

Numerical solutions of malaria models  
– a qualitative approach

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Doctoral Thesis Booklet

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The thesis addresses numerical solutions of malaria models, containing three main parts introduction, qualitative properties of malaria models, and qualitative analysis of discrete malaria models. The objective of the thesis is the investigation of the dynamical consistency, i.e., the discrete schemes preserve the qualitative properties of the continuous models. For this aim, in the first step, some mathematical models of malaria propagation and their important qualitative properties are considered. Then the dynamical consistency of the models is analyzed. The results are summarized as follows.

## 1 Mathematical modeling of malaria propagation

Malaria is an ancient vector-borne disease transmitted through infected bites from female Anopheles mosquitoes. It is endemic in tropical and subtropical regions affecting almost half of the world. Since it is fatal and there is no effective vaccine, it is worth studying. In the propagation process, an infected mosquito bites a susceptible human who becomes infected (or not), and in the next round, if a susceptible mosquito bites this human then it becomes infected, and so on. A variety of mathematical models have been brought up to interpret the mechanism of malaria propagation.

**The Ross–Macdonald model.** The most preliminary model for malaria transmission is the Ross–Macdonald model given by a system of nonlinear ODEs as below

$$\begin{cases} \dot{S}_h(t) = dN_h + rI_h(t) - b\beta_h S_h(t) \frac{I_m(t)}{N_h} - dS_h(t) \\ \dot{I}_h(t) = b\beta_h S_h(t) \frac{I_m(t)}{N_h} - (r + d)I_h(t) \\ \dot{S}_m(t) = \mu N_m - b\beta_m S_m(t) \frac{I_h(t)}{N_h} - \mu S_m(t) \\ \dot{I}_m(t) = b\beta_m S_m(t) \frac{I_h(t)}{N_h} - \mu I_m(t), \end{cases} \quad (1)$$

where  $S_h(t)$ , and  $S_m(t)$  signify the number of susceptible humans and mosquitoes at time  $t$ , respectively.  $I_h(t)$ , and  $I_m(t)$  are the number of infected humans and mosquitoes at time  $t$ , respectively. The parameter  $b$  is the rate of mosquito biting,  $\beta_h$  is the probability that a bite from an infected mosquito will cause infection of a susceptible human, and  $\beta_m$  is the probability that a bite from a susceptible mosquito to an infected human individual will cause infection of the mosquito. The parameter  $r$  is the recovery rate for humans,  $d$  and  $\mu$  are the birth and mortality rates for humans and mosquitoes, respectively. The numbers of the total populations are defined as the sum of the number of susceptible and infected individuals that could be time-dependent in general. Easy

calculations from the system (1) result in that the numbers of the total population for humans and mosquitoes,  $N_h$  and  $N_m$ , are constant

$$S_h(t) + I_h(t) = N_h, \quad \text{and} \quad S_m(t) + I_m(t) = N_m. \quad (2)$$

Using (2) we can normalize and reduce the system (1) to the system

$$\begin{cases} \dot{x}(t) = \alpha y(t)(1 - x(t)) - rx(t) \\ \dot{y}(t) = \beta(1 - y(t))x(t) - \mu y(t), \end{cases} \quad (3)$$

where  $m = \frac{N_m}{N_h}$ ,  $x(t)$  is the density of infected humans, and  $y(t)$  signifies the density of infected mosquitoes at time  $t$ ,  $\alpha = b\beta_h$  and  $\beta = b\beta_m$ . Also,  $r + d$  in the system (1) is replaced by new  $r$  in this system. Since the basic Ross–Macdonald model is highly simple and some important biological factors are not taken into account, some modified models are considered as follows.

**The extended Ross model.** If we assume a different incidence rate and we also take into account of the changing size of the population then one possible model is the following

$$\begin{cases} \dot{S}_h(t) = \Lambda_h - \frac{b\beta_h S_h(t) I_m(t)}{1 + \nu_h I_m(t)} - \mu_h S_h(t) + \omega R_h(t) \\ \dot{E}_h(t) = \frac{b\beta_h S_h(t) I_m(t)}{1 + \nu_h I_m(t)} - (\alpha_h + \mu_h) E_h(t) \\ \dot{I}_h(t) = \alpha_h E_h(t) - (r + \mu_h + \delta_h) I_h(t) \\ \dot{R}_h(t) = r I_h(t) - (\mu_h + \omega) R_h(t) \\ \dot{S}_m(t) = \Lambda_m - \frac{b\beta_m S_m(t) I_h(t)}{1 + \nu_m I_h(t)} - \mu_m S_m(t) \\ \dot{E}_m(t) = \frac{b\beta_m S_m(t) I_h(t)}{1 + \nu_m I_h(t)} - (\alpha_m + \mu_m) E_m(t) \\ \dot{I}_m(t) = \alpha_m E_m(t) - (\mu_m + \delta_m) I_m(t). \end{cases} \quad (4)$$

Here the functions  $E_h(t)$  and  $E_m(t)$  describe the number of the exposed humans and mosquitoes at time  $t$ , respectively. The function  $R_h(t)$  is the number of the recovered humans at time  $t$ .

In the extended model (4), we assume that all children are born healthy, so the birth rate ( $\Lambda_h, \Lambda_m$ ) is only enters into the class of susceptible individuals ( $S_h, S_m$ ).

