# Quasi-Newton type iterative solution of nonlinear elliptic PDEs with non-uniform monotonicity conditions

PhD Dissertation – Outline

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# 1 Introduction

Nonlinear elliptic problems arise in various applications for models that describe stationary states, see, e.g., [12, 14, 19, 24, 29] and the references there. We may mention, for instance, elasticity, glaciology, flow problems in physics and other fields, see, e.g., [6, 15, 31, 33]. As shown by such works as well, a widespread way to solve such problems is to use finite element discretization (FEM) and then to apply a Newton-like iteration, see also [17, 24, 37].

To construct quasi-Newton methods, a general approach to has been given in [27], where approximate Jacobians are defined via spectral equivalence, and hence they can be regarded as variable preconditioners. Thus variable preconditioning provides a transition between fixed preconditioning and Newton's method. With fixed preconditioning, one can define simple iteration that is often able to yield favorable speed of global convergence if supplied with suitable preconditioning, and in these cases its usage can be justified versus Newton's method owing to the extra work of forming the Jacobians (see, e.g., [3, 4] for early work in this direction, further [34] and the references therein for later applications.). However, it might be insufficient for strong nonlinearities. These can favor Newton's method, which is more complicated and costly for one iteration step, but provides better convergence altogether for the strong nonlinearities. Quasi-Newton methods, which can also be regarded as variable preconditioners, may combine the advantages of these methods. Alternatively, one can use Newton's method for outer iteration, and exploit the preconditioner in case of the inner iteration, as follows.

The Newton-type method yields linear problems that can be solved by direct or approximate methods, depending on the scale of the problem. A widely used approximate approach is to use conjugate gradient method (CGM) in the inner iterations. The construction of such inner-outer iterations can be found in [16, 36], their framework for uniformly monotone elliptic problems has been presented in [2, 37], see also [32, 40] for recent applications. Preconditioned CGM can be readily formulated with the help of the variable preconditioners discussed above.

The aim of this dissertation is to extend earlier results that provide theory only for the assumption of uniform ellipticity, which does not hold for many real-life problems, for example, in non-Newtonian flows, nonlinear optics, minimal surface problems, glacier modelling, etc.

We make use of tools of functional analysis to address this task, since Sobolev spaces are the natural underlying spaces of the boundary value problems that are the subjects of our investigation. Given a nonlinear boundary value problem, one can usually formulate an operator equation of the form $F(u) = 0$, where $F$ maps either a Hilbert or Banach space to its dual. Depending on the original nonlinearity, the resulting operator $F$ may or may not exhibit ellipticity with uniform lower and upper bounds.

The structure of the dissertation is the following.

In Section 2, the results prior to the work presented here are discussed, detailing the convergence rates of the quasi-Newton method for problems with uniform lower and upper bound in the ellipticity condition, for well-posed problems in Hilbert function space, for both local and global convergence. Furthermore, a brief insight is given into the related boundary value problems and the place of quasi-Newton methods among Newton-type methods. This section is based on [27].

Sections 3 and 4 are devoted to quasi-Newton methods for problems with stronger nonlinearities.

In Section 3, the uniform upper bound in the ellipticity condition is relaxed, and the derivation of the resulting convergence rates are discussed for local and global convergence. The

Lipschitz condition is also found to be relaxed as a result of the more general assumptions. An example problem is discussed in detail using an equation from nonlinear optics, and numerical results are presented.

In Section 4, Banach space setting is employed for operators with relaxed both lower and upper bounds. Details of a model classification is given, and various examples are shown for equations, based on real-life models, that satisfy our conditions and for efficient variable preconditioners.

Section 5 presents the results for inner-outer iterations in case of inexact Newton methods. The results are illustrated using a nonlinear fluid flow model.

Section 6 studies a one-dimensional fourth-order engineering model, with detailed theoretical and practical comparison of the three methods (simple iteration, quasi-Newton and full Newton).

Section 7 presents a problem with a Stefan-Boltzmann heat radiation boundary nonlinearity in 3D. Nonnegativity result is obtained, finite element approximation is discussed, and the applicability of the quasi-Newton methods is shown.

Sections 3, 4, 5, 6 and 7 are based on the author's papers [7], [8], [9], [10] and [11], respectively. The numerical investigations have been carried out using Matlab.

Section 8 is entitled to give a short summary for developers on the apparent findings they might find interesting. In Section 9, a brief general summary of the presented results can be found.

## 2 Theoretical background

This section is entitled to provide results of work prior (see [27]) to this dissertation for the sake of convenience and to provide reference material for later sections.

Let $H$ be a real Hilbert space with inner product $\langle .,. \rangle$ and corresponding norm $\|\cdot\|$. The following energy inner product is used for positive definite self-adjoint operator $A$:

$$\langle u, v\rangle_A := \langle Au, v\rangle \qquad (\forall u,\ v \in H), \tag{2.1}$$

with correspondingnorm $\|\cdot\|_A$. For the operator $F : H \to H$, the following operator equation is under investigation

$$F(u) = 0. \tag{2.2}$$

### 2.1 Basic concept and motivation

In this subsection, we discuss the basic concept of quasi-Newton methods, and their place as compared to other methods. Algorithms used for local convergence are discussed, but the conceptual illustration holds for more complicated requirements.

