geq 0$.

Proof: Since $\Omega_T \subseteq \Omega_{T+1}$ for any $T \geq 0$, $0 \in \Omega_T$, therefore it follows by lemma 1 that $\hat{V}_T(0) \geq 0$. Because of theorem 1, $\hat{V}_T(0) \leq \hat{V}_{T-1}(0) \leq \cdots \leq \hat{V}_0(0) = g(0) = 0$. □

2.2 Feedback saddle point solution and receding horizon control

The results above are valid for any kind of input strategy. In what follows, we restrict ourselves to feedback strategies, and optimality in assumption 2 is meant in the sense of a feedback saddle point solution.

Definition 3: For any $k \in \{0, 1, \ldots, T-1\}$, let pairs of functions $(\alpha_{k|T-1}, \mu_{k|T-1})$ be defined as $\alpha(s, \cdot): \Omega_{T-s} \to U$, $\mu(s, \cdot): \Omega_{T-s} \to W$, for $s = k, k+1, \ldots, T-1$. It is said that such pairs of functions form an admissible feedback strategy, if for any $s \in \{k, k+1, \ldots, T-1\}$ and for any $\bar{x} \in \Omega_{T-s}$ the recursion:

$$\begin{aligned}\xi(t+1) &= f(\xi(t), \alpha(t, \xi(t)), \mu(t, \xi(t))), \quad t = s, \ldots, T-1 \\ \xi(s) &= \bar{x}\end{aligned} \tag{6}$$

gives such a sequence $\xi(t)$, $t = s, \ldots, T$ that $\xi(t) \in \Omega_{T-t}$.

Definition 4: An admissible feedback strategy $(\alpha^*_{0|T-1}, \mu^*_{0|T-1})$ is called a feedback saddle point solution if for any $k \in \{0, 1, \ldots, T-1\}$ and for any $\bar{x} \in \Omega_{T-k}$, inequalities:

$$\begin{aligned}J(k, T, \bar{x}, \alpha^*_{k|T-1}, \mu_{k|T-1}) &\leq J(k, T, \bar{x}, \alpha^*_{k|T-1}, \mu^*_{k|T-1}) \\ &\leq J(k, T, \bar{x}, \alpha_{k|T-1}, \mu^*_{k|T-1})\end{aligned} \tag{7}$$

hold true for any admissible feedback strategies $(\alpha^*_{k|T-1}, \mu_{k|T-1})$ and $(\alpha_{k|T-1}, \mu^*_{k|T-1})$ such that $\mu(s, \cdot) = \mu^*(s, \cdot)$ and $\alpha(s, \cdot) = \alpha^*(s, \cdot)$ if $s \geq k+1$.

The following proposition formulates a version of theorem 6.6 and corollary 6.2 of [16] for the case when the terminal state is restricted to belong to a given set $\mathcal{G}_\eta$.

Proposition 1: $\{\alpha^*_{0|T-1}, \mu^*_{0|T-1}\}$ is a feedback saddle point solution of the game (1) and (2) if and only if there exist functions $V(k, \cdot): \Omega_{T-k} \to \boldsymbol{R}$, $k = 0, 1, \ldots, T-1$ such that:

$$\begin{aligned}V(k, x) &= \min_{u \in U} \max_{w \in W} \{L(x, u, w) + V(k+1, f(x, u, w))\} \\ &= \max_{w \in W} \min_{u \in U} \{L(x, u, w) + V(k+1, f(x, u, w))\} \\ &= L(x, \alpha^*(k, x), \mu^*(k, x)) \\ &\quad + V(k+1, f(x, \alpha^*(k, x), \mu^*(k, x)))\end{aligned} \tag{8}$$

and

$$V(T, x) = g(x) \quad \text{if } x \in \mathcal{G}_\eta$$

Then $V(0, x)$ is the value of the game, i.e. $V(0, x) = \hat{V}_T(x)$.

Remark 5: The max and min operations in (8) have to be taken for such $u \in U$ and $w \in W$ that the expression in the braces is well-defined.

