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Investigation of depth of subgroups of finite groups

Ph.D. Dissertation Booklet

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1 Introduction

The notion of depth was originally defined for von-Neumann algebras, see [8]. Later it was also defined for Hopf algebras, see [21]. For some recent results in this direction, see [10, 18, 9]. In [20] and later in [6], the depth of semisimple algebra inclusions was studied, by Burciu, Kadison and Külshammer. First results concerned the depth 2 case. Later these were generalized for arbitrary positive integer n as depth. In the case of the group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$, it was shown that the depth is at most 2 if and only if H is normal in G, see [20]. For similar results on group algebras over commutative rings, see [4]. The **ordinary depth** of a group inclusion $H \leq G$ (denoted by $d(H,G)$) is defined as the minimal depth of the group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$, studied in [2]. In general there are several equivalent definitions of ordinary depth. We give details of several definitions in Chapter 2 of the Thesis. One of the definitions is with the help of the **inclusion or Frobenius matrix**. This definition was used in our GAP [7] programs to calculate the depth of a subgroup. To explain this, let G be a finite group, $H \leq G$, Irr(G)= $\{\chi_1, \chi_2, ..., \chi_k\}$ and Irr(H)= $\{\varphi_1, \varphi_2, ..., \varphi_r\}$. Then $M = (m_{i,j}) \in \mathbb{Z}^{rxk}$, where $m_{i,j} = (\varphi_i \uparrow^G, \chi_j) = (\varphi_i, \chi_j \downarrow_H)$, is the Frobenius matrix of $H \leq G$. Define the "Powers" of the Frobenius matrix as $M^{(2s)} = (MM^t)^s \in \mathbb{Z}^{r\times r}$, $M^{(2s+1)} = (MM^t)^s M \in \mathbb{Z}^{r\times k}$ and $M^{(0)} = I_{r\times r}$. The depth of the Frobenius matrix is

$$d(M) = \min\{i \geq 1 \mid \exists q > 0, M^{(i+1)} \leq qM^{(i-1)}\}$$

$$= \min\{i \geq 1 \mid Z(M^{(i+1)}) = Z(M^{(i-1)})\},$$

see [6, Propostion 2.2].(Here Z denotes the set of zero positions of a matrix.) The depth of a subgroup is the depth of its Frobenius matrix. For the theoretical proofs we were using another definition of depth with the help of G**-distance of characters**. We will give it in details in Section 2 below. To get an impression about it, we mention a remark of Dr. Erzsébet Lukács, saying that

$$d(H,G) = \max_{\phi,\psi \in \mathrm{Irr}(H)} d^G(\phi,\psi) + \max_{\chi \in \mathrm{Irr}(G)} m(\chi) + 1.$$

Here the G-distance $d^G(\phi,\psi) = 1$ if ϕ and ψ are constituents of the same irreducible character of G. One can define the G-distance of characters by induction in general. The notation $m(\chi)$ stands for the maximum distance of irreducible characters of H from the set of irreducible constituents of $\chi|_H$. In general $d(H,G) \leq 2$ if and only if H is normal in G, and $d(H,G) = 1$ if and only if $G = HC_G(x)$ for all $x \in H$, see [6, Theorem 3.10]. Moreover by

[3, Lemma 1.7] a subgroup H is of depth 1 if and only if H is normal and each irreducible character of H is G-invariant.

It is known that the ordinary depth of a subgroup in a group can be arbitrarily large if it is odd. E. g., it is shown in [6] that the depth of the symmetric group S_n in S_{n+1} is $2n-1$. Lars Kadison posed the following problem on his homepage, see [19]: Are there subgroups of (minimum) depth $2n$ where $n > 3$? In the Thesis, we show that the depth of a subgroup in a group can also be an arbitrarily large even number.

Our strategy to prove this was the following. First we were writing GAP programs to calculate the depths of a subgroup of a finite group. Then we made experiments with GAP. We were looking for subgroups of even depth bigger than 6 in the Small Groups Library, in the Primitive Groups Library and in the Transitive Groups Library. We found that the subgroup $D_8 \times S_4$ was of depth 8 in $S_4 \wr C_2$. We tried to generalize this. The first generalization was leading to the construction in Section 3.1. Here we show a series of examples of subgroups of ordinary depth 2^n for all positive integers n. This was published in [15]. Another generalization was leading to the constructions in Section 3.2. Here we show a series of examples of ordinary depth $2n$, which is published [16]. In both constructions we were defining a series of groups and subgroups (G_n, H_n). To estimate the depth, we were using an upper bound for the depths in Theorem 3. To prove that it cannot be a strict inequality, we found two irreducible characters of suitable distance in H_n. For that, we considered appropriate characters of the base group of the wreath product and defined a Cartesian product of graphs that encodes the neighbouring relation of characters of H_n.