Nonlinear boundary value problems can often be written in the form of (2.2). If this has a unique solution $u^*$, we may approach it with a sequence $(u_n) \subset H$ using the following algorithm:

$$u_{n+1} := u_n + p_n \qquad (\forall n \in \mathbb{N}),$$

$$\text{where :} \qquad B_n p_n = -F(u_n),$$

with an auxiliary linear operator $B_n: H \to H$ and sequence $(p_n) \subset H$. Different formulations of $B_n$ correspond to different methods.

For the Sobolev gradient method, $B_n = \text{const.} \cdot I$, hence the algorithm reads simply as $u_{n+1} = u_n - \text{const.} \cdot F(u_n)$, allowing swift computation for each $n$, however, the required number of iterations might be large.

In contrast, Newton's method uses $B_n = F'(u_n)$, which may be hard to compute for each $n$, but it provides convergence of quadratic order.

The idea of quasi-Newton methods is to choose auxiliary operators $B_n$ between these two in a sophisticated manner, so that the advantages of these methods can be combined. The 'sophisticated manner', in which $B_n$ is obtained, is similar to the general twofold idea of preconditioning:

*(i)* a symbolic simplification of the operator $F'(u_n)$ by intuition, so that we obtain a significantly simpler operator, moreover,

*(ii)* a spectral equivalence relation can be established between $B_n$ and $F'(u_n)$ which provides a favorable convergence result.

The term variable preconditioning can be used, because the auxiliary operator $B_n$ is allowed to be different for each $n$, otherwise, it could be called constant preconditioning.

In the next two subsections, we summarize the corresponding results in literature prior to this work.

## 2.2 Linear convergence by variable preconditioning

Let us recall the following theorem, which provides a linear convergence result [27].

**Assumptions 2.5** *Let $H$ be a real Hilbert space and $F : H \to H$ a nonlinear operator. Let $F$ have a Gâteaux derivative that satisfies the following properties:*

*(i) For any $u \in H$ the operator $F'(u)$ is self-adjoint.*

*(ii) There exist constants $\Lambda \geq \lambda > 0$ satisfying:*

$$\lambda \|h\|^2 \leq \langle F'(u)h, h\rangle \leq \Lambda \|h\|^2 \qquad (\forall u, h \in H). \tag{2.3}$$

*(iii) There exists $L > 0$ such that*

$$\|F'(u) - F'(v)\| \leq L\|u - v\| \qquad (u,\ v \in H).0 \tag{2.4}$$

*Denote by $u^* \in H$ the unique solution of $F(u) = 0$. Let $M \geq m > 0$ be given constants, and for any $n \in \mathbb{N}$ let us choose a bounded self-adjoint linear operator $B_n : H \to H$ such that*

$$m\langle B_n h, h\rangle \leq \langle F'(u_n)h, h\rangle \leq M\langle B_n h, h\rangle \qquad (\forall h \in H). \tag{2.5}$$

**Algorithm 2.6** *With Assumptions 2.5, starting from a* $u_0 \in H$*, we define a sequence from the following formula:*

$$u_{n+1} := u_n - \frac{2}{M+m} B_n^{-1} F(u_n) \qquad (\forall n \in \mathbb{N}). \tag{2.6}$$

**Theorem 2.7** *With Assumptions 2.5, the sequence generated by Algorithm 2.6 converges locally linearly to* $u^*$*, namely, there exists a neighbourhood* $U$ *of* $u^*$ *and for given* $u_0 \in U$ *there exists a constant* $C > 0$ *such that*

$$\|u_n - u^*\| \le C \left( \frac{M-m}{M+m} \right)^n \qquad (\forall n \in \mathbb{N}). \tag{2.7}$$

Here, *(ii)* is called the ellipticity condition. This condition is too strong for various applications (see details later in e.g. Subsection 3.2), the main aim of this work is to show that similar results can be obtained for certain more general conditions.

In (2.5), the chain of inequalities is called the spectral equivalence of operators $F'(u)$ and $B$, while $m$ and $M$ can be called the spectral equivalence constants, or spectral bounds.

If we drop the Lipschitz-condition *(iii)*, but the fixed spectral bounds can be achieved with fixed preconditioners only, i.e. $B_n \equiv B$, then global convergence can be obtained with (2.7) repeated (see [20]). However, using a different auxiliary operator for every step $n$ (i. e. variable preconditioning) entails local linear convergence for Lipschitz-continuous operators. This variable preconditioning is vital for a useful algorithm, and the convergence can be made global again by damping, as it can be seen below.

We introduce the following energy norms:

$$\|h\|_u := \langle F'(u)^{-1} h, h \rangle^{1/2} \quad \text{(for given } u \in H), \qquad \| \cdot \|_* := \| \cdot \|_{u^*}, \qquad \| \cdot \|_n := \| \cdot \|_{u_n} \tag{2.8}$$

(for given $n \in \mathbb{N}$).

It has been shown that for fixed $u$ the norms $\| \cdot \|_u$ and $\| \cdot \|$ are equivalent here, namely:

$$\lambda^{1/2} \|h\|_u \le \|h\| \le \Lambda^{1/2} \|h\|_u \qquad (\forall h \in H). \tag{2.9}$$

## 2.3 Damped quasi-Newton method as variable preconditioning

We recall the following definitions of norms (see (2.8)), where $(u_n)$ is an iterative sequence and $u^*$ is the solution of $F(u) = 0$:

$$\|h\|_n = \langle F'(u_n)^{-1} h, h \rangle^{1/2} \quad (n \in \mathbb{N}), \qquad \|h\|_* = \langle F'(u^*)^{-1} h, h \rangle^{1/2}. \tag{2.10}$$

The following theorem generalizes Theorem 2.7. Using damped iteration and variable spectral bound preconditioning, it provides global convergence up to second order.