Since malaria has a 2–4 weeks latent period, when the parasite is injected into the blood system with some probability ( $\beta_h$ ), the susceptible human moves to the exposed class  $E_h(t)$ . Exposed humans are not able to transmitting the disease to the susceptible mosquitoes as the parasites are in asexual stages.

When the incubation period is over the exposed human is progressed to the infectious state with some  $\alpha_h$  rate. Then the individual in this class will either die or recover and moves to the recovered ( $R_h$ ) class. The recovered human has some immunity to the disease, however after a few year the individual loses the immunity and become susceptible again.

As the model distinguish natural and disease induced death rate, every class is decreased by natural death rate  $\mu_h$ , and only the infectious class is decreased by disease induced death rate  $\delta_h$ .

In a similar way, when a susceptible mosquito bites an infected human, the parasite enters the mosquito with some probability  $\beta_m$  and moves to the exposed class  $E_m(t)$ . After a given time it becomes infectious. Mosquitoes leave the population by natural death rate  $\mu_m$  or disease induced death rate  $\delta_m$ .

Ratio  $\frac{I_m(t)}{1 + \nu_h I_m(t)}$  denotes a saturating feature that inhibits the force of infection from infectious mosquitoes to susceptible humans. In other words  $\nu_h \in [0, 1]$  is the proportion of antibodies produced by human in response the incidence of antigens produced by infectious mosquito. This interpretation can be used for mosquitoes similarly, whereas  $\nu_m \in [0, 1]$  is the proportion for which antibodies are produced against the antigens contacted from infectious humans.

Since malaria transmission occurs in an inharmonious population, the epidemiological model (4) must partition the population into groups, in which the members have similar characteristics such as mode of transmission, contact patterns, latent period, infectious period, genetic susceptibility or resistance. Accordingly, the total human population size at time t,  $V_h(t)$ , is defined as

$$V_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t). \quad (5)$$

The same definition holds for the total size of the mosquitoes population,  $V_m(t)$ , at time t

$$V_m(t) = S_m(t) + E_m(t) + I_m(t). \quad (6)$$

**The delayed Ross–Macdonald model.** Since malaria parasites run some latent period in humans' and mosquitoes' bodies, another modification of the Ross–Macdonald model is given by the delayed differential equations as

$$\begin{cases} \dot{x}(t) = b\beta_h m y(t - \tau_h)(1 - x(t - \tau_h))e^{-r\tau_h} - r x(t) \\ \dot{y}(t) = b\beta_m(1 - y(t - \tau_m))x(t - \tau_m)e^{-\mu\tau_m} - \mu y(t). \end{cases} \quad (7)$$

Where  $\tau_h$  and  $\tau_m$  are the latent periods for humans and mosquitoes, apiece. The terms  $e^{-r\tau_h}$  and  $e^{-\mu\tau_m}$  represent the probability that an infected human and mosquito respectively, can survive their latent periods.

## 2 Qualitative properties of malaria models

In this section, the qualitative properties of the models are discussed, i.e., positively invariant property for some sets. The findings are based on the research publications [1, 5, 6].

### 2.1 Qualitative properties of the Ross–Macdonald models

Since the solutions of the Ross–Macdonald models are densities, they should belong to the interval  $[0,1]$ . To this end, we prove the interval  $[0,1]$  is positively variant for these models. We will refer to this property as *density preservation* (DP) property and the following results are proven.

**Lemma 2.1.** *The Ross–Macdonald model (3) has the DP property.*

**Lemma 2.2.** *Let the history functions are mapping of type  $[-\tau^*, 0] \mapsto [0, 1]$ , and assume that the conditions*

$$\alpha e^{-r\tau_h} - r \leq 0 \quad \text{and} \quad \beta e^{-\mu\tau_m} - \mu \leq 0 \quad (8)$$

*hold. Then the delayed Ross–Macdonald model (7) possesses the DP property.*

Here  $\tau^* = \max\{\tau_h, \tau_m\}$ .

### 2.2 Qualitative properties of the extended Ross model

In this part, we prove that some sets are positively invariant for the total populations and each component of the solution of the extended Ross model (4).

**Theorem 2.3.** *Assume that  $\Lambda_h, \Lambda_m > 0$  and all initial values in (4) are positive. Then the solution of the model (4) is positive for all  $t$ .*

**Theorem 2.4.** *If  $\Lambda_h, \Lambda_m > 0$  and the initial total populations of the system (4) are in the intervals  $(0, \frac{\Lambda_h}{\mu_h}]$  and  $(0, \frac{\Lambda_m}{\mu_m}]$  for humans and mosquitoes respectively, then these intervals are positively invariant for the total populations and for each compartment.*

We can extend Theorem 2.4 with the notations

$$V_h^* = \max\{V_h(0), \frac{\Lambda_h}{\mu_h}\}, \quad \text{and} \quad V_m^* = \max\{V_m(0), \frac{\Lambda_m}{\mu_m}\} \quad (9)$$

as follows.

