

Computations and comparison of generalized Montréal functors

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Abstract

In this thesis we examine the functors D_{SV} of Schneider and Vigneras ([17]) and D_ξ^\vee of Breuil ([3]) generalizing the so called Montréal functor D of Colmez ([4]).

Let $G = \mathbf{G}(F)$ be the F -points of a F -split reductive group \mathbf{G} defined over \mathbb{Z}_p for a finite extension $F|\mathbb{Q}_p$ with connected centre and split Borel $\mathbf{B} = \mathbf{TN}$. Let \mathfrak{o} be the ring of integers in a finite extension $K|\mathbb{Q}_p$, and $\varpi \in \mathfrak{o}$ be an uniformizer.

In chapter 2 we compute D_{SV} attaching a module over the Iwasawa algebra $\Lambda(N_0)$ of certain compact subgroup $N_0 \leq N$ to a B -representation for irreducible modulo ϖ principal series of the group $G = \mathbf{GL}_n(F)$.

Chapter 3 and some parts of chapter 4 are joint work with Gergely Záradi. We show that Breuil's [3] pseudocompact (φ, Γ) -module $D_\xi^\vee(\pi)$ attached to a smooth \mathfrak{o} -torsion representation π of $B = \mathbf{B}(\mathbb{Q}_p)$ is isomorphic to the pseudocompact completion of the basechange $\mathcal{O}_\mathcal{E} \otimes_{\Lambda(N_0), \ell} \widetilde{D}_{SV}(\pi)$ to Fontaine's ring (via a Whittaker functional $\ell: N_0 = \mathbf{N}(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p$) of the étale hull $\widetilde{D}_{SV}(\pi)$ of D_{SV} .

Both in [17] and [3] the functional ℓ was generic. In the last chapter we examine the case when ℓ is chosen to be $\ell = \ell_\alpha$, the projection of N_0 onto a root subgroup of a simple root α of \mathbf{G} , which is nongeneric. We extend the results of Breuil to this situation, moreover we define an étale action of the submonoid $T_+ \leq T$ on the noncommutative multivariable version $D_{\xi, \ell, \infty}^\vee(\pi)$ of $D_\xi^\vee(\pi)$ enabling us to go backwards to the representations of G . We also show some disadvantages of this choice of ℓ .

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Chapter 1

Introduction

1.1 Local Langlands correspondence

At first, we catch a glimpse of local class field theory (see for example [19]) as an antecedent of the local Langlands conjectures.

Let p be a prime number and \mathbb{Q}_p be the p -adic field. Let $F|\mathbb{Q}_p$ be a field extension—in general it can be any local field—, F^* be the multiplicative group of F , and E be an algebraically closed field.

The main theorem of local class field theory gives the Artin homomorphism $\theta : \mathbf{GL}_1(F) \simeq F^* \rightarrow \mathrm{Gal}(\overline{F}|F)^{ab}$, which induces an isomorphism on the profinite completion $\widehat{F^*}$ of F^* .

Since $\mathbf{GL}_1(F)$ is abelian, the irreducible E -representations of $\mathbf{GL}_1(F)$ are the homomorphisms $\mathbf{GL}_1(F) \rightarrow E^*$, which are this way related to the homomorphisms $\mathrm{Gal}(\overline{F}|F)^{ab} \rightarrow E^*$ corresponding to one dimensional E -representations of the absolute Galois group of F .

The precise statements depend on the field E , and we do not explain them in details here.

The local Langlands conjectures are generalizations of this, namely for \mathbf{GL}_n the aim is to relate certain irreducible E -representations of $\mathbf{GL}_n(F)$ with certain continuous n dimensional E -representations of $\mathrm{Gal}(\overline{F}|F)$. This correspondence shall be compatible with different structures (such as ε - and L -factors) on these representations.

In the situation $E = \overline{\mathbb{Q}_\ell}$ ($\ell \neq p$ is a prime number) and hence also if $E = \mathbb{C}$

Harris and Taylor ([11]), and independently Henniart ([12]) established the correspondence.