**Assumptions 2.14** *Let* $H$ *be a real Hilbert space and* $F : H \to H$ *a nonlinear operator. Let* $F$ *have a Gâteaux derivative that satisfies the properties (i)-(iii) of Assumptions 2.5.*

*Denote* $u^*$ *the unique solution of equation* $F(u) = 0$*.*

*Furthermore, the following conditions hold:*

*(iv)* *For each $n \in \mathbb{N}$, let $M_n \geq m_n > 0$ and let us choose a bounded self-adjoint linear operator $B_n : H \to H$ such that*

$$m_n\langle B_n h, h\rangle \leq \langle F'(u_n)h, h\rangle \leq M_n\langle B_n h, h\rangle \qquad (n \in \mathbb{N},\ h \in H);$$

*further, using notation $\mu(u_n) = L\lambda^{-2}\|F(u_n)\|$, there exist constants $K > 1$ and $\varepsilon > 0$ such that $M_n/m_n \leq 1 + 2/(\varepsilon + K\mu(u_n))$.*

*(v)* *We define*

$$\tau_n = \min\left\{1, \frac{1-Q_n}{2\rho_n}\right\}, \tag{2.11}$$

*where $Q_n = \frac{M_n - m_n}{M_n + m_n}(1 + \mu(u_n))$, $\rho_n = 2LM_n^2\lambda^{-3/2}(M_n + m_n)^{-2}\|F(u_n)\|_n(1 + \mu(u_n))^{1/2}$, $\mu(u_n)$ is as in condition (iv), and $\|\,.\,\|_n$ is defined in (2.10). (This value of $\tau_n$ ensures optimal contractivity in the nth step in the $\|\,.\,\|_*$-norm.)*

**Algorithm 2.15** *With Assumptions 2.14, for arbitrary $u_0 \in H$ let $(u_n)$ be the sequence defined by*

$$u_{n+1} = u_n - \frac{2\tau_n}{M_n + m_n}\, B_n^{-1}F(u_n) \qquad (n \in \mathbb{N}). \tag{2.12}$$

**Theorem 2.16** *With Assumptions 2.14, the sequence generated by Algorithm 2.15 converges globally linearly to $u^*$, namely,*

$$\|u_n - u^*\| \leq \lambda^{-1}\|F(u_n)\| \to 0; \tag{2.13}$$

*namely,*

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq \limsup \frac{M_n - m_n}{M_n + m_n} < 1\,. \tag{2.14}$$

*Moreover, if in addition we assume $M_n/m_n \leq 1 + c_1\|F(u_n)\|^\gamma$ $(n \in \mathbb{N})$ with some constants $c_1 > 0$ and $0 < \gamma \leq 1$, then*

$$\|F(u_{n+1})\|_* \leq d_1\|F(u_n)\|_*^{1+\gamma} \qquad (n \in \mathbb{N}) \tag{2.15}$$

*with some constant $d_1 > 0$.*

Owing to the equivalence of the norms $\|\,.\,\|$ and $\|\,.\,\|_*$, the orders of convergence corresponding to the estimates (2.14) and (2.15) can be formulated with the original norm.

**Corollary 2.17** *(rate of convergence in the original norm)*

(a) *If* $\limsup M_n/m_n = K > 1$, *then*

$$\|u_n - u^*\| \leq \lambda^{-1}\|F(u_n)\| \leq \text{const.} \cdot \rho^n$$

*with $\rho = (K-1)/(K+1)$.*

(b) *In the case* $M_n/m_n \leq 1 + c_1\|F(u_n)\|^\gamma$ *(with constants* $c_1 > 0$, $0 < \gamma \leq 1$*) there holds*

$$\|u_n - u^*\| \leq \lambda^{-1}\|F(u_n)\| \leq \text{const.} \cdot \rho^{(1+\gamma)^n}$$

*with some constant $0 < \rho < 1$.*

## 3 Quasi-Newton variable preconditioning under strong upper growth conditions in Hilbert spaces

The goal of this section is to extend the approach of variable preconditioning, quoted in the previous section, to problems with stronger nonlinearities without an upper uniform boundedness assumption. This situation covers power order growth of nonlinearities, which also appears in various physical models. The results of this section are based on [7].

### 3.1 Variable preconditioning for strongly nonlinear operator equations

Let $H$ be a real Hilbert space with inner product $\langle .,. \rangle$ and corresponding norm $\|\cdot\|$. We study operator equations

$$F(u) = 0$$

for a given nonlinear operator $F : H \to H$. We extend Theorem 2.7 to nonlinearities without an upper uniform boundedness assumption.

The allowed strong nonlinearity of the operator means that both the upper spectral bounds and the Lipschitz constants of the Gâteaux derivatives are allowed to grow up to infinity along with the norms of the arguments. The setting is based on the results recalled in Section 2, however, its technique has to be essentially redone to follow and eliminate the effect of the non-uniform nonlinearities.