Proof: (a) *Necessity.* Let $\{\alpha^*_{0|T-1}, \mu^*_{0|T-1}\}$ denote a feedback saddle point solution and let $V(T, x) = g(x)$, if $x \in \mathcal{G}_\eta$. For $k = T-1, T-2, \ldots, 0$ and $\bar{x} \in \Omega_{T-k}$ define $V(k, \cdot)$ as:

$$V(k, \bar{x}) = J(k, T, \bar{x}, \alpha^*_{k|T-1}, \mu^*_{k|T-1}) \tag{9}$$

By definition of the cost function and the feedback saddle point solution, we have:

$$\begin{aligned}
J(k, T, \bar{x}, \alpha^*_{k|T-1}, \mu_{k|T-1}) &= g(\bar{\xi}(T)) + L(\bar{\xi}(k), \alpha^*(k, \bar{x}), \mu(k, \bar{x})) \\
&\quad + \sum_{t=k+1}^{T-1} L(\bar{\xi}(t), \alpha^*(t, \bar{\xi}(t)), \mu^*(t, \bar{\xi}(t))) \\
&= L(\bar{x}, \alpha^*(k, \bar{x}), \mu(k, \bar{x})) \\
&\quad + V(k+1, f(\bar{x}, \alpha^*(k, \bar{x}), \mu^*(k, \bar{x}))) \qquad (10)
\end{aligned}$$

Analogously:

$$\begin{aligned}
J(k, T, \bar{x}, \alpha^*_{k|T-1}, \mu^*_{k|T-1}) &= L(\bar{x}, \alpha^*(k, \bar{x}), \mu^*(k, \bar{x})) \\
&\quad + V(k+1, f(\bar{x}, \alpha^*(k, \bar{x}), \mu^*(k, \bar{x}))) \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
J(k, T, \bar{x}, \alpha_{k|T}, \mu^*_{k|T-1}) &= L(\bar{x}, \alpha(k, \bar{x}), \mu^*(k, \bar{x})) \\
&\quad + V(k+1, f(\bar{x}, \alpha(k, \bar{x}), \mu^*(k, \bar{x}))) \qquad (12)
\end{aligned}$$

From (10)–(12) it follows that $(\alpha^*(k, \bar{x}), \mu^*(k, \bar{x}))$ is a saddle point of the function:

$$(u, w) \rightarrow L(\bar{x}, u, w) + V(k+1, f(\bar{x}, u, w)) \qquad (13)$$

thus the min and max operations are interchangeable. This together with (9), gives (8). Since k and $\bar{x} \in \Omega_{T-k}$ are arbitrary, the necessity is proven.

(b) *Sufficiency.* Assume that the functions $V(k, \cdot): \Omega_{T-k} \rightarrow \boldsymbol{R}$, $k = 0, 1, \ldots, T-1$ satisfy (8) with $\{\alpha^*_{0|T-1}, \mu^*_{0|T-1}\}$. Let $k = T-1$. From (8), it follows that $\{\alpha^*_{T-1|T-1}, \mu^*_{T-1|T-1}\}$ is admissible, because $V(T, \cdot)$ is only defined for $x \in \mathcal{G}_\eta$. Thus $f(\bar{x}, \alpha^*(T-1, \bar{x}), \mu^*(T-1, \bar{x})) = \xi^*(T) \in \mathcal{G}_\eta$. On the other hand, the interchangeability of the min and max operations implies that $(\alpha^*(T-1, \bar{x}), \mu^*(T-1, \bar{x}))$ is a saddle point of the function given (13) with $k = T-1$.

But for any $(\bar{u}, \bar{w}) \in U \times W$:

$$\begin{aligned}
L(\bar{x}, \bar{u}, \bar{w}) + V(T, f(\bar{x}, \bar{u}, \bar{w})) &= L(\bar{x}, \bar{u}, \bar{w}) + g(x(T; T-1, \bar{x}, u_{T-1|T-1}, w_{T-1|T-1})) \\
&= J(T-1, T, \bar{x}, u_{T-1|T-1}, w_{T-1|T-1})
\end{aligned}$$

where $u(t) = \bar{u}$ and $w(t) = \bar{w}$, therefore (7) holds true, if $k = T-1$, and $V(T-1, \bar{x}) = J(T-1, T, \bar{x}, \alpha^*(T-1, \bar{x}), \mu^*(T-1, \bar{x}))$. Using analogous arguments, one obtains by induction that $\{\alpha^*_{k|T-1}, \mu^*_{k|T-1}\}$ is admissible and (7) holds true for $k = T-2, \ldots, 0$ as well. □