However, the p -adic version $E = \overline{\mathbb{Q}_p}$ of the conjectures (which are closely related to the p -characteristic version) seems to be much more involved. A satisfactory explanation comes from the representation theory of $\mathbf{GL}_n(F)$: there are much more p -adic representation than ℓ -adic. By now the correspondence for $\mathbf{GL}_2(\mathbb{Q}_p)$ is very well understood through the work of Colmez [4], [5] and others (see [1] for an overview). In other cases the conjectural picture is not clear yet.

One can see the problem even for $\mathbf{GL}_2(F)$ with $F \neq \mathbb{Q}_p$ as follows: On the Galois side nothing really different happens as we change from \mathbb{Q}_p to F . On the other hand, the dimension of $\mathbf{GL}_2(F)$ as a p -adic analytic group is bigger than that of $\mathbf{GL}_2(\mathbb{Q}_p)$, consequently the representation theory of $\mathbf{GL}_2(F)$ is much more complicated than that of $\mathbf{GL}_2(\mathbb{Q}_p)$. In particular there is no possible naive 1-1 correspondence (see [2]).

Since that many efforts have been done to generalize parts of Colmez's results. The aim of this thesis is to examine and compare the functors of Schneider-Vigneras ([17]) and Breuil ([3]) going towards the Galois side (we call these "generalized Montréal" functors).

1.2 The correspondence for $\mathbf{GL}_2(\mathbb{Q}_p)$

To review Colmez's work let $K|\mathbb{Q}_p$ be a finite extension with ring of integers o , uniformizer ϖ and residue field k .

The starting point is Fontaine's [13] theorem that the category of o -torsion Galois representations of \mathbb{Q}_p is equivalent to the category of torsion (φ, Γ) -modules over $\mathcal{O}_{\mathcal{E}} = \varprojlim_h o/\varpi^h((X))$.

Recall that a (φ, Γ) -module D is an $\mathcal{O}_{\mathcal{E}}$ -module with additional actions of the Frobenius φ and the group $\Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ which are commutative, satisfying the étale property: the map $\mathcal{O}_{\mathcal{E}} \otimes_{\varphi} D \rightarrow D$, $\lambda \otimes d \mapsto \lambda\varphi(d)$ is an isomorphism or equivalently

$$D \simeq \bigoplus_{\lambda \in \mathcal{O}_{\mathcal{E}}/\varphi(\mathcal{O}_{\mathcal{E}})} \lambda\varphi(D) = \bigoplus_{i=0}^{p-1} (1+X)^i \varphi(D).$$

Let $A_{\mathbb{Q}_p}$ be those elements $f \in \mathcal{O}_{\mathcal{E}}$ which have coefficients in \mathbb{Z}_p (the ring of p -adic integers) and A be the p -adic completion of the maximal unramified

extension $A_{\mathbb{Q}_p}^{nr}$ of $A_{\mathbb{Q}_p}$. We have actions of φ and Γ on A . Let Γ and $\chi : \Gamma \rightarrow \mathbb{Z}_p^*$ be the cyclotomic character with kernel \mathcal{H} .

The category equivalence of Fontaine is realized by these exact functors: For an étale (φ, Γ) -module D , $V(D) = (o \cdot A \otimes_{\mathcal{O}_{\mathcal{E}}} D)^{\varphi=1}$ is a Galois representation of \mathbb{Q}_p . For a Galois representation V , $D(V) = (A \otimes_{\mathbb{Z}_p} V)^{\mathcal{H}}$ is an étale (φ, Γ) -module.

One of Colmez's breakthroughs was that he managed to relate p -adic (and mod p) representations of $G^{(2)} = \mathbf{GL}_2(\mathbb{Q}_p)$ to (φ, Γ) -modules, too.

The so-called "Montréal-functor" D associates to a smooth o -torsion representation π of the standard Borel subgroup $B^{(2)}$ of $G^{(2)}$ a torsion (φ, Γ) -module over $\mathcal{O}_{\mathcal{E}}$. We can construct it in the following way:

Let $T^{(2)} \leq B^{(2)}$ be the maximal torus and $N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ be a compact open subgroup of the unipotent radical of $B^{(2)}$, T_+ be the submonoid $\{t \in T \mid tN_0t^{-1} \subseteq N_0\}$ in T , and $B_+ = N_0T_+$.