**Assumptions 3.1.** *Let $H$ be a real Hilbert space and $F : H \to H$ a nonlinear operator. Let $F$ have a Gâteaux derivative that satisfies the following properties:*

*(i) For any $u \in H$ the operator $F'(u)$ is self-adjoint.*

*(ii) There exists a constant $\lambda > 0$ and a continuous increasing function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following condition is satisfied:*

$$\lambda \|h\|^2 \leq \langle F'(u)h, h\rangle \leq \Lambda(\|u\|)\, \|h\|^2 \qquad (\forall u, h \in H). \tag{3.1}$$

*(iii) There exists a continuous increasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$\|F'(u) - F'(v)\| \leq L(\max\{\|u\|,\, \|v\|\})\, \|u - v\| \qquad (\forall u,\, h \in H). \tag{3.2}$$

*Denote by $u^* \in H$ the unique solution of $F(u) = 0$. Let $M \geq m > 0$ be given constants, and for any $n \in \mathbb{N}$ let us choose a bounded self-adjoint linear operator $B_n : H \to H$ such that*

$$m\langle B_n h, h\rangle \leq \langle F'(u_n)h, h\rangle \leq M\langle B_n h, h\rangle \qquad (\forall h \in H). \tag{3.3}$$

The following algorithm is identical to Algorithm 2.6, however, it is repeated here for the sake of convenience:

**Algorithm 3.2.** *With Assumptions 3.1, starting from a $u_0 \in H$, we define a sequence from the following formula:*

$$u_{n+1} := u_n - \frac{2}{M+m} B_n^{-1} F(u_n) \qquad (\forall n \in \mathbb{N}). \tag{3.4}$$

**Theorem 3.3.** *With Assumptions 3.1, the sequence generated by Algorithm 3.2 converges locally linearly to $u^*$, namely, there exists a neighbourhood $U$ of $u^*$ and for given $u_0 \in U$ there exists a constant $C > 0$ such that*

$$\|u_n - u^*\| \le C \left(\frac{M-m}{M+m}\right)^n \qquad (\forall n \in \mathbb{N}). \tag{3.5}$$

## 3.2 Application to power order nonlinear elliptic problems

In this section we apply the obtained iterative method to the finite element discretization of a strongly nonlinear elliptic problem with power order nonlinearity. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, let $p \ge 3$ and $k_1, k_2 > 0$ be given constants, $g \in L^2(\Omega)$ a given function, and consider the following boundary value problem:

$$\begin{cases} -\mathrm{div}\big((k_1 + k_2|\nabla u|^{p-2})\ \nabla u\big) &= g, \\ u_{|\partial\Omega} &= 0, \end{cases} \tag{3.6}$$

Problem (3.6) has a unique weak solution in the Sobolev space $W_0^{1,p}(\Omega)$, see, e.g., [41].

We apply the finite element method (FEM) for the discretization of the problem. Let $V_h$ be a given FE subpace of certain continuous piecewise polynomial functions, then we look for $u \in V_h$ such that

$$\int_\Omega (k_1 + k_2|\nabla u|^{p-2})\nabla u \cdot \nabla v = \int_\Omega gv \qquad (\forall v \in V_h). \tag{3.7}$$

Our goal is to define the corresponding iterative method for this problem and to prove its convergence.

The operator can be defined in weak form as

$$\langle F(u), v\rangle = \int_\Omega f(\nabla u)\cdot \nabla v - \int_\Omega gv, \tag{3.8}$$

the Gâteaux derivative $F'(u)$ satisfies

$$\langle F'(u)h, v\rangle = \int_\Omega \partial_\eta f(\nabla u)\nabla h \cdot \nabla v, \tag{3.9}$$

We introduce the operators $B_n : V_h \to V_h$ defined by the following weak forms: for given $u_n \in V_h$ in the iteration, let

$$\langle B_n h, v\rangle \equiv \int_\Omega \big(k_1 + k_2|\nabla u_n|^{p-2}\big)\, \nabla h \cdot \nabla v \qquad (\forall h, v \in V_h), \tag{3.10}$$

and we define the following iteration:

$$\begin{cases} \text{solve} \quad B_n z_n = F(u_n), \\ \text{let} \ \ u_{n+1} := u_n - \frac{2}{M+m} z_n\,. \end{cases} \tag{3.11}$$

**Theorem 3.12** *The iteration defined in (3.11) converges locally according to the estimate*

$$\|u_n - u^*\|_{H_0^1} \leq C\Big(1 - \frac{2}{p}\Big)^n \qquad (\forall n \in \mathbb{N}). \tag{3.12}$$

## 3.3 Numerical experiments

Consider the following boundary value problem:

$$\left\{ \begin{array}{rcl} -\mathrm{div}\big((\chi_1 + \chi_2 |\nabla u|^2)\ \nabla u\big) & = & g, \\ u_{|\partial\Omega} & = & 0, \end{array} \right. \tag{3.13}$$

where $\chi_1, \chi_2 > 0$ are given constants. Such a nonlinear operator arises, e.g., in electrorheological fluid models, see [13]. Our test domain is the unit square $\Omega := [0,1]^2$, and we use piecewise linear finite elements. The theoretical convergence factor is 1/2, independently of the constants $\chi_1, \chi_2$ (the susceptibility coefficients)..

We have run the iteration with various physical and mesh parameter values. The initial guess $u_0$ was the solution of the Poisson equation with r.h.s. $g$. We measured the relative residual error

$$\varepsilon_n := \frac{\|F(u_n)\|_{H_0^1}}{\|F(u_0)\|_{H_0^1}}$$

throughout the iteration.

We have observed that the actual convergence follows very closely the expected theoretical error. Further, both the number of iterations and the relative residual errors behave in a robust way w.r.t. the variation of all parameters.

