Investigation of depth of subgroups of finite groups

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Contents

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Abstract

In this thesis we study the notion of depth of subgroups in finite groups. We solve an open problem posed by Lars Kadison: are there subgroups of (minimum) depth $2n$ where $n > 3$? We construct several series of subgroups of even depth. We prove that there are infinitely many examples of subgroups of depth $2n$ for every n. In our work we use some wreath product construction. This construction also shows some new series of subgroups of depth $2n + 1$ for every n. As a corollary we prove that for every positive integer n there are infinitely many triples (H, N, G) of finite solvable groups $H \lhd N \lhd G$ such that G/N is cyclic of order $\lceil n/2 \rceil$, N/H is cyclic of arbitrarily large prime order, and $d(H, G) = n$. We also investigate the construction in some other cases as well.

After that we study the relation between subgroups of depth 3 and TI subgroups. We give a complete list of TI subgroups and depth 3 subgroups of simple Suzuki groups. We also give a new characterization of Suzuki groups.

Introduction

The notion of depth was originally defined for von-Neumann algebras, see [12]. Later it was also defined for Hopf algebras, see [31]. For some recent results in this direction, see [17, 28, 16]. In [30] and later in [6], the depth of semisimple algebra inclusions was studied, by Burciu, Kadison and Külshammer. First results concerned the depth 2 case. If $B \leq A$ are rings with common unit elements then we say B is of depth 2 in A if $A \otimes_B A$ is a direct summand of A^k for some positive integer k as both (A, B) and (B, A) bimodules. Later these were generalized for arbitrary positive integer n as depth. In the case of group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$ it was shown that the depth is at most 2 if and only if H is normal in G, see [30]. For similar results on group algebras over commutative rings, see [4]. The ordinary depth of a group inclusion $H \leq G$ (denoted by $d(H, G)$) is defined as the minimal depth of the group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$, studied in [2]. We will give several equivalent definitions of ordinary depth in Chapter 2. One of the definitions is with the help of the inclusion or Frobenius matrix. This definition was used in our GAP programs to calculate the depth of a subgroup. To explain this, let G be a finite group, $H \leq G$, Irr(G)= $\{\chi_1, \chi_2, ..., \chi_k\}$ and Irr(H)= $\{\varphi_1, \varphi_2, ..., \varphi_r\}$.

Then $M = (m_{i,j}) \in \mathbb{Z}^{rxk}$ where $m_{i,j} = (\varphi_i \uparrow^G, \chi_j) = (\varphi_i, \chi_j \downarrow_H)$ is the Frobenius matrix of $H \leq G$. Define the "Powers" of the Frobenius matrix as $M^{(2s)} = (MM^t)^s \in \mathbb{Z}^{r\times r}$, $M^{(2s+1)} = (MM^t)^s M \in \mathbb{Z}^{r\times k}$ and $M^{(0)} = I_{r\times r}$.

The depth of the Frobenius matrix is

$$d(M) = \min\{i \geq 1 \mid \exists q > 0, M^{(i+1)} \leq qM^{(i-1)}\}$$

$$= \min\{i \geq 1 \mid Z(M^{(i+1)}) = Z(M^{(i-1)})\},$$

see [6, Propostion 2.2]. (Here Z denotes the set of zero positions of a matrix.) The depth of a subgroup is the depth of its Frobenius matrix. For the theoretical proofs we were using another definition of depth with the help of distance of characters. We will give details in Chapter 2. For the sake of simplicity, here we mention a remark of Dr. Erzsébet Lukács, saying that

$$d(H,G) = \max_{\phi,\psi\in \mathrm{Irr}(H)} d^G(\phi,\psi) + \max_{\chi\in \mathrm{Irr}(G)} m(\chi) + 1.$$

Here the distance $d^G(\phi,\psi) = 1$ if ϕ and ψ are constituents of the same irreducible character of G. One can define the distance of characters by induction in general. The notation $m(\chi)$ stands for the maximum distance of irreducible characters of H from the set of irreducible constituents of $\chi|_H$. In general $d(H,G) \leq 2$ if and only if H is normal in G, and $d(H,G) = 1$ if and only if $G = HC_G(x)$ for all $x \in H$, see [6, Theorem 3.10]. Moreover by [3, Lemma 1.7] a subgroup H is of depth 1 if and only if H is normal and each irreducible character of H is G-invariant.

It is known that the ordinary depth of a subgroup in a group can be arbitrarily large if it is odd. E. g., it is shown in [6] that the depth of the symmetric group S_n in S_{n+1} is $2n-1$. Lars Kadison posed the following problem on his homepage, see [29]: Are there subgroups of (minimum) depth $2n$ where $n > 3$? In this thesis, we show that the depth of a subgroup in a group can also be an arbitrarily large even number.

Our strategy to prove this was the following. First we were writing GAP, see [11], programs to calculate the depths of a subgroup of a finite group. Then we made experiments with GAP. We were looking for subgroups of even depth bigger than 6 in the Small Groups Library, in the Primitive Groups Library and in the Transitive Groups Library. We found that the subgroup $D_8 \times S_4$ was of depth 8 in $S_4 \wr C_2$. We tried to generalize this. The first generalization was leading to the construction in Section 3.1. Here we show a series of examples of subgroups of ordinary depth 2^n for all positive integers n. This was published in [25]. Another generalization was leading to the constructions in Section 3.2. Here we show a series of examples of ordinary depth $2n$, which is published [26]. In both constructions we were defining a series of groups and subgroups (G_n, H_n). To estimate the depth, we were using an upper bound for the depths in Theorem 2.1.3. To prove that it cannot be a strict inequality, we found two irreducible characters of suitable distance in H_n. For that, we considered appropriate characters of the base group of the wreath product and defined a Cartesian product of graphs that encodes the neighbouring relation of characters of H_n.

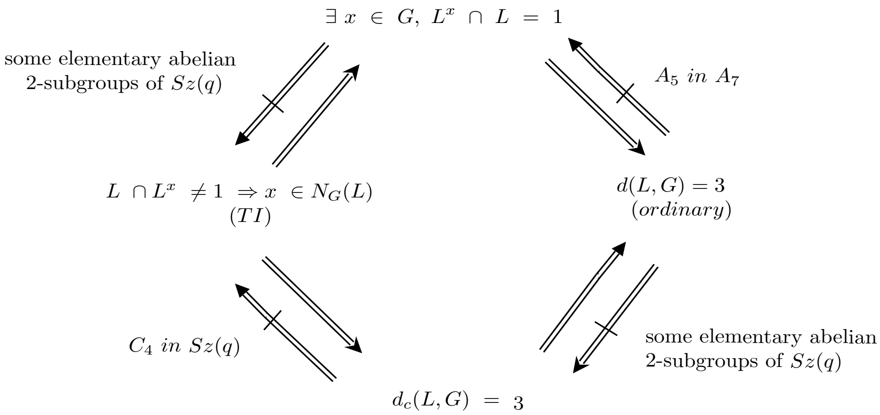
In Chapter 4 we generalize the method of Section 3.2 to construct infinitely many series (G_n, H_n), where the depth $d(H_n, G_n) = 2n$. In fact this always holds if $d(H_1, G_1) = 2$. In the construction we always have $G_1 := G$, $H_1 := H$, where $H \leq G$, furthermore $H_n := H \times G^{n-1}$

and $G_n := G \wr C_n$. We describe all depth series $d(H_n, G_n)$ with this generalized construction where $d(H_1, G_1) = 1$. As a corollary we have that for every positive integer n there are infinitely many triples (H, N, G) of finite solvable groups $H \triangleleft N \triangleleft G$ such that G/N is cyclic of order $\lceil n/2 \rceil$, N/H is cyclic of arbitrarily large prime order, and $d(H, G) = n$. We also investigate the case when $d(H_1, G_1) = 3$. It may also happen that $d(H_1, G_1) = 3$ and for some n, $d(H_n, G_n)$ is even. We also show an example when $d(H_1, G_1) = 4$ and for some n, $d(H_n, G_n)$ is odd. We prove that if $H_1 = S_k$ and $G_1 = S_{k+1}$ then $d(H_n, G_n) = 2nk - 1$, in particular these groups will always give odd depth series with this generalized construction. These results were written up in the paper [20].

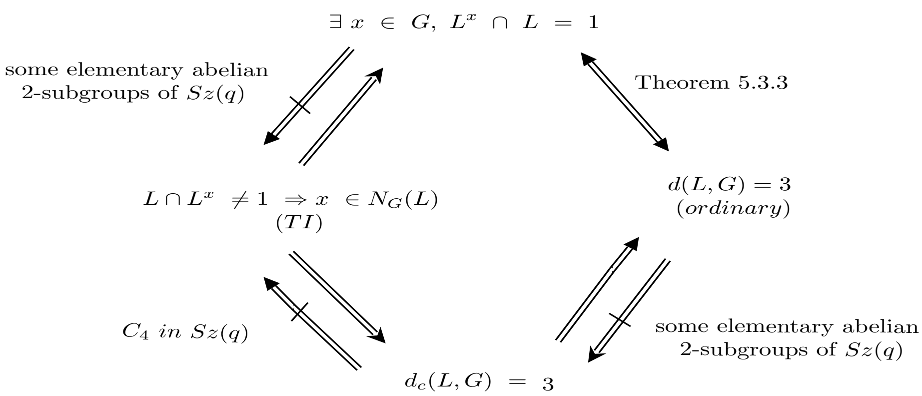
In Chapter 5 we determine the TI subgroups and depth 3 subgroups of the simple Suzuki groups $Sz(q)$. For that we use Suzuki's characterization of subgroups of $Sz(q)$ in Theorem 5.1.1, as well as the result in [14], that every maximal subgroup different from a 2-Sylow normalizer in $Sz(q)$, has a disjoint conjugate.

The following diagrams explain some relations between ordinary depth, combinatorial depth, TI and subgroups having a disjoint conjugate. The cited results are from Chapter 5.

When G is any finite group and L is a non-trivial subgroup:

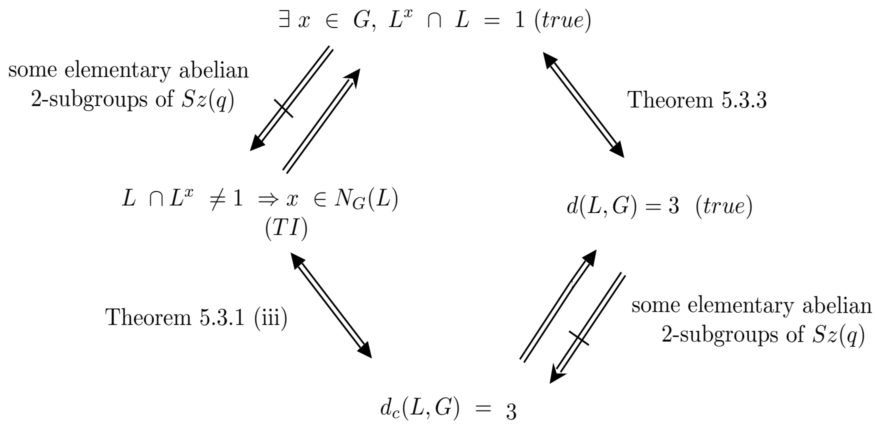


When G is a simple Suzuki group and L is a non-trivial subgroup:



When G is a simple Suzuki group and L is a non-trivial elementary abelian 2-subgroup:

For these subgroups, $d(L,G) = 3$ always holds



In Chapter 5 we determine those nontrivial subgroups of $Sz(q)$ that are disjoint from some of their conjugates. It turns out that the latter ones are exactly those subgroups that have ordinary depth 3. We also get that an elementary abelian 2-subgroup of $Sz(q)$ is TI if and only if it is the center of the Sylow 2-subgroup of a smaller Suzuki subgroup. The Sylow 2-subgroups of simple Suzuki groups belong to the class of so called Suzuki 2-groups, which have been studied extensively by Higman [18]. These results were extended later by Goldschmidt, Shaw, Shult, Gross, Wilkens and Bryukhanova. As a corollary of our investigations we get some interesting results for the Sylow 2-subgroups of Suzuki groups, as well. We also give some characterization of Suzuki groups. These results were published in [27].

In Chapter 1 we gives a short introduction to modules, representations and characters and also list basic definitions and results from group theory that are used later.

In Chapter 2 we give information on depth and in Section 5.1 we give preliminaries on Suzuki groups.

Chapter 1

Basic notions

1.1 Modules

Definition 1.1.1 *A is a F-algebra if A is a ring and vector space over a field F such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$, for every $a,b \in A$ and $\lambda \in F$.*

Definition 1.1.2 *Let F be a field and G a finite group, The group algebra FG, is the set of all finite formal expressions on the form $\lambda_1 g_1 + \cdots + \lambda_n g_n, \lambda_i \in F, g_i \in G$. Addition and multiplication by scalars in the group algebra is defined in the following way.*
If $r, s \in FG$ and $k \in F$ with $r = \sum_{g \in G} \lambda_g g$ and $s = \sum_{g \in G} \mu_g g$ then $r + s = \sum_{g \in G}(\lambda_g + \mu_g)g$ and $kr = \sum_{g \in G} k\lambda_g g$. Thus FG is a vector space with basis elements $g \in G$.

Multiplication of two elements is given by the group multiplication on the basis elements $g \in G$, and extended F-linearly. That is
$(\sum_{g \in G} \lambda_g g)(\sum_{h \in G} \mu_h h) = \sum_{g \in G}(\sum_{h \in G} \lambda_{gh^{-1}} \mu_h)g$, $(\lambda_g, \mu_h \in F)$.

Definition 1.1.3 *Let R be a ring with unit element. Then ${}_RM$ is a* *left R-module if M is an abelian group and for all $m \in M$ and $r \in R$, we have that $r \cdot m \in M$ and:*

$$\left.\begin{aligned} r(m+m^{'}) &= rm + rm^{'} \\ (r+r^{'})m &= rm + r^{'}m \\ (rs)m &= r(sm) \\ 1m &= m \end{aligned}\right\} \text{ for all } m, m^{'} \in M \text{ and } r, r^{'}, s \in R$$

A right module can be defined in a similar way, only we multiply from the right by elements $r \in R$.

In the following, if otherwise not stated, we will always deal with left modules.

Definition 1.1.4 *Let ${}_RM$ be an R-module and let U be a subset of M and suppose that U is also a left R-module with the same operations, then we say U is a submodule of M and we use the notation $U \leq M$.*

Definition 1.1.5 *Let ${}_RM$ be R-module and let X be a subset of M. Then the submodule generated by X is the smallest submodule containing the set X.*

Definition 1.1.6 *Let $U_i \leq {}_RM$ for all $i \in I$ then ${}_RM$ is the direct sum of the submodules U_i, ($M = \oplus_{i \in I} U_i$), if $M = \sum_{i \in I} U_i$ and $U_i \cap \sum_{j \neq i} U_j = 0$ for every $i \in I$, where $\sum_{i \in I} U_i$ denotes the submodule generated by $U_i, i \in I$.*

Definition 1.1.7 *Let ${}_RM$ be a R-module then it is cyclic if it is generated by one element.(${}_RM = Rm$ for some $m \in M$)*

Definition 1.1.8 *A module M is said to be irreducible if it is non-zero* *and has no submodules apart from $\{0\}$ and M. If M has a submodule U, such that U is not equal to $\{0\}$ or M, then M is reducible.*

Definition 1.1.9 *An FG-module M is said to be completely reducible if it is the direct sum of irreducible FG-submodules of M.*
FG is called semisimple if the regular module ${}_{FG}FG$ is completely reducible.

Theorem 1.1.10 *If FG is semisimple then every FG-module is also semisimple.*

Definition 1.1.11 *Let ${}_RM$ be a left module, then ${}_RM$ is said to be decomposable if there exist non-trivial submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$. Otherwise, M is called indecomposable.*

Definition 1.1.12 *Let M and N be R-modules.*

1. *A morphism of R-modules (or R-module homomorphism, or R-linear map) from M to N is a map $\phi : M \to N$ such that $\phi(m + n) = \phi(m) + \phi(n)$ and $\phi(am) = a\phi(m)$ for all $m, n \in M$ and $a \in R$. The set of all such morphisms from M to N will be denoted $Hom_R(M, N)$ or just $Hom(M, N)$;*

2. *A morphism $\phi : M \to N$ of R-modules is called an isomorphism if it is bijective. In this case, the inverse map $\phi^{-1} : N \to M$ is a morphism of R-modules again. We call M and N isomorphic (written $M \cong N$) if there is an isomorphism between them.*

1.2 Representations

Let G be a finite group. If V is a vector space over F, where F is a field, we denote by $GL(V)$ the group of F-linear isomorphisms of V.

Definition 1.2.1 *Let V be a vector space over a field F, a representation of a group G is a homomorphism $\phi : G \to GL(V)$.*
The dimension of V called the degree of ϕ.

Remark 1.2.2 *If V is a vector space over F, then every representation $\phi : G \to GL(V)$ defines a left FG-module structure on V, and every left FG-module gives a representation of G, where G is acting by $\phi(G)$.*

Definition 1.2.3 *Two representations $\phi : G \to GL(V)$ and $\psi : G \to GL(W)$ are equivalent if there exists an isomorphism $T : V \to W$ such that $\psi_g = T\phi_g T^{-1}$ for all $g \in G, i.e, \psi_g T = T\phi_g$ for all $g \in G$. This holds if and only if the modules corresponding to ϕ and ψ are isomorphic. In this case we write $\phi \cong \psi$.*

Definition 1.2.4 *A representation is called irreducible if the corresponding module is irreducible.*

Definition 1.2.5 *We say that ϕ is decomposable if the corresponding module is decomposable, otherwise, it is indecomposable. it is the completely reducible if the corresponding module is completely reducible.*

Lemma 1.2.6 *(Schur)[23, Lemma 1.5]*
Let V, W be irreducible representations of G over the field K.

1. *Every KG-homomorphism $\phi : V \to W$ is either zero or an isomorphism.*

2. *If* $K = \overline{K}$ *then*

$$dim_K\ Hom_{KG}(V,W) = \begin{cases} 1 & if\ \ V \cong W \\ 0 & otherwise. \end{cases}.$$

In particular if $V \cong W$ *then* $Hom_{KG}(V,W) = \{\lambda id : \lambda \in K\}$.

Theorem 1.2.7 *(Maschke)[23, Theorem 1.9]*
Let F *be a field. The group algebra* FG *is semisimple if and only if the characterstic of the field* F *does not divide the order of the group* G.

Theorem 1.2.8 *(Artin-Wedderburn)[36, Theorem 2.1.3]*
The group algebra $\mathbb{C}G \cong \oplus_{i=1}^{r} \mathbb{C}^{n_i \times n_i}$. *Here* r *is the number of simple* $\mathbb{C}G$*-modules, and the* n_i *are determined by* $\mathbb{C}G$ *up to isomorphism, they are the dimensions of the irreducible representations of* G.

1.3 Characters

Definition 1.3.1 *Let* V *be a finite dimensional vector space over the field* $\mathbb{C}$. *Let* $X : G \to GL(V)$ *be a representation of the finite group* G. *Then the character belonging to the representation* X *is a function* $\chi : G \to \mathbb{C}$, *such that* $\chi(g) = tr(X(g))$.

Remark 1.3.2 1. $\chi(x^{-1}gx) = \chi(g)$ *for all* $g, x \in G$.

2. *The characters of equivalent representations are equal. Moreover, the converse is also true.*

Remark 1.3.3 *Let* $I_n \in \mathbb{C}^{n \times n}$ *be the identity matrix. Since* $X(1) = I_n$, *so* $\chi(1) = n$, *the degree of the representation* X. *It is also called the degree of the character* χ.

Definition 1.3.4 *The character of an irreducible representation is called an irreducible character.*

Definition 1.3.5 *Let G be a finite group, then* $\mathrm{Irr}(G)$ *denotes the set of complex irreducible characters of G.*

From Theorem 1.2.8 it follows that

Corollary 1.3.6 $|G| = \sum_{\chi \in Irr(G)} \chi(1)^2$

Definition 1.3.7 *A function $\varphi : G \to \mathbb{C}$ is called a class function of G if it is constant on the conjugacy classes of G, i.e, $\varphi(g) = \varphi(x^{-1}gx)$ for every $x, g \in G$.*
So characters are class functions. The set of class functions of G we will denote by $Cf(G)$.

Proposition 1.3.8 *(Operations with characters)*
Let X and Y be are two representations of the finite group G, and let χ be the character of X and ψ be the character of Y. Let $X \oplus Y$ be the representation of the module belonging to the direct sum of modules belonging to X and Y. Let $X \otimes Y$ be the representation of the module of the tensor product of the modules belonging to X and Y. Then:

- *$\chi + \psi$ is the character of the representations $X \oplus Y$, so* $(\chi + \psi)(g) = tr(X \oplus Y)(g) = tr(X(g)) + tr(Y(g)) = \chi(g) + \psi(g)$
- *$\chi \cdot \psi$ is the character of the representations $X \otimes Y$, so* $(\chi \cdot \psi)(g) = tr(X \otimes Y)(g) = \chi(g) \cdot \psi(g)$.

Proposition 1.3.9 *Every character is the sum of irreducible characters, and conversely, every sum of irreducible characters is a character.*

Definition 1.3.10 *The kernel of a character χ is:*
$Ker\chi = \{g \in G \mid \chi(g) = \chi(1)\}$. *This is the kernel of the corresponding representation.*

Definition 1.3.11 *The trivial character $1_G : G \to \mathbb{C}^*$ belongs to the trivial representations $g \mapsto 1$, for all $g \in G$.*

Definition 1.3.12 *(The regular character)*
Let X be the representation belonging to the regular module ${}_{\mathbb{C}G}CG$. Let ρ be the character of X. Then:

$$\rho(g) = \begin{cases} 0 & \text{if } g \neq 1, \\ |G| & \text{if } g = 1 \end{cases}$$

Theorem 1.3.13 *(1^{st} orthogonality relation)*
Let $\chi_i, \chi_j \in \mathrm{Irr}(G)$. Then $\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \delta_{ij}$.

Definition 1.3.14 *If φ, μ are class functions of G then their scalar product is defined as: $(\varphi, \mu) = \frac{1}{|G|}\sum \varphi(g)\overline{\mu(g)}$*

Theorem 1.3.15 *(2^{nd} orthogonality relation)*

$$\sum_{i=1}^{k} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} 0 & \text{if } g \nsim_G h, \\ |C_G(g)| & \text{if } g \sim_G h \end{cases}$$

Here $g \sim_G h$ means that the elements g and h are conjugate in G and $g \nsim_G h$ means they are not conjugate.