The receding horizon controller is defined as follows. Let $T > 0$ be fixed and, at the current time $t \geq 0$, let the state be $x(t)$. Let $\{\alpha^*_{0|T-1}, \mu^*_{0|T-1}\}$ denote a feedback saddle point solution of the game (1) and (2) with $x_0 = x(t)$. Then:

$$\tilde{u}(x(t)) := \alpha^*(0, x(t)) \qquad (14)$$

Remark 6: The receding horizon controller is defined by means of a feedback to avoid difficulties arising in open-loop min-max model predictive control (see e.g. [6], and also [11–13]). On the other hand, the application of a feedback strategy has the well-known drawback of computational complexity: the optimisation has to be carried out over an infinite dimensional space of functions. Proposition 1 provides a tool for the solution of this task inasmuch as the dynamic programming equation usually do.

Lemma 2: If assumptions 1 to 5 hold true and $\tilde{u}(x(t))$ is defined by (14) then:

$$\hat{V}_T(f(x(t), \tilde{u}(x(t)), \tilde{w}(t))) - \hat{V}_T(x(t)) \leq -L(x(t), \tilde{u}(x(t)), \tilde{w}(t)) \qquad (15)$$

where $\tilde{w}(t) = \mu(0, x(t))$ and $\{\alpha^*_{0|T-1}, \mu_{0|T-1}\}$ is an admissible feedback.

Proof: By definition of admissibility:

$$\bar{x}(t+1) = f(x(t), \tilde{u}(x(t)), \tilde{w}(t)) \in \Omega_{T-1} \qquad (16)$$

Let the left-hand side of (15) be written as:

$$\begin{aligned}
\hat{V}_T(f(x(t), \tilde{u}(x(t)), \tilde{w}(t))) - \hat{V}_T(x(t)) &= \hat{V}_T(\bar{x}(t+1)) - \hat{V}_{T-1}(\bar{x}(t+1)) \\
&\quad + \hat{V}_{T-1}(\bar{x}(t+1)) - \hat{V}_T(x(t)) \qquad (17)
\end{aligned}$$

Using theorem 1 with $(T-1, \bar{x}(t+1))$ instead of (T, x_0), we have:

$$\hat{V}_T(\bar{x}(t+1)) - \hat{V}_{T-1}(\bar{x}(t+1)) \leq 0 \qquad (18)$$

Since $\{\alpha^*_{0|T-1}, \mu^*_{0|T-1}\}$ is a feedback saddle point solution, proposition 1 gives for $k = 0$:

$$\begin{aligned}
\hat{V}_T(x(t)) = V(0, x(t)) &= L(x(t), \alpha^*(0, x(t)), \mu^*(0, x(t))) \\
&\quad + V(1, f(x(t), \alpha^*(0, x(t)), \mu^*(0, x(t)))) \\
&\geq L(x(t), \alpha^*(0, x(t)), \mu(0, x(t))) \\
&\quad + V(1, f(x(t), \alpha^*(0, x(t)), \mu(0, x(t)))) \\
&= L(x(t), \tilde{u}(x(t)), \tilde{w}(t)) + \hat{V}_{T-1}(\bar{x}(t+1)) \qquad (19)
\end{aligned}$$

Equations (16)–(19) combine to produce (15). □

For completeness, we remember the formal definition for dissipativity (see e.g. [1, 2]).