Let Π be a smooth (the action of $G^{(2)}$ is locally constant) o -representation of $G^{(2)}$ of finite length. For a certain (sufficiently small) generating B_+ -subrepresentation M of Π (which is denoted by $I_{\mathbb{Z}_p}^{\Pi}(W)$ in [4]) $D(\Pi)$ is defined as the localization $M^{\vee}[1/X]$ of the Pontryagin dual of M . The functor $\Pi \mapsto D(\Pi)$ is contravariant and exact.

The way Colmez goes back to representations of $G^{(2)}$ requires the following construction.

Let D be an étale (φ, Γ) -module over $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[1/p]$. For all $d \in D$ there are unique $d_i \in D$ such that $d = \sum_{i=0}^{p-1} (1+X)^i \varphi(d_i)$. Set $\psi(d) = d_0$, thus ψ is a left inverse of φ . With the help of that we can define a $\begin{pmatrix} \mathbb{Q}_p \setminus \{0\} & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$ -equivariant sheaf of K -vectorspaces over \mathbb{Q}_p , with global sections

$$D \boxtimes \mathbb{Q}_p = \{(d^{(n)})_{n \in \mathbb{N}} \mid \forall n : d^{(n)} \in D, \psi(d^{(n)}) = d^{(n-1)}\}$$

This can be done for the smallest compact ψ -invariant generating $\mathcal{O}_{\mathcal{E}}^+ = o[[X]]$ -submodule $D^{\natural} \leq D$ as well.

After choosing a character $\delta : \mathbb{Q}_p^* \rightarrow o^*$ we can extend this sheaf to a $G^{(2)}$ -equivariant sheaf $\mathfrak{D} : U \mapsto D \boxtimes_{\delta} U$ ($U \subseteq \mathbb{P}^1$ open) of K -vectorspaces on the

projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong G^{(2)}/B^{(2)}$. This sheaf has the following properties: (i) the centre of $G^{(2)}$ acts via δ on $D \boxtimes_{\delta} \mathbb{P}^1$; (ii) we have $D \boxtimes_{\delta} \mathbb{Z}_p \cong D$ as a module over the monoid $\begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ (where we regard \mathbb{Z}_p as an open subspace in $\mathbb{P}^1 = \mathbb{Q}_p \cup \{\infty\}$).

Whenever D is 2-dimensional and δ is the character corresponding to the Galois representation of $\bigwedge^2 D$ via local class field theory, we set $\Pi(D) = D \boxtimes_{\delta} \mathbb{P}^1 / D^{\natural} \boxtimes_{\delta} \mathbb{P}^1$, where

$$D^{\natural} \boxtimes_{\delta} \mathbb{P}^1 = \{x \in D \boxtimes_{\delta} \mathbb{P}^1 \mid \text{Res}_{\mathbb{Q}_p}(x) \in D^{\natural} \boxtimes_{\delta} \mathbb{Q}_p\}$$

is a G -invariant submodule of $D \boxtimes_{\delta} \mathbb{P}^1$. $\Pi(D)$ is an irreducible smooth representation of $G^{(2)}$.

We have $D(\Pi(D)) = \check{D}$, where $\check{D} = \text{Hom}(D, \mathcal{E})$ is the dual (φ, Γ) -module. Moreover the G -representation of global sections $D \boxtimes_{\delta} \mathbb{P}^1$ admits a short exact sequence

$$0 \rightarrow \Pi(\check{D})^{\vee} \rightarrow D \boxtimes_{\delta} \mathbb{P}^1 \rightarrow \Pi(D) \rightarrow 0.$$

It also turns out, that this relation has the other required properties as well.

1.3 Generalized Montréal functors

By now there are more different approaches to generalize Colmez's functor D to reductive groups G other than $\mathbf{GL}_2(\mathbb{Q}_p)$. We briefly recall these generalized Montréal functors here.