In Chapter 4 of the Thesis we generalize the method of Section 3.2 to construct infinitely many series (G_n, H_n), where the depth $d(H_n, G_n) = 2n$. In fact this always holds if $d(H_1, G_1) = 2$. In the construction we always have $G_1 := G$, $H_1 := H$, where $H \leq G$, furthermore $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$. We describe all depth series $d(H_n, G_n)$ with this generalized construction where $d(H_1, G_1) = 1$. As a corollary we have that for every positive integer n there are infinitely many triples (H, N, G) of finite solvable groups $H \triangleleft N \triangleleft G$ such that G/N is cyclic of order $\lceil n/2 \rceil$, N/H is cyclic of arbitrarily large prime order, and $d(H, G) = n$. We also investigate the case when $d(H_1, G_1) = 3$. It may also happen that $d(H_1, G_1) = 3$ and for some n, $d(H_n, G_n)$ is even. We also show an example when $d(H_1, G_1) = 4$ and for some n, $d(H_n, G_n)$ is odd. We prove that if $H_1 = S_k$ and $G_1 = S_{k+1}$ then $d(H_n, G_n) = 2nk - 1$, in particular these groups will always give odd depth series with this generalized construction. These results were written up in the paper [13].

In Chapter 5 of the Thesis we determine the TI subgroups and depth 3 subgroups of the simple Suzuki groups $Sz(q)$.

In Chapter 1 of the Thesis we recall some basic notions and results from representation theory and group theory.

In Chapter 2 of the Thesis we define ordinary depth and combinatorial depth. In Section 2 below we also give some of these definitions.

In Chapter 3 of the Thesis we solved an open problem posed by Lars Kadison.

2 Preliminaries

The results on characters in [6] help to determine $d(H,G)$. Two irreducible characters $\alpha, \beta \in \mathrm{Irr}(H)$ are called *G-related*, $\alpha \sim^G \beta$, if they are constituents of χ_H, for some $\chi \in \mathrm{Irr}(G)$. The *G-distance* $d^G(\alpha,\beta)$ is the smallest integer m such that there is a chain of irreducible characters of H such that $\alpha = \psi_0 \sim^G \psi_1 \ldots \sim^G \psi_m = \beta$. If there is no such chain then $d^G(\alpha,\beta) = -\infty$ and if $\alpha = \beta$ then the G-distance is 0. If X is the set of irreducible constituents of χ_H then we set $m(\chi) := \max\{\min\{d^G(\alpha,\psi); \psi \in X\}; \alpha \in \mathrm{Irr}(H)\}$. We will use the following result from [6].

Theorem 1. *[6, Theorem 3.6, Theorem 3.10]*
Let H be a subgroup of a finite group G.

(i) Let $m \geq 1$. Then H has ordinary depth $\leq 2m+1$ in G if and only if the G-distance between two irreducible characters of H is at most m.

(ii) Let $m \geq 2$. Then H has ordinary depth $\leq 2m$ in G if and only if $m(\chi) \leq m-1$ for all $\chi \in \mathrm{Irr}(G)$.

We have the following.

Definition 2. *Let H be a subgroup of a finite group G. The* ordinary depth *$d(H,G)$ is the minimal possible positive integer which can be determined from the upper bounds (i) and (ii) of Theorem 1 and from*

(iii) $d(H,G) \leq 2$ if and only if H is normal in G, see [20, Corollary 3.2],

(iv) $d(H,G) = 1$ if and only if $G = HC_G(x)$ for all $x \in H$, see [3, Theorem 1.7].

We will also use the following result from [6].

Theorem 3. *[6, Theorem 6.9] Suppose that H is a subgroup of a finite group G and $N = \mathrm{Core}_G(H)$ is the intersection of m conjugates of H. Then $d(H,G) \leq 2m$. If additionally $N \leq Z(G)$ holds then $d(H,G) \leq 2m-1$.*

In Chapter 4 of the Thesis we will introduce also H-distance of characters. If we say distance, we always mean G-distance.