Definition 5: A discrete-time nonlinear system of the form:

$$\begin{aligned}
x(k+1) &= f(x(k), w(k)) \\
z(k) &= h(x(k), w(k)) \qquad (20)
\end{aligned}$$

is said to be dissipative with supply rate $W: \boldsymbol{R}^s \times \boldsymbol{R}^{m_2} \rightarrow \boldsymbol{R}$ if there exists a non-negative function $V: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ with $V(0) = 0$, called the storage function, such that for all $w \in \boldsymbol{R}^{m_2}$ and all $k \in \boldsymbol{Z}_+$

$$V(x(k+1)) - V(x(k)) \leq W(z(k), w(k))$$

Theorem 2: Suppose that assumptions 1 to 5 hold true. Then the receding horizon controller (14) results in a closed-loop system:

$$\begin{aligned}
x(t+1) &= f(x(t), \tilde{u}(x(t)), w(t)) \\
x(0) &= x_0, \quad x_0 \in \Omega_T \qquad (21)
\end{aligned}$$

which is dissipative for any admissible input $w(\cdot)$, with a storage function $x \mapsto \hat{V}_T(x)$ and supply rate function $(x, w) \mapsto -L(x, \tilde{u}(x), w)$.

Proof: The proof immediately follows from lemmas 1 and 2, and corollary 1. □

3 Receding horizon formulation of discrete-time nonlinear H_∞ control

3.1 H_∞ receding horizon control

In this Section the results of the previous section are applied to the problem of disturbance attenuation with internal stability. For the solution of this problem a game theoretic approach is frequency used [1–3, 6, 11–15].

Before formulating the main result of this Section, for completeness, we give the formal definitions for L_2-gain [1, 2].

Definition 6: Suppose γ is a given positive real number. A discrete-time nonlinear system of the form of (20) is said to have L_2-gain less than or equal to γ if:

$$\sum_{k=0}^{N} \|z(k)\|^2 \leq \gamma^2 \sum_{k=0}^{N} \|w(k)\|^2$$

for all $N \in \boldsymbol{Z}_+$ and all $w(\cdot) \in l_2([0, N], \boldsymbol{R}^{m_2})$ with the output $z(\cdot)$ resulting by $w(\cdot)$ from initial state $x_0 = 0$.

Consider the nonlinear system governed by (1) together with the penalty output:

$$z(k) = h(x(k), u(k), w(k)) \quad k \geq 0 \tag{22}$$

where u is considered as control input and w as disturbance input. The problem is to find a controller in feedback form, which renders the unperturbed closed-loop system (i.e. the system with $w(t) \equiv 0$) locally or globally asymptotically stable so that the L_2-gain is less than or equal to the prescribed number γ at least for a given class of perturbation. In order to define such a class of disturbances, for any $x \in \boldsymbol{R}^n$ and $u \in U$, let a set $\mathcal{W}(x, u) \subseteq W$ be given, which represents the possible values of the disturbance at x and u. In what follows, the following modification of assumption 3 is presumed:

Assumption 3′: $U(x) \neq \emptyset$, and $\mathcal{W}(x, u) \subseteq W(x)$ for all $x \in \mathcal{G}_\eta$.

Definition 7: A control function $u_{0|T-1}$ is said to be an admissible control for x_0, if $(u_{0|T-1}, w_{0|T-1})$ is admissible for x_0 for all $w_{0|T-1}$ with $w(t) \in \mathcal{W}(\xi(t), u(t))$, $t = 0, \ldots, T-1$, where $\xi_{0|t}$ is the trajectory corresponding to $(u_{0|T-1}, w_{0|T-1})$ and starting from x_0. The set of admissible inputs is restricted to pairs with admissible controls.

Let $\gamma > 0$ be a given real number and let the function L be defined as:

$$L(x, u, w) = \|h(x, u, w)\|^2 - \gamma^2 \|w\|^2 \tag{23}$$

Theorem 3: Assume that for some $\gamma > 0$ the game problem (1) and (2) with the function L defined by (22) and (23) and with a set of admissible inputs in the sense of definition 7 satisfies assumptions 1 to 5 when assumption 3 is substituted by assumption 3′. If $\tilde{a}(\cdot)$ is defined by (14) according to the receding horizon method, the closed-loop system (21) is internally stable and has an L_2-gain less than or equal to γ, in Ω_T.