The approach by Schneider and Vigneras [17] starts with the set $\mathcal{B}_+(\pi)$ of generating B_+ -subrepresentations $W \leq \pi$. The Pontryagin dual $W^{\vee} = \text{Hom}_o(W, K/o)$ of each W admits a natural action of the inverse monoid B_+^{-1} . Moreover, the action of $N_0 \leq B_+^{-1}$ on W^{\vee} extends to an action of the Iwasawa algebra $\Lambda(N_0) = o[[N_0]]$. For $W_1, W_2 \in \mathcal{B}_+(\pi)$ we also have $W_1 \cap W_2 \in \mathcal{B}_+(\pi)$ (Lemma 2.2 in [17]) therefore we may take the inductive limit $D_{SV}(\pi) = \varinjlim_{W \in \mathcal{B}_+(\pi)} W^{\vee}$. In [17] it is denoted by $D(\pi)$, however, in order to avoid confusion we denote it by $D_{SV}(\pi)$ (also note that the notation V is used for the o -torsion representation that we denote by π). In general, $D_{SV}(\pi)$ does not have good properties: for instance it may not admit a

canonical right inverse of the T_+ -action making $D_{SV}(\pi)$ an étale T_+ -module over $\Lambda(N_0)$. However, by taking a resolution of π by compactly induced representations of B , one may consider the derived functors D_{SV}^i of D_{SV} for $i \geq 0$ producing étale T_+ -modules $D_{SV}^i(\pi)$ over $\Lambda(N_0)$. Note that the functor D_{SV} is neither left- nor right exact, but takes injective (resp. surjective) maps to surjective (resp. injective) maps. The fundamental open question of [17] whether the topological localizations $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}^i(\pi)$ are finitely generated over $\Lambda_\ell(N_0)$ in case when π comes as a restriction of a smooth admissible representation of G of finite length. One can pass to usual 1-variable étale (φ, Γ) -modules—still not necessarily finitely generated—over $\mathcal{O}_\mathcal{E}$ via the map $\ell: \Lambda_\ell(N_0) \rightarrow \mathcal{O}_\mathcal{E}$ which step is an equivalence of categories for finitely generated étale (φ, Γ) -modules (Thm. 8.20 in [18]).

More recently, Breuil [3] managed to find a different approach, producing a pseudocompact (ie. projective limit of finitely generated) (φ, Γ) -module $D_\xi^\vee(\pi)$ over $\mathcal{O}_\mathcal{E}$ when π is killed by a power ϖ^h of the uniformizer ϖ . In [3] (and also in [17]) ℓ is a *generic* Whittaker functional, namely ℓ is chosen to be the composite map

$$\ell: N_0 \rightarrow N_0/(N_0 \cap [N, N]) \cong \prod_{\alpha \in \Delta} N_{\alpha,0} \xrightarrow{\sum_{\alpha \in \Delta} u_\alpha^{-1}} \mathbb{Z}_p .$$

Breuil passes right away to the space of H_0 -invariants π^{H_0} of π where H_0 is the kernel of the group homomorphism $\ell: N_0 \rightarrow \mathbb{Z}_p$. By the assumption that π is smooth, the invariant subspace π^{H_0} has the structure of a module over the Iwasawa algebra $\Lambda(N_0/H_0)/\varpi^h \cong \mathcal{O}/\varpi^h[[X]]$. Moreover, it admits a semilinear action of F which is the Hecke action of $s = \xi(p)$: For any $m \in \pi^{H_0}$ we define

$$F(m) = \mathrm{Tr}_{H_0/sH_0s^{-1}}(sm) = \sum_{u \in J(H_0/sH_0s^{-1})} usm .$$