**Theorem 2.5.** *Assume that  $\Lambda_h, \Lambda_m > 0$  and the initial total populations of the system (4) are on the intervals  $(0, V_h^*)$  and  $(0, V_m^*)$  for humans and mosquitoes, respectively. Then the intervals  $(0, V_h^*)$  and  $(0, V_m^*)$  are positively invariant for each component and for the total populations for humans and mosquitoes respectively.*

**Corollary 2.6.** *The limit behavior of the total populations and each component of the solution of the extended Ross model (4) can be considered on the intervals  $(0, \frac{\Lambda_h}{\mu_h}]$  and  $(0, \frac{\Lambda_m}{\mu_m}]$  for humans and mosquitoes, respectively. Particularly when there is no death due to the disease ( $\delta_h(t) = \delta_m(t) = 0$ ), the limit of the total population for humans and mosquitoes is  $\frac{\Lambda_h}{\mu_h}$  and  $\frac{\Lambda_m}{\mu_m}$ , respectively.*

### 3 Qualitative analysis of discrete malaria models

In this section, we consider the dynamical consistency of the models. For this aim, we apply the explicit and implicit Euler methods to the Ross–Macdonald models and the standard  $\theta$ -method to the extended Ross model. The results are based on the papers [2, 3, 4, 5, 7].

#### 3.1 The discrete Ross–Macdonald models

In this part, we seek the DP property for the discrete Ross–Macdonald models by applying the explicit and implicit Euler methods.

**The Ross–Macdonald model.** By applying the explicit and implicit Euler methods to the Ross–Macdonald (3) we attain

$$\begin{cases} x_{i+1} = x_i + \Delta t (\alpha y_i (1 - x_i) - r x_i) \\ y_{i+1} = y_i + \Delta t (\beta x_i (1 - y_i) - \mu y_i), \end{cases} \quad (10)$$

and

$$\begin{cases} x_{i+1} = x_i + \Delta t (\alpha y_{i+1} (1 - x_{i+1}) - r x_{i+1}) \\ y_{i+1} = y_i + \Delta t (\beta x_{i+1} (1 - y_{i+1}) - \mu y_{i+1}), \end{cases} \quad (11)$$

respectively and the following results are proven. We have the following results for the above discrete models.

**Theorem 3.1.** • *The explicit Euler method applied to the Ross–Macdonald model (10) possesses the DP property for any step size  $\Delta t \in (0, h^*]$ , where*

$$h^* = \min \left\{ \frac{1}{r}, \frac{1}{\mu}, \frac{1}{\alpha}, \frac{1}{\beta} \right\}. \quad (12)$$

- *The implicit Euler method applied to the Ross–Macdonald model (11) possesses the DP property unconditionally.*

**The delayed Ross–Macdonald model.** Using the explicit and implicit Euler methods to the delayed Ross–Macdonald (7) results in

$$\begin{cases} x_{i+1} = x_i + \Delta t (\alpha y_{i,\tau_h} (1 - x_{i,\tau_h}) e^{-r\tau_h} - r x_i) \\ y_{i+1} = y_i + \Delta t (\beta x_{i,\tau_m} (1 - y_{i,\tau_m}) e^{-\mu\tau_m} - \mu y_i), \end{cases} \quad (13)$$

and

$$\begin{cases} x_{i+1} = x_i + \Delta t (\alpha y_{i+1,\tau_h} (1 - x_{i+1,\tau_h}) e^{-r\tau_h} - r x_{i+1}) \\ y_{i+1} = y_i + \Delta t (\beta x_{i+1,\tau_m} (1 - y_{i+1,\tau_m}) e^{-\mu\tau_m} - \mu y_{i+1}), \end{cases} \quad (14)$$

respectively with the following results.

**Theorem 3.2.** *We assume that (8) holds.*

- *If*

$$h^{**} = \min \left\{ \frac{1}{r}, \frac{1}{\mu}, \frac{e^{r\tau_h}}{\alpha}, \frac{e^{\mu\tau_m}}{\beta} \right\}, \quad (15)$$

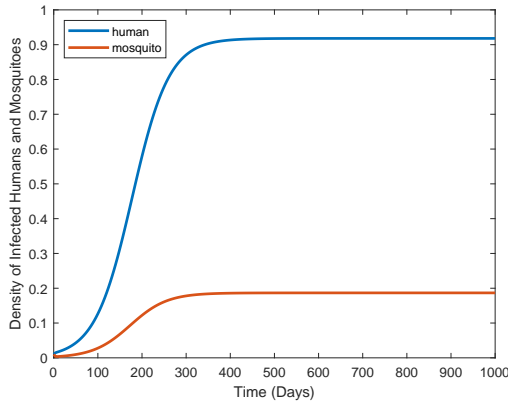
*then the explicit Euler method applied to the delayed Ross–Macdonald model (13) possesses the DP property for any step size  $\Delta t \in (0, h^{**}]$ .*

- *The implicit Euler method applied to the delayed Ross–Macdonald model (14) possesses the DP property provided that*

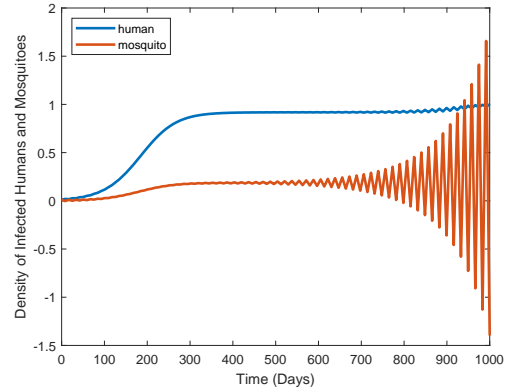
$$\Delta t \leq \min \{ \tau_h, \tau_m \}. \quad (16)$$

### 3.2 Numerical simulations

We demonstrate the theoretical results by examples.