Proof: The L_2-gain property of (21) follows from nonlinear version of the bounded real lemma (proposition 1 [1], see also proposition 1 [2]). The asymptotic stability of the closed-loop system with $w = 0$ follows easily from LaSalle's invariance principle and assumption 1(b). □

Remark 7: In order to show that theorem 3 extends the class of problems for which the H_∞ control problem is solvable, it has to be verified that the assumptions of the present paper hold true under the conditions postulated in [1–3, 13, 14]. In this respect, the most important issues are assumptions 3′, 4 and the set of admissible disturbances. In [1] and [2], the existence of a sufficiently smooth function V satisfying a Hamilton–Jacobi–Isaacs equation/inequality in a neighbourhood of the origin is required, and, under some further natural conditions, the local solvability of the H_∞ control problem is proven. Then, with the choice of $g = V$ and $\mathcal{G}_\eta$ to be any level set contained in the neighbourhood of the origin considered in the papers above, assumptions 3′ and 4 hold true. In [3], the solvability of the H_∞ control problem for the linearised system is assumed, and the local solution of the original problem is given. Under the conditions of [3], g and $\mathcal{G}_\eta$ can be chosen as $g(x) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x}$, where $\boldsymbol{P}$ is a positive definite matrix satisfying the usual algebraic Riccati inequality, and $\mathcal{G}_\eta$ as a level set contained in a suitable neighbourhood of the origin. Then assumptions 3′ and 4 hold again true, moreover, $W(x)$ contains a ball with radius $\delta > 0$. It can also be verified that, under a suitable controllability condition, any $l_2[0, \infty)$ function with values in $W \cap \mathcal{B}_\delta(0)$, where δ is sufficiently small, is admissible. Since these three papers deal with the H_∞ control problem in a small neighbourhood of the origin, the other assumptions of the present paper are not relevant for them.

Receding horizon H_∞ control algorithms are proposed for nonlinear systems in [11] and [12]. Both of them assume that the H_∞ control problem is solvable for the linearised system, and uses a final state constraint set and a final state penalty function determined on the basis of the linearisation, moreover, in both papers the set of admissible disturbances is of the type (25). The algorithm presented in [11] involves a precompensation and an open-loop model predictive strategy, While the algorithm of [12] uses a feedback strategy based on inverse optimality.

Similarly to [3], the starting point of [13] is also the assumption that the H_∞ control problem is solvable for the linearised system, but here the domain of attraction is extended by using a variant of the receding horizon approach, while the set of admissible disturbances is taken to be the set of outputs of an arbitrary dynamic system with input z, having a finite L_2-gain smaller than $1/\gamma$. (Sometimes such disturbances are said to be structured.) It can be verified that under the assumptions of [13], assumptions 3′ and 4 are satisfied. In [14] the H_∞ controller is developed by applying the receding horizon approach to a dynamic game problem with a final state constraint and final state penalty function. This final state penalty function is assumed to satisfy an inequality analogous to that of assumption 4. The set of admissible disturbances is also taken to be of the type (25). Due to this specific form, assumption 3′ is also valid. Differently from the present paper, $l_2[0, \infty)$ function with values in $W \cap \mathcal{B}_\delta(0)$, are not admissible in this work.

Remark 8: It should be noted that the function g and the set $\mathcal{G}_\eta$ have to be determined in advance. Undoubtedly, the easiest and, from a practical point of view, most important way for constructing them is linearisation, but they can also be obtained e.g. by using the method presented in [1]. Thus, the assumptions of the present paper can also be satisfied in cases, when the linearisation technique does not work. The example of Section 4 also shows such a case.

On the basis of the above considerations, one can make the following conclusion: the proposed method can be applied in all of the cases discussed in the literature. Likewise, the set of admissible disturbances is not more restrictive than the ones considered in the works mentioned above. Although it also gives a local result, in this case its stability region is larger than that of the methods of [1–3], roughly speaking, inasmuch as Ω_T is larger then the set $\mathcal{G}_\eta$. A further advantage of the proposed method is that, in contrast to methods based on linearisation techniques or on the implicit function theorem, it creates the possibility of deriving a global result.

3.2 Global solution

To obtain a global solution for the H_∞ control problem, one needs the following assumption.