Theorem 1.3.16 *The set of class functions $Cf(G)$ form a complex Euclidean space with the scalar product $(\ ,\ )$, and $\mathrm{Irr}(G)$ is an orthonormal basis in it.*

Definition 1.3.17 *Let $H \leq G$,and let φ be a class function of H.*
$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xgx^{-1})$ *where*

$$\varphi^\circ(g) = \begin{cases} \varphi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

This is a class function of G and $\varphi^G(1) = |G : H|\varphi(1)$.

Remark 1.3.18 *1. If ν is a class function of G then its restriction ν_H to a subgroup H, is a class function of H.*

2. If ν is a character of G then ν_H is a character of H(since the restriction of a representation is a representation).

Theorem 1.3.19 *(Frobenius reciprocity)*
Let $H \leq G, \varphi \in Cf(H), \nu \in Cf(G)$, then $(\varphi^G, \nu)_G = (\varphi, \nu_H)_H$.

Proposition 1.3.20 *If φ is a character of H then φ^G is a character of G.*

Definition 1.3.21 *Let $H \lhd G$. If ν is a class function of H and $g \in G$, we define $\nu^g : H \to \mathbb{C}$ by $\nu^g(h) = \nu(ghg^{-1})$. We say that ν^g is conjugate to ν in G.*

Remark 1.3.22 *If ν is a character of a normal subgroup H of G, then it can be proved that each conjugate ν^g, for $g \in G$, is also a character of H. Moreover, G permutes $\mathrm{Irr}(H)$ by $g : \nu \mapsto \nu^g$.*

Theorem 1.3.23 *(Clifford's theorem)[23, Theorem 6.2]*
Let $H \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Let ν be an irreducible constituent of

χ_H and suppose $\nu = \nu_1, \nu_2, \ldots, \nu_t$ are the distinct conjugates of ν in G. Then

$$\chi_H = e\sum_{i=1}^{t} \nu_i$$

where $e = (\chi_H, \nu)$.

Theorem 1.3.24 *[23, Theorem 6.11]*
Let $H \lhd G$ and let $\nu \in \mathrm{Irr}(H)$, and let $T = I_G(\nu) = \{g \in G \mid \nu^g = \nu\}$ be the inertia subgroup of ν. Let $A = \{\psi \in \mathrm{Irr}(T) \mid (\psi_H, \nu) \neq 0\}$, $B = \{\chi \in \mathrm{Irr}(G) \mid (\chi_H, \nu) \neq 0\}$. Then

1. *If $\psi \in A$, then ψ^G is irreducible;*

2. *The map $\psi \mapsto \psi^G$ is a bijection of A onto B;*

3. *If $\psi^G = \chi$, with $\psi \in A$, then ψ is the unique irreducible constituent of χ_T which lies in A;*

4. *If $\psi^G = \chi$, with $\psi \in A$, then $(\psi_H, \nu) = (\chi_H, \nu)$.*

Definition 1.3.25 *Let H and K be subgroups of a group G. An (H, K)-double coset is a set of the form HgK for $g \in G$. Here HgK is defined in the way as $HgK = \{hgk | h \in H \text{and } k \in K\}$.*

Theorem 1.3.26 *(Mackey)*
Let H and K be subgroups of G and let T be a set of representatives of (H, K) double cosets so that $G = \cup_{t\in T} HtK$ is a disjoint union.
Let ϕ be a character of H. Then $\phi^G|_K = \sum_{t\in T}(\phi^t_{H^t\cap K})^K$, where $\phi^t(x) = \phi(txt^{-1})$ is a character of H^t.

1.4 Some definitions and results from group theory

Definition 1.4.1 *A subgroup $L \leq G$ is called a* TI *subgroup if for every $x \in G$ from $L^x \cap L \neq \{1\}$ it follows that $x \in N_G(L)$.*

Definition 1.4.2 *A group $G \leq S_\Omega$ is called a Frobenius group if, it is transitive on Ω, every $1 \neq g \in G$ fixes at most one point and there exists a non-trivial element $g \in G$ that fixes a point. (So the action is not regular.)*

Definition 1.4.3 *Let G be a Frobenius group on a set Ω. Let N be the subset of G containing all fixed-point-free elements and the identity. This is called the Frobenius kernel of G. The point stabilizer $H = Stab_G(\omega)$ for an element $\omega \in \Omega$ is called the Frobenius complement of G.*

Theorem 1.4.4 *The Frobenius kernel is a normal subgroup of the Frobenius group G and $G = HN, H \cap N = \{1\}$. The subgroup H is a TI set, namely $H \cap H^g = 1$ if $g \in G \setminus N_G(H)$. Moreover, $N_G(H) = H$ and H is acting fixed point freely on the nonidentity elements of N by conjugation. By a result of Thompson, the Frobenius kernel is nilpotent.*

Theorem 1.4.5 *Properties of Frobenius groups G with kernel N and complement H*

1. $C_G(n) \leq N$ *for all* $1 \neq n \in N$.

2. $C_H(n) = 1$ *for all* $1 \neq n \in N$.

3. $C_G(h) \leq H$ *for all* $1 \neq h \in H$.

4. *Every* $x \in G \setminus N$ *is conjugate to an element of* H *with an element in* N.

5. $(|N|, |H|) = 1$ *and* $|H|$ *divides* $(|N| - 1)$.

Definition 1.4.6 *[22, Ch XI.]*
A group $G \leq S_\Omega$ *is called a Zassenhaus group, if it is a doubly transitive permutation group without any regular normal subgroups, where any non-identity element has at most two fixed points.*

Definition 1.4.7 *(Wreath product)*
Let $D \leqslant S_\Lambda$, $Q \leqslant S_\Omega$ *be permutation groups, let* $D_\omega \cong D$ *for all* $\omega \in \Omega$. *Then* $K = \prod_{\omega \in \Omega} D_\omega$, *is the base group of the wreath product. Let:* $D \wr Q := \left(\prod_{\omega \in \Omega} D_\omega \right) \rtimes Q$, *where* Q *acts on* $\prod_{\omega \in \Omega} D_\omega$ *by permuting components:* $(d_{\omega_1}, d_{\omega_2}, \ldots)^q = (d_{\omega_1{}^q}, d_{\omega_2{}^q}, \ldots)$. *This group is called the wreath product of* D *and* Q.

Chapter 2

Preliminaries

2.1 Depth

2.1.1 Ordinary Depth

There are several ways to define a notion of *ordinary depth* of a subgroup in the finite group:

Definition with the help of tensor products:

Let F be a field. We say that the *depth of the group algebra inclusion* $FH \subseteq FG$ is $2n$, for a positive integer n, if $FG \otimes_{FH} \cdots \otimes_{FH} FG$ ($n+1$-times FG) is isomorphic to a direct summand of $\oplus_{i=1}^{a} FG \otimes_{FH} \cdots \otimes_{FH} FG$ (n times FG) as FG-FH-bimodules (or equivalently as FH-FG-bimodules) for some positive integer a. Furthermore, FH is said to have depth $2n+1$, for a positive integer n, in FG if the same assertion holds for FH-FH-bimodules, for some positive integer a. Finally FH has depth 1 in FG if FG is isomorphic to a direct summand of $\oplus_{i=1}^{a} FH$ as FH-FH bimodules, for some positive integer a.

The *ordinary depth* $d(H, G)$ of a subgroup H in a finite group G is defined as the minimal depth of the group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$. This is well defined. It is shown in [2, Remark 4.5] by Boltje, Danz and Külshammer that the depth of a group algebra inclusion does not depend on the field, only on the characteristic. If the characteristic is prime then we get the notion of modular depth. From now on we will always consider group algebras over $\mathbb{C}$.

Definition with the help of the inclusion matrix:

The ordinary depth can be obtained from the so called *inclusion or Frobenius matrix* M. If $\chi_1, \ldots, \chi_s$ are the irreducible characters of G and $\psi_1, \ldots, \psi_r$ are the irreducible characters of H, then $m_{i,j} := (\psi_i^G, \chi_j)$. Let $M = (m_{i,j})$. Some kinds of powers of M are defined by $M^{(1)} := M$, $M^{(2l)} := M^{(2l-1)} M^T$, $M^{(2l+1)} := M^{(2l)} M$, for positive integers l, and $M^{(0)}$ is the $r \times r$ unit matrix. The ordinary depth $d(H, G)$ can be obtained as the smallest positive integer n such that $M^{(n+1)} \leq a M^{(n-1)}$ for some positive integer a, where the inequality of matrices means that this inequality holds componentwise.

Definition with the help of distance of irreducible characters:

The results on characters in [6] help to determine $d(H, G)$. Two irreducible characters $\alpha, \beta \in \mathrm{Irr}(H)$ are called *related*, $\alpha \sim_G \beta$, if they are constituents of χ_H, for some $\chi \in \mathrm{Irr}(G)$. The *distance* $d_G(\alpha, \beta)$ is the smallest integer m such that there is a chain of irreducible characters of H such that $\alpha = \psi_0 \sim_G \psi_1 \ldots \sim_G \psi_m = \beta$. If there is no such chain then $d_G(\alpha, \beta) = -\infty$ and if $\alpha = \beta$ then the distance is 0. If X is the set of irreducible constituents of χ_H then we set $m(\chi) := \max\{\min\{d_G(\alpha, \psi); \psi \in X\}; \alpha \in \mathrm{Irr}(H)\}$. We will use the following result from [6].

Theorem 2.1.1 *[6, Theorem 3.6, Theorem 3.10]*

Let H be a subgroup of a finite group G.

(i) Let $m \geq 1$. Then H has ordinary depth $\leq 2m+1$ in G if and only if the distance between two irreducible characters of H is at most m.

(ii) Let $m \geq 2$. Then H has ordinary depth $\leq 2m$ in G if and only if $m(\chi) \leq m-1$ for all $\chi \in \mathrm{Irr}(G)$.

Thus we have the following.

Corollary 2.1.2 *Let H be a subgroup of a finite group G. The* ordinary depth *$d(H,G)$ is the minimal possible positive integer which can be determined from the upper bounds (i) and (ii) of Theorem 2.1.1 and from*

(iii) $d(H,G) \leq 2$ if and only if H is normal in G, see [30, Corollary 3.2],

(iv) $d(H,G) = 1$ if and only if $G = HC_G(x)$ for all $x \in H$, see [3, Theorem 1.7].

We will also use the following result from [6].

Theorem 2.1.3 *[6, Theorem 6.9] Suppose that H is a subgroup of a finite group G and $N = \mathrm{Core}_G(H)$ is the intersection of m conjugates of H. Then $d(H,G) \leq 2m$. If additionally $N \leq Z(G)$ holds then $d(H,G) \leq 2m-1$.*

In Chapter 4 we will call the distance of characters $d(\phi, \psi)$ as the G-distance and it will be denoted by $d^G(\phi, \psi)$. In that chapter we will introduce also H-distance of characters. In the other chapters the simpler notation will be used.

Two irreducible characters $\chi, \psi \in \mathrm{Irr}(G)$ are *H-related*, denoted by $\chi \sim_H \psi$, if $(\chi|_H, \psi|_H) \neq 0$. The *$H$-distance* $d_H(\chi, \psi)$ is the smallest integer m such that there is a chain of irreducible characters of G such that $\chi = \chi_0 \sim_H \chi_1 \sim_H \cdots \sim_H \chi_m = \psi$. If there is no such chain then $d_H(\chi, \psi) = -\infty$ and if $\chi = \psi$ then their H-distance is 0.

In recent publications, several authors determined the ordinary depth of certain subgroups in some special series of groups, e.g. $PSL(2, q)$, Suzuki groups, Ree groups, symmetric and alternating groups, see [6], [8], [10], [14], [15]. In [7], twisted group algebra inclusions for symmetric and alternating groups are studied.

2.1.2 Combinatorial Depth

The combinatorial depth can be defined as follows, see [2]:

Definition 2.1.4 *Let L be a subgroup of the finite group G and let $i \geq 1$. Then the* combinatorial depth $d_c(L, G)$ *of the subgroup L in G is defined in the following way:*

(i) $d_c(L, G) \leq 2i$ if and only if for every $x_1, \ldots, x_i \in G$, there exist some $y_1, \ldots, y_{i-1} \in G$ with $L \cap L^{x_1} \cap \ldots L^{x_i} = L \cap L^{y_1} \cap \ldots \cap L^{y_{i-1}}$.

(ii) Let $i > 1$. Then $d_c(L, G) \leq 2i - 1$ if and only if for every $x_1, \ldots, x_i \in G$ there exist some $y_1, \ldots, y_{i-1} \in G$ with $L \cap L^{x_1} \cap$

$\ldots L^{x_i} = L \cap L^{y_1} \cap \ldots \cap L^{y_{i-1}}$ *and* $x_1 h x_1^{-1} = y_1 h y_1^{-1}$ *for all* $h \in L \cap L^{x_1} \cap \ldots \cap L^{x_i}$.

(iii) $d_c(L, G) = 1$ *if and only if for every* $x \in G$ *there exists some* $y \in L$ *with* $xhx^{-1} = yhy^{-1}$ *for all* $h \in L$*. This holds if and only if* $G = LC_G(L)$.

Remark 2.1.5 *1. If* $d_c(L, G) \leq 2$*, then by Definition 2.1.4 (i) and (iii),* L *is a normal subgroup of* G*.*

2. It is easy to see from Definition 2.1.4 that if a non-normal subgroup $L \leq G$ *is TI then* $d_c(L, G) = 3$*.*

Proof. Since L *is not normal, then* $d_c(L, G) \geq 2$*. We have to prove that for every* $x_1, x_2 \in G$ *there exists* $y_1 \in G$ *such that* $L \cap L^{x_1} \cap L^{x_2} = L \cap L^{y_1}$ *and* $x_1 h x_1^{-1} = y_1 h y_1^{-1}$ *for all* $h \in L \cap L^{x_1} \cap L^{x_2}$*. Since* L *is a TI subgroup, this intersection is either* $\{1\}$ *or* L*. If it is* $\{1\}$*, then we choose* y_1 *from* $G \setminus N_G(L)$*. If it is* L*, then we choose* y_1 *to be* x_1*. Thus* $d_c(L, G) = 3$*.* □

3. If $d_c(L, G) = 3$*, then* L *is not necessarily TI. We will see later that* C_4 *is not TI in Sz(q), however* $d_c(C_4, Sz(q)) = 3$ *by [14, Theorem 4.1].*

4. By Theorem [2, Theorem 4.1] we also know that $d(L, G) \leq d_c(L, G)$ *for every subgroup* $L \leq G$*.*

5. For a non-normal subgroup $L \leq G$ *to have a disjoint conjugate is a weaker property than being TI, e.g. in the case of simple Suzuki groups every nontrivial* 2*-subgroup has a disjoint conjugate, however by Theorem 5.3.1, only some of them are TI.*

Chapter 3

Solution of the problem posed by Lars Kadison

Lars Kadison posed the following problem on his homepage, see [29]: Are there subgroups of (minimum) depth $2n$ where $n > 3$?
If one looks at the results of the related papers or the calculations presented in [19], one has the impression that in most cases the depth of subgroups is odd. However still one can find examples of arbitrarily large even depth. In our examples wreath products will play an important role.

3.1 A series of subgroups of depth 2^n

Here we show the first example that shows that exist subgroups of arbitrary even depth. The results of this section are published in [25]:

In this section, we construct groups $H_n \leq G_n$, for nonnegative integers n, such that $d(H_n, G_n) = 2^{n+2}$ holds. Examples of subgroups of depth 8 had been constructed earlier by E. Horváth with the help of the

GAP system [11], see [19]. We found, again using GAP, more examples of depth 8 and 16, including the cases $n \in \{1,2\}$ from Theorem 3.1.5. The generalization shown below is based on the computations of characters of maximal distance and shortest paths between these characters in these cases.

Let G_0 be a permutation group on the set $\{1, 2, \ldots, N\}$ and H_0 be a subgroup of G_0. Define inductively, for $n \geq 1$,

$$\begin{aligned} \sigma_n &= \prod_{j=1}^{2^{n-1}N} (j, j + 2^{n-1}N), \\ G_n &= \langle G_{n-1}, \sigma_n \rangle, \\ H_n &= \langle H_{n-1}, G_{n-1}^{\sigma_n} \rangle. \end{aligned}$$

Let C_2 denote the cyclic group of order two. Then $G_n \cong G_{n-1} \wr C_2$ and

$$H_n \cong H_{n-1} \times G_{n-1} \leq G_{n-1} \times G_{n-1} < G_n.$$

Let $N_n = \mathrm{Core}_{G_n}(H_n)$, the largest normal subgroup of G_n that is contained in H_n. If the set $\Sigma_n \subseteq G_n$ satisfies $N_n = \bigcap_{\sigma \in \Sigma_n} H_n^\sigma$ then

$$N_{n+1} = \left(\bigcap_{\sigma \in \Sigma_n} H_{n+1}^\sigma \right) \cap \left(\bigcap_{\sigma \in \Sigma_n} H_{n+1}^\sigma \right)^{\sigma_{n+1}}$$

holds, which means that we may choose $\Sigma_{n+1} = \Sigma_n \cup \Sigma_n \cdot \sigma_{n+1}$. In other words, N_n can be written as the intersection of $2^n |\Sigma_0|$ conjugates of H_n in G_n.

Let $\mathrm{Irr}(G_0)$ be parametrized by the set I_0, that is, $\mathrm{Irr}(G_0) = \{\Theta(i); i \in I_0\}$ and $|\mathrm{Irr}(G_0)| = |I_0|$. For $\psi, \tau \in \mathrm{Irr}(G_n)$, $\psi \times \tau \in \mathrm{Irr}(G_n \times G_n)$ satisfies the following.

- The induced character $(\psi \times \tau)^{G_{n+1}}$ is irreducible if and only if $\psi \neq \tau$.

- $\psi \times \psi$ extends twofold to an irreducible character of G_{n+1}.

If $\mathrm{Irr}(G_n)$ is indexed by the set I_n then $\mathrm{Irr}(G_{n+1})$ is indexed by the set

$$I_{n+1} = \{(\{i,j\},0); i,j \in I_n, i \neq j\} \cup \{(\{i\},\epsilon); i \in I_n, \epsilon \in \{\pm\}\},$$

where $\Theta((\{i,j\},0)) = (\Theta(i) \times \Theta(j))^{G_{n+1}}$ and $\Theta((\{i\},\pm))$ consists of the two extensions of $\Theta(i) \times \Theta(i)$, where $\mathrm{Irr}(G_n) = \{\Theta(i); i \in I_n\}$, for $n \geq 0$.

Example 3.1.1 *Choose $G_0 = S_4$, the symmetric group on four points, and $H_0 \leq G_0$ a Sylow 2-subgroup of G. (It is mentioned in [6] that $d(H_0, G_0) = 4$.) The character tables of H_0 and G_0 are as follows, where the columns are indexed by the conjugacy classes of the elements $g_1 = ()$, $g_2 = (1,3)(2,4)$, $g_3 = (1,2)(3,4)$, $g_3' = (1,2,3)$, $g_4 = (1,3)$, $g_5 = (1,2,3,4)$.*

	g_1	g_2	g_3	g_4	g_5
φ_1	1	1	1	1	1
φ_2	1	1	1	-1	-1
φ_3	1	1	-1	1	-1
φ_4	1	1	-1	-1	1
φ_5	2	-2	0	0	0

	g_1	g_2	g_3'	g_4	g_5
$\Theta(1)$	1	1	1	1	1
$\Theta(2)$	1	1	1	-1	-1
$\Theta(3)$	2	2	-1	0	0
$\Theta(4)$	3	-1	0	1	-1
$\Theta(5)$	3	-1	0	-1	1

$$\begin{aligned} \Theta(1)|_{H_0} &= \varphi_1 \\ \Theta(2)|_{H_0} &= \varphi_2 \\ \Theta(3)|_{H_0} &= \varphi_1 + \varphi_2 \\ \Theta(4)|_{H_0} &= \varphi_3 + \varphi_5 \\ \Theta(5)|_{H_0} &= \varphi_4 + \varphi_5 \end{aligned}$$

We have chosen $I_0 = \{1,2,3,4,5\}$ and get

$$\begin{aligned} I_1 = \{&(\{1,2\},0), (\{1,3\},0), (\{1,4\},0), (\{1,5\},0), (\{2,3\},0), (\{2,4\},0), (\{2,5\},0), \\ &(\{3,4\},0), (\{3,5\},0), (\{4,5\},0), (\{1\},+), (\{1\},-), (\{2\},+), (\{2\},-), \\ &(\{3\},+), (\{3\},-), (\{4\},+), (\{4\},-), (\{5\},+), (\{5\},-)\}. \end{aligned}$$

The subgroup $N_0 = \mathrm{Core}_{G_0}(H_0)$ is the Klein four group. It is maximal in H_0, thus we can choose Σ_0 of cardinality 2, and N_n can be written *as an intersection of 2^{n+1} conjugates of H_n in G_n. By Theorem 2.1.3, we have $d(H_n, G_n) \leq 2^{n+2}$.*

For $(\{i,j\},\epsilon) \in I_{n+1}$, the restriction of $\Theta((\{i,j\},\epsilon))$ to $G_n \times G_n$ is equal to $\Theta(i) \times \Theta(j)$ if $i = j$ holds, and otherwise is equal to $\Theta(i) \times \Theta(j) + \Theta(j) \times \Theta(i)$. Thus the restriction $\Theta((\{i,j\},\epsilon))|_{H_{n+1}}$ to H_{n+1} is equal to $\Theta(i)|_{H_n} \times \Theta(i)$ if $i = j$, and is equal to $\Theta(i)|_{H_n} \times \Theta(j) + \Theta(j)|_{H_n} \times \Theta(i)$ otherwise.

The characters $\Theta((\{i,j\},\epsilon))|_{H_{n+1}}$, $\Theta((\{i',j'\},\epsilon'))|_{H_{n+1}}$ have a common constituent if and only if one of the following holds.

1. $i = i'$ and $\Theta(j)|_{H_n}$, $\Theta(j')|_{H_n}$ have a common constituent,
2. $i = j'$ and $\Theta(j)|_{H_n}$, $\Theta(i')|_{H_n}$ have a common constituent,
3. $j = i'$ and $\Theta(i)|_{H_n}$, $\Theta(j')|_{H_n}$ have a common constituent,
4. $j = j'$ and $\Theta(i)|_{H_n}$, $\Theta(i')|_{H_n}$ have a common constituent.