# 4 Quasi-Newton variable preconditioning under non-uniform monotonicity conditions in Banach spaces

In contrast to previous sections, where Hilbert space was used, the theoretical results are proven in this section in Banach space level, since the latter is a more natural underlying space for the corresponding problems. Non-uniform lower bounds are also allowed in the ellipticity condition, in addition to the strong upper growth. The results of this section are based on [8].

## 4.1 The abstract iterations in Banach spaces

The main theoretical results are presented in two stages. First a simpler version is developed with fixed spectral bounds and without damping, that generalizes the results of Theorem 3.3. The main point is that, in addition to the strong upper growth, non-uniform lower bounds are also allowed in the ellipticity condition. This generalization is motivated by various real models. Then a general version is presented, which generalizes Theorem 2.16 similarly.

Our results involve a complete rewriting of proofs of Sections 2-3: besides going to a Banach space setting, careful sequences of recursive estimates are needed to avoid the uniform lower bound that was exploited throughout the previous proofs.

### 4.1.1 The quasi-Newton method with fixed spectral bounds

Let $X$ be a real Banach space with dual space $X'$. The action of a $v \in X'$ on $u \in X$ will be denoted by $\langle v, u\rangle$. The norm sign $\|\cdot\|$ will be used both in $X$ and $X'$, this never makes a confusion thanks to the context. We study an operator equation

$$F(u) = 0 \tag{4.1}$$

where $F: X \to X'$ is a nonlinear operator.

To replace the previous energy norm formulation of (2.8) with one more convenient in our setting, in the sequel, we will use the following energy norms in $X'$:

$$\|v\|_u := \langle v, F'(u)^{-1}v\rangle^{1/2} \quad (\text{for given } u \in X), \quad \|\cdot\|_* := \|\cdot\|_{u^*}, \quad \|\cdot\|_n := \|\cdot\|_{u_n} \tag{4.2}$$

(for given $n \in \mathbb{N}$).

**Assumptions 4.2** *Let $X$ be a real Banach space and $F : X \to X'$ a nonlinear operator. Let $F$ have a bihemicontinuous Gâteaux derivative that satisfies the following properties:*

*(i) For any $u \in X$ the operator $F'(u)$ is symmetric.*

*(ii) There exists a continuous nonincreasing function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\int_0^{+\infty} \lambda(t)\, dt = +\infty \tag{4.3}$$

*and*

$$\langle F'(u)h, h\rangle \geq \lambda(\|u\|)\, \|h\|^2 \qquad (\forall u, h \in X). \tag{4.4}$$

*(iii) There exists a continuous nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|F'(u) - F'(h)\| \leq L(\max\{\|u\|, \|h\|\})\, \|u - h\| \qquad (\forall u,\, h \in X). \tag{4.5}$$

*Denote by $u^* \in X$ the unique solution of (4.1). Let $M \geq m > 0$ be given constants, and for any $n \in \mathbb{N}$ let us choose a bounded symmetric linear operator $B_n : X \to X'$ such that*

$$m\langle B_n h, h\rangle \leq \langle F'(u_n)h, h\rangle \leq M\langle B_n h, h\rangle \qquad (\forall h \in X). \tag{4.6}$$

**Algorithm 4.4** *With Assumptions 4.2, starting from a $u_0 \in H$, we obtain a sequence from the following formula:*

$$u_{n+1} := u_n - \frac{2}{M+m} B_n^{-1} F(u_n) \qquad (\forall n \in \mathbb{N}). \tag{4.7}$$

**Theorem 4.5** *With Assumptions 4.2, the sequence generated by Algorithm 4.4 converges locally linearly to $u^*$, namely, there exists a neighbourhood $U$ of $u^*$ and for given $u_0 \in U$ there exists a constant $C > 0$ such that*

$$\|u_n - u^*\| \leq C \left(\frac{M-m}{M+m}\right)^n \qquad (\forall n \in \mathbb{N}). \tag{4.8}$$

### 4.1.2 Damped quasi-Newton method with variable spectral bounds

In this subsection the previous result is extended to a damped version which exhibits global convergence, further, variable spectral bounds are allowed which yield convergence up to quadratic order.

**Assumptions 4.11** *Let $X$ be a real Banach space and $F : X \to X'$ a nonlinear operator. Let $F$ have a bihemicontinuous Gâteaux derivative that satisfies conditions (i)–(iii) of Assumptions 4.2.*

*Denote by $u^* \in X$ the unique solution of equation $F(u) = 0$.*

*Furthermore, the following conditions hold:*

*(iv)* $M_n \geq m_n > 0$ *and the symmetric linear operators* $B_n : X \to X'$ *satisfy*

$$m_n \langle B_n h, h \rangle \leq \langle F'(u_n)h, h \rangle \leq M_n \langle B_n h, h \rangle \qquad (n \in \mathbb{N},\ h \in X); \tag{4.9}$$

*further, there exist constants* $K > 1$ *and* $\varepsilon > 0$ *such that* $M_n/m_n \leq 1+2/(\varepsilon+KR^*(\|F(u_n)\|_*))$.

*(v) We define*

$$\tau_n := \min\Big\{1, \frac{1-Q_n}{2\rho_n}\Big\}, \tag{4.10}$$

*where* $Q_n := \frac{M_n-m_n}{M_n+m_n}(1 + R^*(\|F(u_n)\|_*))$, $\rho_n := H^*(\|F(u_n)\|_*)$.