Two irreducible characters $\chi, \psi \in \mathrm{Irr}(G)$ are *H-related*, denoted by $\chi \sim_H \psi$, if $(\chi|_H, \psi|_H) \neq 0$. The *H-distance* $d_H(\chi,\psi)$ is the smallest integer m such that there is a chain of irreducible characters of G such that $\chi = \chi_0 \sim_H \chi_1 \sim_H \cdots \sim_H \chi_m = \psi$. If there is no such chain then $d_H(\chi,\psi) = -\infty$ and if $\chi = \psi$ then their H-distance is 0.

The combinatorial depth can be defined as follows, see [2]:

Definition 4. *Let L be a subgroup of the finite group G and let $i \geq 1$. Then the* combinatorial depth *$d_c(L,G)$ of the subgroup L in G is defined in the following way:*

(i) $d_c(L,G) \leq 2i$ *if and only if for every* $x_1, \ldots, x_i \in G$, *there exist some* $y_1, \ldots, y_{i-1} \in G$ *with* $L \cap L^{x_1} \cap \ldots L^{x_i} = L \cap L^{y_1} \cap \ldots \cap L^{y_{i-1}}$.

(ii) *Let* $i > 1$. *Then* $d_c(L,G) \leq 2i-1$ *if and only if for every* $x_1, \ldots, x_i \in G$ *there exist some* $y_1, \ldots, y_{i-1} \in G$ *with* $L \cap L^{x_1} \cap \ldots L^{x_i} = L \cap L^{y_1} \cap \ldots \cap L^{y_{i-1}}$ *and* $x_1 h x_1^{-1} = y_1 h y_1^{-1}$ *for all* $h \in L \cap L^{x_1} \cap \ldots \cap L^{x_i}$.

(iii) $d_c(L,G) = 1$ *if and only if for every* $x \in G$ *there exists some* $y \in L$ *with* $xhx^{-1} = yhy^{-1}$ *for all* $h \in L$. *This holds if and only if* $G = LC_G(L)$.

Definition 5. *A subgroup $L \leq G$ is called a* TI *subgroup if for every $x \in G$ from $L^x \cap L \neq \{1\}$ it follows that $x \in N_G(L)$.*

Definition 6. *A group $G \leq S_\Omega$ is called a Frobenius group if, it is transitive on Ω, every $1 \neq g \in G$ fixes at most one point and there exists a non-trivial element $g \in G$ that fixes a point. (So the action is not regular.)*

Definition 7. *[14, Ch XI.]*
A group $G \leq S_\Omega$ is called a Zassenhaus group, if it is a doubly transitive permutation group without any regular normal subgroups, where any non-identity element has at most two fixed points.

Main results of Chapters 3 and 4

The first half of the dissertation solves the open problem posed by Lars Kadison, see [19]: Are there subgroups of (minimum) depth $2n$ where $n > 3$?

3 Solution of the problem posed by Lars Kadison

An example of a subgroup H of depth 6 in a group G is mentioned in [6] as found with GAP, see [7]: one takes $G = AGL(2,3)$ and $H = N_G(P)$, where $P \in Syl_3(G)$. Note that $|G| = 432$ and $|H| = 108$.

The smallest examples of depth 6 are G of structure $C_2 \times C_4^2 \rtimes C_3$ and $H \cong C_4^2$, and G of structure $C_2 \times C_2^4 \rtimes C_3$ and $H \cong C_2^4$, see [12]. The groups G can be found in the Small groups library of , see [7], as SmallGroup(96, 68) and SmallGroup(96, 229), respectively.

More examples of depth 6 were found with GAP, see [7] among maximal subgroups of some alternating groups, see [12]: $d(2^4 : (S_3 \times S_3), A_8) = 6$ and $d(S_7, A_9) = 6$.

The following examples of subgroups of depth 8 had been constructed earlier by the E. Horváth with the help of the GAP system [7], see [12]: $d(A_{15} \cap (S_{12} \times S_3), A_{15}) = 8$, $d(2^6 : U_4(2), O_8^-(2)) = 8$, and $d(G \cap (A_8 \times A_8), G) = 8$, for $G = C_2 \wr C_2 \wr C_2 \wr C_2$.