Let $\Gamma_n = (I_n, E_n)$ be the undirected simple graph with vertex set I_n and edge set

$$E_n = \{\{i,j\}; i,j \in I_n, i \neq j, \Theta(i)|_{H_n} \text{and} \Theta(j)|_{H_n} \text{have a common constituent}\}.$$

For $n > 0$, the set of neighbors of $(\{i,j\},\epsilon)$ in Γ_n consists of those $(\{i',j'\},\epsilon')$ which have the property that $j = j'$ and i, i' are neighbors in Γ_{n-1} (perhaps after exchanging i and j resp. i' and j').

Note that the ϵ components are irrelevant for the neighborship relation. In particular, $(\{i,j\},\epsilon)$ and $(\{i,j\},\epsilon')$ have the same neighbors. We define the graph $\tilde{\Gamma}_n = (\tilde{I}_n, \tilde{E}_n)$ in which the ϵ parts are omitted, as follows.

We set $\tilde{I}_0 = I_0$,

$$\tilde{I}_{n+1} = \Big\{\{i,j\}; i,j \in \tilde{I}_n, i \neq j\Big\} \cup \Big\{\{i\}; i \in \tilde{I}_n\Big\},$$

and define the map $I_n \to \tilde{I}_n, i \mapsto \tilde{i}$ recursively as the identity map on I_0, and mapping $(\{i,j\},\epsilon)$ to $\{\tilde{i},\tilde{j}\}$. We set $\tilde{E}_n = \big\{\{\tilde{i},\tilde{j}\}; \{i,j\} \in E_n\big\}$.

Then Γ_n and $\tilde{\Gamma}_n$ have the same number of connected components, and the distance of two vertices i,j in Γ_n is either equal to the distance of the corresponding vertices $\tilde{i},\tilde{j}$ in $\tilde{\Gamma}_n$, or $\tilde{i} = \tilde{j}$ holds.

Lemma 3.1.2 *(i) If $\tilde{\Gamma}_n$ has exactly j connected components, with vertex sets $J_1, J_2, \ldots, J_j$, then $\tilde{\Gamma}_{n+1}$ has exactly $j(j+1)/2$ connected components, with vertex sets $J_{k,l}$, $1 \leq k \leq l \leq j$, where $J_{m,m} = \{\{\rho\}; \rho \in J_m\} \cup \{\{\rho,\sigma\}; \rho,\sigma \in \mathrm{J}_m, \rho \neq \sigma\}$ and, for $1 \leq m,m' \leq j$ with $m \neq m'$, $J_{m,m'} = \{\{\rho,\sigma\}; \rho \in J_m, \sigma \in J_{m'}\}$.*

(ii) If $i,j \in J_m$ have distance d in $\tilde{\Gamma}_n$ then $\{i\},\{j\} \in J_{m,m}$ have distance $2d$.

(iii) If $i,j \in J_m$ have distance d in $\tilde{\Gamma}_n$ and $i',j' \in J_{m'}$ have distance d' in $\tilde{\Gamma}_n$ then $\{i,i'\},\{j,j'\} \in J_{m,m'}$ have distance $d+d'$ in $\tilde{\Gamma}_{n+1}$.

Proof. Fix n and consider the Cartesian product C of two copies of $\tilde{\Gamma}_n$. This is a graph with vertex set $I_n^2 = \{(i,j); i,j \in I_n\}$, the ordered pairs of vertices in $\tilde{\Gamma}_n$, and with edge set

$$\{\{(i_1,j_1),(i_2,j_2)\}; (i_1 = i_2, j_1 \neq j_2, \{j_1,j_2\} \in E_n) \text{ or } (j_1 = j_2, i_1 \neq i_2, \{i_1,i_2\} \in E_n)\}.$$

The graph $\tilde{\Gamma}_{n+1}$ is obtained from C by identifying the vertices (i,j) and (j,i) if $i \neq j$. We may arrange the vertices of C in a two-dimensional array in such a way that the x-direction indexes the first coordinate, and the y-direction indexes the second coordinate. Each edge of C

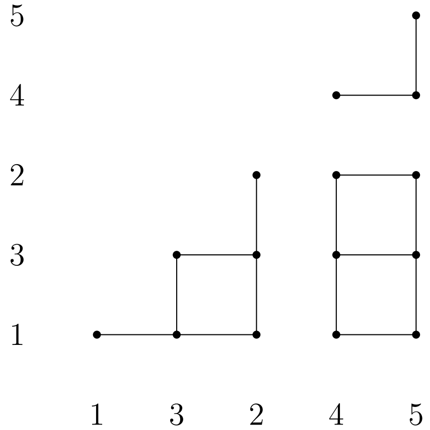
connects two vertices with the same x-coordinate or with the same y-coordinate. We may choose the ordering of vertices along the two coordinate axes such that the vertices from each connected component of $\tilde{\Gamma}_n$ are grouped together, and such that the ordering of vertices is the same for both axes. With this arrangement, $\tilde{\Gamma}_{n+1}$ can be obtained from C by discarding all vertices and edges above the main diagonal.

The connected components of C are the Cartesian products of the connected components of $\tilde{\Gamma}_n$, thus the identification of vertices and edges described above yields the connected components claimed for $\tilde{\Gamma}_{n+1}$ in (i).

Each edge of C connects two vertices with the same x-coordinate or with the same y-coordinate, and any path of length two which goes first in y-direction and then in x-direction can be replaced by a path of length two which goes first in x-direction and then in y-direction. Thus any path in C can be replaced by a path of (at most) the same length which consists of an initial path in x-direction followed by a path in y-direction. If we choose the two end vertices of the path in question already on or below the main diagonal then these statements also hold in $\tilde{\Gamma}_{n+1}$, that is, after the identification of vertices and edges described above. This implies (ii) and (iii). □

Example 3.1.3 *For $G = S_4$ and $I = \{1, 2, 3, 4, 5\}$ as above, the edges of $\Gamma_0 = \tilde{\Gamma}_0$ are $E_0 = \tilde{E}_0 = \{\{1, 3\}, \{2, 3\}, \{4, 5\}\}$. In particular, Γ_0 has two connected components.*

The graph $\tilde{\Gamma}_1$ has three connected components, and can be depicted as follows.



Lemma 3.1.4 *Let* $G_0 = S_4$*,* $H_0 \in \mathrm{Syl}_2(G_0)$*,* $I_0 = \{1, 2, 3, 4, 5\}$ *as above, and choose* $\lambda_0 = 1, \mu_0 = 2$*,* $x_0 = 4, y_0 = 5$*, and define recursively, for* $n = 0, 1, 2, \ldots$*,* $\lambda_{n+1} = (\{\lambda_n\}, +)$*,* $\mu_{n+1} = (\{\mu_n\}, +)$*,* $x_{n+1} = (\{x_n, \lambda_n\}, 0)$*, and* $y_{n+1} = (\{y_n, \mu_n\}, 0)$*. Then the following holds.*

(i) The distance of $\tilde{\lambda}_n$ *and* $\tilde{\mu}_n$ *in* $\tilde{\Gamma}_n$ *is* 2^{n+1}*,*

(ii) The distance of $\tilde{x}_n$ *and* $\tilde{y}_n$ *in* $\tilde{\Gamma}_n$ *is* $2^{n+1} - 1$*.*

(iii) The characters $\Theta(x_n), \Theta(y_n) \in \mathrm{Irr}(G_n)$ *are induced from the characters*

$$\alpha_n := \varphi_3 \times \Theta(\lambda_0) \times \Theta(\lambda_1) \times \cdots \times \Theta(\lambda_{n-1})$$

and

$$\omega_n := \varphi_4 \times \Theta(\mu_0) \times \Theta(\mu_1) \times \cdots \times \Theta(\mu_{n-1})$$

of H_n*, respectively.*

(iv) The distance of α_n *and* ω_n *is* 2^{n+1}*.*

Proof. (Note that x_n and y_n are well-defined because $x_n \neq \lambda_n$ and $y_n \neq \mu_n$ for all $n \geq 0$.) Part (i) follows from Lemma 3.1.2 (ii) and the fact that the distance of $\tilde{\lambda}_0$ and $\tilde{\mu}_0$ in $\tilde{\Gamma}_0$ is 2.

Let d_n denote the distance of $\tilde{x}_n$ and $\tilde{y}_n$ in $\tilde{\Gamma}_n$. Lemma 3.1.2 (iii) implies that $d_{n+1} = d_n + 2^{n+1}$. Since $d_0 = 1$, we get $d_n = 2^{n+1} - 1$, which shows part (ii).

Part (iii) follows from $\Theta(x_0) = \varphi_3^{G_0}$, $\Theta(y_0) = \varphi_4^{G_0}$, and

$$\Theta(x_n) = (\Theta(x_{n-1}) \times \Theta(\lambda_{n-1}))^{G_n},\ \Theta(y_n) = (\Theta(y_{n-1}) \times \Theta(\mu_{n-1}))^{G_n}.$$

Note that we can choose $\Theta(\lambda_n)$ to be the trivial character of G_n, and $\Theta(\mu_n)$ to be an extension of the sign character of G_0 to G_n. For part (iv), fix n and let $\alpha_n \sim_{G_n} \psi_1 \sim_{G_n} \psi_2 \sim_{G_n} \cdots \sim_{G_n} \psi_m \sim_{G_n} \omega_n$ be a shortest path of related characters in $\mathrm{Irr}(H_n)$, of length $m+1$. Then ψ_1 is a constituent of $\Theta(x_n)|_{H_n}$ and ψ_m is a constituent of $\Theta(y_n)|_{H_n}$, by part (iii). Thus the distance of x_n and y_n in $\tilde{\Gamma}_n$ is at most m, and part (ii) implies $m+1 \geq 2^{n+1}$. Conversely, any path of length $2^{n+1}-1$ between x_n and y_n yields a path of related characters from α_n to ω_n, of length 2^{n+1}, hence $m+1 = 2^{n+1}$. □

Theorem 3.1.5 *Let $G_0 = S_4$ and $H_0 \in \mathrm{Syl}_2(G)$, Then $d(H_n, G_n) = 2^{n+2}$.*

Proof. By Example 3.1.1, we know that $d(H_n, G_n) \leq 2^{n+2}$, and by Definition 2.1.2, it suffices to show that $\mathrm{Irr}(H_n)$ contains two characters of distance 2^{n+1}. The characters α_n, ω_n constructed in Lemma 3.1.4 have this property. □

3.2 A series of subgroups of depth 2n

In this section we improved our results in Section 3.1 to show that every even integer can be the depth of a subgroup. These results were published in [26].

Theorem 3.2.1 *There exists a series of groups and subgroups* (G_n, H_n) *such that* $d(H_n, G_n) = 2n$ *for every positive integer* n*.*

Constructing examples

An example of a subgroup H of depth 6 in a group G is mentioned in [6] as found with GAP [11]: One takes $G = AGL(2, 3)$ and $H = N_G(P)$, where $P \in Syl_3(G)$. Note that $|G| = 432$ and $|H| = 108$.

The smallest examples of depth 6 are G of structure $C_2 \times C_4^2 \rtimes C_3$ and $H \cong C_4^2$, and G of structure $C_2 \times C_2^4 \rtimes C_3$ and $H \cong C_2^4$, see [19]. The groups G can be found in the Small groups library of GAP [11] as SmallGroup(96, 68) and SmallGroup(96, 229), respectively.

More examples of depth 6 were found with GAP [11] among maximal subgroups of some alternating groups, see [19]: $d(2^4 : (S_3 \times S_3), A_8) = 6$ and $d(S_7, A_9) = 6$.

The following examples of subgroups of depth 8 had been constructed earlier by the E. Horváth with the help of the GAP system [11], see [19]: $d(A_{15} \cap (S_{12} \times S_3), A_{15}) = 8$, $d(2^6 : U_4(2), O_8^-(2)) = 8$, and $d(G \cap (A_8 \times A_8), G) = 8$, for $G = C_2 \wr C_2 \wr C_2 \wr C_2$.

It was shown already in [6] that $d(D_8, S_4) = 4$ holds. We found with GAP that $d(D_8 \times S_4, S_4 \wr C_2) = 8$. Continuing this process, we obtained that $d((D_8 \times S_4) \times (S_4 \wr C_2), S_4 \wr C_2 \wr C_2) = 16$. In general, we can define

- $G_0 := S_4$, $H_0 := D_8$,
- $G_n := G_{n-1} \wr C_2$, $H_n := H_{n-1} \times G_{n-1} < G_{n-1} \times G_{n-1} < G_n$,

and get $d(H_n, G_n) = 2^{n+2}$. This was proved in our paper [25] and written down in section 3.1.

We wanted to simplify the construction of our previous results. Our aim was also to construct as depth more even numbers. We can generalize the first two steps of the former construction in another way as follows:

- $d(D_8, S_4) = 4$,
- $d(D_8 \times S_4, S_4 \wr C_2) = 8$,
- $d(D_8 \times S_4 \times S_4, S_4 \wr C_3) = 12$.

In general, we take

- $G_1 := S_4$, $H_1 := D_8$,
- $G_n := G_1 \wr C_n$, $H_n := H_1 \times G_1^{n-1} < G_1^n < G_n$.

Then we have that $d(H_n, G_n) = 4n$. The proof is again using Theorem 2.1.3 to prove that $d(H_n, G_n) \leq 4n$. If $d(H_n, G_n) \leq 4n - 1 = 2(2n - 1) + 1$, then by Corollary 2.1.2 any two irreducible characters of H_n have distance at most $2n - 1$. However, one can show that there exist irreducible characters of H_n of distance $2n$.

If we want to get every even number then we can use a modified construction. We take the Klein four group $V_4 \triangleleft S_4$ instead of D_8 and get:

- $d(V_4, S_4) = 2$,

- $d(V_4 \times S_4, S_4 \wr C_2) = 4$,
- $d(V_4 \times S_4 \times S_4, S_4 \wr C_3) = 6$.

In general, we have a series of groups and subgroups such that $d(H_n, G_n) = 2n$ holds. The idea of the proof will be the same as before, for the inequality we will use again Theorem 2.1.3, and to prove that it cannot be a strict inequality, we find two irreducible characters of distance n in H_n. For that, we consider suitable characters of the base group of the wreath product and define a Cartesian product of graphs that encodes the relation $\sim$.

Proof of Theorem 3.2.1

Let G be the symmetric group on four points, and N be its normal Klein four subgroup. Set $G_1 = G$, $H_1 = N$. Then $d(H_1, G_1) = 2$, by Corollary 2.1.2. Define for $n \geq 2$

$$\begin{aligned} \sigma_n &= \prod_{j=1}^{4}(j, j+4, j+8, \cdots, j+4(n-1)), \\ G_n &= \langle G, \sigma_n \rangle, \\ H_n &= \langle N, G^{\sigma_n}, G^{\sigma_n^2}, \ldots, G^{\sigma_n^{n-1}} \rangle. \end{aligned}$$

Let C_n denote the cyclic group of order n. Then $H_n < G_n \cong G \wr C_n$ and

$$H_n \cong N \times G^{n-1} \leq G^n < G \wr C_n.$$

Let $N_n = \mathrm{Core}_{G_n}(H_n)$, the largest normal subgroup of G_n that is contained in H_n. Then $N_1 = N$, and

$$N_n = \langle N, N^{\sigma_n}, \ldots, N^{\sigma_n^{n-1}} \rangle = \bigcap_{i=0}^{n-1} H_n^{\sigma_n^i}$$

is an intersection of n conjugates of H_n, and Theorem 2.1.3 yields $d(H_n, G_n) \leq 2n$. Set

$$K_n = \langle G, G^{\sigma_n}, \ldots, G^{\sigma_n^{n-1}} \rangle \leq G_n.$$

Then $H_n \leq K_n \cong G^n$.

The character tables of N and G are as follows, where the columns are indexed by the conjugacy classes of the elements $g_1 = ()$, $g_2 = (1,3)(2,4)$, $g_3 = (1,2)(3,4)$, $g_3' = (1,2,3)$, $g_4 = (1,4)(2,3)$, $g_4' = (1,3)$, $g_5 = (1,2,3,4)$.

	g_1	g_2	g_3	g_4
ν_1	1	1	1	1
ν_2	1	1	-1	-1
ν_3	1	-1	1	-1
ν_4	1	-1	-1	1

	g_1	g_2	g_3'	g_4'	g_5
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

$$\begin{aligned}
\chi_1|_N &= \nu_1 \\
\chi_2|_N &= \nu_1 \\
\chi_3|_N &= 2\nu_1 \\
\chi_4|_N &= \nu_2 + \nu_3 + \nu_4 \\
\chi_5|_N &= \nu_2 + \nu_3 + \nu_4
\end{aligned}$$

Set

$$X_n = \{\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n} \in \mathrm{Irr}(K_n); i_1 \in \{4,5\}, i_j \in \{1,2,3\} \text{ for } 2 \leq j \leq n\}$$

and

$$Y_n = \left\{\chi^{G_n}; \chi \in X_n\right\}.$$

Let Γ_1 be the undirected graph with vertex set $\{4,5\}$ and edge set $\{\{4,5\}\}$, Γ_0 be the undirected graph with vertex set $\{1,2,3\}$ and edge set $\{\{1,3\},\{2,3\},\{1,2\}\}$. For $n \geq 2$, let Γ_n be the Cartesian product of Γ_1 and $n-1$ copies of Γ_0, that is, Γ_n has vertex set

$$\{(i_1, i_2, \ldots, i_n); i_1 \in \{4,5\}, i_j \in \{1,2,3\} \text{ for } 2 \leq j \leq n\},$$

and there is an edge between $(i_1, i_2, \ldots, i_n)$ and $(i'_1, i'_2, \ldots, i'_n)$ if and only if there is a (unique) j such that $i_k = i'_k$ for $k \neq j$ and $i_j \neq i'_j$ and either $\{i_j, i'_j\} = \{4, 5\}$ or $\{i_j, i'_j\} \subset \{1, 2, 3\}$.

Lemma 3.2.2

(i) $Y_n \subseteq \mathrm{Irr}(G_n)$*, and mapping* χ *to* χ^{G_n} *defines a bijection from* X_n *to* Y_n*.*

(ii) For $\psi \in Y_n$ *and* $\psi' \in \mathrm{Irr}(G_n)$*, if* $\psi|_{H_n}$ *and* $\psi'|_{H_n}$ *have a common constituent then* $\psi' \in Y_n$*.*

(iii) Let $\psi = \chi^{G_n}$*,* $\psi' = (\chi')^{G_n}$ *for* $\chi, \chi' \in X_n$*, with* $\psi \neq \psi'$*. Then* $\psi|_{H_n}$ *and* $\psi'|_{H_n}$ *have a common constituent if and only if there is an edge between* $(i_1, i_2, \ldots, i_n)$ *and* $(i'_1, i'_2, \ldots, i'_n)$ *in* Γ_n*, where* $\chi = \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ *and* $\chi' = \chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$*.*

(iv) The distance of the vertices $(4, 1, 1, \ldots, 1)$ *and* $(4, 2, 2, \ldots, 2)$ *of* Γ_n *is* $n - 1$*.*

(v) The distance $d_{G_n}(\alpha_n, \omega_n)$ *of the characters* $\alpha_n := \nu_2 \times \chi_1 \times \cdots \times \chi_1$ *and* $\omega_n := \nu_2 \times \chi_2 \times \cdots \times \chi_2$ *of* H_n *is* n*.*

Proof. Let $\psi = \chi^{G_n}$, where $\chi = \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n} \in X_n$, that is, χ_{i_1} is faithful and the other χ_{i_j} are not.

For part (i), χ has inertia subgroup K_n inside G_n. Hence by Theorem 1.3.24, χ^{G_n} is irreducible. The irreducible constituents of the restriction $\psi|_{K_n}$ are the n conjugates of χ by σ_n, i. e., those characters where the n components of χ are cyclically permuted. Thus each constituent has exactly one faithful component. Hence χ is the only constituent of $\psi|_{K_n}$ that lies in X_n. Thus we get an inverse to the map $\chi \mapsto \chi^{G_n}$.

For part (ii), consider the restriction of the constituents of $\psi|_{K_n}$ to H_n. We get irreducible constituents where the first component is a nontrivial character of N and all other components are non-faithful characters of G, and irreducible constituents where the first component is the trivial character of N and exactly one other component is faithful. Let $\psi' \in \mathrm{Irr}(G_n)$ have the property that $\psi'|_{H_n}$ and $\psi|_{H_n}$ have a common irreducible constituent, which means that $0 \neq (\psi|_{H_n}, \psi'|_{H_n}) = ((\psi|_{H_n})^{G_n}, \psi')$. If this constituent is of the first kind then inducing it to K_n yields a character with first component $\chi_4 + \chi_5$ and all other components non-faithful. If the common constituent is of the second kind then inducing it to K_n yields a character with first component $\chi_1 + \chi_2 + 2\chi_3$ and exactly one other component faithful. (Here we used that $(\mu \times \theta_2 \times \cdots \times \theta_n)^{K_n} = (\mu^G \times \theta_2 \times \cdots \times \theta_n)$, where $\theta_i \in \mathrm{Irr}(G)$, for $i = 2 \ldots n$, $\mu \in \mathrm{Irr}(N)$.)

In both cases, the irreducible constituents are cyclic shifts of characters in X_n, thus inducing further from K_n to G_n yields characters all whose irreducible constituents lie in Y_n. Now note that ψ' is one of them.

For part (iii), note that there is an edge between $(i_1, i_2, \ldots, i_n)$ and $(i'_1, i'_2, \ldots, i'_n)$ in Γ_n if and only if $\chi := \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and $\chi' := \chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$ differ in exactly one component χ_{i_j}, $\chi_{i'_j}$, such that $\chi_{i_j}|_N$ and $\chi_{i'_j}|_N$ have a common constituent. Let $\psi := (\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n})^{G_n}$, and $\psi' := (\chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n})^{G_n}$. Then $\psi|_{K_n}$ contains as a constituent $\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and all its cyclic shifts, ψ'_{K_n} contains as a constituent $\chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$ and all its cyclic shifts. When restricted further to H_n

the scalar product can be nonzero if and only if some of cyclic shifts of χ and some of cyclic shifts of χ' have in the first component a restriction that have a common component and all other components are equal. But then they must be shifted in the same way, since otherwise the faithful components were in different place. Thus $\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and $\chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$ differ in exactly one component χ_{i_j}, $\chi_{i'_j}$, such that $\chi_{i_j}|_N$ and $\chi_{i'_j}|_N$ have a common constituent.

For part (iv), observe that any shortest path from $(4, 1, \ldots, 1)$ to $(4, 2, \ldots, 2)$ in Γ_n replaces in each step exactly one 1 by a 2.