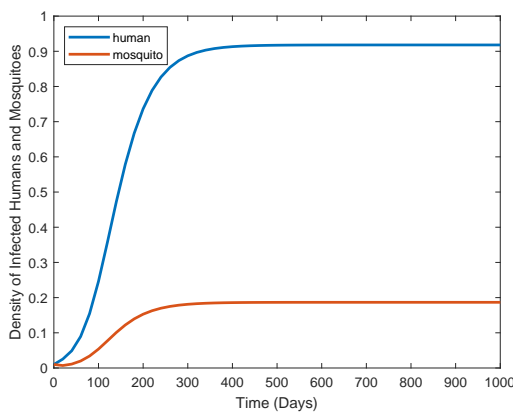
(a)  $\Delta t = 4$



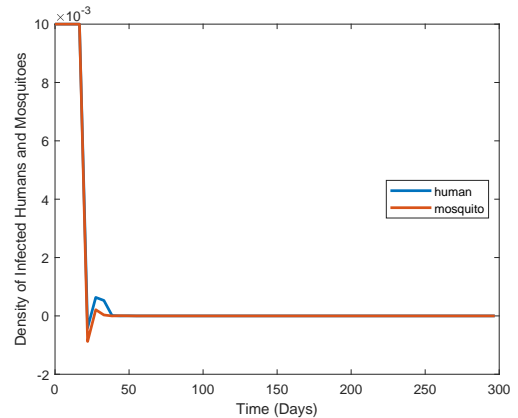
(b)  $\Delta t = 6$

Figure 1: EEM, Density of Infected Humans and Mosquitoes of the Ross–Macdonald Model

In this example, the step size tolerance for the explicit Euler scheme (10) is  $h^* = 5$ . Fig.1.(a) confirms the results of Theorem 3.1 for which with  $\Delta t \in (0, h^*]$  the solution possesses DP property. Fig.1.(b) shows when  $\Delta t \notin (0, h^*]$ , the DP property is not preserved for the explicit Euler method applied to the Ross–Macdonald model.



(a) IEM,  $\Delta t = 20$



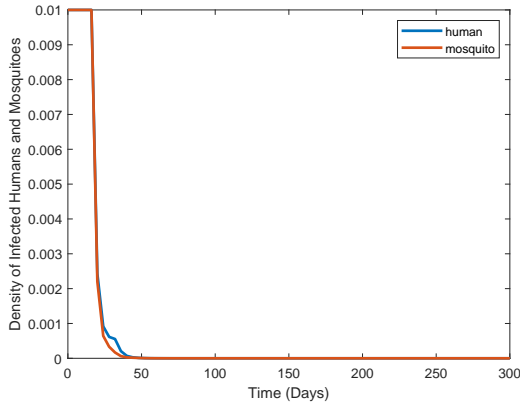
(b) EEM-Delayed model,  $\Delta t = 5.1$

Figure 2: Density of Infected Individuals of the Basic and Delayed Ross–Macdonald Models

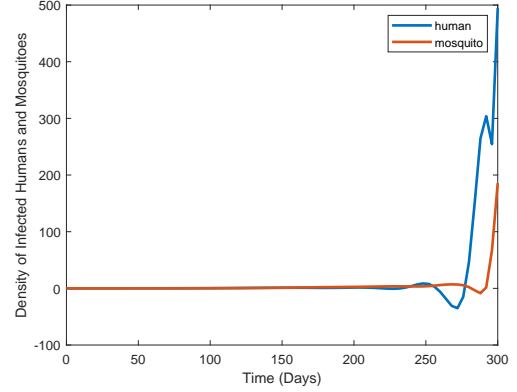
According to fig.2.(a), by applying the implicit Euler method to the basic Ross–Macdonald model, the DP property holds for for the adequately large step size. Fig.2.(b) visualizes the results of the explicit Euler method applied to the delayed Ross–Macdonald



model and depicts that when the condition (8) holds for the humans and mosquitoes and  $\Delta t \notin (0, h^{**}]$ , the DP property is not satisfied, here  $h^{**} = 5$ .



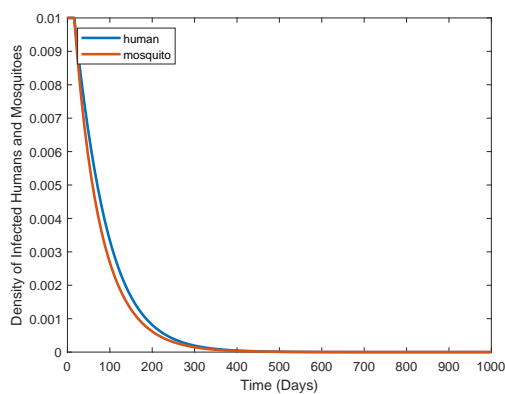
(a) Conditions (8) hold,  $\Delta t = 4$