Assumption 6: (a) $\Omega_T = \boldsymbol{R}^n$.
(b) Function g is proper.
(c) There exist constants $K > 0$, $r > 0$ such that:

$$\|f(x, u, 0) - x\| \leq K\|h(x, u, 0)\|^2 \text{ for every } (x, u)$$
$$\|x\|^2 + \|u\|^2 > r^2, \quad u \in U, x \in \boldsymbol{R}^n.$$

Remark 9: Assumption 6 is the discrete-time counterpart of assumption A5 of [8]. As lemma 3 shows, the role of this assumption is to ensure the radial unboundedness of $\hat{V}_T$. Since we want to achieve global stability, condition (a) is necessarily hard, but conditions (b) and (c) restrict the g and h, functions which are design parameters. We note that all of these conditions are fulfilled in the example of Section 4.

Lemma 3: Suppose that assumptions 1 to 6 are satisfied. Then $\hat{V}_T$ is radially unbounded, i.e. $\hat{V}_T(x) \to \infty$, if $\|x\| \to \infty$.

Proof: Let $x_0 \in \boldsymbol{R}^n$ be arbitrary. Because of assumptions 2 and 6(a) $\hat{V}_T(x_0)$ is well-defined and:

$$\hat{V}_T(x_0) \geq J(0, T, x_0, u^*_{0|T-1}, w^0_{0|T-1}) \geq \sum_{t=0}^{T-1} \|h(\xi^*_0(t), u^*(t), 0)\|^2$$

where $\xi^*_0(\cdot)$ is the trajectory of system (1) corresponding to $(u^*_{0|T-1}, w^0_{0|T-1})$ and $x(0) = x_0$. Since $\xi^*_0(T) \in \mathcal{G}_\eta$ and, according to assumption 6(b), $\mathcal{G}_\eta$ is compact, there is a constant $\tilde{r} > 0$ (independent of x_0) such that $\|\xi^*_0(T)\| \leq \tilde{r}$. Assume that x_0 is such that $\|x_0\| > \max\{r, 2\tilde{r}\}$. Then:

$$\frac{1}{2}\|x_0\| < \|\xi^*_0(T) - \xi^*_0(0)\| \leq \sum_{t=0}^{T-1} \|f(\xi^*_0(t), u^*(t), 0) - \xi^*_0(t)\| \tag{24}$$

Let

$$I_r = \{k \in \boldsymbol{Z} : 0 \leq k \leq T, \|\xi^*_0(k)\|^2 + \|u^*(k)\|^2 > r^2\}$$

and let $\bar{I}_r = \{0, 1, \ldots, T\} \backslash I_r$. With these notations, it follows from (24) that:

$$\frac{1}{2}\|x_0\| < \sum_{t \in I_r} \|h(\xi^*_0(t), u^*(t), 0)\|^2 + \sum_{t \in \bar{I}_r} M$$

where

$$M = \max_{\|x\|^2 + \|u\|^2 \leq r^2} \|f(x, u, 0) - x\|$$

Therefore:

$$\hat{V}_T(x_0) \geq \frac{1}{K}\left(\frac{1}{2}\|x_0\| - TM\right) \to \infty, \text{ if } \|x_0\| \to \infty. \quad \square$$

Theorem 4: Assume that for some $\gamma > 0$ the game problem (1) and (2) with the function L defined by (22) and (23) satisfies assumptions 1 to 6 (with assumption 3′ instead of assumption 3). If $\tilde{u}(\cdot)$ is defined by (14) according to the receding horizon method, then the closed-loop system (21) is globally internally stable and has an L_2-gain less than or equal to γ.

Proof: The global asymptotic stability of system (21) at $x = 0$ with $w(t) \equiv 0$ follows from lemmas 1, 2 and 3 and also from LaSalle's invariance principle, while the L_2-gain property is the same as before. $\square$

4 An illustrative example

Let

$$f(x, u, w) = \boldsymbol{A}(x) + \boldsymbol{B}(x)u + \boldsymbol{E}(x)w$$
$$\boldsymbol{A}(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$
$$\boldsymbol{E}(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \boldsymbol{U} = [-1, 1], \quad \boldsymbol{W} = [-1/\sqrt{2}, 1/\sqrt{2}]$$
$$L(x, u, w) = \frac{1}{2}\|\boldsymbol{x}\|^4 + u^2 - w^2$$
$$g(x) = \|\boldsymbol{x}\|^2 \quad \mathcal{G}_1 = \{x : \|\boldsymbol{x}\|^2 \leq 1\}$$

