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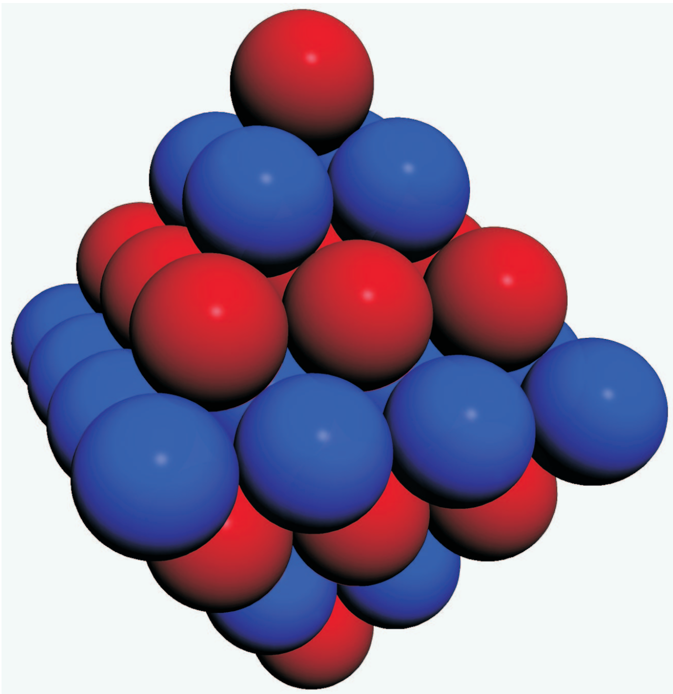
Finite ball packings and coverings

Ph.D. thesis summary



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Introduction

The thesis consists of six chapters. The Introduction briefly summarizes the earlier results about packing of equal balls into three polytopes.

Chapter 1 discusses the known optimal packings of equal balls into the three polyhedron.

Chapter 2 considers packings of equal balls into the d-dimensional crosspolytope for $d \geq 4$.

Chapter 3 verifies the local stability of an infinite family of lattice-like packings of equal balls into the octahedron and into the four-dimensional crosspolytope.

Chapter 4 discusses the coverings of the crosspolytope by equal balls.

Chapter 5 solves the problem of covering S^d by $d+3$ equal spherical balls.

Chapter 6 provides a good density bound for coverings of S^d by equal spherical balls.

Let me summarize the main results of each chapter.

1. Chapter

The octahedron

Let $v_1, \dots, v_3$ be an orthonormal base of the 3–space, and we write O to denote the 3–dimensional octahedron whose vertices are $\pm v_1, \dots, \pm v_3$, and hence the edge length of O is $\sqrt{2}$. We write r_n to denote the maximal radius of n non-overlapping balls in O.

1. Theorem: *If $n = 3$ than the points form a regular triangle. One of the points is a vertex of O (for example v_1), the other two points lie on the edges which correspond with v_{d+1} and the plane of the triangle cuts O in a square.*

2. Theorem: *If $n = 4$ than the optimal packing is the four vertices of O or the three vertices which correspond with a two dimensional face plus the central point of the opposite face. $r_4 = r_5 = r_6 = \sqrt{3} - \sqrt{2}$, $\rho_4 = 4\pi(9\sqrt{3} - 11\sqrt{2}) \approx 0,4035$, $\rho_5 = 5\pi(9\sqrt{3} - 11\sqrt{2}) \approx 0,5044$, $\rho_6 = 6\pi(9\sqrt{3} - 11\sqrt{2}) \approx 0,6053$.*

3. Theorem: If $n = 7$ than the optimal packing of the n points are the center point and the vertices of O. $r_7 \approx 0,1895$.

2. Chapter

The d-dimensional regular crosspolytope

Let $v_1, \dots, v_d$ be an orthonormal base of the d–space, and we write O^d to denote the d–dimensional regular crosspolytope whose vertices are $\pm v_1, \dots, \pm v_d$, and hence the edge length of O^d is $\sqrt{2}$.