3.1 A series of subgroups of depth 2^n

It was shown already in [6] that $d(D_8, S_4) = 4$ holds. We found with GAP, see [7], that $d(D_8 \times S_4, S_4 \wr C_2) = 8$. Continuing this process, we obtained that $d((D_8 \times S_4) \times (S_4 \wr C_2), S_4 \wr C_2 \wr C_2) = 16$. In general, we proved

Theorem 8. *(Theorem 3.1.5 of Thesis) Let $G_0 = S_4$ and $H_0 \in Syl_2(G)$, $G_n := G_{n-1} \wr C_2$, $H_n := H_{n-1} \times G_{n-1} < G_{n-1} \times G_{n-1} < G_n$, Then $d(H_n, G_n) = 2^{n+2}$.*

This was our first example that shows that there exist subgroups of arbitrary large even depth. The results are published in [15]:

3.2 A series of subgroups of depth 2n

We wanted to simplify the construction of our previous results. Our aim was also to construct as depth more even numbers. We can generalize the first two steps of the former construction in another way as follows:

- $d(D_8, S_4) = 4$,
- $d(D_8 \times S_4, S_4 \wr C_2) = 8$,
- $d(D_8 \times S_4 \times S_4, S_4 \wr C_3) = 12$.

In general, we take

- $G_1 := S_4$, $H_1 := D_8$,
- $G_n := G_1 \wr C_n$, $H_n := H_1 \times G_1^{n-1} < G_1^n < G_n$.

Then we have that $d(H_n, G_n) = 4n$.

If we want to get every even number then we can use a modified construction. We take the Klein four group $V_4 \lhd S_4$ instead of D_8 and get:

- $d(V_4, S_4) = 2$,
- $d(V_4 \times S_4, S_4 \wr C_2) = 4$,
- $d(V_4 \times S_4 \times S_4, S_4 \wr C_3) = 6$.

In general, we proved

Theorem 9. *(Theorem 3.2.1 of Thesis) There exists a series of groups and subgroups (G_n, H_n) such that $d(H_n, G_n) = 2n$ for every positive integer n.*

The idea of the proof: for the inequality we use Theorem 3, and to prove that it cannot be a strict inequality, we find two irreducible characters of distance n in H_n. For that, we consider suitable characters of the base group of the wreath product and define a Cartesian product of graphs that encodes the relation $\sim$.

These results were published in [16].

4 Results on the generalization of methods of Section 3.2

The results of this chapter are written up in the paper [13].

4.1 Infinitely many pairs (G_n, H_n) where $d(H_n, G_n) = 2n$

In this section we prove the following

Theorem 10. *(Theorem 4.1.1 of Thesis) Let G_1 be a permutation group on k points. Let $H_1 := N$ be a normal subgroup of depth two in it, and C_n is the cyclic group of order n. Then the series $G_n = G_1 \wr C_n$, $H_n = H_1 \times G_1^{n-1}$ has the property that $d(H_n, G_n) = 2n$.*

4.2 Depth series coming from subgroups of depth one of any group

We study further the series $G_n = G \wr C_n$ and $H_n = H \times G^{n-1}$ for a subgroup H in G. Let us suppose now that $d(H, G) = 1$. Then by [3, Theorem 1.7], H is normal and every irreducible character of H is G-invariant.

If $\{1\} < H = H_1 = G_1 = G$, then the construction of the series G_n and H_n gives us that $H_n = G^n$ is the base group of the wreath product $G_n = G \wr C_n$, hence $d(H_n, G_n) = 2$ for $n > 1$.

If $\{1\} = H_1 = G_1$ then $H_n = \{1\}^n$ and $G_n = \{1\}^n \rtimes C_n$. Hence $d(H_n, G_n) = 1$.

