



# Theoretical study of excitations in Bose-Einstein condensates

Thesis of Ph.D. thesis

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- [IV.] Schumayer Dániel és Apagyi Barnabás,  
*Gerjesztések Bose-Einstein kondenzátumban*  
Seminar Talk at KFKI Research Institute for Particle and Nuclear  
Physics, (2003)
- [V.] Apagyi Barnabás és Schumayer Dániel,  
*Assessment of values of interspecies scattering length  $a_{12}$  from solitonic  
stability*  
ICAP 2004, Rio de Janeiro, Poster session, (2004)
- [VI.] Schumayer Dániel és Apagyi Barnabás,  
*Nonlinear evolution equations, Inverse Scattering Transformation and  
Bose-Einstein condensates*  
Seminar talk at the Institute of Theoretical Physics of the Justus-  
Liebig-Universität Gießen (2004)

### 3.1 Additional publications

- [ a. ] András Szilágyi  
*Nemlineáris evolúciós egyenletek megoldása Inverz Szórás Transzformációval,*  
Students' Scientific Work (2003), First prize, Supervisors: Barnabás  
Apagyi and Dániel Schumayer

### 3 Results

1. Exploring the integrability of coupled Gross-Pitaevskii (CGP) equations I applied the Painlevé-analysis [1]. This method is based on the classification of singularities of the solutions of nonlinear partial differential equations. The test of the CGP equations has been carried out with the result [1] that the coupled equations can be integrable if the relation below is satisfied

$$\frac{2T_{11}T_{22} - (\vartheta/\lambda) T_{11}T_{12} - (\lambda/\vartheta) T_{21}T_{22}}{T_{11}T_{22} - T_{12}T_{21}} = \frac{1}{16}[(2m + 1)^2 + 7],$$

where  $m$  is a non-negative integer number,  $\lambda$  and  $\vartheta$  are in inverse proportion to the masses of the species, and  $T_{ij}$  is related with the scattering lengths. Moreover, the value of  $m$  classifies the external potentials. In this way the system parameters determine the form of the applicable external potential without destroying the integrability and vice versa.

2. Computer algebra is applied [1] to evaluate a compatibility condition arisen in Painlevé test for admissible external potentials. This requirement results in a constraint on the external potential. I found that a general form of the trapping potential (in case of  $m = 2$ ) which does not break the integrability property is ( $j = 1, 2$ )

$$V_j = \frac{1}{\lambda} \left( \frac{1}{2} \frac{d\beta_j(t)}{dt} - \beta_j^2(t) \right) x^2 + V_j^{(1)}(t)x + V_j^{(0)}(t) + i\beta_j(t),$$

where  $V_j^{(1)}$ ,  $V_j^{(0)}$  and  $\beta_j$  are arbitrary real functions of time  $t$ . The integrability of CGP equations requires a special combination of the real parts of the potentials  $V_j$  which may depend on the spatial coordinate  $x$ , at the most, quadratically. A stringent relationship can be established between the coefficient of the quadratic term and the imaginary part interpreted as a source term. Although such potentials cannot model a simple harmonic field (which is usual in forming BECs) absorption/feeding related to negative/positive  $\beta(t)$  is always being present in the experiments. In the linear case the potentials are real, and may have an arbitrary time dependence. Such potentials can be produced in miniaturized BEC apparatus.

3. Two-component Bose-Einstein condensates are usually composed of two different hyperfine states of rubidium. Using Painlevé-analysis I can derive some constraints on the parameters in which cases the integrability of the system is preserved, if one does not allow conversing between the two

states. In this case the masses of the two components are equal and the constraint can be written in a very simple way:

$$a_{11} < a_{12} < a_{22} \quad \text{or} \quad a_{22} < a_{12} < a_{11}.$$

This inequalities tell us that if the interspecies scattering length is greater or lesser than that of the intraspecies ones then the sign of the external potentials will be different. This case corresponds to untrapping one of the BEC components.