So π^{H_0} is a module over the skew polynomial ring $\Lambda(N_0/H_0)/\varpi^h[F]$ (defined by the identity $FX = (sXs^{-1})F = ((X + 1)^p - 1)F$). We consider those (i) finitely generated $\Lambda(N_0/H_0)/\varpi^h[F]$ -submodules $M \subset \pi^{H_0}$ that are (ii) invariant under the action of Γ and are (iii) *admissible* as a $\Lambda(N_0/H_0)/\varpi^h$ -module, ie. the Pontryagin dual $M^\vee = \mathrm{Hom}_\mathcal{O}(M, \mathcal{O}/\varpi^h)$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h$. Note that this admissibility condition (iii) is equivalent to the usual admissibility condition in smooth representation theory, ie. that

for any (or equivalently for a single) open subgroup $N' \leq N_0/H_0$ the fixed points $M^{N'}$ form a finitely generated module over o . We denote by $\mathcal{M}(\pi^{H_0})$ the—via inclusion partially ordered—set of those submodules $M \leq \pi^{H_0}$ satisfying (i), (ii), (iii). Note that whenever M_1, M_2 are in $\mathcal{M}(\pi^{H_0})$ then so is $M_1 + M_2$. It is shown in [4] (see also [6] and Lemma 2.6 in [3]) that for $M \in \mathcal{M}(\pi^{H_0})$ the localized Pontryagin dual $M^\vee[1/X]$ naturally admits a structure of an étale (φ, Γ) -module over $o/\varpi^h((X))$. Therefore Breuil [3] defines

$$D_\xi^\vee(\pi) = \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} M^\vee[1/X].$$

By construction this is a projective limit of usual (φ, Γ) -modules. Moreover, D_ξ^\vee is right exact and compatible with parabolic induction [3]. It can be characterized by the following universal property: For any (finitely generated) étale (φ, Γ) -module over $o/\varpi^h((X)) \cong o/\varpi^h[[\mathbb{Z}_p]][([1] - 1)^{-1}]$ (here [1] is the image of the topological generator of \mathbb{Z}_p in the Iwasawa algebra $o/\varpi^h[[\mathbb{Z}_p]]$) we may consider continuous $\Lambda(N_0)$ -homomorphisms $\pi^\vee \rightarrow D$ via the map $\ell: N_0 \rightarrow \mathbb{Z}_p$ (in the weak topology of D and the compact topology of π^\vee). These all factor through $(\pi^\vee)_{H_0} \cong (\pi^{H_0})^\vee$. So we may require these maps be ψ_s - and Γ -equivariant where $\Gamma = \xi(\mathbb{Z}_p \setminus \{0\})$ acts naturally on $(\pi^{H_0})^\vee$ and $\psi_s: (\pi^{H_0})^\vee \rightarrow (\pi^{H_0})^\vee$ is the dual of the Hecke-action $F: \pi^{H_0} \rightarrow \pi^{H_0}$ of s on π^{H_0} . Any such continuous ψ_s - and Γ -equivariant map f factors uniquely through $D_\xi^\vee(\pi)$. However, it is not known in general whether $D_\xi^\vee(\pi)$ is nonzero for smooth irreducible representations π of G (restricted to B).

Even more recently Scholze and Grosse-Klönne proposed different methods, which are just mentioned here. For $G = \mathbf{GL}_n(F)$ Scholze ([20]) uses a finiteness result of the p -adic cohomology of the Lubin-Tate tower to get a representation of the Galois group Gal_F , he also gets an additional action of a central division algebra D/F . Grosse-Klönne ([14]) uses the G -equivariant coefficient system on the Bruhat Tits building attached to π with some additional information to construct a functor of this type, which is also exact and for $\mathbf{GL}_2(\mathbb{Q}_p)$ is the same as the classical functor D .

1.4 Summary of results

The thesis is mostly based on the papers [9] and [10].

In chapter 2 we compute D_{SV} for principal series representations of $G = \mathbf{GL}_n(F)$.

In order to that, we need to understand the B_+ -module structure of the principal series. In section 2.2 we decompose G into open N_0 -invariant subsets U_w , indexed by the elements w of Weyl group. The action of B_+ respects this structure in the following sense: if $w, w' \in W$, $y \in U_w$ and $b \in B_+$ such that $b^{-1}y \in U_{w'}$, then $w' \preceq w$ for certain ordering on W .

With the help of this we prove in section 2.3 that there exists a minimal element M_0 in the set of generating B_+ -subrepresentations of π : namely the B_+ -submodules generated by the "characteristic functions" of the sets $U_w w$ for w in W .