Proposition 11. *(Proposition 4.2.1 of Thesis) Let $H = H_1 = \{1\}$ be the trivial subgroup of $G = G_1$. Then $d(H_1, G_1) = 1$. Suppose that G is a nontrivial group. Then for the series $G_n = G \wr C_n$, $H_n = H \times G^{n-1}$ we have that $d(H_n, G_n) = 2n - 1$.*

Corollary 12. *(Corollary 4.2.2 of Thesis) For every positive integer n there are infinitely many triples (H, N, G) of finite solvable groups $H \lhd N \lhd G$ such that G/N is cyclic of order $\lceil n/2 \rceil$, N/H is cyclic of arbitrarily large prime order, and $d(H, G) = n$.*

Theorem 13. *(Theorem 4.2.3 of Thesis) Let $H = H_1$ be a nontrivial proper subgroup of the group $G = G_1$ and let us suppose that $d(H_1, G_1) = 1$. Then by the construction of the series of $G_n = G \wr C_n$ and $H_n = H \times G^{n-1}$, we have that $d(H_n, G_n) = 2n$, for $n > 1$.*

4.3 Depth series coming from subgroups of depth three, all whose different irreducible characters have G-distance one.

If the subgroup H has depth three in G then every two irreducible characters of H have G-distance at most one, by Definition 2. By the second part of

Theorem 3, if $H \neq \{1\}$ and has a disjoint conjugate, then H has depth three in G. In this case, however every two distinct irreducible characters of H have G-distance exactly one, by Mackey's theorem: for, if $\phi_1, \phi_2 \in \mathrm{Irr}(H)$ then $(\phi_1^G, \phi_2^G) = (\phi_1^G{}_H, \phi_2) = (\sum_{x \in H\backslash G/H}((\phi_1^x{}_{H^x \cap H})^H, \phi_2) \neq 0$ since the regular character of H occurs in the sum. However, if for a subgroup H of G every two different irreducible characters of $\mathrm{Irr}(H)$ have G-distance exactly one, then H need not have a disjoint conjugate in G, see e.g. A_5 in A_7.

Theorem 14. *(Theorem 4.3.3 of Thesis) Let G be a finite group and let $H \leq G$ be a subgroup of ordinary depth three. Suppose that any two different characters of $\mathrm{Irr}(H)$ have G-distance one. (E.g. this is the case if H has a disjoint conjugate.) Then any two different characters of $\mathrm{Irr}(G)$ have H-distance one or two.*

Let us construct the series of groups G_n and H_n from G and H as before, namely $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$, $G_n := G \wr C_n$. If any two different irreducible characters of $\mathrm{Irr}(G)$ have H-distance one then $d(H_n, G_n) = 2n+1$, if there exist two different irreducible characters of $\mathrm{Irr}(G)$ that have H-distance two then $d(H_n, G_n) = 4n-1$.

4.4 Depth series coming from general subgroups of depth three

Theorem 15. *(Theorem 4.4.11 of Thesis) Let G be a finite group and let H be an arbitrary subgroup of G of depth three. Then the above method, where $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$, gives a series of groups H_n and G_n with the following properties: if the H-distance of any two different characters of $\mathrm{Irr}(G)$ is at most one, then $2n+1 \leq d(H_n, G_n) \leq 2n+3$, if there exist characters of $\mathrm{Irr}(G)$ of H-distance two then $2n+1 \leq d(H_n, G_n) \leq 4n+1$.*

4.5 Depth series coming from S_k in S_{k+1}

Theorem 16. *(Theorem 4.5.1 of Thesis) Let $H = H_1 := S_k$ and $G = G_1 := S_{k+1}$. Then if $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$ then $d(H_n, G_n) = 2kn - 1$.*

5 TI subgroups and depth 3 subgroups of Suzuki groups

5.1 Suzuki groups

Suzuki groups $\mathrm{Sz}(q)$ are twisted groups of Lie type ${}^2B_2(q)$, where $q := 2^{2m+1}$. If $m > 0$, then they are simple. Suzuki groups are also doubly transitive permutation groups on q^2+1 points, they belong to the class of Zassenhaus groups. Suzuki groups also can be defined as subgroups of $\mathrm{GL}(4,q)$. The order of $\mathrm{Sz}(q)$ is $(q^2+1)(q-1)q^2$. The order of $\mathrm{Sz}(q)$ is congruent to 2 mod 3, however it is always divisible by 5. For further information see [11].