**Algorithm 4.12** *With Assumptions 4.11, for arbitrary $u_0 \in X$, let $(u_n) \subset X$ be the sequence defined by*

$$u_{n+1} = u_n - \frac{2\tau_n}{M_n + m_n} B_n^{-1} F(u_n) \qquad (n \in \mathbb{N}). \tag{4.11}$$

**Theorem 4.13** *We take Assumptions 4.11 and the sequence definition of Algorithm 4.12. Then there holds*

$$\|u_n - u^*\| \leq C_0 \|F(u_n)\| \ \to\ 0 \tag{4.12}$$

*with a certain $C_0$, and with the $*$-norm from (4.2):*

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq \limsup \frac{M_n - m_n}{M_n + m_n} < 1\,. \tag{4.13}$$

*Moreover, if in addition we assume* $M_n/m_n \leq 1 + c_1 \|F(u_n)\|^\gamma$ $(n \in \mathbb{N})$ *with some constants* $c_1 > 0$ *and* $0 < \gamma \leq 1$, *then*

$$\|F(u_{n+1})\|_* \leq d_1 \|F(u_n)\|_*^{1+\gamma} \qquad (n \in \mathbb{N}) \tag{4.14}$$

*with some constant* $d_1 > 0$.

## 4.2 Applications to elliptic problems

The ellipticity framework (4.3)–(4.5) covers problems arising in various applications, such as non-Newtonian flow models with Carreau type laws, rheology, nonlinear optics, minimal surfaces or subsonic flow models. A proper choice of variable preconditioners $B_n$, satisfying the given spectral equivalences, can significantly reduce the cost of the iteration. We are interested in finite element solution of these problems, hence the iterations are constructed in the considered FEM subspace $V_h \subset W^{1,p}(\Omega)$.

In this thesis, a classification has been given for the problems falling under our assumptions, the widest class of problems being described by the following general nonlinearity with divergence form

$$\begin{cases} -\text{div } f(x, \nabla u) & = & \omega, \\ u_{|\partial\Omega} & = & g \end{cases} \tag{4.15}$$

with a nonlinear vector field $f \in C^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ (that is, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $f$ is $C^1$) which has symmetric Jacobians w.r.t. $\eta$ and satisfies

$$c_1 \left(k_0 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi|^2 \le \frac{\partial f}{\partial \eta}(x, \eta)\, \xi \cdot \xi \le \widetilde{c}_1 \left(k_0 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi|^2, \tag{4.16}$$

$$\left\| \frac{\partial f}{\partial \eta}(x, \eta_1) - \frac{\partial f}{\partial \eta}(x, \eta_2) \right\| \le d_1 \max_{\eta \in [\eta_1, \eta_2]} \left\{ (k_0 + |\eta|^2)^{\frac{p-3}{2}} \right\} |\eta_1 - \eta_2| \tag{4.17}$$

$(\forall x \in \Omega,\ \xi, \eta, \eta_1, \eta_2 \in \mathbb{R}^n)$ for some proper constants

$$1 < p < \infty, \quad \widetilde{c}_1 \ge c_1 > 0, \quad k_0 > 0\,.$$

We assume $\omega \in L^{p'}(\Omega)$ and that $g$ has a Dirichlet lift $\widetilde{g} \in W^{1,p}(\Omega)$.

Additionally, mixed boundary conditions have been discussed. Furthermore, a guide has been presented on finding preconditioners for the operators including analogues of (3.10) and generalizations. Moreover, various examples have been presented in the dissertation. Here let us just illustrate it with the following non-Newtonian fluid flow model model: a parallel sided slab model of stationary motion in glaciology provides the equation

$$-\text{div}\left( \frac{2}{T_0 + \sqrt{T_0^2 + |\nabla u|}}\ \nabla u \right) = P,$$

see [25], where $\Omega$ is the domain occupied by the glacier and $P$ is the hydrostatic pressure.

## 4.3 Numerical experiments

We have used four of these models for numerical investigation, namely, a subsonic flow model, a cubic nonlinearity, minimal surface equation, and an example in glaciology. As a result, we have observed that three of the model problems produce robust behaviour (that is, the number of iterations is bounded uniformly w.r.t the mesh size $h$), whereas the minimal surface equation shows a slight sensitivity to $h$. This is in accordance with the theoretical estimates.

Additionally, the quasi-Newton steps allowed considerably simpler coding.

In most of the tests the total computational time of our quasi-Newton method was less than for a full Newton method, that is, the cheaper linear systems in the iteration steps were able to compensate for the larger number of iterations and yielded less overall computational work.

Altogether, the numerical tests confirm the convergence and efficiency of the method, moreover, they indicate wider applicability for problems not yet fully covered by the theory.

# 5 Outer-inner iterations: inexact Newton method coupled with preconditioned CG

## 5.1 Introduction

Besides the quasi-Newton methods seen in the previous sections, a widely used approach to solve a discretized nonlinear elliptic problem is to apply a Newton-type iterative solver with conjugate gradient method used in inner iteration.

This section, based on [9], provides an inexact Newton method, coupled with preconditioned conjugate gradient method in inner iterations, for non-uniformly elliptic problems based on the setting of [7, 8, 27]. The preconditioners are based on spectrally equivalent operators. Additionally, the results of a numerical experiment for a subsonic flow model (see [5]) are provided as an example.