Now we have $D_{SV}(\pi) = M_0^\vee$ - the dual of this minimal B_+ -subrepresentation. We do not know whether it is finitely generated or it has rank 1 as a module over the modulo p Iwasawa algebra $\Omega(N_0)$. However, we show that in some sense only a rank 1 quotient of $D_{SV}(\pi)$ is relevant if we want to get an étale (φ, Γ) -module.

In the last section we point out some properties of M_0 , which sheds some light on why the picture for principal series is more difficult compared to the case of subquotients defined by the Bruhat filtration.

In chapter 3 we relate the functors D_{SV} and D_ξ^\vee .

Our first result is the construction of a noncommutative multivariable version of $D_\xi^\vee(\pi)$. Let π be a smooth \mathfrak{o} -torsion representation of B such that $\varpi^h \pi = 0$. The idea here is to take the invariants π^{H_k} for a family of open normal subgroups $H_k \leq H_0$ with $\bigcap_{k \geq 0} H_k = \{1\}$. Now Γ and the quotient group N_0/H_k act on π^{H_k} (we choose H_k so that it is normalized by both Γ and N_0). Further, we have a Hecke-action of s given by $F_k = \text{Tr}_{H_k/sH_k s^{-1}} \circ (s \cdot)$. As in [3] we consider the set $\mathcal{M}_k(\pi^{H_k})$ of finitely generated $\Lambda(N_0/H_k)[F_k]$ -submodules of π^{H_k} that are stable under the action of Γ and admissible as a representation of N_0/H_k . In section 3.1 we show that for any $M_k \in \mathcal{M}_k(\pi^{H_k})$ there is an étale (φ, Γ) -module structure on $M_k^\vee[1/X]$ over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$. So the projective limit

$$D_{\xi, \ell, \infty}^\vee(\pi) = \varprojlim_{k \geq 0} \varprojlim_{M_k \in \mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X]$$

is a pseudocompact étale (φ, Γ) -module over $\Lambda_\ell(N_0)/\varpi^h = \varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X]$. Moreover, we also give a natural isomorphism $D_{\xi, \ell, \infty}^\vee(\pi)_{H_0} \cong D_\xi^\vee(\pi)$ showing that $D_{\xi, \ell, \infty}^\vee(\pi)$ corresponds to $D_\xi^\vee(\pi)$ via (the

projective limit of) the equivalence of categories in Thm. 8.20 in [18]. Moreover, the natural map $\pi^\vee \rightarrow D_\xi^\vee(\pi)$ factors through the projection map $D_{\xi,\ell,\infty}^\vee(\pi) \rightarrow D_\xi^\vee(\pi) = D_{\xi,\ell,\infty}^\vee(\pi)_{H_0}$. Note that this shows that $D_{\xi,\ell,\infty}^\vee(\pi)$ is naturally attached to π —not just simply via the equivalence of categories (loc. cit.)—in the sense that any ψ - and Γ -equivariant map from π^\vee to an étale (φ, Γ) -module over $o/\varpi^h((X))$ factors uniquely through the corresponding multivariable (φ, Γ) -module.

In section 3.2 we develop these ideas further and show that the natural map $\pi^\vee \rightarrow D_{\xi,\ell,\infty}^\vee(\pi)$ factors through the map $\pi^\vee \rightarrow D_{SV}(\pi)$. In fact, we show (Prop. 3.2.4) that $D_{\xi,\ell,\infty}^\vee(\pi)$ has the following universal property: Any continuous ψ_s - and Γ -equivariant map $f: D_{SV} \rightarrow D$ into a finitely generated étale (φ, Γ) -module D over $\Lambda_\ell(N_0)$ factors uniquely through $\text{pr} = \text{pr}_\pi: D_{SV}(\pi) \rightarrow D_{\xi,\ell,\infty}^\vee(\pi)$. The association $\pi \mapsto \text{pr}_\pi$ is a natural transformation between the functors D_{SV} and $D_{\xi,\ell,\infty}^\vee$. One application is that Breuil's functor D_ξ^\vee vanishes on compactly induced representations of B (see Corollary 3.2.3).