For part (v), fix n and let $\alpha_n \sim_{G_n} \psi_1 \sim_{G_n} \psi_2 \sim_{G_n} \cdots \sim_{G_n} \psi_m \sim_{G_n} \omega_n$ be a shortest path of related characters in $\mathrm{Irr}(H_n)$, of length $m+1$. This means that there are irreducible characters $\Phi_1, \Phi_2, \ldots, \Phi_{m+1}$ of G_n such that α_n and ψ_1 are constituents of $\Phi_1|_{H_n}$, ψ_i and ψ_{i+1} are constituents of $\Phi_{i+1}|_{H_n}$, for $1 \leq i \leq m-1$, and ψ_m and ω_n are constituents of $\Phi_{m+1}|_{H_n}$. By Frobenius reciprocity we have that $(\alpha_n^{G_n}, \Phi_1) \neq 0$. Since $\alpha_n^{K_n} = (\chi_4 + \chi_5) \times \chi_1 \times \cdots \times \chi_1$ is a sum of characters in X_n, we know that $\Phi_1 \in Y_n$, and part (ii) implies that $\Phi_i \in Y_n$ for all $i \in \{1, 2, \ldots, m+1\}$. Let Θ_i be the unique character in X_n with the property $\Phi_i = \Theta_i^{G_n}$, for $1 \leq i \leq m+1$. By part (iii), Θ_i and Θ_{i+1} differ in at most one component. Now Θ_1 has $n-1$ components χ_1, and Θ_{m+1} has $n-1$ components χ_2, thus $m \geq n-1$ holds. Conversely, any path of length $n-1$ between $(4, 1, 1, \ldots, 1)$ and $(4, 2, 2, \ldots, 2)$ in Γ_n yields a path of related characters from α_n to ω_n, of length n, hence $m + 1 = n$. □

In order to prove that $d(H_n, G_n) = 2n$, it remains to show that $d(H_n, G_n) \geq 2n$ holds. If $d(H_n, G_n) \leq 2n - 1 = 2(n-1) + 1$, then by Corollary 2.1.2 we have that every two irreducible characters of H_n have distance at most $n-1$. However, the characters α_n and ω_n constructed in Lemma 4.1.2 have distance n, which is a contradiction. So we are done.

Chapter 4

Results on the generalization of methods of Section 3.2

The results of this chapter are written down in the paper [20].

4.1 Infinitely many pairs (G_n, H_n) where $d(H_n, G_n) = 2n$

In the following we will denote the cyclic group of order n by C_n. In this section we will prove the following

Theorem 4.1.1 *Let G_1 be a permutation group on k points. Let $H_1 := N$ be a normal subgroup of depth two in it. Then the series $G_n = G_1 \wr C_n$, $H_n = H_1 \times G_1^{n-1}$ has the property that $d(H_n, G_n) = 2n$.*

Our construction is a generalization of that of [25].

4.1.1 Construction of a series of groups G_n and subgroups H_n

Let G be a permutation group on the points $\Delta = \{1, 2 \ldots, k\}$ and let $\Sigma = \{1, 2, \ldots, n\}$. Then $S_\Delta \wr S_\Sigma$ acts on $\Omega = \Delta \times \Sigma$, which associated to $\{1, 2, \ldots, nk\}$. Let $H \leq G$. Set $G_1 = G$, $H_1 = H$. Define for $n \geq 2$

$$\begin{aligned} \sigma_n &= \prod_{j=1}^{k} (j, j+k, j+2k, \cdots, j+(n-1)k), \\ C_n &= \langle \sigma_n \rangle, \\ G_n &= \langle G, \sigma_n \rangle \simeq G \wr C_n, \\ H_n &= \langle H, G^{\sigma_n}, G^{\sigma_n^2}, \ldots, G^{\sigma_n^{n-1}} \rangle \simeq H \times G^{n-1}. \end{aligned}$$

Then $H_n < G_n \cong G \wr C_n$ and

$$H_n \cong H \times G^{n-1} \leq G^n < G \wr C_n.$$

Set

$$K_n = \langle G, G^{\sigma_n}, \ldots, G^{\sigma_n^{n-1}} \rangle \leq G_n.$$

So K_n is the base group of the wreath product $G \wr C_n$.

Then $H_n \leq K_n \cong G^n$.

In the special situation of Theorem 4.1.1, H_1 is a normal subgroup of depth two in G. Then $G_1 \neq H_1$, otherwise $d(H_1, G_1) = 1$.

Let $N_n = \mathrm{Core}_{G_n}(H_n)$, the largest normal subgroup of G_n that is contained in H_n. Then $N_1 = H_1$, and

$$N_n = \bigcap_{x \in G_n} H_n^x = \bigcap_{i=0}^{n-1} H_n^{\sigma_n^i} = \langle N_1^{\sigma_n}, \ldots, N_1^{\sigma_n^{n-1}} \rangle$$

is an intersection of n conjugates of H_n. By Theorem 2.1.3, we have $d(H_n, G_n) \leq 2n$.

4.1.2 Construction of the graph Γ_n

Let $N = H_1 \lhd G$, let $\chi_1 = 1_G, \dots, \chi_r \in \mathrm{Irr}(G)$ all irreducible characters of G lying above $\phi_1 = 1_N$, i.e. $N \leq Ker(\chi_i)$, for $i = 1, \dots, r$. Then $r \geq 2$, since $d(N, G) = 2$ implies that $G \neq N$. Let $\chi_{r+1}, \dots, \chi_s$ all irreducible characters of G lying above a non-invariant character $\phi_2 \in \mathrm{Irr}(N)$ and all its conjugates. Such a character exists, since N is not of depth one, see [3, Theorem 1.7]. Then all other irreducible characters of G do not have common constituents with $\chi_1, \dots, \chi_s$, when restricted to N. Set

$$X_n = \{\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n} \in \mathrm{Irr}(K_n); i_1 \in \{r+1, \dots, s\}, i_j \in \{1, \dots, r\} \text{ for } 2 \leq j \leq n\}$$

and

$$Y_n = \left\{\chi^{G_n}; \chi \in X_n\right\}.$$

Let Γ_0 be the undirected complete graph with vertex set $\{r+1, \dots, s\}$, Γ_1 be the undirected complete graph with vertex set $\{1, \dots, r\}$. For $n \geq 2$, let Γ_n be the Cartesian product of Γ_0 and $n-1$ copies of Γ_1, that is, Γ_n has vertex set

$$\{(i_1, i_2, \dots, i_n); i_1 \in \{r+1, \dots s\}, i_j \in \{1, \dots r\} \text{ for } 2 \leq j \leq n\},$$

and there is an edge between $(i_1, i_2, \dots, i_n)$ and $(i'_1, i'_2, \dots, i'_n)$ if and only if there is a unique j such that $i_k = i'_k$ for $k \neq j$ and $i_j \neq i'_j$ and either $\{i_j, i'_j\} \subseteq \{r+1, \dots, s\}$, (in this case $j = 1$), or $\{i_j, i'_j\} \subseteq \{1, \dots, r\}$, (in this case $j \neq 1$).

Lemma 4.1.2 *(i) $Y_n \subseteq \mathrm{Irr}(G_n)$. The map $\chi \mapsto \chi^{G_n}$ defines a bijection between X_n and Y_n.*

(ii) *Let* $\mu \times \theta_2 \times \cdots \times \theta_n \in \mathrm{Irr}(H_n)$*, where* $\mu \in \mathrm{Irr}(H)$ *and* $\theta_i \in \mathrm{Irr}(G)$*, for* $i = 2, \ldots, n$*. Then* $(\mu \times \theta_2 \times \cdots \times \theta_n)^{K_n} = \mu^G \times \theta_2 \times \cdots \times \theta_n$*.*

(iii) *For* $\psi \in Y_n$ *and* $\psi' \in \mathrm{Irr}(G_n)$*, if* $\psi|_{H_n}$ *and* $\psi'|_{H_n}$ *have a common constituent then* $\psi' \in Y_n$*.*

(iv) *Let* $\psi = \chi^{G_n}$*,* $\psi' = (\chi')^{G_n}$ *with* $\psi' \neq \psi$*, for* $\chi, \chi' \in X_n$*. Then* $\psi|_{H_n}$ *and* $\psi'|_{H_n}$ *have a common constituent if and only if there is an edge between* $(i_1, i_2, \ldots, i_n)$ *and* $(i'_1, i'_2, \ldots, i'_n)$ *in* Γ_n*, where* $\chi = \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ *and* $\chi' = \chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$*. This holds if and only if* χ *and* χ' *differ exactly in one component* χ_{i_j}*,* $\chi_{i'_j}$*, such that* $\chi_{i_j}|_N$ *and* $\chi_{i'_j}|_N$ *have a common constituent.*

(v) *The distance of the vertices* $(r+1, 1, 1, \ldots, 1)$ *and* $(r+1, 2, 2, \ldots, 2)$ *of* Γ_n *is* $n - 1$*.*

(vi) *The* G_n*-distance* $d^{G_n}(\alpha_n, \omega_n)$ *of the characters*

$$\alpha_n := \phi_2 \times \chi_1 \times \cdots \times \chi_1$$

and

$$\omega_n := \phi_2 \times \chi_2 \times \cdots \times \chi_2$$

of H_n*, is* n*. Here* $\phi_2 \in \mathrm{Irr}(N)$ *is the non-invariant character defined in the construction of the graph* Γ_n *above,* $\chi_1 = 1_G, \chi_2 \in \mathrm{Irr}(G)$ *two different characters lying over* 1_N*.*

Proof. For part (i), any $\chi \in X_n$ has inertia subgroup K_n inside G_n, since the first component does not contain N in the kernel the others contain, hence characters in X_n cannot be invariant under the cyclic shift. Hence by Theorem 1.3.24, χ^{G_n} is irreducible. Let $\psi := \chi^{G_n}$, for a character $\chi \in X_n$. The irreducible constituents of the restriction

$\psi|_{K_n}$ are the n conjugates of χ by σ_n, so those characters where the n components of χ are cyclically permuted. Each consituent has just one component that is nontrivial on N. Hence χ is the only consituent of $\psi|_{K_n}$ that lies in X_n. Thus we get an inverse to the map $\chi \mapsto \chi^{G_n}$.
For part (ii): this can be proved by direct calculation.
For part (iii), let $\psi = \chi^{G_n}$, where $\chi = \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n} \in X_n$, where, for χ_{i_1} we have that $i_1 \in \{r+1, \ldots, s\}$ and for the other χ_{i_j} we have that $i_j \in \{1, \ldots r\}$. The irreducible constituents of the restriction $\psi|_{K_n}$ are the n conjugates of χ by σ_n, i. e., those characters where the n components of χ are cyclically permuted. Thus each constituent has exactly one component from $\{\chi_{r+1}, \ldots, \chi_s\}$. Restricting further to H_n, we get irreducible constituents where the first component is ϕ_2 or one of its G-conjugates, and all other components are from $\{\chi_1, \ldots, \chi_r\}$, and irreducible constituents where the first component is $\phi_1 = 1_N$ and exactly one other component is from $\{\chi_{r+1}, \ldots, \chi_s\}$. Let $\psi' \in \mathrm{Irr}(G_n)$ have the property that $\psi'|_{H_n}$ and $\psi|_{H_n}$ have a common constituent. It means that $0 \neq (\psi|_{H_n}, \psi'|_{H_n}) = (\psi|_{H_n}{}^{G_n}, \psi')$. If the common constituent of $\psi'|_{H_n}$ and $\psi|_{H_n}$ is of the first kind, then inducing it to K_n yields a character where the first component has irreducible constituents in $\{\chi_{r+1}, \ldots, \chi_s\}$ and all other components are in $\{\chi_1, \ldots, \chi_r\}$. If the common constituent is of the second kind, then inducing it to K_n yields a character where the first component has irreducible constituents from $\{\chi_1, \ldots, \chi_r\}$ and exactly one other component is from $\{\chi_{r+1} \ldots, \chi_s\}$, the others are from $\{\chi_1, \ldots, \chi_r\}$. Here we used that $(\mu \times \theta_2 \times \cdots \times \theta_n)^{K_n} = (\mu^G \times \theta_2 \times \cdots \times \theta_n)$, where $\theta_i \in \mathrm{Irr}(G)$ for $i = 2, \ldots, n$ and $\mu \in \mathrm{Irr}(N)$, by part (ii).

In both cases, the irreducible constituents are cyclic shifts of characters in X_n, thus inducing further from K_n to G_n yields characters all whose irreducible constituents lie in Y_n. Now note that ψ' is one of them.

For part (iv), note that there is an edge between $(i_1, i_2, \ldots, i_n)$ and $(i'_1, i'_2, \ldots, i'_n)$ in Γ_n if and only if $\chi := \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and $\chi' := \chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$ differ in exactly one component χ_{i_j}, $\chi_{i'_j}$, such that $\chi_{i_j}|_N$ and $\chi_{i'_j}|_N$ have a common constituent. Let $\psi := \chi^{G_n}$ and $\psi' := \chi'^{G_n}$. The $\psi|_{K_n}$ has constituents χ and all its cyclic shifts and $\psi'|_{K_n}$ has constituents χ' and all its cyclic shifts. When restricted further to H_n the scalar product can be nonzero if and only if some cyclic shifts of χ and some cyclic shifts of χ' have in the first component a restriction that have a common constituent and all other components are equal. But then χ and χ' must be shifted in the same way, otherwise there was a component where one of χ and χ' would contain a character that has N in its kernel and the other not. Thus χ and χ' differ exactly in one component $\chi_{i_j} \neq \chi_{i'_j}$ and $\chi_{i_j}|_N$ and $\chi_{i'_j}|_N$ have a common constituent.

For part (v), observe that any shortest path from $(r+1, 1, \ldots, 1)$ to $(r+1, 2, \ldots, 2)$ in Γ_n replaces in each step exactly one 1 by a 2.

For part (vi), fix n and let $\alpha_n \sim^{G_n} \psi_1 \sim^{G_n} \psi_2 \sim^{G_n} \cdots \sim^{G_n} \psi_m \sim^{G_n} \omega_n$ be a shortest path of G_n-related characters in $\mathrm{Irr}(H_n)$, of length $m+1$. This means that there are irreducible characters $\Phi_1, \ldots, \Phi_{m+1} \in \mathrm{Irr}(G_n)$ such that α_n and ψ_1 are constituents of $\Phi_1|_{H_n}$, ψ_i and ψ_{i+1} are constituents of $\Phi_{i+1}|_{H_n}$, for $i = 1, \ldots, m-1$, and ψ_m and ω_n are constituents of $\Phi_{m+1}|_{H_n}$. By Frobenius reciprocity we have that $(\alpha_n^{G_n}, \Phi_1) \neq 0$. Since $\alpha_n^{K_n}$ has constituents $\chi_i \times \chi_1 \times \cdots \times \chi_1$, where $i \in \{r+1, \ldots, s\}$, these are characters in X_n, hence by part (i) $\alpha_n^{G_n}$

has constituents in Y_n, thus $\Phi_1 \in Y_n$. By part (iii), then $\Phi_i \in Y_n$ for $i = 1, \ldots, m+1$. Let Θ_i be the unique character in X_n such that $\Theta_i^{G_n} = \Phi_i$, for $i = 1, \ldots, m+1$, which exist by part (i). By part (iv) Θ_i and Θ_{i+1} differ only in one component. Now Θ_1 has $n-1$ components χ_1 and Θ_{m+1} has $n-1$ components χ_2. So $m \geq n-1$ holds. Conversely, any path of length $n-1$ between $(r+1, 1, 1, \ldots, 1)$ and $(r+1, 2, 2, \ldots, 2)$ in Γ_n yields a path of G_n-related characters from α_n to ω_n, of length n, hence $m+1 = n$. To see this, let $\Theta_1, \Theta_2, \ldots, \Theta_n$ be characters of X_n belonging to vertices in the path of length $n-1$ between $(r+1, 1, \ldots, 1)$ and $(r+1, 2, \ldots, 2)$, where the componets of Θ_i are characters of $\mathrm{Irr}(G)$ indexed by the components of the vertices in the path. Let $\Phi_i := \Theta_i^{G_n}$ for $i = 1, \ldots, n$. By part (iv) we know that then $\Phi_i|_{H_n}$ and $\Phi_{i+1}|_{H_n}$ have a common constituent for $i = 1, \ldots, n-1$. Thus $\Phi_1, \ldots, \Phi_n$ gives a path of length n of G_n-related characters between α_n and ω_n. □

Now we prove Theorem 4.1.1:

Proof. (of Theorem 4.1.1)

We know that $d(H_n, G_n) \leq 2n$. If $d(H_n, G_n) \leq 2n-1 = 2(n-1)+1$, then by Definition 2.1.2, every pair of irreducible characters of H_n has G_n-distance at most $n-1$. However, $\mathrm{Irr}(H_n)$ contains two characters of G_n-distance n: the characters α_n, ω_n constructed in Lemma 4.1.2 have this property. Thus $d(H_n, G_n) = 2n$. □

The next two examples show that there are infinitely many series of groups and subgroups (G_n, H_n) such that $d(H_n, G_n) = 2n$.

Example 4.1.3 *Let $k \geq 3$, $G_1 := S_k$, the symmetric group of degree k, and $H_1 := A_k$, the index two normal subgroup of S_k. Then* *the depth $d(A_k, S_k)$ is at most two. However, $d(A_k, S_k)$ cannot be one, since then, by [3, Lemma 1.7], every irreducible character of A_k would be invariant in S_k. Each irreducible character of S_k belongs to a partititon of k. If it is a symmetric partition, then the restriction of the corresponding irreducible character of S_k to A_k splits into two conjugate irreducible characters. These characters of A_k are not invariant. Thus $d(H_1, G_1) = 2$ and so the above construction gives that $d(H_n, G_n) = 2n$, where $H_n = A_k \times S_k^{n-1}$ and $G_n = S_k \wr C_n$.*

We observe that due to the Cayley representation every group can be considered to be a permutation group, so our construction can be applied to any group and subgroup.

Example 4.1.4 *Let G be a non-abelian p-group. Then it has at least one non-linear character $\chi \in \mathrm{Irr}(G)$. By [23, Corollary 6.14] every p-group is monomial, so every irreducible character of G can be induced from a linear character of a subgroup. Let $\chi = \lambda^G$, for $\lambda \in \mathrm{Irr}(H)$ a linear character. Let N be a maximal subgroup of G containing H. Then $\phi := \lambda^N \in \mathrm{Irr}(N)$.*

Thus χ can be induced from $\phi \in \mathrm{Irr}(N)$, where N is a normal subgroup of index p in G. Then $d(N, G) = 2$, since ϕ is not an invariant character of N. Thus if $G_1 := G$ and $H_1 := N$ then the above construction gives a series of groups with $d(H_n, G_n) = 2n$, where $H_n = N \times G^{n-1}$, $G_n = G \wr C_n$.

Corollary 4.1.5 *For every positive integer n there are infinitely many triples (H, N, G) of finite solvable groups $H \lhd N \lhd G$ such that G/N is cyclic of order n, N/H is cyclic of arbitrarily large prime order, and $d(H, G) = 2n$.*

Proof. Let G_1 and H_1 be as in Example 4.1.4. Let $H := H_n$, $N := K_n$ and $G := G_n$. Then $G_n \simeq G_1 \wr C_n$ is solvable, $G/N \simeq C_n$, $N/H \simeq C_p$, and $d(H,G) = 2n$. □

The following example shows that one cannot generalize Theorem 4.1.1 from $d(H_1, G_1) = 2$ for $d(H_1, G_1)$ even.

Example 4.1.6 *Let $G_1 := A_6$ and $S_4 \simeq H_1 \leq A_6$. There are two conjugacy classes of such subgroups. Calculations with* GAP *[11] show that $d(H_1, G_1) = 4$, however $d(H_2, G_2) = 7$ and $d(H_3, G_3) = 11$, where $H_n = H_1 \times A_6^{n-1}$, $G_n = A_6 \wr C_n$. Thus for a subgroup H_1 of even depth bigger than two in G_1, the above method can also give odd depth subgroups H_n in G_n for some n.*

4.2 Depth series coming from subgroups of depth one of any group

We study further the series $G_n = G \wr C_n$ and $H_n = H \times G^{n-1}$ for a subgroup H in G. Let us suppose now that $d(H,G) = 1$. Then by [3, Theorem 1.7], H is normal and every irreducible character of H is G-invariant.

If $\{1\} < H = H_1 = G_1 = G$, then the construction of the series G_n and H_n gives us that $H_n = G^n$ is the base group of the wreath product $G_n = G \wr C_n$, hence $d(H_n, G_n) = 2$ for $n > 1$.

If $\{1\} = H_1 = G_1$ then $H_n = \{1\}^n$ and $G_n = \{1\}^n \rtimes C_n$. Hence $d(H_n, G_n) = 1$.

Proposition 4.2.1 *Let $H = H_1 = \{1\}$ be the trivial subgroup of $G = G_1$. Then $d(H_1, G_1) = 1$. Suppose that G_1 is a nontrivial group. Then* *for the series* $G_n = G \wr C_n$, $H_n = H \times G^{n-1}$ *we have that* $d(H_n, G_n) = 2n - 1$.

Proof. We have a group G and its trivial subgroup H, and define $G_n = G \wr C_n$, $K_n = G^n \leq G_n$ (the base group of the wreath product), $H_n = H \times G^{n-1} \leq K_n$.

Consider $\phi, \psi \in \mathrm{Irr}(H_n)$ with $\phi \sim^{G_n} \psi$. Then there is $\Phi \in \mathrm{Irr}(G_n)$ such that $(\Phi_{H_n}, \phi) \neq 0 \neq (\Phi_{H_n}, \psi)$, which means that $(\Phi_{K_n}, \phi^{K_n}) \neq 0 \neq (\Phi_{K_n}, \psi^{K_n})$.

We have $\phi = 1_H \times \chi_2 \times \cdots \times \chi_n$, for $\chi_i \in \mathrm{Irr}(G)$, and thus $\phi^{K_n} = \rho_G \times \chi_2 \times \cdots \times \chi_n$, where ρ_G is the regular character of G. Here we used again that $(\mu \times \theta_2 \times \cdots \times \theta_n)^{K_n} = (\mu^G \times \theta_2 \times \cdots \times \theta_n)$, where $\theta_i \in \mathrm{Irr}(G)$ for $i = 2, \ldots, n$ and $\mu \in \mathrm{Irr}(N)$, by part (ii) of Lemma 4.1.2. We also used the fact that the regular character is induced from the trivial character of the trivial subgroup.