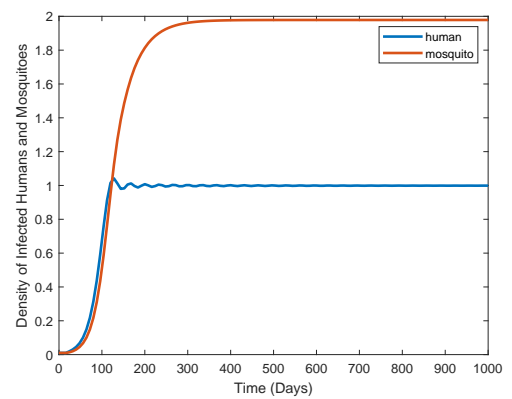
(b)  $\Delta t = 4$

Figure 3: EEM, Density of Infected Humans and Mosquitoes of the Delayed Ross–Macdonald Model

Fig.3 illustrates the numerical results of the explicit Euler method applied to the delayed Ross–Macdonald model. Fig.3.(a) confirms the first part of Theorem 3.2 where the conditions (8) are satisfied for humans and mosquitoes with  $r = 0.2$  and the step size tolerance  $h^{**} = 5$ . Fig.3.(b) displays the non-desired results occur when the conditions (8) do not hold neither for humans nor mosquitoes when we apply  $\mu = 0.002$  with  $h^{**} = 8.536$ .



(a) Conditions (8) hold,  $\Delta t = 8$



(b)  $\Delta t = 8$

Figure 4: IEM Applied to the Delayed Ross–Macdonald Model

Fig.4 depicts the results of the implicit Euler method applied to the delayed Ross–Macdonald method. In fig.4.(a) we consider the parameter  $r = 0.2$  to have the condi-

tions (8) for humans and mosquitoes, then the second part of Theorem 3.2 validates for  $\Delta t = 8$ . Fig.4.(b) illustrates if the conditions (8) do not hold while (16) is satisfied, the second part of Theorem 3.2 is not valid.

### 3.3 The discrete extended Ross model

In this part, similar to the continuous model, we prove that some sets are positively invariant for the discrete extended Ross model by applying the standard  $\theta$ -method. Due to the structure of the model, we use a nonlocal discretization to the right side of the system resulting in a semi-implicit method as follows.

$$\begin{cases} \frac{S_h^{i+1} - S_h^i}{\Delta t} = (1 - \theta)(\Lambda_h - \frac{b\beta_h S_h^i I_m^i}{1 + \nu_h I_m^i} - \mu_h S_h^i + \omega R_h^i) + \theta(\Lambda_h - \frac{b\beta_h S_h^{i+1} I_m^i}{1 + \nu_h I_m^i} - \mu_h S_h^{i+1} + \omega R_h^i) \\ \frac{E_h^{i+1} - E_h^i}{\Delta t} = (1 - \theta)(\frac{b\beta_h S_h^i I_m^i}{1 + \nu_h I_m^i} - (\alpha_h + \mu_h)E_h^i) + \theta(\frac{b\beta_h S_h^{i+1} I_m^i}{1 + \nu_h I_m^i} - (\alpha_h + \mu_h)E_h^{i+1}) \\ \frac{I_h^{i+1} - I_h^i}{\Delta t} = (1 - \theta)(\alpha_h E_h^i - (r + \mu_h + \delta_h)I_h^i) + \theta(\alpha_h E_h^{i+1} - (r + \mu_h + \delta_h)I_h^{i+1}) \\ \frac{R_h^{i+1} - R_h^i}{\Delta t} = (1 - \theta)(rI_h^i - (\mu_h + \omega)R_h^i) + \theta(rI_h^{i+1} - \mu_h R_h^{i+1} - \omega R_h^i) \\ \frac{S_m^{i+1} - S_m^i}{\Delta t} = (1 - \theta)(\Lambda_m - \frac{b\beta_m S_m^i I_h^i}{1 + \nu_m I_h^i} - \mu_m S_m^i) + \theta(\Lambda_m - \frac{b\beta_m S_m^{i+1} I_h^{i+1}}{1 + \nu_m I_h^{i+1}} - \mu_m S_m^{i+1}) \\ \frac{E_m^{i+1} - E_m^i}{\Delta t} = (1 - \theta)(\frac{b\beta_m S_m^i I_h^i}{1 + \nu_m I_h^i} - (\alpha_m + \mu_m)E_m^i) + \theta(\frac{b\beta_m S_m^{i+1} I_h^{i+1}}{1 + \nu_m I_h^{i+1}} - (\alpha_m + \mu_m)E_m^{i+1}) \\ \frac{I_m^{i+1} - I_m^i}{\Delta t} = (1 - \theta)(\alpha_m E_m^i - (\mu_m + \delta_m)I_m^i) + \theta(\alpha_m E_m^{i+1} - (\mu_m + \delta_m)I_m^{i+1}). \end{cases} \quad (17)$$

Since this system is linear, we can obtain the solution explicitly and the following results are proven.