## 5.2 Abstract inner-outer iteration in Banach spaces

The theorem below shows convergence results for outer iteration in Banach space. The involved assumptions repeat exactly the setting of Section 4.

**Algorithm 5.2** *For arbitrary $u_0 \in X$ let $(u_n) \subset X$ be the sequence defined by*

$$u_{n+1} = u_n + p_n \qquad (n \in \mathbb{N}), \tag{5.1}$$

*where $p_n$ satisfies*

$$\|F'(u_n)\, p_n + F(u_n)\|_n \le \delta_n \|F(u_n)\|_n, \qquad (0 < \delta_n \le \delta_0 < 1), \tag{5.2}$$

*and*

$$\exists c_\gamma > 0,\ 0 < \gamma \le 1,\ \text{such that } \delta_n \le c_\gamma \|F(u_n)\|_n^\gamma. \tag{5.3}$$

**Theorem 5.3** *We impose Assumptions 4.2 without the preconditioners. Then the sequence defined by Algorithm 5.2 converges locally to $u^*$ with order $(1+\gamma)$, namely, there exists a neighbourhood $U$ of $u^*$ that for a given $u_0 \in U$ there exists constants $C > 0$ and $0 < Q < 1$ such that*

$$\|u_n - u^*\| \le C Q^{(1+\gamma)^n} \quad (n \in \mathbb{N}).$$

We have specified the inner-iteration as a preconditioned conjugate gradient method, and obtained a linear convergence result for the inner iteration. For brevity the theorem is not quoted here.

## 5.3 Numerical experiments

We have run numerical tests for the subsonic flow example, namely, to study the number of inner and outer iterations. We have found robustness of the method for both the inner and outer iterations. Additionally, we have observed that, although the theory gives a guarantee of 8 iterations to reach below the prescribed limit, in fact far less iterations can be sufficient as well.

# 6 Robust iterative solvers for nonlinear Gao beam models in elasticity

## 6.1 Preliminaries

The numerical study of the deformation of thin beams and plates is a widespread problem in elasticity theory and engineering practice, since such elastic structures regularly appear in several real applications, see, e.g., [1, 26, 35, 38, 39].

For the description of the bending of a beam resting on a foundation, to treat deformations beyond the infinitesimal region, models have been developed in [21, 23] derived from the presence of lateral stress. The following model considers a beam with classical Winkler foundation, which is a widespread concept in civil engineering, also with a profound effect on the field of adhesion mechanics, see [18]. Here the deflection $u$ of the beam is described by the following equation:

$$EIu^{IV} - E\alpha(u')^2u'' + k_F u = f \qquad \text{in } J := [0,b] \tag{6.1}$$

with the following constants: $E > 0$ is the elastic modulus, $I > 0$ is the moment of inertia for the beam's cross-section, $\alpha = 3h(1-\nu^2)$ where $h$ is thickness measured from the axis and $\nu > 0$ denotes the Poisson ratio, and $k_F > 0$ is the foundation stiffness coefficient. Further, the transverse distributed load function is denoted by $q$, and $f = (1-\nu^2)q$.

The results of this section are based on [10].

## 6.2 Numerical solution of the beam problem

### 6.2.1 Weak formulation and finite elements

The weak formulation uses the Sobolev space $H^2(J)$, moreover, using the boundary conditions, we work in the space

$$H_0^2(J) = \{u \in H^2(J) : \ u(0) = u'(0) = u(b) = u'(b) = 0\},$$

which has the standard inner product and corresponding norm

$$\langle u, v\rangle_{H_0^2} := \int_0^b u''v'', \qquad \|u\|_{H_0^2}^2 = \int_0^b (u'')^2 \,. \tag{6.2}$$

Then the weak solution of the problem is defined as $u_* \in H_0^2(J)$ satisfying

$$\int_0^b \left(u_*''v'' + \beta(u_*')^3v' + ku_*v\right) = \int_0^b gv \qquad (\forall v \in H_0^2(J)). \tag{6.3}$$

### 6.2.2 Properties of the linearized operator

The following properties hold. We have relied heavily here on generalized Hölder inequalities and Sobolev embeddings.

**Proposition 6.5** *There exists a continuous increasing function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$, independently of $h$, such that*

$$\|h\|_{H_0^2}^2 \le \langle F'(u)h, h\rangle_{H_0^2} \le \Lambda(\|u\|_{H_0^2})\, \|h\|_{H_0^2}^2 \qquad (\forall u, h \in V_h). \tag{6.4}$$

*Namely,* $\Lambda(t) = 1 + kC_2^4 + 3\beta C_4^4 t^2$.

**Proposition 6.6** *There exists a continuous increasing function* $L : \mathbb{R}^+ \to \mathbb{R}^+$, *independently of* $h$, *such that*

$$\|F'(u) - F'(v)\| \le L(\max\{\|u\|_{H_0^2}, \|v\|_{H_0^2}\}) \, \|u - v\|_{H_0^2} \qquad (\forall u,\ h \in V_h). \tag{6.5}$$

*Namely,* $L(t) = 6C_4^4 \beta t$.