Theorem 17 (Suzuki)**.** *[22, Theorem 4.12]*
Let $G = Sz(q)$, where $q = 2^{2m+1}$, for some positive integer m. Then G has the following subgroups:

1. *The Hall subgroup $N_G(F) = FH$, which is a Frobenius group of order $q^2(q-1)$, where $F \in Syl_2(G)$ and H is cyclic of order $q-1$.*
2. *The dihedral group $B_0 = N_G(H)$ of order $2(q-1)$ where H is the same subgroup as in part 1.*
3. *The cyclic Hall subgroups A_1, A_2 of orders $q+2r+1, q-2r+1$, respectively, where $r = 2^m$ and $|A_1||A_2| = q^2+1$.*
4. *The Frobenius subgroups $B_1 = N_G(A_1), B_2 = N_G(A_2)$ of orders $4|A_1|, 4|A_2|$, respectively.*
5. *The subgroups of form $Sz(s)$, where s is an odd power of 2, $s \geq 8$, and $q = s^n$ for some positive integer n. Moreover, for every odd 2-power s, where $s^n = q$ for some positive integer n, there exists a subgroup isomorphic to $Sz(s)$.*
6. *Subgroups (and the conjugates of the subgroups) of the above groups.*

Theorem 18. *[14, Theorem 3.10 Chapter XI]*
Let $q = 2^{2m+1}$, $m > 0$, $r = 2^m$ and $G = Sz(q)$. Let A_1, A_2 be the same subgroups as in the Theorem 17 part 3.

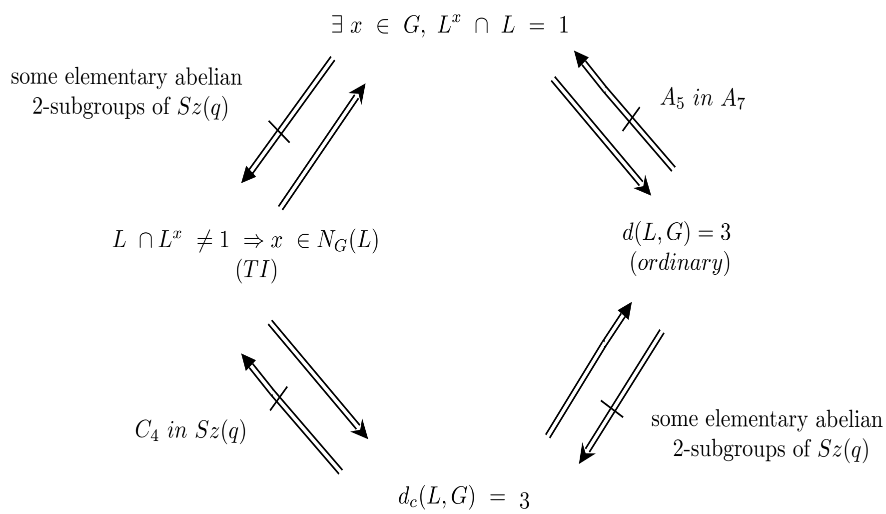
a) Let $i \in \{1,2\}$ and let $u_i \in A_i$, $u_i \neq 1$. Then $C_G(u_i) = A_i$. If $B_i = N_G(A_i)$ then $B_i = \langle A_i, t_i \rangle$, where t_i is an element of order 4, and $u^{t_i} = u^q$, for all $u \in A_i$. Moreover, $N_G(A_i)$ is a Frobenius group with kernel A_i.

b) Let F, H, A_1, A_2 as in Theorem 17. Then the conjugates of F, H, A_1, A_2 form a partition of G. In particular F, H, A_1, A_2, their conjugates and the conjugates of their characteristic subgroups are TI sets in G.

5.2 Motivation

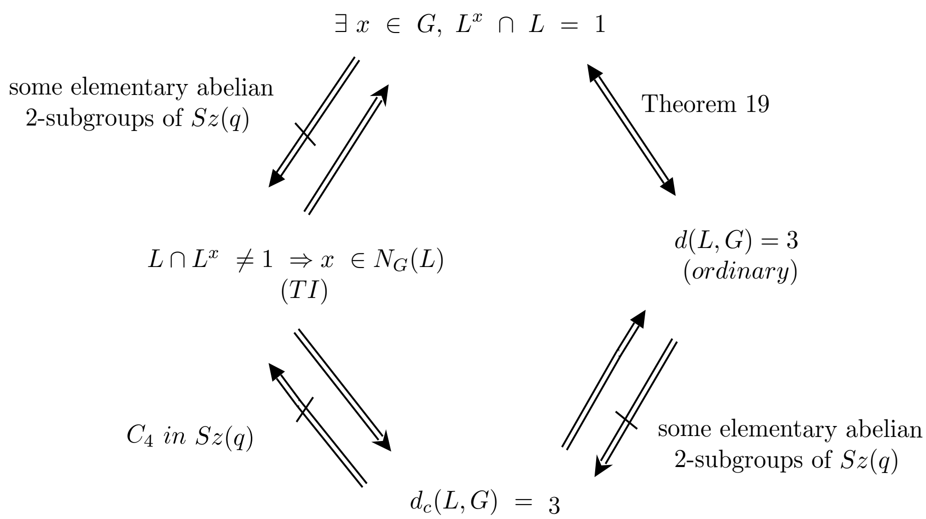
It is an open problem, see [19], how to characterize subgroups of ordinary depth 3 in a group theoretical way in an arbitrary group. The following diagrams explain some relations between ordinary depth, combinatorial depth, TI and subgroups having a disjoint conjugate.