**Theorem 3.3.** *Suppose  $\Lambda_h, \Lambda_m > 0$ . If  $\Delta t \in (0, h^*(\theta)]$  then the intervals  $(0, \frac{\Lambda_h}{\mu_h}]$  and  $(0, \frac{\Lambda_m}{\mu_m}]$  are positively invariant for the total number of humans and mosquitoes and each component of the solution of the system (17), respectively. Where*

$$h^*(\theta) = \begin{cases} \frac{1}{1-\theta} \min\{h_1, h_2\} & \theta \neq 1 \\ \frac{1}{\omega} & \theta = 1, \end{cases} \quad (18)$$

with

$$A_1 = \frac{\mu_m + \nu_h \Lambda_m}{b\beta_h \Lambda_m - \frac{\theta}{1-\theta} \mu_h (\mu_m + \nu_h \Lambda_m)}, \quad A_2 = \frac{\mu_m + \nu_h \Lambda_m}{b\beta_h \Lambda_m - \frac{\theta}{1-\theta} \mu_h (\mu_m + \nu_h \Lambda_m)},$$

$$B_1 = \min\left\{\frac{1}{\alpha_h + \mu_h}, \frac{1}{r + \mu_h + \delta_h}, \frac{1}{\mu_h + \frac{\omega}{1-\theta}}\right\}, \quad \text{and} \quad B_2 = \min\left\{\frac{1}{\alpha_m + \mu_m}, \frac{1}{\mu_m + \delta_m}\right\}.$$

- If  $A_1 > 0$ , then  $h_1 = \min\{A_1, B_1\}$ , otherwise  $h_1 = B_1$ .

- If  $A_2 > 0$ , then  $h_2 = \min\{A_2, B_2\}$ , otherwise  $h_2 = B_2$ .

Similar to the continuous model, we can extend the above results with the following theorem.

**Theorem 3.4.** *Assume that  $\Lambda_h, \Lambda_m > 0$ . If  $\Delta t \in (0, h^{**}(\theta)]$ , then the intervals  $(0, V_h^*)$  and  $(0, V_m^*)$  are positively invariant for each component of the solution of the system (17) and for the total populations for humans and mosquitoes, respectively. Here*

$$h^{**}(\theta) = \begin{cases} \frac{1}{1-\theta} \min\{h_1^*, h_2^*\} & \theta \neq 1 \\ \frac{1}{\omega} & \theta = 1, \end{cases} \quad (19)$$

with

$$A_1^* = \frac{V_h^*(1 + \nu_h V_m^*)}{V_h^*(b\beta_h V_m^* + \mu_h(1 + \nu_h V_m^*)) - \frac{\Lambda_h(1 + \nu_h V_m^*)}{1-\theta}},$$

$$A_2^* = \frac{V_h^*(1 + \nu_h V_m^*)}{V_h^*(b\beta_h V_m^* + \mu_h(1 + \nu_h V_m^*)) - \frac{\Lambda_h(1 + \nu_h V_m^*)}{1-\theta}},$$

$$B_1^* = \min\left\{\frac{1}{\alpha_h + \mu_h}, \frac{1}{r + \mu_h + \delta_h}, \frac{1}{\mu_h + \frac{\omega}{1-\theta}}\right\}, \quad \text{and} \quad B_2^* = \min\left\{\frac{1}{\alpha_m + \mu_m}, \frac{1}{\mu_m + \delta_m}\right\}.$$

- If  $A_1^* > 0$ , then  $h_1^* = \min\{A_1^*, B_1^*\}$ , otherwise  $h_1^* = B_1^*$ .
- If  $A_2^* > 0$ , then  $h_2^* = \min\{A_2^*, B_2^*\}$ , otherwise  $h_2^* = B_2^*$ .

**Remark 3.5.** Corollary 2.6 is valid for the discrete model (17).

### 3.4 Numerical simulations

In this section, we present numerical examples to illustrate the theoretical results regarding the explicit Euler method applied to the extended Ross model.

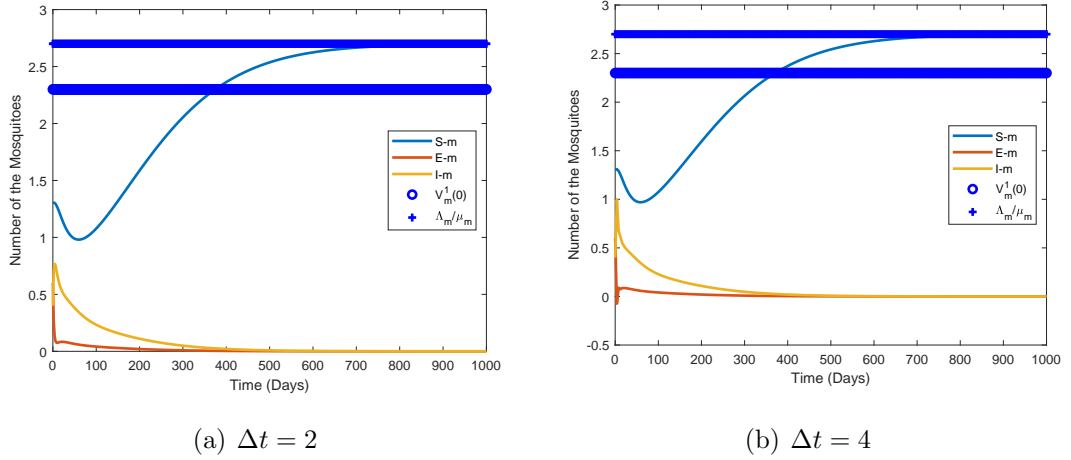


Figure 5: Explicit Euler Method, Mosquito

Fig.5.(a) shows the positively invariant property of the interval  $(0, \frac{\Lambda_m}{\mu_m}]$  for each mosquito component of the system (17), when  $\Delta t \in (0, h^*(0)]$ ,  $h^*(0) = 3.33$ , while in fig.5.(b) this property is not preserved when  $\Delta t > h^*(0)$ .