### 6.2.3 Finite elements using splines

We apply piecewise cubic functions $f_1,\ f_2 : [-1,1] \to \mathbb{R}$, which are defined as

$$f_i(x) = \begin{cases} f_i^*(x) & x \in [0,\ 1] \\ (-1)^{(i-1)} f_i^*(-x) & x \in [-1,\ 0) \end{cases} \qquad (i = 1,\ 2),$$

where $f_1^*(x) = 2x^3 - 3x^2 + 1$, $f_2^*(x) = x^3 - 2x^2 + x$. The basis functions are obtained via affine transformations $L_k$ $(k \in K)$, such that the domain of $L_k(f_i)$ is $[h(k-1), h(k+1)]$ for $i = 1,\ 2$.

### 6.2.4 The iterative solvers: construction, convergence and mesh-independence

**Theorem 6.12** *The iterative methods provide the following convergence estimates:*

- *Simple iteration/gradient method:*

$$\frac{\|F(u_{n+1})\|_{H_0^2}}{\|F(u_n)\|_{H_0^2}} \le \frac{\Lambda_0 - \lambda}{\Lambda_0 + \lambda}, \quad \textit{where } \Lambda_0 = 1 + kC_2^4 + 3\beta C_4^4 \left( \|u_0\| + \frac{1}{\lambda}\|F(u_0)\| \right)^2.$$

- *Newton's method:*

$$\frac{\|F(u_{n+1})\|_{H_0^2}}{\|F(u_n)\|_{H_0^2}^2} \le \frac{L_0}{2\lambda^2}, \quad \textit{where } L_0 = 6C_4^4 \beta \left( \|u_0\| + \frac{2}{\lambda}\|F(u_0)\| \right).$$

- *Quasi-Newton/variable preconditioning:*

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \le \limsup \frac{M_n - m_n}{M_n + m_n}.$$

*These hold globally for the simple iteration and locally for the Newton and quasi-Newton methods.*

Global convergence for the Newton and quasi-Newton methods can be achieved via damped versions (using e.g. Theorem 4.13), which are not detailed here for brevity.

## 6.3 Numerical experiments, conclusion

We have performed various numerical experiments on this nonlinear model. One can conclude that all three examined methods are robust, and quasi-Newton method can replace full Newton method for this nonlinear model.

The Sobolev gradient method is very efficient for thousands of elements and more, otherwise, one should use quasi-Newton method. These coarse meshes appear to be of significance, as [22] states that 32 elements already suffice for accurate computations.

## 7 Stefan-Boltzmann heat radiation problems in 3D

In this section we briefly introduce a stationary heat conduction problem, involving nonlinear Stefan-Boltzmann radiation boundary conditions, on a bounded domain in $\mathbb{R}^3$. We present here the results of [11], which was motivated by the paper [30], where this problem was treated carefully, and our goal was to extend their results in two directions.

The problem consists of the elliptic heat conduction equation

$$-\mathrm{div}\big(A\nabla u\big) = f \quad \text{in } \Omega \tag{7.1}$$

equipped with mixed Dirichlet and nonlinear Stefan-Boltzmann radiation boundary conditions

$$u_{|\partial\Omega} = \overline{u} \quad \text{on } \Gamma_D, \tag{7.2}$$

$$\alpha u + \nu^{\mathrm{T}} A\nabla u + \beta u^4 = g \quad \text{on } \Gamma_N, \tag{7.3}$$

We obtained our results for the general anisotropic case with nondiagonal full matrix $A$. The problem (7.1)–(7.3) in its original form does not allow to involve a convex potential, since the term $u \mapsto u^4$ is not monotone. Consider now the problem where this term is replaced with $|u|^3 u$. With the help of [28], the nonnegativity result has been obtained for this model, i.e., for the case of general anisotropy, allowing the finite element approximation theory of [30] to work here.

Further, the application of the previous theorems to our elliptic problem require that the solution and the test functions be in the same Banach space, hence we homogenize BVP by letting $z := u - \bar{u}$. This allowed us to use a special case of Theorem 4.13, with a certain auxiliary operator.

The numerical experiments followed the setting of [30] but involved anisotropic heat conduction matrices, and quasi-Newton method was studied. The results showed the robustness of the methods, and an apparent supremacy of the quasi-Newton methods over Newton's method in case of the investigated combinations of certain parameters.

## 8 Conclusion

In this dissertation, it is shown that previous results for the convergence of quasi-Newton methods can be extended to much more general settings, with relaxed ellipticity and Lipschitz conditions, and replacing the Hilbert space with a Banach space setting. Namely, similar favorable results hold as for the strict conditions.

Firstly, the upper ellipticity condition is relaxed, together with the relaxation of the Lipschitz condition in Hilbert space.

Then, a Banach space setting is employed, and the lower bound is relaxed. The results for both local and global convergence hold. A detailed classification is presented for the models falling under the assumptions used.

Additionally, inner-outer iterations are investigated in Banach space with preconditioned conjugate gradient method used in inner iterations without damping, yielding a favorable local convergence result.

A one-dimensional fourth-order nonlinear beam model is studied with a detailed presentation of the usage of Sobolev gradient method, quasi-Newton method, and full Newton's method.

A heat radiation problem with boundary nonlinearity is presented with results of non-negativity and finite element approximation. Additionally, the applicability of quasi-Newton methods is shown and corresponding preconditioners are suggested.

The theoretical achievements are supported with several simulation results. One can observe that the studied quasi-Newton methods perform perfect robustness for essentially all of the investigated nonlinearities.

Experimental damping coefficients are crucial to such work. With these coefficients involved in the methods under investigation, we can conclude that quasi-Newton methods can be faster than full-Newton method regarding overall runtime, i.e. more efficient with respect to computational cost.

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