Obviously, these functions satisfy assumption 1. Let $T > 0$ be fixed. Then $\{u_{0|T-1}, w_{0|T-1}\}$ with $u(t) = -1$, $w(t) \in \boldsymbol{W}$ for $t = 0, \ldots, T-1$ is an admissible input pair for any x with $\|\boldsymbol{x}\| < \sqrt{(2)}^T$. If, in contrary, an arbitrary x is given, then, by taking $T > \ln\|\boldsymbol{x}\|/\ln\sqrt{2}$, the same input pair remains admissible.

An easy computation shows that $u^*(x) = -\|\boldsymbol{x}\|^2/(1 + \|\boldsymbol{x}\|^2)$, $w^*(x) = 0$ give the min–max of $H(x, u, w) = L(x, u, w) + g(f(x, u, w))$ in $u \in \boldsymbol{U}$, $w \in \boldsymbol{W}$, if $x \in \mathcal{G}_1$ and:

$$H(x, u^*(x), w^*(x)) = \|\boldsymbol{x}\|^2\left(\frac{1}{2}\|\boldsymbol{x}\|^2 + \frac{1}{1 + \|\boldsymbol{x}\|^2}\right) \leq \|\boldsymbol{x}\|^2 \quad \text{if } \|\boldsymbol{x}\| \leq 1$$

because $\|\boldsymbol{x}\|^2/2 + 1/(1 + \|\boldsymbol{x}\|^2) < 1$, if $\|\boldsymbol{x}\| \leq 1$. Let u_λ be defined as $u_\lambda(x) = (1 - \lambda)u^*(x) - \lambda$. Then $u_\lambda(x) \in [-1, 1]$ for $\lambda \in [0, 1]$ and $\|f(x, u_\lambda(x), w\| \leq 1$ for any $x \in \mathcal{G}_1$ and $w \in \boldsymbol{W}$. Therefore assumption 3′ and 4 are satisfied with a $U(x) \supset [-1, -\|\boldsymbol{x}\|^2/(1 + \|\boldsymbol{x}\|^2)]$ and $\mathcal{W}(x, u) = W(x) = \boldsymbol{W}$.

To illustrate the way of computations and to verify assumption 5, consider $T = 2$. According to proposition 1, $V(2, x) = \|\boldsymbol{x}\|^2$, if $x \in \mathcal{G}_1$. Then:

$$V(1, x) = \min_u \max_w \{L(x, u, w) + V(2, f(x, u, w))\}$$

The solution of the min–max problem is:

$$\alpha^*(1, \boldsymbol{x}) = -\frac{\|\boldsymbol{x}\|^2}{1 + \|\boldsymbol{x}\|^2}$$
$$\mu^*(1, \boldsymbol{x}) = \begin{cases} 0 & \text{if } \|\boldsymbol{x}\| < 1 \\ \pm\sqrt{2}/2 & \text{if } \|\boldsymbol{x}\| \geq 1 \end{cases}$$

for such vectors $\boldsymbol{x}$, for which $\|f(x, \alpha^*(1, \boldsymbol{x}), \mu^*(1, \boldsymbol{x}))\| \leq 1$. This inequality holds true at least for $\|\boldsymbol{x}\|^2 \leq 3/2$, thus $V(1, \cdot)$ is defined on $\Omega_1 \supseteq \{x : \|\boldsymbol{x}\|^2 \leq 3/2\}$ and there:

$$V(1, \boldsymbol{x}) = \|\boldsymbol{x}\|^2\left(\frac{1}{2}\|\boldsymbol{x}\|^2 + \frac{1}{1 + \|\boldsymbol{x}\|^2}\right) + (\|\boldsymbol{x}\|^2 - 1)\mu^*(1, \boldsymbol{x}).$$

Now,

$$V(0, \boldsymbol{x}) = \min_u \max_w \{L(\boldsymbol{x}, u, w) + V(1, f(\boldsymbol{x}, u, w))\}$$