The irreducible constituents of Φ_{K_n} are the G_n-conjugates of some irreducible constituent of ϕ^{K_n}, that is, the cyclic shifts of a character $\chi \times \chi_2 \times \cdots \times \chi_n$, with $\chi \in \mathrm{Irr}(G)$.

One of them is an irreducible constituent of ψ^{K_n}, thus all irreducible constituents of ψ^{K_n} are given by taking this one cyclic shift and then replacing the first component by arbitrary irreducible characters of G. (This is because the first component of ψ is also 1_H, hence the first component of ψ^{K_n} is also ρ_G and the other components are the same as the respective components of ψ.) This means that in each constituent, at least $n-2$ of the characters $\chi_2, \chi_3, \ldots, \chi_n$ occur in the components 2 to n, and the same holds for the constituents of the restriction to H_n. Hence the set of characters occurring in the components of ψ can differ from the set of characters in components of ϕ in at most one character.

Now consider $\alpha_n = 1_H \times 1_G \times 1_G \times \cdots \times 1_G$ and $\omega_n = 1_H \times \chi \times \chi \times \cdots \times \chi$, with $\chi \in \mathrm{Irr}(G_n)$, $\chi \neq 1_G$. If $\alpha_n = \phi_0 \sim^{G_n} \phi_1 \sim^{G_n} \cdots \sim^{G_n} \phi_m = \omega_n$ then the above argument implies that $m \geq n-1$ holds, i. e., $d^{G_n}(\alpha_n, \omega_n) \geq n-1$, in particular $m(1_{G_n}) \geq n-1$.

According to Definition 2.1.1 (ii), we have $d(H_n, G_n) \geq 2n-1$.

Since $\mathrm{Core}_{G_n}(H_n) = \{1\}$ is the intersection of n conjugates of H_n, $d(H_n, G_n) \leq 2n-1$ holds by Theorem 2.1.3, so we get equality. □

Corollary 4.2.2 *For every positive integer n there are infinitely many triples (H, N, G) of finite solvable groups $H \lhd N \lhd G$ such that G/N is cyclic of order $\lceil n/2 \rceil$, N/H is cyclic of arbitrarily large prime order, and $d(H, G) = n$.*

Proof. We already proved the statement for even depth in Corollary 4.1.5.
Let $G_1 := C_p$, for a prime p, and $H_1 := \{1\}$. Then the triple (H_n, K_n, G_n) has the property that $H_n \lhd K_n \lhd G_n$ are solvable groups, $G_n/K_n \simeq C_n$, $K_n/H_n \simeq C_p$ and $d(H_n, G_n) = 2n-1$. □

If $\{1\} \neq H < G$ then we have:

Theorem 4.2.3 *Let $H = H_1$ be a nontrivial proper subgroup of the group $G = G_1$ and let us suppose that $d(H_1, G_1) = 1$. Then by the construction of the series of $G_n = G \wr C_n$ and $H_n = H \times G^{n-1}$, we have that $d(H_n, G_n) = 2n$, for $n > 1$.*

Proof. For $n = 1$, $d(H_1, G_1) = 1$ holds. Since $Core_{G_n}(H_n) = H^n$, and this is an intersection of n conjugates of H_n, hence by Theorem 2.1.3

we have that $d(H_n, G_n) \leq 2n$. We construct a similar graph as Γ_n as in the graph construction part of Theorem 4.1.1.

Let $\phi_1 = 1_H$, let $\chi_1, \ldots, \chi_r$ be those irreducible characters of G that lie above ϕ_1. Let $\phi_2 \neq \phi_1$ be another irreducible character of H, which exists since $|H| > 1$. Let $\chi_{r+1}, \ldots, \chi_s$ be the irreducible characters of G lying above ϕ_2. With this numbering the definition of the new Γ_n is the same as in the proof of Theorem 4.1.1. We could state a similar lemma as Lemma 4.1.2. The only difference in the proof is that ϕ_2 is now invariant, so one need not consider its conjugates. Hence we get in a very similar way as in Theorem 4.1.1 that $d(H_n, G_n) = 2n$ for $n > 1$. $\square$

4.3 Depth series coming from subgroups of depth three, all whose different irreducible characters have G-distance one.

If the subgroup H has depth three in G then every two irreducible characters of H have G-distance at most one, by Definition 2.1.2. By the second part of Theorem 2.1.3, if $H \neq \{1\}$ and has a disjoint conjugate, then H has depth three in G. In this case, however every two distinct irreducible characters of H have G-distance exactly one, by Mackey's theorem: for, if $\phi_1, \phi_2 \in \mathrm{Irr}(H)$ then $(\phi_1^G, \phi_2^G) = (\phi_1^G{}_H, \phi_2) = (\sum_{x \in H\backslash G/H}((\phi^x_{1\,H^x \cap H})^H, \phi_2) \neq 0$ since the regular character of H occurs in the sum. However, if for a subgroup H of G every two different irreducible characters of $\mathrm{Irr}(H)$ have G-distance exactly one, then H need not have a disjoint conjugate in G.

Example 4.3.1 *Let $G := A_7$, let $H \leq G$ and $H \simeq A_5$. Calculations with* GAP *[11] show that: there are two conjugacy classes of such subgroups, we can take H from any of these classes, $d(H, G) = 3$ will hold, and every two distinct irreducible characters of H have G-distance exactly one. However, H does not have a disjoint conjugate in G.*

Lemma 4.3.2 *Let G be a non-trivial group, let $H \leq G$ a nontrivial subgroup. Suppose that for any two different irreducible characters $\phi, \psi \in \mathrm{Irr}(H)$ we have that $d^G(\phi, \psi) = 1$. Then for any two different irreducible characters $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ we have that $0 < d_H(\chi_1, \chi_2) \leq 2$.*

Proof. Let $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ be two different characters. Let $(\chi_1|_H, \phi) \neq 0$ and $(\chi_2|_H, \psi) \neq 0$, for some $\phi, \psi \in \mathrm{Irr}(H)$. If $\phi = \psi$ then $d_H(\chi_1, \chi_2) = 1$. Otherwise, since $d^G(\phi, \psi) = 1$, so there is an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $(\phi, \chi|_H) \neq 0 \neq (\psi, \chi|_H)$. Then $\chi_1 \sim_H \chi \sim_H \chi_1$. So $0 < d_H(\chi_1, \chi_2) \leq 2$. □

The main result of this section is:

Theorem 4.3.3 *Let G be a finite group and let $H \leq G$ be a subgroup of ordinary depth three. Suppose that any two different characters of $\mathrm{Irr}(H)$ have G-distance one. (E.g. this is the case if H has a disjoint conjugate.) Let us construct the series of groups G_n and H_n from G and H as before, namely $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$, $G_n := G \wr C_n$. If any two different irreducible characters of $\mathrm{Irr}(G)$ have H-distance one then $d(H_n, G_n) = 2n + 1$, if there exist two different irreducible characters of $\mathrm{Irr}(G)$ that have H-distance two then $d(H_n, G_n) = 4n - 1$.*

First we prove some lemmas:

Lemma 4.3.4 *Let us suppose that* $H \leq G$. *Let* $H_1 = H, G_1 = G$ *and let* $G_n = G \wr C_n$ *and* $H_n = H \times G^{n-1}$ *as before. Let* $\Phi := \phi_1 \times \chi_2 \times \cdots \times \chi_n \in \mathrm{Irr}(H_n)$ *and let* $\Phi' := \phi_2 \times \chi_3 \times \chi_4 \times \cdots \times \chi_n \times \chi \in \mathrm{Irr}(H_n)$, *where* $(\phi_1^G, \chi) \neq 0$ *and* $(\phi_2^G, \chi_2) \neq 0$, *in other words let* χ *be above* ϕ_1 *and* χ_2 *above* ϕ_2. *Then* $0 \leq d^{G_n}(\Phi, \Phi') \leq 1$.

Proof. By Lemma 4.1.2 (ii), $\Phi^{K_n} = \phi_1^G \times \chi_2 \times \cdots \times \chi_n$ and $\Phi'^{K_n} = \phi_2^G \times \chi_3 \times \cdots \times \chi_n \times \chi$. Let $X := \chi \times \chi_2 \times \cdots \times \chi_n$ and let $X' := \chi_2 \times \chi_3 \times \cdots \times \chi_n \times \chi$. Then $X, X' \in \mathrm{Irr}(K_n)$ and $(\Phi^{K_n}, X) \neq 0 \neq (\Phi'^{K_n}, X')$, since $(\phi_1^G, \chi) \neq 0 \neq (\phi_2^G, \chi_2)$. Then $(\Phi^{G_n}, X^{G_n}) \neq 0 \neq (\Phi'^{G_n}, X'^{G_n})$. However, $0 \neq (\Phi^{G_n}, X^{G_n}) = (\Phi^{G_n}|_{K_n}, X) = (\Phi^{G_n}|_{K_n}, X') = (\Phi^{G_n}, X'^{G_n})$, since X and X' are conjugate characters. Hence $(\Phi^{G_n}, \Phi'^{G_n}) \neq 0$. This is equivalent to $0 \leq d^{G_n}(\Phi, \Phi') \leq 1$, by the Frobenius reciprocity. □

Lemma 4.3.5 *Let* $H \leq G$, $H_1 = H$, $G_1 = G$, $G_n = G \wr C_n$ *and* $H_n = H \times G^{n-1}$. *Let* $\Phi := \phi_1 \times \chi_2 \times \cdots \times \chi_n \in \mathrm{Irr}(H_n)$ *and* $\Phi' := \phi'_2 \times \chi_3 \times \cdots \times \chi_n \times \chi'_1 \in \mathrm{Irr}(H_n)$. *Assume that* $(\phi_1^G, \chi'_1) \neq 0$. *Let* $\chi'_2 \in \mathrm{Irr}(G)$ *such that* $(\phi_2'^G, \chi'_2) \neq 0$. *Suppose further that there exists a character* $\phi_2 \in \mathrm{Irr}(H)$ *such that* $(\phi_2^G, \chi_2) \neq 0 \neq (\phi_2^G, \chi'_2)$. *Then* $0 \leq d^{G_n}(\Phi, \Phi') \leq 2$.

Proof. Let $\Psi := \phi_2 \times \chi_3 \times \cdots \times \chi_n \times \chi'_1$. We apply Lemma 4.3.4 to Φ and Ψ. Then we have that $0 \leq d^{G_n}(\Phi, \Psi) \leq 1$.
Let $X := \chi'_2 \times \chi_3 \times \cdots \times \chi_n \times \chi'_1$. Then by Lemma 4.1.2 (ii) and using that $(\phi_2'^G, \chi'_2) \neq 0 \neq (\phi_2^G, \chi'_2)$, we have that $(\Phi'^{K_n}, X) \neq 0 \neq (\Psi^{K_n}, X)$. Hence $(\Phi'^{G_n}, X^{G_n}) \neq 0 \neq (\Psi^{G_n}, X^{G_n})$. Hence $(\Psi^{G_n}, \Phi'^{G_n}) \neq 0$, which is equivalent to $0 \leq d^{G_n}(\Psi, \Phi') \leq 1$. Since $0 \leq d^{G_n}(\Phi, \Psi) \leq 1$, we have that $0 \leq d^{G_n}(\Phi, \Phi') \leq 2$. □

Lemma 4.3.6 *Let us suppose that $H \leq G$ and any two different irreducible characters of H have G-distance one and any two different irreducible characters of G have H-distance one. Let $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$. Then $0 \leq d^{G_n}(\Phi, \Psi) \leq n$ for any $\Phi, \Psi \in \mathrm{Irr}(H_n)$.*

Proof. We apply Lemma 4.3.4 to prove that if $\Phi := \phi_1 \times \chi_2 \times \cdots \times \chi_n \in \mathrm{Irr}(H_n)$ and $\Psi := \phi_1' \times \chi_2' \times \cdots \times \chi_n' \in \mathrm{Irr}(H_n)$ then $0 \leq d^{G_n}(\Phi, \Psi) \leq n$. Let $\chi \in \mathrm{Irr}(G)$ be a character above ϕ_1 and ϕ_1'. Let $\phi_i \in \mathrm{Irr}(H)$ be a character below χ_i and χ_i', for $i = 2, \ldots, n$. Let $\Phi^{(1)} := \phi_2 \times \chi_3 \times \chi_4 \times \cdots \times \chi_n \times \chi$, $\Phi^{(k)} := \phi_{k+1} \times \chi_{k+2} \times \cdots \times \chi_n \times \chi \times \chi_2' \times \cdots \times \chi_k'$ for $k = 2, \ldots, n-1$. By Lemma 4.3.4 we have that $0 \leq d^{G_n}(\Phi^{(k)}, \Phi^{(k+1)}) \leq 1$ for $k = 0, \ldots, n-1$, where $\Phi^{(0)} := \Phi$ and $\Phi^{(n)} := \Psi$. Hence $0 \leq d^{G_n}(\Phi, \Psi) \leq n$. □

Lemma 4.3.7 *Let $H \leq G$. Suppose that any two different irreducible characters of H have G-distance one. Let us suppose that there exist irreducible characters of G of H-distance two. Let $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$. Then $0 \leq d^{G_n}(\Phi, \Psi) \leq 2n - 1$, for every $\Phi, \Psi \in \mathrm{Irr}(H_n)$.*

Proof. Let $\Phi = \Phi^{(1)} := \phi_1 \times \chi_2 \times \ldots \times \chi_n \in \mathrm{Irr}(H_n)$ and $\Psi := \phi_1' \times \chi_2'' \times \ldots \times \chi_n'' \in \mathrm{Irr}(H_n)$ be arbitrary characters. By Lemma 4.3.2 we have that $0 \leq d_H(\chi_i, \chi_i'') \leq 2$, there are characters $\chi_i' \in \mathrm{Irr}(G)$ and $\phi_i, \phi_i' \in \mathrm{Irr}(H)$ such that $\chi_i, \chi_i' \in \mathrm{Irr}(G)$ are above $\phi_i \in \mathrm{Irr}(H)$ for $i = 2, \ldots, n$ and $\chi_i', \chi_i'' \in \mathrm{Irr}(G)$ are above ϕ_i', for $i = 2, \ldots, n$. Let $\chi_1' \in \mathrm{Irr}(G)$ be a character above ϕ_1 and ϕ_1'. Let $\Phi^{(2)} := \phi_2' \times \chi_3 \times \cdots \times \chi_n \times \chi_1'$, let $\Phi^{(k)} := \phi_k' \times \chi_{k+1} \times \cdots \times \chi_n \times \chi_1' \times \chi_2'' \times \cdots \times \chi_{k-1}''$, for $k = 3, \ldots, n$. By Lemma 4.3.5 we have that $0 \leq d^{G_n}(\Phi^{(1)}, \Phi^{(2)}) \leq 2$,

$0 \leq d^{G_n}(\Phi^{(k)}, \Phi^{(k+1)}) \leq 2$ for $k = 2, \ldots, n-1$ and by Lemma 4.3.4 we have that $0 \leq d^{G_n}(\Phi^{(n)}, \Psi) \leq 1$. Hence $0 \leq d^{G_n}(\Phi, \Psi) \leq 2(n-1)+1 = 2n-1$. □

Proof. **(of Theorem 4.3.3)** By Lemma 4.3.2 we know that the H-distance of any two different irreducible characters of $\mathrm{Irr}(G)$ is either one or two.

Let $\Phi, \Psi \in \mathrm{Irr}(H_n)$. If every two different characters of $\mathrm{Irr}(G)$ have H-distance one then Lemma 4.3.6 yields $0 \leq d^{G_n}(\Phi, \Psi) \leq n$. If there exist irreducible characters of G which have H-distance two then Lemma 4.3.7 yields $0 \leq d^{G_n}(\Phi, \Psi) \leq 2n-1$. We show that these upper bounds are attained for suitable Φ, Ψ.

In the first situation, we fix $\phi_2 \in \mathrm{Irr}(H)$ different from 1_H. Since $d^G(1_H, \phi_2) = 1$, there is a character $\chi_2 \neq 1_G$ above both 1_H and ϕ_2.

In the second situation, if $d_H(1_G, \chi) = 1$ would hold for all $\chi \in \mathrm{Irr}(G)$ then we would have $d_H(\chi, \psi) = 1$ for all $\chi, \psi \in \mathrm{Irr}(G)$. Thus we may fix $\chi_2, \chi_3 \in \mathrm{Irr}(G)$ with the properties $d_H(1_G, \chi_2) = 2$, $d_H(1_G, \chi_3) = 1$, and $d_H(\chi_2, \chi_3) = 1$. Moreover, there is a character $1_H \neq \phi_2 \in \mathrm{Irr}(H)$ below χ_2 and χ_3.

In both situations, let $\alpha_n := 1_H \times 1_G \times \cdots \times 1_G = 1_{H_n}$, let $\omega_n := \phi_2 \times \chi_2 \times \cdots \times \chi_2 \in \mathrm{Irr}(H_n)$, and let $\alpha_n = \mu_0 \sim^{G_n} \mu_1 \sim^{G_n} \cdots \sim^{G_n} \mu_k = \omega_n$ be a shortest chain between α_n and ω_n.

Then there are characters $\tilde{X}_1, \cdots, \tilde{X}_k \in \mathrm{Irr}(G_n)$ such that $\tilde{X}_i$ lies above both μ_{i-1} and μ_i, for $i = 1, \ldots, k$. Consider the restriction of $\tilde{X}_i$ to K_n. By Clifford's theorem, its irreducible constituents are cyclic shifts of each other. At least one of them lies above μ_{i-1}, let us call it Θ_i; and some conjugate of it lies above μ_i, let us call it Θ_i'.

Each irreducible character of K_n is an n-fold tensor product of characters of G. Thus Θ_i and Θ'_i have the same irreducible components, with multiplicities. Since both Θ'_i and Θ_{i+1} lie above μ_i, their components at the positions 2 to n are equal, that is, they differ in at most one component. Now Θ_1 and Θ'_1 have $n-1$ components 1_G, and Θ_k and Θ'_k have $n-1$ components χ_2.

This implies directly that $k-1 \geq n-1$ holds, which is enough in the first situation: The G_n-distance of α_n and ω_n is at least n. Using the upper bound above, we see that equality holds. Thus by Definition 2.1.2, we have $d(H_n, G_n) \leq 2n+1$. If $d(H_n, G_n) \leq 2n$ would hold then $m(X) \leq n-1$ for all $X \in \mathrm{Irr}(G_n)$, however $m(1_{G_n}) = n$, since $d^{G_n}(1_{H_n}, \omega_n) = n$. Thus $d(H_n, G_n) = 2n+1$.

In the second situation, we know that $n-1$ steps are needed to replace all components 1_G by other irreducible characters of G, and since none of these other characters can be equal to χ_2, at least $n-1$ steps are needed to replace these characters by χ_2, which means that $k-1 \geq 2(n-1)$ holds. Thus the G_n-distance of α_n and ω_n is at least $2n-1$. Using the upper bound above, we see that equality holds. Thus by Definition 2.1.2, we have $d(H_n, G_n) \leq 2(2n-1)+1 = 4n-1$. If $d(H_n, G_n) \leq 4n-2$ would hold then $m(X) \leq 2n-2$ for all $X \in \mathrm{Irr}(G_n)$, however $m(1_{G_n}) = 2n-1$, since $d^{G_n}(1_{H_n}, \omega_n) = 2n-1$. Thus $d(H_n, G_n) = 4n-1$. □

Example 4.3.8 *In the case of S_4, nontrivial subgroups that are disjoint from their conjugates are: the cyclic subgroups C_2 of order two, generated by a transposition, the cyclic groups C'_2 of order two, generated by a product of disjoint transpositions, the cyclic subgroups C_3*

of order three, the cyclic subgroups C_4 of order four, the non-normal Klein four subgroups T_4. The subgroups C_2' generated by double transpositions are examples for subgroups giving a depth series $2n + 1$, all the others mentioned here are examples giving depth series $4n - 1$. We mentioned in [25] that the depth series coming from $D_8 \leq S_4$ is $4n$. We will see in section 7 that the depth series coming from $S_3 \leq S_4$ is $6n - 1$. Subgroups A_4 and the normal Klein four subgroup V_4 are of depth two, so they give depth series $2n$. The trivial subgroup of S_4 gives depth series $d(H_n, G_n) = 1$ and the whole group gives depth series $d(H_1, G_1) = 1$, $d(H_n, G_n) = 2$ for $n > 1$. These come from our results on subgroups of depth one.

4.4 Depth series coming from general subgroups of depth three

Example 4.4.1 *In the nonabelian group $G := C_3 \rtimes C_4$ a subgroup $H \simeq C_4$ has depth three. The two faithful characters of $\mathrm{Irr}(H)$ have G-distance one and the two nonfaithful characters of $\mathrm{Irr}(H)$ also have G-distance one. However, a faithful and a nonfaithful character of $\mathrm{Irr}(H)$ have G-distance $-\infty$.*

Remark 4.4.2 *If G is a simple group and H is a subgroup of depth three, then every two different irreducible characters of H have G-distance one, since by [6, Corollary 3.6], the characters of $\mathrm{Irr}(H)$ that have non-negative G-distance from 1_H are exacly those belonging to $\mathrm{Irr}(H/Core_G(H)) = \mathrm{Irr}(H)$.*

In the following we will deal with subgroups of depth three, where there exist characters in $\mathrm{Irr}(H)$ which are of G-distance $-\infty$.