When G is any finite group and L is a non-trivial subgroup:



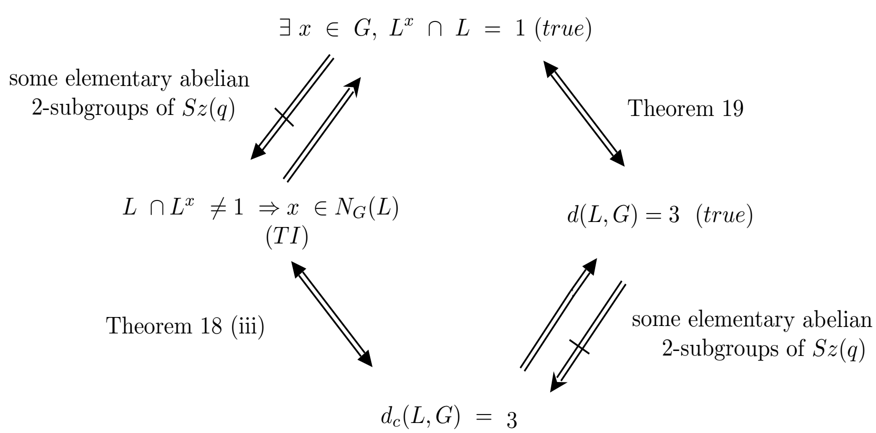
With results of Chapter 5 we will get the following diagrams:

When G is a simple Suzuki group and L is a non-trivial subgroup:



When G is a simple Suzuki group and L is a non-trivial elementary abelian 2-subgroup:

For these subgroups, $d(L,G) = 3$ always holds



In an arbitrary group, non-normal TI subgroups are always of combinatorial depth three, hence also of ordinary depth three. Moreover, nontrivial subgroups having a disjoint conjugate are always of ordinary depth three in every group. However, in general the converse is not true, e.g. $L = A_5$ has in $G = A_7$ ordinary depth 3, but there is no element $x \in G$ such that $L \cap L^x = \{1\}$. (A similar example is A_6 in A_{10}).

We show in Theorem 20 that in the case of simple Suzuki groups, subgroups of ordinary depth 3 are exactly those nontrivial subgroups that have a disjoint conjugate.

In Theorem 19 we also characterize those nontrivial TI subgroups of Sz(q) which are noncyclic elementary abelian 2-subgroups. These are exactly those subgroups that are conjugate to the centre of a Sylow 2-subgroup of a smaller Suzuki subgroup $Sz(s) \leq Sz(q)$. This property can help the recognition of those subgroups of Sz(q) which are isomorphic to a Suzuki subgroup $Sz(s) \leq Sz(q)$. Recognition of Suzuki groups in GL$(4, q)$ is considered in some recent papers, see [1] and [5]. In another paper, see [23], Suzuki groups were used to construct some block designs. So results about intersections in Suzuki groups might also be helpful in combinatorial investigations.

In this section we use the following notation of Theorem 17:

We denote by K_{2^n} an elementary abelian subgroup of order 2^n. The center $Z(F)$ of the Sylow 2-subgroup F of the Suzuki group Sz(q) is of order $q = 2^{2m+1}$, and we denote $2m + 1$ by f. We also suppose that $m > 0$.

5.3 Main results of Chapter 5

Theorem 19. *(Theorem 5.3.1 of Thesis) If $G = Sz(q)$ is a simple Suzuki group then G has the following TI subgroups:*

(i) Cyclic subgroups of prime order and the trivial subgroup.