In order to be able to compute $D_{\xi,\ell,\infty}^\vee(\pi)$ (hence also $D_\xi^\vee(\pi)$) from $D_{SV}(\pi)$ we introduce the notion of the *étale hull* of a $\Lambda(N_0)$ -module with a ψ -action of T_+ (or of a submonoid $T_* \leq T_+$). Here a $\Lambda(N_0)$ -module D with a ψ -action of T_+ is the analogue of a (ψ, Γ) -module over $o[[X]]$ in this multivariable noncommutative setting. The étale hull \widetilde{D} of D (together with a canonical map $\iota: D \rightarrow \widetilde{D}$) is characterized by the universal property that any ψ -equivariant map $f: D \rightarrow D'$ into an étale T_+ -module D' over $\Lambda(N_0)$ factors uniquely through ι . It can be constructed as a direct limit $\varinjlim_{t \in T_+} \varphi_t^* D$ where $\varphi_t^* D = \Lambda(N_0) \otimes_{\varphi_t, \Lambda(N_0)} D$ (Prop. 3.3.4). We show (Thm. 3.3.9 and the remark thereafter) that the pseudocompact completion of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}(\pi)}$ is canonically isomorphic to $D_{\xi,\ell,\infty}^\vee(\pi)$ as they have the same universal property.

In order to go back to representations of G we need an étale action of T_+ on $D_{\xi,\ell,\infty}^\vee(\pi)$, not just of $\xi(\mathbb{Z}_p \setminus \{0\})$. This is only possible if $tH_0t^{-1} \leq H_0$ for all $t \in T_+$ which is not the case for generic ℓ . So in the last chapter we equip $D_{\xi,\ell,\infty}^\vee(\pi)$ with an étale action of T_+ (extending that of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$) in case $\ell = \ell_\alpha$ is the projection of N_0 onto a root subgroup $N_{\alpha,0} \cong \mathbb{Z}_p$ for some simple root α in Δ . Moreover, we show (Prop. 4.1.5) that the map $\text{pr}: D_{SV}(\pi) \rightarrow D_{\xi,\ell,\infty}^\vee(\pi)$ is ψ -equivariant for this extended action, too. Note that $D_{\xi,\ell,\infty}^\vee(\pi)$ may not be the projective limit of finitely generated étale

T_+ -modules over $\Lambda_\ell(N_0)$ as we do not necessarily have an action of T_+ on $M_\infty^\vee[1/X]$ for $M \in \mathcal{M}(\pi^{H_0})$, only on the projective limit.

Let $P \leq G$ be a parabolic subgroup with Levi decomposition $P = L_P N_P$. We show in section 4.2 that the compatibility with parabolic induction [3] Theorem 6.1 goes through in this situation:

$$D_\xi^\vee(\text{Ind}_{P^-}^G \pi_P) \cong \begin{cases} D_\xi^\vee(\pi_P) & \text{if } N_\alpha \subseteq L_P \\ o/\varpi^h((X)) \widehat{\otimes}_{o/\varpi^h} \text{Ord}_{s^{\mathbb{Z}} N_{L_P}}(\pi_P)^\vee & \text{if } N_\alpha \subseteq N_P \end{cases},$$

where Ord is the ordinary part similar to the definition of Emerton (cf Definition 3.1.9 in [7]).

We present the results of section 4 in [10], where a G -equivariant sheaf \mathfrak{Y} on G/B is attached to $D_{\xi, \ell, \infty}^\vee(\pi)$ and a natural transformation $\beta_{G/B}$ from $(\cdot)^\vee$ to $\pi \rightarrow \mathfrak{Y}$ is constructed, which is compatible with a reverse functor.

In section 4.4 we show some disadvantages of the choice $\ell = \ell_\alpha$: D_ξ^\vee vanishes for the twist of a modulo p supercuspidal representation $\pi^{(2)}$ of $\mathbf{GL}_2(\mathbb{Q}_p)$ by a character χ . Moreover D_ξ^\vee is not exact even for extensions of principal series $\pi_P = \pi^{(2)} \otimes \chi$.

The mostly folklore computation with (φ, Γ) -modules which is needed for the latter result is carried out in section 4.5.