Definition 4.4.3 *Let $H \leq G$. We define a bipartite graph Γ whose vertices are $\mathrm{Irr}(G) \cup \mathrm{Irr}(H)$ and $(\phi, \chi) \in \mathrm{Irr}(H) \times \mathrm{Irr}(G)$ is an edge in this graph if and only if $(\phi^G, \chi) \neq 0$. We will call it the* Frobenius or inclusion graph *of (G, H)*

Example 4.4.4 *Let G be a nonabelian group of order 12, where $G = C_3 \rtimes C_4$, $H = C_4$ and let $\mathrm{Irr}(H) = \{1_H, \phi, \zeta, \overline{\zeta}\}$ and $\mathrm{Irr}(G) = \{1_G, \phi, \overline{\zeta}, \zeta, \chi, \zeta\chi\}$. Here we use the same name for those characters of $\mathrm{Irr}(G)$ that can be lifted from $\mathrm{Irr}(H)$. Here the edges of the Frobenius graph are: $(1_H, 1_G), (1_H, \chi), (\phi, \phi), (\phi, \chi)$ and $(\zeta, \zeta), (\zeta, \chi\zeta), (\overline{\zeta}, \overline{\zeta}), (\overline{\zeta}, \chi\zeta)$. So the graph has two connected components.*

Remark 4.4.5 *Let $G = G_1 := SL(2, 3)$ and let $H = H_1$ be a cyclic subgroup of order four. Let $H_n = H \times G^{n-1}$ and $G_n = G \wr C_n$. Calculations with* GAP *[11] show that $d(H_1, G_1) = 3$, $d(H_2, G_2) = 5$ and $d(H_3, G_3) = 8$. This example shows that in the case of a disconnected Frobenius graph of (G, H), the fact that the maximal H_1-distance in G_1 is one does in general not imply that $d(H_n, G_n)$ is equal to $2n + 1$ (cf. Theorem 4.3.3). It also shows that our method can produce $d(H_n, G_n)$ even for some n also when $d(H_1, G_1)$ is odd. This group is SmallGroup$(24, 3)$ in the Small Groups Library of* GAP *[11].*

We want to prove some estimates for $d(H_n, G_n)$ if the Frobenius graph has more than one connected components. First we prove some lemmas:

Lemma 4.4.6 *Let G be a non-trivial group, let $H < G$ be a non-trivial subgroup. Suppose that for any two different irreducible charac-*

ters $\phi, \psi \in \text{Irr}(H)$ *that belong to the same connected component of the Frobenius graph* Γ *of* (G, H)*, we have that* $d^G(\phi, \psi) = 1$*. Then for any two different irreducible characters* $\chi_1, \chi_2 \in \text{Irr}(G)$*, that belong to the same connected component of* Γ*, we have that* $0 < d_H(\chi_1, \chi_2) \leq 2$*.*

Proof. The proof is similar to that of Lemma 4.3.2. □

Lemma 4.4.7 *Let* $G = G_1$ *be a finite group and* $H = H_1 \leq G_1$ *a subgroup. Let* Γ *be the Frobenius graph of* (G, H)*. Let* $H_n = H \times G^{n-1}$ *and* $G_n = G \wr C_n$*. Let* $K_n = G^n$ *be the base group of this wreath product. Let* $\Phi := \phi_1 \times \chi_2 \times \cdots \times \chi_n \in \text{Irr}(H_n)$ *and* $\Psi := \phi_1' \times \chi_2' \times \cdots \times \chi_n' \in \text{Irr}(H_n)$*. Let* $X := \chi_1 \times \chi_2 \times \cdots \times \chi_n \in \text{Irr}(K_n)$ *above* Φ *and let* $X' := \chi_1' \times \chi_2' \times \cdots \times \chi_n' \in \text{Irr}(K_n)$ *above* Ψ*. Suppose that* $d^{G_n}(\Phi, \Psi) \neq -\infty$*. Then there is a cyclic permutation of the direct components of* X*, let us denote the new character by* X''*, such that the direct components of* X'' *and* X' *are pairwise in the same connected components of* Γ*.*

Proof. We will use induction on the distance $d^{G_n}(\Phi, \Psi)$. If the distance is one, then Φ^{G_n} has an irreducible constituent $\tilde{X}$ such that $(\tilde{X}|_{H_n}, \Psi) \neq 0$. And also $(\tilde{X}|_{H_n}, \Phi) \neq 0$. Thus some irreducible constituent X_1 of $\tilde{X}|_{K_n}$ is above Φ and some irreducible constituent X_2 of $\tilde{X}|_{K_n}$ is above Ψ. Since the irreducible constituents of $\tilde{X}|_{K_n}$ are G_n-conjugate, one can get X_2 as a cyclic permutation of the direct components of X_1. Since the direct components of X_1 are in the same connected components of Γ as those of X and the direct components of X_2 are in the same connected components of Γ as those of X', we have that a cyclic permutation of the direct components of X gives a character X'' whose direct components are in the same connected components of Γ as those of X'.

Let us suppose now that $d^{G_n}(\Phi, \Psi) = k$ and for characters of smaller distance the statement is true. If there is a path of G_n-related characters $\Phi \sim^{G_n} \Phi^{(2)} \sim \cdots \sim^{G_n} \Phi^{(k)} = \Psi$, then by induction the character $X^{(2)} \in \mathrm{Irr}(K_n)$ above $\Phi^{(2)}$ has a cyclic shift that the direct components of that are in the same connected components of Γ as those of X'. This is also true for X and $X^{(2)}$, so composing the two cyclic shifts, we get the required result. □

Remark 4.4.8 *Let G be a finite group and let $H \leq G$ be a subgroup of depth three.*

(i) Let Γ_0 denote the connected component of the Frobenius graph of (G, H) containing 1_G. Then it contains 1_H and all irreducible constituents of 1_H^G. Since H is not normal, $Core_G(H) < H$. There exists an irreducible character $\psi \in \mathrm{Irr}(G)$ such that $(1_H, \psi_H) = (1_H^G, \psi) \neq 0$. Not every such character ψ contains H in its kernel, otherwise $Ker(1_H^G) \geq H$, and H would be normal, which is not the case. So we can find a character $\phi \in \mathrm{Irr}(H)$ in Γ_0 such that $\phi \neq 1_H$.

(ii) If the maximal H-distance of irreducible characters of G in Γ_0 is two, then $d_H(1_G, \chi_2) = 2$ for some $\chi_2 \in \mathrm{Irr}(G)$, since otherwise each character in $\mathrm{Irr}(G) \cap \Gamma_0$ would have H-distance one from 1_G, and then all these characters would lie above 1_H, hence the H-distance between every two different characters of $\mathrm{Irr}(G) \cap \Gamma_0$ would be one. So there is also a non-trivial character $\phi_2 \in \mathrm{Irr}(H) \cap \Gamma_0$ below a character χ_2 with $d_H(1_G, \chi_2) = 2$.

Lemma 4.4.9 *Let us suppose that $H \leq G$. Let Γ be the Frobenius* *graph of (G, H). Let us suppose that any two different irreducible characters of H belonging to the same connected component of Γ have G-distance one and any two different irreducible characters of G belonging to the same connected component of Γ have H-distance one. Let $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$. Let $\Phi, \Psi \in \mathrm{Irr}(H_n)$, whose components of the same index belong to the same connected component of Γ. Then $0 \leq d^{G_n}(\Phi, \Psi) \leq n$.*

Proof. The proof is the same as that of Lemma 4.3.6. □

Lemma 4.4.10 *Let $H \leq G$. Let Γ be the Frobenius graph of (G, H). Suppose that any two different irreducible characters of H belonging to the same connected component of Γ have G-distance one. Suppose further that there exist irreducible characters of G of H-distance two. Let $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$. Let $\Phi, \Psi \in \mathrm{Irr}(H_n)$, whose components of the same index belong to the same connected component of Γ. Then $0 \leq d^{G_n}(\Phi, \Psi) \leq 2n - 1$.*

Proof. The proof is similar to that of Lemma 4.3.7. Instead of Lemma 4.3.2 now we use Lemma 4.4.6. □

The main result of this section is the following

Theorem 4.4.11 *Let G be a finite group and let H be an arbitrary subgroup of G of depth three. Then the above method, where $H_1 := H$, $G_1 := G$, $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$, gives a series of groups H_n and G_n with the following properties: if the H-distance of any two different characters of $\mathrm{Irr}(G)$ is at most one then $2n + 1 \leq d(H_n, G_n) \leq 2n + 3$, if there exist characters of $\mathrm{Irr}(G)$ of H-distance two then $2n + 1 \leq d(H_n, G_n) \leq 4n + 1$.*

Proof. If the Frobenius graph Γ of (G, H) is connected then any two distinct irreducible characters of H have G-distance one, by Definition 2.1.2, and we are done by Theorem 4.3.3.

So let us suppose that the Frobenius graph of (G, H) has at least two connected components, and set $\Phi := \phi_1 \times \chi_2 \times \cdots \times \chi_n \in \mathrm{Irr}(H_n)$ and $\Psi := \phi_1' \times \chi_2' \times \cdots \times \chi_n' \in \mathrm{Irr}(H_n)$.

Let $\Theta := \chi_1 \times \chi_2 \times \cdots \times \chi_n \in \mathrm{Irr}(K_n)$ above Φ, and let $\Theta' := \chi_1' \times \chi_2' \times \cdots \times \chi_n' \in \mathrm{Irr}(K_n)$ above Ψ. If $d^{G_n}(\Phi, \Psi) \neq -\infty$ then Lemma 4.4.7 yields that there exists a permutation τ of the components of Θ such that each component of the permuted character Θ'' is in the same connected component as the component of Θ' with the same index.

Consider $\Phi'' \in \mathrm{Irr}(H_n)$ below Θ''. By Lemma 4.4.9 and Lemma 4.4.10, $d^{G_n}(\Phi'', \Psi) \leq n$ if every two different irreducible characters in the same connected component of Γ have H-distance one, and $d^{G_n}(\Phi'', \Psi) \leq 2n - 1$ if there exist two characters of $\mathrm{Irr}(G)$ of H-distance two.

Now observe that $d^{G_n}(\Phi, \Phi'') \leq 1$ holds, since any $\tilde{X} \in \mathrm{Irr}(G_n)$ above Θ lies also above Θ'', by Clifford's theorem, hence also above Φ and Φ''.

Thus, if every two different irreducible characters in the same connected component of Γ have H-distance one then $d^{G_n}(\Phi, \Psi) \leq n + 1$, and Definition 2.1.2 yields $d(H_n, G_n) \leq 2(n+1) + 1 = 2n + 3$. If there exist two characters of $\mathrm{Irr}(G)$ of H-distance two then $d^{G_n}(\Phi, \Psi) \leq 2n$, and Definition 2.1.2 yields $d(H_n, G_n) \leq 2(2n) + 1 = 4n + 1$.

In order to establish the lower bounds, we show that $m(1_{G_n})$ is sufficiently large. For that, we consider $\omega_n := \phi_2 \times \chi_2 \times \cdots \times \chi_2 \in \mathrm{Irr}(H_n)$, where ϕ_2 and χ_2 are chosen as follows. If the maximal H-

distance of characters of $\mathrm{Irr}(G)$ in Γ_0 is one then let $\phi_2 \in \mathrm{Irr}(H)$ be a nontrivial character in Γ_0 (which exists by Remark 4.4.8 (i)), and let $\chi_2 \in \mathrm{Irr}(G)$ be a character above 1_H and ϕ_2. Then the proof of Theorem 4.3.3 yields $m(1_{G_n}) \geq d^{G_n}(1_{H_n}, \omega_n) = n$ and thus $d(H_n, G_n) \geq 2n + 1$. If the maximal H-distance of characters of $\mathrm{Irr}(G)$ in Γ_0 is two then let χ_2 and ϕ_2 be as in Remark 4.4.8 (ii). Again the proof of Theorem 4.3.3 yields $m(1_{G_n}) \geq d^{G_n}(1_{H_n}, \omega_n) = 2n - 1$ and thus $d(H_n, G_n) \geq 4n - 1$. So in any case $d(H_n, G_n) \geq 2n + 1$. □

Remark 4.4.12 *If the maximal H-distance of characters of G in Theorem 4.4.11 is two then in one of the connected components of the Frobenius graph (G, H) the maximal H-distance is two. We have seen in the proof of Theorem 4.4.11 that if it happens in Γ_0 then we have a better lower bound $d(H_n, G_n) \geq 4n - 1$. However, if this occurs not in Γ_0 but in another connected component then we only have the weaker lower bound $d(H_n, G_n) \geq 2n + 1$, since then we only know that $m(1_{G_n}) \geq n$.*

4.5 Depth series coming from S_k in S_{k+1}

We have the following

Theorem 4.5.1 *Let $H = H_1 := S_k$ and $G = G_1 := S_{k+1}$. Then if $H_n := H \times G^{n-1}$ and $G_n := G \wr C_n$ then $d(H_n, G_n) = 2kn - 1$.*

Before proving the theorem we recall some definitions and notation: A *partition* λ of the natural number n is a sequence

$$\lambda = (\lambda_1, \ldots, \lambda_k),$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, and $\sum_{i=1}^k \lambda_i = n$. The set of all partitions of n we will denote by $\mathcal{P}_n$. The *Young diagram* $[\lambda]$ of the partition λ is a table, with rows of lengths $(\lambda_1, \ldots, \lambda_k)$. We will also denote the elements of the table by (r, s), meaning that the node is at the r^{th} row and s^{th} column. We tell that a node (i, λ_i) is *removable* from $[\lambda]$, if we remove the last node from the i^{th} row, we still get again a diagram of a partition of $n-1$. A node (r, s) is *addable* to $[\lambda]$ if it is not in $[\lambda]$, and by adding it to $[\lambda]$, we get the diagram of a partition of $n+1$. The difference $[\lambda] \setminus [\mu]$ denotes the set of nodes belonging to the diagram of λ but not belonging to the diagram of μ, $[\lambda] \cap [\mu]$ is the set of common nodes in the diagram of λ and μ. The *distance* of partitions λ and μ of n is $d(\lambda, \mu) := |[\lambda] \setminus [\mu]| + |[\mu] \setminus [\lambda]| = 2(n - |[\lambda] \cap [\mu]|)$. The character $\chi_\lambda \in \mathrm{Irr}(S_n)$ denotes the character belonging to the partition λ. Let us note that if $\lambda, \mu, \nu \in \mathcal{P}_n$ then $d(\lambda, \mu) \leq d(\lambda, \nu) + d(\nu, \mu)$, since $|[\lambda] \setminus [\mu]| \leq |[\lambda] \setminus [\nu]| + |[\nu] \setminus [\mu]|$ and $|[\mu] \setminus [\lambda]| \leq |[\mu] \setminus [\nu]| + |[\nu] \setminus [\lambda]|$.

Theorem 4.5.2 *[Branching rule, [24, Theorem 2.4.3]] Let $\chi_\lambda \in \mathrm{Irr}(S_{k+1})$. Then the constituents of $\chi_\lambda|_{S_k}$ are exactly those $\chi_\mu \in \mathrm{Irr}(S_k)$, where $[\mu]$ can be obtained from $[\lambda]$ by the removal of a removable node. Let $\chi_\mu \in \mathrm{Irr}(S_k)$. Then the constituents of $\chi_\mu^{S_{k+1}}$ are exactly those characters $\chi_\lambda \in \mathrm{Irr}(S_{k+1})$, where $[\lambda]$ can be obtained from $[\mu]$ by adding an addable node.*

Lemma 4.5.3 *Let $G := S_{k+1}$ and let $H := S_k$. Let χ_λ and χ_μ be characters of $\mathrm{Irr}(G)$ belonging to partitions λ and μ of $k+1$, respectively. Then the H-distance $d_H(\chi_\lambda, \chi_\mu)$ is equal to $k+1-|[\lambda] \cap [\mu]| = d(\lambda, \mu)/2$.*

Proof. The proof is similar to that of Proposition A.2 in [6]. We use induction on the distance of partitions $2m := d(\lambda, \mu) = |[\lambda] \setminus$

$[\mu]| + |[\mu] \setminus [\lambda]| = 2(k+1-|[\lambda] \cap [\mu]|)$. If $m = 0$, then $\lambda = \mu$, so $d_H(\chi_\lambda, \chi_\mu) = 0$. We construct a partition λ^1. Suppose $(i, \lambda_i) \in [\lambda] \setminus [\mu]$ a removable node of $[\lambda]$, so $(i+1, \lambda_i) \notin [\lambda]$. Let us suppose that $(r, s) \in [\mu] \setminus [\lambda]$ an addable node to $[\lambda]$, so $(t, s) \in [\lambda]$ for $t = 1, ..., r-1$ and $(r, u) \in [\lambda]$ for $u = 1, ..., s-1$. Let $[\nu] := [\lambda] \setminus (i, \lambda_i)$. If (r, s) would not be addable to $[\nu]$ then $(r, s) = (i, \lambda_i + 1)$ or $(r, s) = (i+1, \lambda_i)$. However then $(i, \lambda_i) \in [\mu]$, which is a contradiction. So if $[\lambda^1] := [\lambda] \setminus (i, \lambda_i) \cup (r, s)$ then λ^1 is of distance two from λ and λ^1 and μ have distance $2m-2$, since $[\lambda^1] \cap [\mu]$ has one more node than $[\lambda] \cap [\mu]$. Also $d_H(\chi_\lambda, \chi_{\lambda^1}) = 1$, by the Branching rule. By induction we have that $d_H(\chi_{\lambda^1}, \chi_\mu) = m - 1$. So $d_H(\chi_\lambda, \chi_\mu) \leq m$. Let $d_H(\chi_\lambda, \chi_\mu) = r$. So $r \leq m$ and thus $2r \leq 2m = d(\lambda, \mu)$. If $r = 0$ then $\lambda = \mu$. If $r = 1$ then μ can be obtained from λ by first removing a node (i, j) and then adding a node $(r, s) \neq (i, j)$, by the Branching rule. Thus $d(\lambda, \mu) = 2$. If $r \geq 2$ then if $\chi_\lambda \sim_H \chi_{\mu^1} \sim_H \cdots \sim_H \chi_{\mu^r} = \chi_\mu$ then $d(\mu^{r-1}, \mu) = 2$ and by induction $d(\lambda, \mu^{r-1}) = 2(r-1)$. So $d(\lambda, \mu) \leq d(\lambda, \mu^{r-1}) + d(\mu^{r-1}, \mu) = 2r$, and hence $2m = d(\lambda, \mu) = 2r$, which implies that $d_H(\chi_\lambda, \chi_\mu) = r = m = d(\lambda, \mu)/2$ and we are done. □

Proof. **(of Theorem 4.5.1)**

Since $S_k \cap S_k^{(1,k+1)} \cap \cdots \cap S_k^{(k-1,k+1)} = \{1\}$, so $\{1\} = Core_G(H) = \cap_{i=1}^k H^{x_i}$ is the intersection of k conjugates of H, for suitable elements $x_i \in G$, for $i = 1, \ldots, k$. Then $\{1\} = Core_{G_n}(H_n) = \cap_{j=1}^n \cap_{i=1}^k H_n^{x_i \sigma_n^j}$, is the intersection of kn conjugates of H_n. Here σ_n is the cyclic shift in $G \wr C_n$, generating C_n. Thus by the second part of Theorem 2.1.3, we have that $d(H_n.G_n) \leq 2kn - 1 = 2(kn-1) + 1$. If it would be at most $2(kn-1)$, then by Definition 2.1.2, for every character $\chi \in \mathrm{Irr}(G_n)$ the inequality $m(\chi) \leq kn-2$ would hold. We will show that $m(1_{G_n}) \geq kn-1$, which shows that $d(H_n.G_n) = 2kn-1$.

Let $\alpha_n := \phi_1 \times \chi_1 \times \cdots \times \chi_1$ be the trivial character of H_n and let $\omega_n := \phi_2 \times \chi_2 \times \cdots \times \chi_2$, where $\phi_2 \in \mathrm{Irr}(H)$ the sign character, and $\chi_2 \in \mathrm{Irr}(G)$ the sign character. Let us suppose that $\alpha_n = \psi_0 \sim^{G_n} \psi_1 \sim^{G_n} \cdots \sim^{G_n} \psi_{m+1} = \omega_n$ is a shortest path between α_n and ω_n. Then there exist characters $\Phi_1, \ldots, \Phi_{m+1} \in \mathrm{Irr}(G_n)$ such that Φ_1 is above ψ_0 and ψ_1, etc Φ_{m+1} is above ψ_m and ψ_{m+1}. Then $\Phi_i|_{K_n}$ is a sum of conjugate irreducible characters of $K_n = G^n$, the base group of the wreath product $G \wr C_n$. Let Θ_i be a constituent of $\Phi_{i+1}|_{K_n}$ above ψ_i. Then Θ_0 cannot be the trivial character of K_n. The character 1_{K_n} can be extended to 1_{G_n}, hence $(\chi_1 \times \cdots \times \chi_1)^{G_n}$ has irreducible constituents $1_{G_n}\beta$, where $\beta \in \mathrm{Irr}(C_n)$, by Gallagher's theorem, see [23, Cor. 6.17]. Thus Φ_1 would have such a form. Thus the restriction $\Phi_1|_{H_n}$ would be a multiple of 1_{H_n}, and it cannot happen in a shortest path. Thus Θ_0 is of the form $\chi_\mu \times \chi_1 \times \cdots \times \chi_1$, where $\chi_1 \neq \chi_\mu \in \mathrm{Irr}(G)$ and μ is the partititon $(k,1)$, by the Branching rule. Thus in a shortest path one can get Θ_1 as a cyclic shift of Θ_0 and afterwards the first component is exchanged to irreducible character $\chi' \in \mathrm{Irr}(G)$, which is of H-distance one from the first component of the shifted Θ_0. Suppose that $\psi_i = \phi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$. Then $\Theta_i = \chi^{(i)} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$, for some $\chi^{(i)} \in \mathrm{Irr}(G)$ that can be got from ϕ_{i_1} by adding one node to the partition of ϕ_{i_1}, by the Branching rule. Since $\Phi_{i+1}|_{K_n}$ is a sum of possible conjugates of Θ_i, above ψ_{i+1} there is a conjugate of Θ_i, say $\chi_{i_j} \times \cdots \times \chi_{i_n} \times \chi^{(i)} \times \chi_{i_2} \times \cdots \times \chi_{i_{j-1}}$. Then $\psi_{i+1} = \phi' \times \chi_{i_{j+1}} \times \cdots \times \chi_{i_n} \times \chi^{(i)} \times \chi_{i_2} \times \cdots \times \chi_{i_{j-1}}$, for some $\phi' \in \mathrm{Irr}(H)$ below χ_{i_j}. And so $\Theta_{i+1} = \chi^{(i+1)} \times \chi_{i_{j+1}} \times \cdots \times \chi_{i_n} \times \chi^{(i)} \times \chi_{i_2} \times \chi_{i_{j-1}}$, where $\chi^{(i+1)}$

is of H-distance one from χ_{i_j}. (If it would be of distance zero, then we could shorten the path.) In general, from Θ_i one can get Θ_{i+1} by making a cyclic shift on it and exchanging the new first component by a character $\chi^{(i+1)} \in \mathrm{Irr}(G)$, which is of H-distance one from the new first component. In Θ_m one component can be the character $\chi_{\mu'} \in \mathrm{Irr}(G)$ (different from χ_2, since then it would not be a shortest path), and the others are χ_2. Here μ' is the conjugate partition to μ, by the Braching rule above the sign character of H, there is the sign character of G and $\chi_{\mu'}$. So to get from $\chi_\mu \times \chi_1 \times \cdots \times \chi_1$ to $\chi_{\mu'} \times \chi_2 \times \cdots \times \chi_2$, in each step one can get at most one new component of H-distance one from a previous component. Now $d_H(\chi_\mu, \chi_{\mu'}) = k+1-3$, $d_H(\chi_1, \chi_2) = k+1-1$, $d_H(\chi_1, \chi_{\mu'}) = k+1-2$, $d_H(\chi_\mu, \chi_2) = k+1-2$. So if we change χ_μ to $\chi_{\mu'}$ and χ_1 to χ_2, it needs at least $k+1-3+(n-1)(k+1-1) = k-2+(n-1)k = nk-2$ steps, if one changes χ_μ to χ_2 and one χ_1 to $\chi_{\mu'}$ and $n-2$ χ_1 to χ_2 then it needs at least $2(k+1-2)+(n-2)(k+1-1) = 2k-2+(n-2)k = nk-2$ steps. Thus $d^{G_n}(\alpha_n, \omega_n) \geq nk-1$. Thus $m(1_{G_n}) \geq nk-1$, so we get a contradiction. Hence $d(H_n, G_n) = 2nk-1$.