In this chapter, we determine the maximal radius $r(n, d)$ of n non–overlapping spheres within O^d for $n \le 2d + 1$.

We note that it is equivalent to consider the maximum $\varphi(n, d)$ of the minimal distance of n points in O^d.

We present the results and arguments in terms of point sets in O^d because the formulation and the proofs are more transparent this way.

It is probably surprising but the 3-dimensional results generalizable to any dimension d; namely, we verify that

$$\varphi(n, d) = \begin{cases} 2\left(\sqrt{3} - 1\right) & \text{ha } n = 3 \\ \sqrt{2} & \text{ha } 4 \le n \le 2d \\ 1 & \text{ha } n = 2d + 1. \end{cases}$$

1. Theorem: *For $d \geq 3$, let the minimal distance among three points in O^d be maximal. Then one of the points is a vertex, say v_i, and the three points form a regular triangle that is contained in the square with vertices $\pm v_i, \pm v_j$ for some $v_j \neq v_i$.*

Now if the number n of the points in O^d is between 4 and $2d$ then one can not do better than placing them at the vertices:

2. Theorem: *For $d \geq 3$, let the minimal distance among n points in O^d be maximal where $4 \leq n \leq 2d$. Then either each point is a vertex of O^d, or $n = 4$, three of the points are the vertices of a two–face of O^d, and the fourth point is the centroid of the opposite two–face.*

In case of ball packings, Theorem 2 has the following interesting corollary:

If a d–dimensional regular crosspolytope contains four equal solid balls then it can host even $2d$ solid balls of the same radius.

Finally there exists only one optimal configuration if $n = 2d + 1$:

3. Theorem: *For $d \geq 3$, let the minimal distance among $2d + 1$ points in O^d be maximal. Then one of the points is the centre of O^d, and the other points are the vertices of O^d.*

3. Chapter

Local stability

For $d = 3, 4$ it is widely believed that the lattice $D_d = \{(x_1, \ldots, x_d) \in Z^d : x_1 + \ldots + x_d \equiv 0 \bmod 2\}$ provides a densest packing of equal balls. Since the structure of the D_d lattice corresponds quite satisfactory to O^d, we have a real chance that $\lambda \cdot D_d \cap O^d$, for suitable $\lambda \in R$ is an optimal configuration.

Definition: *The set of points is a lattice-like in O^d if the point set is*

$$\left(\frac{1}{k-1} D_d + v_1\right) \cap O^d, \qquad \textit{for some} \quad k \geq 2, \ \ k \in Z.$$

The minimal distance in this lattice is $\frac{\sqrt{2}}{k-1}$. If we reduce this point system in $\frac{\frac{\sqrt{2}}{k-1}}{\frac{\sqrt{2}}{k-1}\sqrt{d}+2} = \frac{1}{\sqrt{d}+\sqrt{2}(k-1)}$ then we obtain the centres of the corresponding lattice-like ball packing.

We will show that the lattice packings in the 3 and 4-dimensional crosspolytopes are locally stable.

Definition: *A packing of balls of radius r in O^d is called locally stable if there exists $\varepsilon > 0$ with the following property: if the balls are perturbed in a way that the resulting balls form a packing in O^d and the distance of the corresponding centres is at most ε then the new and the old packing coincide.*

Theorem: *Any lattice-like packing of equal balls in O^d, $d = 3, 4$ is locally stable.*

4. Chapter

The covering of the d-dimensional crosspolytope

This section discusses coverings of the d-dimensional crosspolytope O^d by equal balls. As before, $v_1, \dots v_d$ form an orthonormal basis of E^d, and the vertices of O^d are $\pm v_1, \dots \pm v_d$. In addition, R^d_n denotes the minimal radius of the equal balls covering O^d.