4. Based on the similarity of the coupled non-linear Schrödinger equations with and without external potentials, a relation is established between their solutions [2]. The external potentials are assumed to modify both amplitudes and phases together with the spatial and temporal variables appearing in the transformation relation. On examples of trap potentials depending linearly and quadratically on spatial coordinate it is demonstrated how the transformation ansatz is working in practice for obtaining atomic solitons of Bose-Einstein condensates consisting of sodium and rubidium atoms when one starts from known optical soliton solution of the non-linear Schrödinger equation without potential.
5. A simple method is presented [3] to analyze the stability of static solitonic excitations of two-component Bose-Einstein condensates described within the Gross-Pitaevskii approximation. In case of one-component condensate there is only one interaction parameter, the  $s$ -wave scattering length which can be tuned, and this parameter is totally determine the stability of the system in a given trap potential. In case of the two-component condensates there are three scattering lengths, the intra- and interspecies ones. The question of general stability is yet an unresolved problem.
6. I have derived inequalities [4] for a quick assessment for the values of interspecies scattering length allowing existence of static solitons independent of the particle numbers. The inequalities depend only on the intra- and interspecies scattering lengths. If the physical value of interspecies scattering length falls into that acceptable interval, then a fine tuning is performed to get the ratio of particle numbers in each component at which the static solitons may be created. The technique is illustrated for four two-component systems, e.g., for the lithium-rubidium and sodium-rubidium condensates [4].
7. During the period of my Ph.D. course I gave from time to time partial account on the status of the research in the form of conference talks [I, II, V] or seminar talks [III, IV, VI]. Furthermore, besides the analytical

calculations we developed a numerical code for the Lax-representation of the Inverse Scattering Transform [a].

## List of publications

### Publications related to the Ph.D. thesis

- [ 1.] Dániel Schumayer and Barnabás Apagyi,  
*Painlevé test of coupled Gross-Pitaevskii equations*,  
J. Phys. A.: Math. Gen., **34**(23) 4969 (2001)
- [ 2.] Dániel Schumayer and Barnabás Apagyi,  
*Relation between optical and atomic solitons*,  
Phys. Rev. A., **65**(5) 053614 (2002)
- [ 3.] Dániel Schumayer and Barnabás Apagyi,  
*Stability of static solitonic excitations of two-component Bose-Einstein condensates in finite range of interspecies scattering length  $a_{12}$* ,  
Phys. Rev. A., **69**(4) 043620 (2004)
- [ 4.] Barnabás Apagyi and Dániel Schumayer,  
*Assessment of interspecies scattering lengths  $a_{12}$  from stability of two-component Bose-Einstein condensates*,  
submitted for publication

### Conferences, Seminar talks

- [I.] Schumayer Dániel és Apagyi Barnabás,  
*Painlevé test of coupled nonlinear Schrödinger equations and Bose-Einstein condensates*  
Q-Math8, Taxco, Oral presentation, (2001)
- [II.] Schumayer Dániel és Apagyi Barnabás,  
*On the existence of soliton excitations in multicomponent Bose-Einstein condensation*  
ICAP 2002, Cambridge, MIT-Harvard, Poster session, (2002)
- [III.] Schumayer Dániel és Apagyi Barnabás,  
*Painlevé test of Coupled Gross-Pitaevskii Equations and Bose-Einstein Condensates*  
Seminar talk at Physikalisches Institut der Universität Tübingen, (2002)

# 1 Introduction

On the basis of quantum mechanics and statistical physics it is easy to verify that a condensation phenomenon occurs in bosonic systems at a given low temperature. This means that a macroscopical part of the particles fills the same quantum state. Despite the early stage of theoretical forecast of this effect in 1924 the experimental observation and verification has been achieved only seven decades later in 1995 for finite alkali systems.

After observing this phenomenon an extensive theoretical study of such systems has been started and diversified by this time. The great interest can be explained by the fact that the study of Bose-Einstein condensates (BEC) extends our knowledge in different branches of theoretical and experimental physics, such as in optics, statistical physics, thermodynamics, atomic collision theory, quantum properties of mesoscopic systems

One of our basic endeavour is to describe the dynamical behaviour of this system and thus examine the type of excitations in it. In case of the Bose-Einstein condensates this aim requires using many-body formalism in which only two-particle hard-sphere interaction is taken into account. This simplification, however, reduces the problem to some extent, but the equation of motion remains nonlinear. This is the so called Gross-Pitaevskii equation.

## 2 Objective

Examining such many particle systems one of our fundamental aims is to deduce or determine the energy spectrum, viz. we should determine what kind of excitations can occur. The objective of my Ph.D. work is to find special type of excitations, called solitons, in two-component Bose-Einstein condensation.

Firstly, my intention was to investigate whether or not the coupled Gross-Pitaevskii equation has solitonic solutions. Secondly, I wanted to present a method for generating such solutions (if the answer is affirmative). This question is not so trivial being closely related with the notion of integrability. The examination requires thus some preliminary mathematical calculations. If the equation passes the test for integrability then we may try to solve it by using the Inverse Scattering Transform.

I was motivated by the experimental observations which have been made several years ago in one-component systems. In Bose-Einstein condensates two groups were able to produce both *bright* and *dark* solitonic excitations which are, perhaps, the two most simple types of solitons.