□

Chapter 5

TI subgroups and depth 3 subgroups of Suzuki groups

The results of this chapter were published in [27].

5.1 Suzuki groups

Suzuki groups Sz(q) are twisted groups of Lie type ${}^2B_2(q)$, where $q := 2^{2m+1}$. If $m > 0$, then they are simple. Suzuki groups are also doubly transitive permutation groups on q^2+1 points, they belong to the class of Zassenhaus groups. Suzuki groups also can be defined as subgroups of GL($4, q$). The order of Sz(q) is $(q^2+1)(q-1)q^2$. The order of Sz(q) iscongruent to 2 mod 3, however it is always divisible by 5. For further information see [14].

We will use the following results on Suzuki groups, see [22, Theorem 3.10 Chapter XI] and [32, Theorem 4.12].

Theorem 5.1.1 (Suzuki) *[32, Theorem 4.12]*
Let $G = Sz(q)$, where $q = 2^{2m+1}$, for some positive integer m. Then G

has the following subgroups:

1. *The Hall subgroup $N_G(F) = FH$, which is a Frobenius group of order $q^2(q-1)$, where $F \in Syl_2(G)$ and H is cyclic of order $q-1$.*

2. *The dihedral group $B_0 = N_G(H)$ of order $2(q-1)$ where H is the same subgroup as in part 1.*

3. *The cyclic Hall subgroups A_1, A_2 of orders $q+2r+1, q-2r+1$, respectively, where $r = 2^m$ and $|A_1||A_2| = q^2+1$.*

4. *The Frobenius subgroups $B_1 = N_G(A_1), B_2 = N_G(A_2)$ of orders $4|A_1|, 4|A_2|$, respectively.*

5. *The subgroups of form $Sz(s)$, where s is an odd power of 2, $s \geq 8$, and $q = s^n$ for some positive integer n. Moreover, for every odd 2-power s, where $s^n = q$ for some positive integer n, there exists a subgroup isomorphic to $Sz(s)$.*

6. *Subgroups (and the conjugates of the subgroups) of the above groups.*

Remark 5.1.2 *The group ${}^2B_2(2) := Sz(2)$ is not simple, it is a Frobenius group of order 20. It is a subgroup of each simple $Sz(q)$. Since 5 divides either $|A_1|$ or $|A_2|$, the group $Sz(2)$ is a subgroup of B_1 or B_2 up to isomorphism.*

Remark 5.1.3 *By [35, Theorem 10] If $Sz(s) \leq Sz(q)$ then every subgroup of $Sz(q)$ isomorphic to $Sz(s)$ is also conjugate to it in $Sz(q)$. The subgroups $Sz(s) \leq Sz(q)$ we will call later in the thesis* Suzuki subgroups *of $Sz(q)$. In the case of matrix generators of $Sz(q)$ in $GL(4,q)$, it is the subgroup generated by the same matrices when the matrix entries are restricted to $GF(s)$.*

Theorem 5.1.4 *[22, Theorem 3.10 Chapter XI]*
Let $q = 2^{2m+1}$*,* $m > 0$*,* $r = 2^m$ *and* $G = Sz(q)$*. Let* A_1, A_2 *be the same subgroups as in the Theorem 5.1.1 part 3.*

a) Let $i \in \{1, 2\}$ *and let* $u_i \in A_i$*,* $u_i \neq 1$*. Then* $C_G(u_i) = A_i$*. If* $B_i = N_G(A_i)$ *then* $B_i = \langle A_i, t_i\rangle$*, where* t_i *is an element of order* 4*, and* $u^{t_i} = u^q$*, for all* $u \in A_i$*. Moreover,* $N_G(A_i)$ *is a Frobenius group with kernel* A_i*.*

b) Let F, H, A_1, A_2 *as in Theorem 5.1.1. Then the conjugates of* F, H, A_1, A_2 *form a partition of* G*. In particular* F, H, A_1, A_2*, their conjugates and the conjugates of their characteristic subgroups are TI sets in* G*.*

The Sylow 2-subgroups of G are Suzuki 2-groups. This means that $F \in Syl_2(G)$ is a non-abelian 2-group, having more than one involution, and having a solvable group of automorphisms which permutes the set of involutions of F transitively. See [21, p. 299] for details.

The group F is a class 2 group of order q^2 and exponent 4. Moreover its center $Z(F) = F' = \Phi(F)$ is of order q. The involutions in F together with the identity element constitute $Z(F)$, and F does not contain any quaternion subgroups. A nontrivial element of F is real in G if and only if it is an involution. (An element of a group G is called *real* element in G if it is conjugate in G to its inverse, see [13, p. 303.])

The subgroup H acts sharply 1-transitively on the involutions of F, and on the nontrivial elements of $F/Z(F)$. The centralizer in G of every nontrivial element of F is a subgroup of F.

Zassenhaus groups, see [22, Chapter XI.], are doubly transitive permutation groups without any regular normal subgroup, where any nonidentity element has at most two fixed points. Zassenhaus groups are always of degree $q+1$, where q is a prime power. If q is even then there are two series of Zassenhaus groups $PSL(2,2^n)$ for $n>1$, and $Sz(2^{2m+1})$ for $m>0$. If $q=p^f$ is odd and f odd then Zassenhaus groups are $PGL(2,p^f)$ and $PSL(2,p^f)$ if $p^f \neq 3$. If $f=2m$ then beside $PGL(2,p^f)$ and $PSL(2,p^f)$ there exists a third type of Zassenhaus group $M(p^f)$. It can be constructed in the following way: let $H = PGL(2,p^f)\langle\alpha\rangle \leq P\Gamma L(2,p^f)$, where $\alpha \in$Aut$(GF(p^f))$ of order 2, raising elements of $GF(p^f)$ to the m^{th} power. There are 3 subgroups of H of index 2 containing $PSL(2,p^f)$: $PGL(2,p^f)$, $PSL(2,p^f)\langle\alpha\rangle$ and $M(p^f)$.

5.2 Motivation

In an arbitrary group, non-normal TI subgroups are always of combinatorial depth three, hence also of ordinary depth three. Moreover, nontrivial subgroups having a disjoint conjugate are always of ordinary depth three in every group. However, in general the converse is not true, e.g. $L=A_5$ has in $G=A_7$ ordinary depth 3, but there is no element $x \in G$ such that $L \cap L^x = \{1\}$. (A similar example is A_6 in A_{10}).

It is an open problem, see [29], how to characterize subgroups of ordinary depth 3 in a group theoretical way in an arbitrary group.

We will show in Theorem 5.3.3 that in the case of simple Suzuki groups, subgroups of ordinary depth 3 are exactly those nontrivial subgroups that have a disjoint conjugate.

In Theorem 5.3.1 we also characterized those nontrivial TI subgroups of $\mathrm{Sz}(q)$ which are noncyclic elementary abelian 2-subgroups. These are exactly those subgroups that are conjugate to the centre of a Sylow 2-subgroup of a smaller Suzuki subgroup $Sz(s) \leq Sz(q)$. This property can help the recognition of those subgroups of $\mathrm{Sz}(q)$ which are isomorphic to a Suzuki subgroup $Sz(s) \leq Sz(q)$. Recognition of Suzuki groups in $\mathrm{GL}(4, q)$ is considered in some recent papers, see [1] and [5]. In another paper, see [33], Suzuki groups were used to construct some block designs. So results about intersections in Suzuki groups might also be helpful in combinatorial investigations.

In this chapter we will use the following notation: let G be simple Suzuki group $\mathrm{Sz}(q)$, where $q = 2^{2m+1}$. Let F be a fixed Sylow 2-subgroup of G. Then $N_G(F) = FH$, where $|H| = q-1$. Let A_1 and A_2 be Hall subgroups of G of orders $q+2r+1$ and $q-2r+1$, respectively, where $r = 2^m$, see Theorem 5.1.1 and also [22, Theorem 3.10, Ch XI] and [32, Theorem 4.12].

We will denote by K_{2^n} an elementary abelian subgroup of order 2^n. The center $Z(F)$ of the Sylow 2-subgroup F of the Suzuki group $\mathrm{Sz}(q)$ will be of order $q = 2^{2m+1}$, and we will denote $2m+1$ by f. We also suppose that $m > 0$.

5.3 Main results

Theorem 5.3.1 *If $G = Sz(q)$ is a simple Suzuki group then G has the following TI subgroups:*

(i) Cyclic subgroups of prime order and the trivial subgroup.

(ii) Subgroups F, H, A_1, A_2, their characteristic subgroups and the conjugates of these.

(iii) *An elementary abelian subgroup* K_{2^n} *of order* $2^n > 2$ *is a TI-subgroup if and only if it is the centre of a Sylow* 2*-subgroup of a simple Suzuki subgroup* $G_1 \leq G$*, or a conjugate to it. This holds if and anly if* $n > 1$ *and* $n|f$*. (Remember:* $|Z(F)| = 2^f$*). These subgroups are exactly those non-cyclic elementary abelian* 2*-subgroups of* G *that have combinatorial depth* 3*.*

All other nontrivial subgroups are not TI.

Proof.

We prove Theorem 5.3.1 in 9 steps.

0. *It is obvious that every subgroup of prime order is TI.*

Now we examine the subgroups of maximal subgroups of G to establish which are TI and which are not TI.

1. *The subgroups* F*,* H*,* A_1*,* A_2*, their characteristic subgroups and their conjugates are TI. In particular, all subgroups of* H, A_1, A_2 *are TI, however not every subgroup of* F *is TI:*

   The first part follows from Theorem 5.1.4. Since H, A_1, A_2 are cyclic and TI, each subgroup of them is characteristic, hence TI. We will see below that not every subgroup of F is TI.

2. *If* $U \leq N_G(F)$*, and* $U = F_1H_1$*, where* $F_1 \leq F$*,* $H_1 \leq H$ *are nontrivial subgroups, then* U *is not TI:*

   Let $x \in N_G(H) \setminus FH$. Then $H_1^x = H_1$, hence $U^x \cap H \geq H_1 \neq \{1\}$. Since $x \notin N_G(F)$, thus $F_1^x \not\leq F$, since F is TI. Hence $U^x \cap U$ is a proper subgroup of U, thus U is not TI.

3. *For every nontrivial subgroup* $H_1 \leq H$ *and* $u \in N_G(H) \setminus H$ *the subgroup* $\langle u, H_1 \rangle$ *is not TI:*

   Let $u \in N_G(H) \setminus H$, then this is an element of order 2, by part 2. of Theorem 5.1.1. So we may suppose that $u \in F_1$ for some subgroup $F_1 \in Syl_2(G)$. Since we know, see e.g. [32, Theorem 4.1 (b)], that all involutions in a Sylow 2-subgroup of $\mathrm{Sz}(q)$ are in the centre of the Sylow 2-subgroup, thus we have that $u \in Z(F_1)$. Let $x \in F_1 \setminus N_G(H)$. Then $(H_1\langle u \rangle)^x \cap H_1\langle u \rangle$ contains u. However, x does not normalize H_1. Otherwise it would also normalize H, since H is TI. Hence $\langle u, H_1 \rangle$ is not TI.

4. *If we take a nontrivial subgroup* $\tilde{A}$ *of* A_i*, and a nontrivial subgroup* $C_1 \leq C$ *of a cyclic complement of order* 4 *in the Frobenius group* $N_G(A_i)$*, then* $C_1\tilde{A}$ *is not TI, in particular* $Sz(2)$ *is not TI:*

   Let $\tilde{A} \leq A_i$, $C \leq F_1$, where $F_1 \in Syl_2(G)$. Let $n \in N_{F_1}(C) \setminus C = N_{F_1}(C) \setminus N_G(A_i)$ be an involution. Then $(C_1\tilde{A})^n \cap C_1\tilde{A} \neq \{1\}$, since C_1 is characteristic in C. However, it does not contain $\tilde{A}$, since then $n \in N_G(\tilde{A}) \leq N_G(A_i)$, which is not the case. Hence $C_1\tilde{A}$ is not TI.

5. *Let* $G_1 := Sz(s)$ *be a simple smaller Suzuki subgroup in* G*. The subgroups* U *of* G_1*, whose order is divisible both by* 2 *and by some odd integer greater than* 1*, are not TI:*

   Let $U_2 \in Syl_2(U)$ and let $U_2 \leq P_2 \in Syl_2(G)$. Then $Z(P_2) \leq N_G(U_2)$, however since $Z(P_2) \not\leq G_1$, we have that $Z(P_2) \not\leq U$. If $N_{P_2}(U_2) \leq N_G(U)$ then $UZ(P_2)$ is a subgroup of G. It cannot be a subgroup of any maximal subgroup of type $N_G(A_i)$, $N_G(H)$ and $Sz(s_1)$, since the orders of these subgroups are not divisible by

q. If $U \leq N_G(S)$ for some $S \in Syl_2(G)$ then U is conjugate to a subgroup discussed in point 2., hence it is not TI. The subgroup $UZ(P_2)$ cannot be G, since U is normal in it. So we can suppose that $N_{P_2}(U_2) \not\leq N_G(U)$. Let $x \in N_{P_2}(U_2) \setminus N_G(U)$. Then $U_2 \leq U^x \cap U < U$, hence U is not TI.

6. *Let $C := \langle c \rangle$ be a cyclic subgroup of F of order 4. This is not TI:*

   By the proof of [22, Lemma 5.9 Ch XI] we have that $C_F(C) = Z(F)C$ is of order $2q$. Since c is not real in F, we have that $N_F(C) = C_F(C)$. Thus there is an element $x \in F \setminus N_F(C)$. Let $u := c^2$. Then $C_G(u) = F$ and hence $C^x \cap C$ contains u. Hence C is not TI.

7. Let us denote an elementary abelian subgroup of order 2^n in F by K_{2^n}. The main results in this step are e) and f). This is exactly the content of (iii) in Theorem 5.3.1, which we will prove with the help of the statements a)-d):

   a) *Let $K_{2^n} = Z(S_1)$, where $S_1 \in Syl_2(G_1)$ and G_1 is a simple Suzuki subgroup of G. Then K_{2^n} is TI in G:*

      We may assume that $S_1 \leq F$. Suppose that $Z(S_1)^x \cap Z(S_1) \neq \{1\}$ for some $x \in G$. Then there exist involutions $a, b \in Z(S_1)$ with $a^x = b$. Since F is TI, thus $x \in N_G(F) = FH$. Since F acts trivially on $Z(S_1)$, we may suppose that $x \in H$. Moreover, H acts sharply 1-transitively on the involutions of F. We also have that $N_{G_1}(S_1) \leq N_G(F) = FH$, since F is TI. For some complement H_1 in $N_{G_1}(S_1)$, $H_1 \leq H$. However H_1 also acts sharply 1-transitively on the involutions of $Z(S_1)$.

Thus $x \in H_1 \leq N_{G_1}(S_1) \leq N_{G_1}(Z(S_1))$. Hence $Z(S_1)$ is TI in G.

b) *If a non-cyclic elementary abelian subgroup of G is TI, and its order is equal to $|Z(S_1)|$ for a Sylow subgroup $S_1 \in Syl_2(G_1)$ for some simple Suzuki subgroup G_1 of G, then this elementary abelian subgroup is the centre of a Sylow 2-subgroup of a subgroup conjugate to a simple Suzuki subgroup of G:*

Let us suppose that a non-cyclic elementary abelian subgroup $K_{2^n} \leq F$ of order 2^n in G is a TI set. We suppose that $2^n = |Z(S_1)|$ for $S_1 \in Syl_2(G_1)$, where G_1 is a simple Suzuki subgroup of G. The involutions of K_{2^n} and their H-conjugates form blocks in $Z(F)$, since K_{2^n} is TI in G. Since H acts sharply 1-transitively on the involutions of $Z(F)$, hence $Z(F)$ is the disjoint union of different H-conjugates of K_{2^n}. We claim that the normalizers in H of the elementary abelian TI subgroups $\tilde{K}_{2^n}$ of order 2^n of G contained in $Z(F)$ are the same. To see this, we note that no element of H fixes any element in F hence the elements of the cyclic subgroup $N_G(\tilde{K}_{2^n}) \cap H$ move the nonunit elements of $\tilde{K}_{2^n}$ sharply 1-transitively, hence $N_G(\tilde{K}_{2^n}) \cap H$ has order $2^n - 1$. Since H is cyclic it has only 1 subgroup of this order, thus each elementary abelian TI-subgroup of order 2^n in $Z(F)$ has the same normalizer in H. We know that $\tilde{K}_{2^n} \subseteq \cup_{h \in H} K_{2^n}^h$. We claim that $\tilde{K}_{2^n}$ is one of the conjugates of K_{2^n}. Suppose that $a \in \tilde{K}_{2^n} \cap K_{2^n}^{h_1}$ and $b \in \tilde{K}_{2^n} \cap K_{2^n}^{h_2}$ are two different involutions and $h_1, h_2 \in H$. Then there exists a unique

$h \in H \cap N_G(\tilde{K}_{2^n}) = H \cap N_G(K_{2^n}^{h_1})$ with $a^h = b \in K_{2^n}^{h_1 h} \cap K_{2^n}^{h_2}$. However then $K_{2^n}^{h_1} = (K_{2^n}^{h_1})^h = K_{2^n}^{h_2}$. Hence $\tilde{K}_{2^n} = K_{2^n}^{h_1}$. Thus all elementary abelian TI subgroups of order 2^n in F are conjugate to $Z(S_1)$, for a Sylow subgroup $S_1 \in Syl_2(G_1)$, for some Suzuki subgroup $G_1 \leq G$.

c) *Let $K_{2^r} \leq F$ be an elementary abelian subgroup of order 2^r in G. Then it is TI if and only if $N_H(K_{2^r}) = H_1$ is of order $2^r - 1$:*

Let $K_{2^r} \leq F$ be an elementary abelian subgroup of order 2^r with the property that $N_H(K_{2^r}) = H_1$ and $|H_1| = 2^r - 1$. Then H_1 permutes the elements of $K_{2^r} \setminus \{1\}$ sharply 1-transitively, and every $h \in H \setminus H_1$ transports each involution of K_{2^r} outside this group. Thus K_{2^r} is TI.

Conversely let $K_{2^r} \leq F$ be TI. Since by [32, Theorem 4.1 (e),(f)] H acts sharply 1-transitively on the involutions of F, if $a \in K_{2^r} \setminus \{1\}$ then for every $b \in K_{2^r} \setminus \{1\}$ there exists a unique $h \in H$ with $a^h = b$. Then $b \in K_{2^r}^h \cap K_{2^r}$. Since K_{2^r} is TI, then $h \in N_H(K_{2^r})$. Thus $N_H(K_{2^r})$ is regular on $K_{2^r} \setminus \{1\}$. Hence it follows that $|N_H(K_{2^r})| = 2^r - 1$.

d) *Let $r > 1$ and let $N_H(K_{2^r}) = H_1$ be of order $2^r - 1$. Then $2^r - 1 | q - 1 = 2^f - 1$, which happens if and only if $|K_{2^r}| = |Z(S_1)|$ for the Sylow subgroup $S_1 \in Syl_2(G_1)$ for a simple Suzuki subgroup $G_1 \leq G$. This happens if and only if $r|f$ and $r > 1$:*

Suppose that $r > 1$ and $|H_1| = 2^r - 1$. This divides $|H| = q - 1 = 2^{2m+1} - 1$. Then $r | 2m+1$, hence $(2^r)^k = 2^{2m+1}$ for some positive integer k. Thus if $G_1 = Sz(2^r)$ then $S_1 \in Syl_2(G_1)$ has centre $Z(S_1)$ of order 2^r. Conversely if 2^r is the size of $Z(S_1)$ for some $S_1 \in Syl_2(G_1)$ for a simple Suzuki subgroup $G_1 \leq G$, then $r > 1$ and $(2^r)^k = 2^{2m+1}$ for some positive integer k, hence $2^r - 1 | 2^{2m+1} - 1$.

e) *A non-cyclic elementary abelian* 2*-subgroup of* G *is TI if and only if it is the centre of a Sylow subgroup* $S_1 \in Syl_2(G_1)$ *for some simple Suzuki subgroup* $G_1 \leq G$ *or conjugate to it, in particular Klein four subgroups of* G *are not TI:*

One direction follows from a) the other direction follows from b) using c) and d).

f) *A non-cyclic elementary abelian* 2*-subgroup of* G *is TI if and only if it is of combinatorial depth* 3*.:*

If K_{2^n} is TI then it is of combinatorial depth 3 by Remark 2.1.5. Let $n > 1$. If $K_{2^n} \leq F$ is not TI then we have that there exists an element $x_2 \notin N_G(K_{2^n})$ such that $K_{2^n}^{x_2} \cap K_{2^n} \neq \{1\}$. Let $x_1 \in F$. Then x_1 centralizes K_{2^n} and we cannot find an element y with $K_{2^n} \cap K_{2^n}^{x_1} \cap K_{2^n}^{x_2} = K_{2^n} \cap K_{2^n}^{y}$ such that y also centralizes the intersection, since then $y \in F$ and it also centralizes K_{2^n}. Hence the combinatorial depth of K_{2^n} in G is bigger than 3.

8. *If* $L < F$ *is not elementary abelian then* L *is not TI:*

Suppose by contradiction that L is TI and not elementary abelian. Let I be the subgroup generated by the involutions of L. If $|I| = 2$ then since F does not contain quaternion subgroups, $L \simeq C_4$, hence it is not TI by step 6. Let $|I| > 2$.