1. The case of 2 balls

Theorem: $R^3_2 = \frac{\sqrt{11}}{4}$*, the centroids of the balls are e.g.* $\pm(1/4, 1/4, 1/4)$. $R^d_2 = \sqrt{1 - \frac{1}{d}}$ *, if $d \geq 4$ then the centroids of the balls are the centroids of two opposite hyperfaces of O^d.*

2. The case of d balls

Theorem:

$$R^d_d = \begin{cases} \frac{\sqrt{5}}{3}, & \text{if } d = 3 \\ \sqrt{\frac{11}{20}}, & \text{if } d \geq 4. \end{cases}$$

1. Assertion: *The three balls with radii $R = \frac{\sqrt{5}}{3}$ and centers with* $(1/3, -1/3, 0)$*;* $(0, 1/3, -1/3)$*;* $(-1/3, 0, 1/3)$ *cover O^3, so $R^3_3 \leq \frac{\sqrt{5}}{3}$.*

2. Assertion: 3 *balls with radii smaller then $\frac{\sqrt{5}}{3}$ can not cover O.*

The 1. and 2. Assertions proof the Theorem if $d = 3$.

3. Assertion: *d balls ($d \geq 4$) with radii* $\sqrt{\frac{11}{20}}$ *cover* O^d *so* $R_d^d \leq \sqrt{\frac{11}{20}}$.

4. Assertion: *If the* $B_1, \ldots, B_d$ *balls with radii* R *cover the edges of* O^d *then* $R \geq \sqrt{\frac{11}{20}}$.

The 3. and 4. Assertions proof the Theorem if $d \geq 4$.

3. The case of 2d balls

Theorem: $R_d^{2d} = 1/2$.

5. Chapter

Covering S^d by $d+3$ balls

We deal with the covering of S^d by n equal spherical balls, where $n = d+3$. Nagyon kevés esetben ismert az S^d-t fedő egybevágó szférikus gömbök minimális sugara.

Conjecture: *For* $d \geq 3$ *and* $d+2 \leq n \leq 2d+2$ *let* n *equal spherical balls of minimal radius cover* S^d. *Then the convex hull of the centres in* E^{d+1} *is the direct sum of mutually orthogonal* $\lceil \frac{d+1}{n-d-1} \rceil$ *and* $\lfloor \frac{d+1}{n-d-1} \rfloor$ *dimensional regular simlicies of circumradius one, and the tital number of these simplicies is* $n-d-1$.

The case $d = 2$ has been actually handled earlier by L. Fejes Tóth [**7**]). We now consider the case of $n = d+3$ balls.

Theorem: *If* $d+3$ *equal spherical balls with minimal possible radius cover* S^d *then the convex hull of the centres in* E^{d+1} *is the convex hull of a* $\lfloor \frac{d+1}{2} \rfloor$ *and a* $\lceil \frac{d+1}{2} \rceil$*-dimensional regular simplex of circumradius one.*

The theorem requires the following lemma.

Lemma: *Spherical balls of radius* $\varphi < \frac{\pi}{2}$ *cover* S^d *if and only if the convex hull of the centres in* E^{d+1} *containes a ball that is centred at the origin and of radius* $\cos \varphi$.

6. Chapter

The density of the covering of the d-dimensional sphere

C.A. Rogers [**12**] verified the existence of a covering of S^d by spherical balls of radius φ whose density is at most

$$(1 + c \cdot \tfrac{\ln \ln d}{\ln d}) \cdot d \cdot \left(\ln d + \ln \frac{1}{\sin \varphi} \right),$$

for some absolute constant c.

We give an argument for a better density bound.

Theorem: *If $\varphi < \frac{\pi}{2}$ then there exists a covering of S^d by spherical balls of radius φ whose density is at most*

$$(1 + c \cdot \tfrac{\ln \ln d}{\ln d}) \cdot d \ln d,$$

for some absolute constant c

Remark: K. Böröczky Jr. and G. Wintsche [**4**], verifies that S^d can covered by equal spherical balls of any raius such that any point of S^d is covered at most $400 \cdot d \ln d$ times.

Using the classical Coxeter-Few-Rogers [**5**] bound, it follows that if φ is small then any covering of S^d by spherical balls of radius φ has density at least $c \cdot d$ for some absolute constant c.

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