(ii) Subgroups F, H, A_1, A_2, their characteristic subgroups and the conjugates of these. (see Theorem 18. part b))

(iii) An elementary abelian subgroup K_{2^n} of order $2^n > 2$ is a TI-subgroup if and only if it is the centre of a Sylow 2-subgroup of a simple Suzuki subgroup $G_1 \leq G$, or a conjugate to it. This holds if and anly if $n > 1$ and $n|f$. (Remember: $|Z(F)| = 2^f$). These subgroups are exactly those non-cyclic elementary abelian 2-subgroups of G that have combinatorial depth 3.

All other nontrivial subgroups are not TI.

In the paper [11] it was proved that every maximal subgroup of $Sz(q)$ different from $N_G(F)$ is of ordinary depth 3. We know from Theorem 19. part (ii) that F and all its subgroups are of depth 3. We extend these results in the following:

Theorem 20. *(Theorem 5.3.3 of Thesis) The subgroups of ordinary depth 3 of a simple Suzuki group $G = Sz(q)$ are the following:*

(i) *Every nontrivial subgroup contained in a maximal subgroup different from a conjugate of* $N_G(F)$.

(ii) *F and all its nontrivial subgroups, and the conjugates of these.*

(iii) *All nontrivial subgroups* $U = F_1K$ *of* $N_G(F)$ *where* $F_1 < F$ *and* $K \leq H$. *And the conjugates of these subgroups.*

Moreover, a nontrivial subgroup $L \leq G = Sz(q)$ *is of ordinary depth* 3 *if and only if there exists an element* $x \in G$ *with* $L \cap L^x = \{1\}$.

Corollary 21. *(Corollary 5.3.4 of Thesis) Let* G *be a simple Zassenhaus group acting on* $q+1$ *points, where* q *is a* 2*-power. Then the following are equivalent:*

(i) $G \simeq Sz(q)$.

(ii) *If* $F \in Syl_2(G)$, *then every subgroup not containing a conjugate of the subgroup* F *has a disjoint conjugate.*

Corollary 22. *(Corollary 5.3.5 of Thesis) Let* G *be a simple Zassenhaus group acting on* $q+1$ *points, where* q *is an odd power of* 2. *Then the following are equivalent for a nontrivial subgroup* $L \leq G$:

(i) *The subgroup* $L \leq G$ *has ordinary depth* 3.

(ii) *The subgroup* $L \leq G$ *has a disjoint conjugate.*

Corollary 23. *(Corollary 5.3.6 of Thesis) Every Sylow* 2*-subgroup of* $G = Sz(q)$ *is the union of conjugates of the Sylow* 2*-subgroups of a smallest simple Suzuki subgroup* G_1 *contained in* G. *If* $F \in Syl_2(G)$ *and* $F_1 \in Syl_2(G_1)$ *contained in* F, *then* $F = \cup_{x \in N_G(F)} {F_1}^x$. *Every element of order* 4 *of* F *is in exactly one conjugate of* F_1 *and any two of the conjugates of* F_1 *in this union either have trivial intersection or their intersection is their center, which is a conjugate of* $\Omega_1(F_1)$.

5.4 GAP Codes

1. Program to find the Frobenius matrix of a subgroup.
2. Program to find the power of a rectangular matrix.
3. Program to find the zero position number of a matrix.
4. Program to find the depth of a matrix.
5. Program to find the depth of subgroups in the small groups library.

6. Program to find the depth of subgroups in the transitive groups library.
7. Program to find the depth of subgroups in the primitive groups library.
8. Program to find the distance matrix of characters of a subgroup.
9. Program to find the maximal value in a matrix.
10. Program to find the shortest distance between irreducible characters ϕ and ψ by using Dijkstra algorithm.
11. Making some programs in GAP to test subgroups of depth 3 in groups in the (small group, primitive, transitive) libraries in GAP.

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List of Publications during the PhD studies

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[2] Hayder Abbas Janabi, Thomas Breuer, and Erzsébet Horváth, Construction of subgroups of ordinary depth 2^n, (2020) XXIII. Spring Wind Conference Volume. Association of Hungarian PHD and DLA Students, Budapest. ISBN 9786155586729 (2020) 764–771.

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