Suppose that $I \cap I^x \neq \{1\}$ then $L \cap L^x \neq \{1\}$ and $x \in N_G(L) \leq N_G(I)$. Thus I is also TI, hence it is the centre of a Sylow 2-subgroup of a simple Suzuki subgroup of G or conjugate to it. If $x \in N_G(I) \setminus N_G(L)$ then $I^x = I \leq L^x \cap L \neq \{1\}$ and hence L is not TI. So we may assume that $N_G(I) = N_G(L)$. Since $I \leq Z(F)$, $F \leq N_G(I) = N_G(L)$. Thus L is normal in F and $Z(F)L \lhd F$. If $x \in L$ with $o(x) = 4$, then, by the proof of [22, Lemma 5.9, Ch XI, p. 216], the number of conjugates of x in F is $|F : C_F(x)| = |F|/(2|Z(F)|) = q/2$. Since $F/Z(F)$ is abelian, all F-conjugates of x are in the coset $Z(F)x$, moreover, using that $L \lhd F$, we have that they are also in $(Z(F) \cap L)x$. Thus $|Z(F) \cap L| \geq q/2$. $Z(F) \cap L$ cannot be the centre of a Sylow 2-subgroup of a smaller Suzuki subgroup, since the order of that subgroup was smaller than $q/2$. Thus $Z(F) \leq L$. Since $Z(F)$ is normalized by H, we have that $L \cap L^h \neq \{1\}$ holds for every $h \in H$. Hence H must normalize L. However H acts fixed point freely on $L/Z(F)$. Hence $|L/Z(F)| = q$, and $L = F$. We are done.

Thus we considered all the possible subgroups up to conjugacy, see Theorem 5.1.1 □

Remark 5.3.2 *This classification of TI subgroups also shows, that [1, Theorem 2.1 (7)] is wrong, the centralizer of an element of order* 4 *cannot be TI.*

Theorem 5.3.3 *The subgroups of ordinary depth* 3 *of a simple Suzuki group* $G = Sz(q)$ *are the following:*

(i) Every nontrivial subgroup contained in a maximal subgroup different from a conjugate of $N_G(F)$

(ii) F and all its nontrivial subgroups, and the conjugates of these.

(iii) All nontrivial subgroups $U = F_1K$ of $N_G(F)$ where $F_1 < F$ and $K \leq H$. And the conjugates of these subgroups.

Moreover, a nontrivial subgroup $L \leq G = Sz(q)$ is of ordinary depth 3 *if and only if there exists an element $x \in G$ with $L \cap L^x = \{1\}$.*

Proof.

(i) Since by [14], except for $N_G(F)$, all maximal subgroups have a disjoint conjugate, this holds also for their subgroups, thus any such nontrivial subgroup has ordinary depth 3.

(ii) This holds since F is TI.

(iii) Now we have to consider the subgroups of $N_G(F) = FH$. We want to prove that if the subgroup does not contain F, then it has a disjoint conjugate.

Let U be such a subgroup. Let us suppose now that $U = F_1K$, where $F_1 < F$ and $K \leq H$. Then by [34, Theorem 17.3], we have that U is a Frobenius group with kernel $F_1 = F \cap U$ and complement K. Moreover, since K is a characteristic subgroup in the cyclic group H, and H is a TI set in G, so K is also a TI set in G. Let $x \notin FH = N_G(F)$. If $U^x \cap U = \{1\}$ then we are done.

Otherwise $U^x \cap U \neq \{1\}$. The subgroup $U^x \cap U$ cannot contain a non-trivial element $f_1 \in F_1$, since then $f_1 \in F_1 \cap F_1^x \leq F \cap F^x$.

Since F is TI, we have that $x \in N_G(F)$, which is not the case. Then $|U^x \cap U|$ is a divisor of $|K|$. Since U is solvable, by Hall's theorem $U^x \cap U$ is contained in a complement of F_1 and it can be conjugated in U, moreover in F_1, to a subgroup of any other complement of F_1 in U. We may suppose that $U^x \cap U \leq K \leq H$. Otherwise let $s \in F_1$ be an element such that $(U^x \cap U)^s \leq K$. Then $K \geq (U^x \cap U)^s = U^{xs} \cap U$ and $xs \notin N_G(F)$. Thus we can exchange if necessary x to xs, to have that $U^x \cap U \leq K$. Let $K_1 := U^x \cap U$. So we may suppose that $K_1 \leq K$.

The Frobenius complements of U^x are of the form $K^{f_1 x}$ for some $f_1 \in F_1$. Since $U^x = F_1^x \cup (\cup_{f_1 \in F_1} K^{f_1 x})$, so for some element $f_1 \in F_1$, we have that
$\{1\} \neq K^{f_1 x} \cap K \leq H^{f_1 x} \cap H$. Since H is TI, we have that $f_1 x \in N_G(H) \leq N_G(K)$, as K is characteristic in the cyclic group H. Thus $K^{f_1 x} = K \leq U^x \cap U = K_1$. Hence $U^x \cap U = K$. Let $l \in N_{F^x}(F_1^x) \setminus F_1^x$. Then $(F_1^x K)^l = F_1^x K^l \leq F^{f_1 x} K = (FK)^{f_1 x}$, since $l \in F^x$ and $f_1 x \in N_G(K)$.

Now, since $l \in N_{F^x}(F_1^x)$ and $f_1 x \in N_G(K)$, moreover $(F_1^x K)^l \leq (FK)^{f_1 x}$, we have that $F_1 K \cap (F_1 K)^{f_1 x l} = F_1 K \cap F_1^x K^l \leq FK \cap (FK)^{f_1 x}$. The subgroup $FK \cap (FK)^{f_1 x}$ contains K, however it cannot contain any elements of F, otherwise, as F is TI, this would imply that $f_1 x \in N_G(F)$, which is not the case. So $FK \cap (FK)^{f_1 x} = K$. If $K \cap (F_1^x K^l) \neq \{1\}$ then, since $F_1^x K^l = (F_1^x K)^l$, so for some element $f_2^x \in F_1^x$ we have that $K \cap K^{f_2^x l} \neq \{1\}$. Hence $f_2^x l \in N_G(K)$, since K is a TI set in G. However $l \in F^x \setminus F_1^x$ so $f_2^x l \in F^x \setminus F_1^x$. Thus a nontrivial element of F^x normalizes a

Frobenius complement of F^xK, which cannot happen. Thus $K \cap (F_1^x K^l) = \{1\}$ and so $F_1K \cap (F_1K)^{f_1xl} = \{1\}$. Hence $d(F_1K, G) = 3$. (If $F_1 = F$ this proof does not work, since then such l does not exist.)

Let us suppose now that $U = FK$, where $K < H$. We want to prove that $d(U, G) = 5$. We know, see the proof of [14, Proposition 4.3], that there exist elements $x_1, x_2 \in G$ with $FH \cap (FH)^{x_1} \cap (FH)^{x_2} = \{1\}$. Hence by the second part of Theorem 2.1.3 we have that $d(U, G) \leq 5$. Suppose by contradiction that $d(U, G) \leq 4$. Then by Definition 2.1.2 we have that $m(\chi) \leq 1$ for each irreducible character $\chi \in \mathrm{Irr}(G)$. We will prove that for $\chi = 1_G$ this is not true. For, let us take a nontrivial irreducible character $\psi \in \mathrm{Irr}(FK/F)$. We will prove that $d(\psi, 1_{FK}) = 2$.

Since ψ can be extended to a nontrivial irreducible character θ of $N_G(F)$ containig F in its kernel, we know by the proof of [14, Corollary 5.1] that $d(\theta, 1_{N_G(F)}) = 2$. Hence $d(\psi, 1_{FK}) \leq 2$.

Suppose by contradiction that this distance was 1. Then $(\psi^G, 1_{FK}^G) \neq 0$. However, ψ can be extended in $|H : K|$ ways to $FH = N_G(F)$. Let us suppose that these extensions are $\psi_1, \ldots \psi_{|H:K|}$. These are all the constituents of ψ^{FH}. Then $(\psi^G, 1_{FK}^G) = ((\psi^{FH})^G, 1_{FK}^G) = \sum (\psi_i^G, 1_{FK}^G)$. However, $1_{FK}^G = (1_{FK}^{FH})^G$, and the irreducible constituents of 1_{FK}^{FH} are exactly those characters $\phi \in \mathrm{Irr}(FH)$ whose kernel contains FK. Thus if $(\psi^G, 1_{FK}^G) \neq 0$ then for some $\phi \in \mathrm{Irr}(FH/FK)$, $(\psi_i^G, \phi^G) \neq 0$. However, according to [22, Ch. XI., Lemma 5.3], $(\phi^G, \psi_i^G) \neq 0$ iff $\phi = \psi_i$ or $\phi = \overline{\psi}_i$. We know that ψ_i is an extension of a nontrivial irreducible character of

FK/F, hence neither ψ_i nor $\overline{\psi_i}$ can contain K in its kernel. Thus $(\psi_i^G, 1_{FK}^G) = 0$ for all i in the above sum, hence $(\psi^G, 1_{FK}^G) = 0$, and $d(U,G) \not\leq 4$. Thus it is 5 and we are done.

In [14] its proved that subgroups of type (i) and (ii) have disjoint conjugates. In (iii) its proved above this property for subgroups of type (iii). Conversely if a non-normal subgroup has a disjoint conjugate, then by the second part of Theorem 2.1.3, it is of ordinary depth 3. □

As a Corollary we get a characterization of Suzuki groups:

Corollary 5.3.4 *Let G be a simple Zassenhaus group acting on $q+1$ points, where q is a 2-power. Then the following are equivalent:*

(i) $G \simeq Sz(q)$.

(ii) If $F \in Syl_2(G)$, then every subgroup not containing a conjugate of the subgroup F has a disjoint conjugate.

Proof. By Theorem 5.3.3 and by the description of subgroups of $\mathrm{Sz}(q)$ in Theorem 5.1.1 we have that (i) implies (ii). For the other direction observe that the only simple Zassenhaus groups with q even are $\mathrm{Sz}(q)$ and $PSL(2,q)$. In $PSL(2,q)$ there are dihedral subgroups of order $2(q+1)$, that are not disjoint from their conjugates, see [9]. □

We also have a wider class of groups where ordinary depth 3 is equivalent to having a disjoint conjugate:

Corollary 5.3.5 *Let G be a simple Zassenhaus group acting on $q+1$ points, where q is an odd power of 2. Then the following are equivalent for a nontrivial subgroup $L \leq G$:*

(i) The subgroup $L \leq G$ has ordinary depth 3.

(ii) The subgroup $L \leq G$ has a disjoint conjugate.

Proof. Since these Zassenhaus groups are $\mathrm{Sz}(q)$ and $PSL(2,q)$, we have to prove the equivalence for these groups. In Theorem 5.3.3 we proved the equivalence for $\mathrm{Sz}(q)$. In [9] this is proved for $PSL(2, 2^{2n+1})$. $\square$

We also get some information about Sylow 2-subgroups of Suzuki groups

Corollary 5.3.6 *Every Sylow* 2*-subgroup of $G = Sz(q)$ is the union of conjugates of the Sylow* 2*-subgroups of a smallest simple Suzuki subgroup G_1 contained in G. If $F \in Syl_2(G)$ and $F_1 \in Syl_2(G_1)$ contained in F, then $F = \cup_{x \in N_G(F)} {F_1}^x$. Every element of order* 4 *of F is in exactly one conjugate of F_1 and any two of the conjugates of F_1 in this union either have trivial intersection or their intersection is their center, which is a conjugate of $\Omega_1(F_1)$.*

Proof. By the part 7. b) in the proof of Theorem 5.3.1 we have that $\cup_{h \in H} Z(F_1)^h = Z(F)$, and the subgroups $Z(F_1)^h$ for $h \in H$ intersect trivially. Let $a \in F \setminus Z(F)$ such that $a \in F_1$. Since H acts sharply 1-transitively on the nontrivial elements of $F/Z(F)$, we have that each coset of $Z(F)$ contains an element of a conjugate of F_1. By the proof of [22, Lemma 5.9, Ch XI.] the element a has $q/2 = |F : C_G(a)| = |F : Z(F)\langle a \rangle|$ conjugates and a^{-1} also. The F-conjugates of these elements all lie in $Z(F)a$. Thus these give all the elements of $Z(F)a$ since this coset has q elements. Hence each element of F is in a conjugate of F_1. Since F is a TI set, the conjugating elements are in $N_G(F)$.

For the second part, see [14, Proposition 4.13] or the following shorter argument: suppose that $x \in G \backslash N_G(F_1)$ and $a \in F_1 \cap F_1^x$ is an element of order 4. Then there exists an element $b \in F_1$ such that $b^x = a$. By [22, Lemma 11.7 Ch XI] we have that in G there are two conjugacy classes of elements of order 4. We know that $K_G(a) \neq K_G(a^{-1})$, otherwise an element of H would centralize a^2, similarly $K_G(b) \neq K_G(b^{-1})$. They are also in different conjugacy classes in G_1. Thus a and b are conjugate also in G_1. Hence they are conjugate in $N_{G_1}(F_1)$. Thus there exists an element $y \in N_G(F_1)$ such that $b^y = a$. Hence $xy^{-1} \in C_G(b)$. Since $x \notin N_G(F_1)$ and $y \in N_G(F_1)$, then $xy^{-1} \notin N_G(F_1)$. However $xy^{-1} \in C_G(b) = Z(F)\langle b \rangle \leq N_G(F_1)$. This contradiction shows that there cannot be an order 4 element in two conjugates of F_1. If $F_1 \cap F_1^x \neq \{1\}$ then they have common elements of order 2 hence $Z(F_1) \cap Z(F_1)^x \neq \{1\}$. We have seen that $Z(F_1)$ is a TI subgroup of G, hence $Z(F_1) = Z(F_1)^x$. Since F_1 and F_1^x do not have common elements of order 4 and all their elements of order 2 are central, hence $F_1 \cap F_1^x = Z(F_1)$. □

Remark 5.3.7 *It is easy to see that* $F = \cup_{x \in N_G(F)} F_2^x$*, where* $F_2 \simeq C_4$*, and in this union any two conjugates of* F_2 *either have trivial intersection, or the intersection is a conjugate of* $\Omega_1(F_2)$*. This is because there are two conjugacy classes of order* 4 *elements in* G*. Let they be* $K_G(a)$ *and* $K_G(a^{-1})$*, where* $a \in F$*. Consider* $\cup_{x \in N_G(F)} \langle a \rangle^x$*. This will contain all elements of order* 4 *in* F*, i.e. all elements of* $F \setminus Z(F)$*. However, each involution in* F *is a square of an element of order* 4*, since this is true for at least* 2 *involutions and the other involutions are conjugate by elements of* $H \leq N_G(F)$*. Hence we are done.*

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Appendix: GAP programs

A program to find the Frobenius matrix for a subgroup h of g:

```
 Input=Group g, subgroup h, Output=Frobenius matrix of h

--------------------------------------------

FrobeniusMatrix:=function(g,h)

   local c,d,e,f;

      c:=Irr(g);

      d:=Irr(h);

      e:=RestrictedClassFunctions(c,h);

      f:=MatScalarProducts(e,d);

   return f;

end;
```

Program to find the powers of a rectangular matrix:

```
 Input=matrix m, power of matrix k, Output=Power Matrix of k

------------------------------------

PowerOfMatrix:=function(m,k)

   local mt,m2,k1,mk;

      mt:=TransposedMat(m);
```

```
      m2:=m*mt;
      k1:=k mod 2;
            if k1=1 then mk := m2^((k-1)/2) * m;
            fi;
            if k1=0 then mk := (m2^((k-2)/2) * m) * mt;
            fi;
   return  mk;
end;
```

Program to find the number of zero position of a matrix:

```
 Input=matrix m, Output=zero position number of Matrix m
-------------------------------
ZeroPositionNo := function(m)
   local z, i, j;
      z := 0;
            for i in [1..Length(m)]
            do
                  for j in [1..Length(m[1])]
                  do
                        if m[i][j]=0 then z:=z+1;
                        fi;
                  od;
            od;
   return z;
end;
```

Program to find the depth of a matrix:

```
 Input=matrix m, Output=depth of matrix m
-----------------------------
DepthOfMatrix:=function(m)
   local k,s,r,u,v;
      k:=1;;
            repeat
                  k := k + 1;
                  s:=PowerOfMatrix(m,k-1);
                  r:=PowerOfMatrix(m,k+1);
                  u:=ZeroPositionNo(s);
                  v:=ZeroPositionNo(r);
            until u=v;
   return k;
end;
```

Program to find subgroups of even depth ≥ 8 in the small groups library:

 Input=y,u is integer numbers, Output=subgroups of even depth ≥ 8 in small groups with degree between y and u.

```
--------------------------------------------
TestDepthSmallGroup:=function(y,u)
 local n,R,k,g,cc,cr,s,h,m,d,p,d1;
  for n in [y..u]
  do
    R:=AllSmallGroups(Size,n,IsAbelian,false);
       for k in [1..Length(R)]
```

```
      do
        g:=R[k];
        cc:=ConjugacyClassesSubgroups(g);
        cr:=List(cc,Representative);
          for s in [1..Length(cr)]
          do
            h:=cr[s];
              if IsNormal(g,h)= false
                then
                m:=FrobeniusMatrix(g,h);
                d:=DepthOfMatrix(m);
                d1:=d mod 2;
                  if d>7
                    then
                    if d1=0
                      then
                      p:=List(h,x->DepthOfMatrix(m));
                      Print("n=",n," k=",k,"s=",s,"depth=",d,"\n");
                    fi;
                  fi;
              fi;
          od;
      od;
  od;
end;
```

Program to find subgroups of even depth ≥ 8 in the primitive groups library:

Input=y,u is integer numbers, Output=subgroups of even depth ≥ 8 in the primitive groups library with degree between y and u.

```
----------------------------------
TestDepthPrimitiveGroup:=function(y,u)
 local n,k,g,cc,cr,s,h,m,d,p,d1;
   for n in [y..u]
   do
     for k in [1..NrPrimitiveGroups(n)]
     do
       g:=PrimitiveGroup(n,k);
       cc:=ConjugacyClassesSubgroups(g);
       cr:=List(cc,Representative);
         for s in [1..Length(cr)]
         do
           h:=cr[s];
             if IsNormal(g,h)= false
               then
               m:=FrobeniusMatrix(g,h);
               d:=DepthOfMatrix(m);
               d1:=d mod 2;
                 if d>7
                   then
                     if d1=0
                       then
```

```
                  p:=List(h,x->DepthOfMatrix(m));
                  Print("n=",n," k=",k,"s=",s," depth=",d,"\n");
                fi;
             fi;
          fi;
       od;
    od;
  od;
end
```

Program to find subgroups of even depth ≥ 8 in the transitive groups library:

 Input=y,u is integer numbers;

Output=subgroups of even depth ≥ 8 in transitive groups library with degree between y and u.

```
----------------------------------------
TestDepthTransitiveGroup:=function(y,u)
 local n,k,g,cc,cr,s,h,m,d,p,d1;
  for n in [y..u] do
    for k in [1..NrTransitiveGroups(n)]
    do
      g:=TransitiveGroup(n,k);
      cc:=ConjugacyClassesSubgroups(g);
      cr:=List(cc,Representative);
        for s in [1..Length(cr)]
        do
```

```
          h:=cr[s];
            if IsNormal(g,h)= false
              then
              m:=FrobeniusMatrix(g,h);
              d:=DepthOfMatrix(m);
              d1:=d mod 2;
                if d>7
                  then
                    if d1=0
                      then
                      p:=List(h,x->DepthOfMatrix(m));
                      Print("n=",n," k=",k," s=",s,"depth=",d,"\n");
                    fi;
                fi;
            fi;
        od;
    od;
  od;
end;
```

Program to find the distance matrix of characters of a subgroup h of g:

```
 Input=h subgroup, g group, c character table of group g, c1 character table
of subgroup h;
Output=matrix of distance between irreducible characters of subgroup h.
----------------------------------------
```

```
DistanceMatrix2:=function(g,h,c,c1)
 local m,m2,nccl, result, i,j,d,m1,k,n,D;
   m:=FrobeniusMatrix(g,h);
   m2:=m*TransposedMat(m);
   D:=DepthOfMatrix(m);
   k:=0;
   nccl:= NrConjugacyClasses( c1 );
   result:= List( [ 1 ..  nccl ], i -> 0 * [ 1 ..  nccl ] -1 );
     for i in [ 1 ..  nccl ]
     do
       result[i][i]:= 0;
     od;
   m1:= m2;
     repeat
       k:=k+1;
         for i in [ 1 ..  nccl ]
         do
           for j in [ 1 ..  nccl ]
           do
             if result[i][j] = -1 and m1[i][j]<>0 then result[i][j]:= k;
             fi;
           od;
         od;
       m1:= m1 * m2;
     until k > D;
 return result;
end;
```

Program to find the maximal value in a matrix:

```
 Input=matrix m;
Output=maximal value in matrix m.
----------------------------------------
MaxOfMatrix:=function(m)
 local Max, i,j;
   Max:=-1;
     for i in [1..Length(m)]
     do
       for j in [1..Length(m[i])]
       do
         if m[i][j]> Max then Max:=m[i][j];
         fi;
       od;
     od;
 return Max;
end;
```

Program to find the shortest distance between irreducible characters u and v by using Dijkstra algorithm:

```
 Input=restrected Frobenius U,index irr.  character(h) u, index irr.  character(h)
v
Output=distance between characters u and v by using Dijkstra algorthem.
----------------------------------------
Dijkstra:=function(U,u,v)
 local d,w,D,G,H,ready,x,y,s,j,k,n,R,R1,min,i,path,lastind;
```

```
R:=[];
R1:=[];
n:=[];
D:=[];
path:=[];
ready:=[];
d:=DistanceMatrix2(g,h,c,c1);
  for x in U
  do
    n[x]:=[];
      for y in U
      do
        n[x][y]:=9;
          if x=y then n[x][y]:=0;
          fi;
          if d[x][y]=1 then n[x][y]:=1;
          fi;
      od;
  od;
  for w in U
  do
    D[w]:=[];
    D[w]:=n[u][w];
      if D[w]=1 then
        path[w]:=[u,w];
      fi;
  od;
```

```
  ready:=[u];
  path[u]:=[u];
    repeat
      R:=Difference(U,ready);
      min:=9;
        for i in R
        do
          if D[i]< min then
            s:=i;
            min:=D[s];
          fi;
        od;
      ready:=Union2(ready,[s]);
      R1:=Difference(U,ready);
        for k in R1
        do
          D[k]:=Minimum(D[k],D[s]+n[s][k]);
            if D[k]=D[s]+n[s][k] then
              path[k]:=Concatenation(path[s],[k]);
            fi;
        od;
    until v=s;
 return path[s];
end;
```