### 3.5 Nonstandard discretization of malaria models

According to the research findings, by applying the standard Runge–Kutta methods to the models, the dynamical consistency highly depends on the step size. As we treat the dynamical systems, we are interested in sufficiently large time intervals. Hence, small step sizes are not useful. To rectify this difficulty, we apply the nonstandard Runge–Kutta methods based on Mickens’s nonstandard finite difference (NSFD) method. Step-size functions and nonlocal discretization are two main features of the NSFD methods. The thesis mainly focuses on the step-size functions summarized in this part.

**Construction the step-size function.** Discretizing the continuous models on the fixed mesh

$$W_{\Delta t} = \{t_i = i\Delta t, i = 0, 1, 2, \dots\}, \quad (20)$$

requires the approximation of the time derivative of the unknown functions of the solution at the mesh points. This can be done by the formula

$$\dot{u}(t_i) \approx \frac{u^{i+1} - u^i}{\Phi(\Delta t)}. \quad (21)$$

Here  $u^i$  denotes the approximation to  $u(t)$  at the mesh point  $t = t_i$  and  $\Phi(\Delta t)$  is some suitably chosen step-size function. In the following, we consider the properties

of the non-negative and sufficiently smooth step-size function  $\Phi(\Delta t)$  in such a way to approximate  $\dot{u}(t)$  properly. The approximation (21) is first-order consistent if

$$\Phi(0) = 0, \quad \text{and} \quad \dot{\Phi}(0) = 1. \quad (22)$$

The step-size functions

$$\Phi(\Delta t) = \Delta t, \quad (23)$$

constructing the standard finite difference methods and

$$\Phi_C(\Delta t) = \frac{1 - e^{-C\Delta t}}{C}, \quad C \in \mathbb{R}_{\neq 0}, \quad (24)$$

using in Mickens's nonstandard finite difference method, are satisfying the conditions (22).

**Full discretization of the models.** We construct the nonstandard discrete models by applying the step-size function  $\Phi(\Delta t)$  instead of  $\Delta t$  to the systems (10) and (13) and we obtain

$$\begin{cases} \frac{x_{i+1} - x_i}{\Phi(\Delta t)} = b\beta_h m y_i (1 - x_i) - r x_i \\ \frac{y_{i+1} - y_i}{\Phi(\Delta t)} = b\beta_m x_i (1 - y_i) - \mu y_i, \end{cases} \quad (25)$$

and

$$\begin{cases} \frac{x_{i+1} - x_i}{\Phi(\Delta t)} = b\beta_h m y_{i,\tau_h} (1 - x_{i,\tau_h}) e^{-r\tau_h} - r x_i \\ \frac{y_{i+1} - y_i}{\Phi(\Delta t)} = b\beta_m x_{i,\tau_m} (1 - y_{i,\tau_m}) e^{-\mu\tau_m} - \mu y_i. \end{cases} \quad (26)$$

Similar results are valid for the extended Ross model (17). The immediate and nice consequence is the following.

**Theorem 3.6.** *The results of Theorems 3.1-3.4 are valid if we replace  $\Delta t$  with the step-size function  $\Phi(\Delta t)$ .*

The above theorem states that if we use the step-size function  $\Phi(\Delta t)$  instead of  $\Delta t$  in Theorems 3.1-3.4, we get  $\Phi(\Delta t) \leq H$  type restrictions instead of  $\Delta t \leq H$ . Here  $H$  is  $h^*$ ,  $h^{**}$ ,  $h^*(\theta)$ , and  $h^{**}(\theta)$ .

### Domain of the step-size functions.

- It is clear that by applying the function  $\Phi(\Delta t) = \Delta t$  we regain the results of the previous sections.
- The step-size function

$$\Phi_C(\Delta t) = \frac{1 - e^{-C\Delta t}}{C} \quad (27)$$

with  $C > 0$  is monotonically decreasing in  $C$ , consequently,

1. if  $\Delta t < H$ , then  $\Phi_C(\Delta t) \in (0, H]$  for all  $C > 0$ .
2. If  $\Delta t \geq H$ , there exists  $C_0$  such that  $\Phi_C(\Delta t) \in (0, H]$  for  $C \geq C_0$ .

Here  $C_0$  is the solution of

$$e^{-C_0\Delta t} + C_0H - 1 = 0, \quad (28)$$

implying the choice

$$C_0 \geq \frac{1}{H} \quad (29)$$

is a sufficient condition for the suitable choice of the parameter  $C_0$ . The exact solution of (28) lies on the principal branch of the Lambert W function as

$$C_0 = \frac{HW_0\left(\frac{-\Delta t}{H}e^{\frac{-\Delta t}{H}}\right) + \Delta t}{H\Delta t}, \quad (30)$$

when  $\Delta t \geq H$ .

- The step-size function (27) with  $C < 0$  is equivalent to the function

$$\Phi_C(\Delta t) = \frac{e^{C\Delta t} - 1}{C}, \quad (31)$$

accordingly, if

1.  $\Delta t \leq H$ , then there exists  $C_0$  such that  $\Phi_C(\Delta t) \in (0, H]$  for all  $C \leq C_0$ .
2.  $\Delta t > H$ , then there is no  $C$  such that  $\Phi_C(\Delta t) \in (0, H]$ .