It can be seen that the function in the braces has a unique minimum which belongs to the interval $[-1, -\|\boldsymbol{x}\|^2/(1 + \|\boldsymbol{x}\|^2)]$. On the other hand, the function in the braces is monotonically increasing with $|w|$ independently of u, if $\|\boldsymbol{x}\| \geq 1$ and it is monotonically decreasing with $|w|$, if $\|\boldsymbol{x}\| < 1$ and $u \in [-1, -\|\boldsymbol{x}\|^2/(1 + \|\boldsymbol{x}\|^2)]$. Therefore:

$$\alpha^*(0, \boldsymbol{x}) \in \left[-1, -\frac{\|\boldsymbol{x}\|^2}{1 + \|\boldsymbol{x}\|^2}\right]-$$
$$\mu^*(0, \boldsymbol{x}) = \begin{cases} 0, & \text{if } \|\boldsymbol{x}\| < 1, \\ \pm\sqrt{2}/2, & \text{if } \|\boldsymbol{x}\| \geq 1 \end{cases}$$

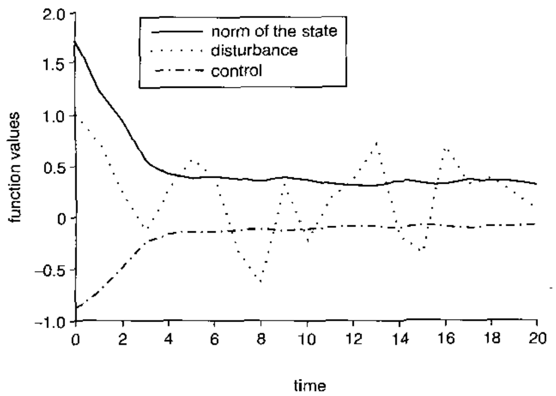

Fig. 1 *Results of simulation*

for any x such that $\|f(x, \alpha^*(0, x), \mu^*(0, x)\|^2 \leq 3/2$. This will be true at least for any x with $\|x\|^2 \leq 5/2$, thus $V(0, \cdot)$ is defined on $\Omega_2 \supseteq \{x: \|x\|^2 \leq 5/2\}$. We note that the value of $\alpha^*(0, x)$ and $V(0. x)$ can only be determined numerically. Nevertheless, the computations above show that both assumptions 2 and 5 are valid with the time interval in consideration. Fig. 1 shows the results of computer simulation, when the proposed controller $u(t) = \alpha^*(0, x(t))$ was applied. The disturbance was simulated by $w(t) = r(t)/\sqrt{2}$, where the parameters $r(t)$ were uniformly distributed random numbers on $[-1, 1]$.

If the set of admissible disturbances is determined as e.g. in [14] by:

$$\|w(t)\|^2 \leq \delta^2 \|z(t)\|^2 \tag{25}$$

where $\delta < 1$, then the considerations above remain true inside of $\mathcal{G}_1$, but this set can only insignificantly be enlarged by using any control strategy, if δ is close to one: in fact, if $\bar{w} = \delta\sqrt{(\|x\|^4/2 + u^2)}$ and $\|x\| \geq 1$, then $\|f(x, u, \bar{w})\|^2 \geq \|x\|^2(\delta^2\|x\|^4/2 + \delta^2/(1 + \delta^2))$, thus all trajectories diverge from initial point with norm greater than $R(\delta)$, where, $R(\delta) = (2/(\delta^2 + \delta^4))^{1/4} \to 1$, if $\delta \to 1$. On the other hand, if the set of admissible disturbances is restricted to:

$$\mathcal{W}(x, u) = \left\{ w: |w| \leq \frac{1}{\sqrt{2}}, \text{ if } \|x\| \geq 1 \text{ and } \right.$$
$$\left. |w| \leq \frac{1}{\sqrt{2}\|x\|}, \text{ if } \|x\| \geq 1 \right\}$$

then assumption 6 holds true with $K = 4$ and $r = 2$, thus the proposed method gives a global solutions for the H_∞ control problem.

This example supports that the flexibility of the notion of admissible inputs adapted in the present paper is useful from the point of view of applications.

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