The exact solution for the function (31) is on the negative branch of the Lambert W function as

$$C_0 = \frac{-HW_{-1}\left(\frac{-\Delta t}{H}e^{\frac{-\Delta t}{H}}\right) - \Delta t}{H\Delta t}. \quad (32)$$

### 3.6 Numerical simulations

In this section, we give some numerical examples to illustrate the theoretical results obtained in section 3.5. We investigate the sharpness of the upper bounds  $h^*$ ,  $h^{**}$ ,  $h^*(\theta)$ ,  $h^{**}(\theta)$  for sufficiently large step sizes.

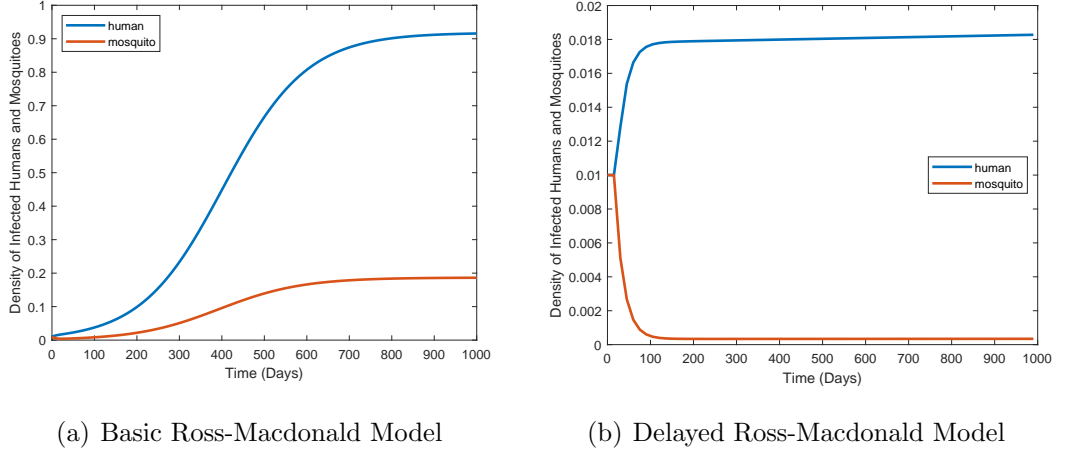


Figure 6:  $\Phi_{\Delta t}(0.4)$ , Explicit Euler Method,  $\Delta t=15$

Fig. 6 visualizes the step-size function  $\Phi_{\Delta t}(0.4)$  applied to the explicit Euler schemes of the basic and delayed Ross-Macdonald models preserves the DP property for sufficiently large step size  $\Delta t$ , while fig.1.(b) and 2.(b) denote that the DP property is not preserved for sufficiently large step sizes  $\Delta t$ . Here  $h^* = h^{**} = 5$ .

In the following, we consider the numerical results for the discretized extended Ross model (17).

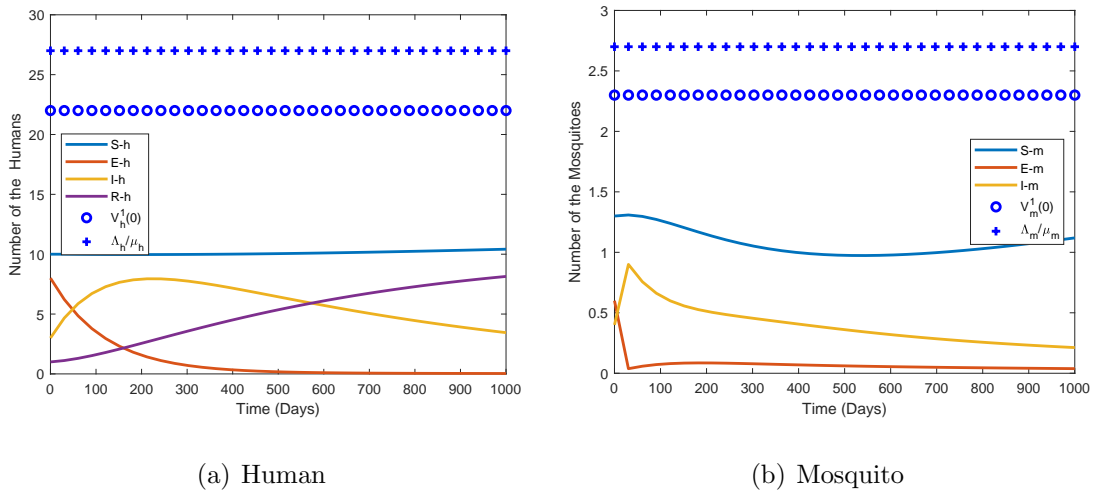


Figure 7:  $\Phi_{\frac{1}{h^*(0)}}(\Delta t)$ , Explicit Euler Method,  $\Delta t=30$

Fig.7 depicts that by applying the nonstandard explicit Euler method to the extended Ross model, the intervals  $(0, \frac{\Lambda_h}{\mu_h}]$  and  $(0, \frac{\Lambda_m}{\mu_m}]$  are positively invariant for each component of the solution with sufficiently large step size  $\Delta t$ .

## 4 Conclusion

According to the research findings, by applying the standard Runge–Kutta methods to the models, the dynamical consistency highly depends on the step sizes, while this property is valid for any large step size when we treat the models with the nonstandard Runge–Kutta methods.

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