1.5 Notations

Let $F, K \leq \overline{\mathbb{Q}_p}$ finite extensions of \mathbb{Q}_p . Let o_F , respectively o_K be the rings of integers in F , respectively in K , $\varpi_F \in o_F$ and $\varpi_K \in o_K$ be the uniformizers, ν_F and ν_K be the standard valuations and $k_F = o_F/\varpi_F o_F$, $k_K = o_K/\varpi_K o_K$ be the residue fields.

Let $G = \mathbf{G}(F)$ be the F -points of a F -split connected reductive group \mathbf{G} defined over \mathbb{Z}_p with connected centre and a fixed split Borel subgroup $\mathbf{B} = \mathbf{TN}$. Put $B = \mathbf{B}(F)$, $T = \mathbf{T}(F)$, and $N = \mathbf{N}(F)$. We denote by Φ_+ the set of roots of T in N , by $\Delta \subset \Phi_+$ the set of simple roots, and by $u_\alpha : \mathbb{G}_a \rightarrow N_\alpha$, for $\alpha \in \Phi_+$, a F -homomorphism onto the root subgroup N_α of N such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in F$ and $t \in \mathbf{T}(\mathbb{Q}_p)$, and $N_0 = \prod_{\alpha \in \Phi_+} u_\alpha(o_F)$ is a subgroup of N . We put $N_{\alpha,0} = u_\alpha(o_F)$ for the image of u_α on o_F .

Let $W = N_G(T)/Z_G(T)$ denote the Weyl group of G and \prec denote the strong Bruhat ordering of W (see [15] II. 13.7): we say $w' \prec w$ for $w \neq w' \in W$ if there exist transpositions $w_1, w_2, \dots, w_i \in W$ such that $w' = ww_1w_2 \dots w_i$ and $l(w) > l(ww_1) > l(ww_1w_2) > \dots > l(ww_1w_2 \dots w_i)$.

We denote by T_+ the monoid of dominant elements t in $\mathbf{T}(\mathbb{Q}_p)$ such that $\nu_F(\alpha(t)) \geq 0$ for all $\alpha \in \Phi_+$, by $T_0 \subset T_+$ the maximal subgroup, by T_{++} the subset of strictly dominant elements, i.e. $\nu_F(\alpha(t)) > 0$ for all $\alpha \in \Phi_+$, and we put $B_+ = N_0T_+, B_0 = N_0T_0$. The natural conjugation action of T_+ on N_0 extends to an action on the Iwasawa \mathfrak{o}_K -algebra $\Lambda(N_0) = \mathfrak{o}_K[[N_0]]$. For $t \in T_+$ we denote this action of t on $\Lambda(N_0)$ by φ_t . The map $\varphi_t: \Lambda(N_0) \rightarrow \Lambda(N_0)$ is an injective ring homomorphism with a distinguished left inverse $\psi_t: \Lambda(N_0) \rightarrow \Lambda(N_0)$ satisfying $\psi_t \circ \varphi_t = \text{id}_{\Lambda(N_0)}$ and $\psi_t(u\varphi_t(\lambda)) = \psi_t(\varphi_t(\lambda)u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ and $\lambda \in \Lambda(N_0)$.

Each simple root α gives a F -homomorphism $x_\alpha: N \rightarrow \mathbb{G}_a$ with section u_α . We denote by $\ell_\alpha: N_0 \rightarrow F \xrightarrow{\text{Tr}_{F/\mathbb{Q}_p}} \mathbb{Z}_p$, resp. $\iota_\alpha: \mathfrak{o}_F \rightarrow N_0$, the restriction of x_α , resp. u_α , to N_0 , resp. \mathfrak{o}_F .

Since the centre of G is assumed to be connected, there exists a cocharacter $\xi: F^* \rightarrow T$ such that $\alpha \circ \xi$ is the identity on F^* for each $\alpha \in \Delta$. If $F = \mathbb{Q}_p$ we put $\Gamma = \xi(\mathbb{Z}_p^*) \leq T$ and often denote the action of $s = \xi(p)$ by $\varphi = \varphi_s$.

For an \mathfrak{o}_K -representation π let $\pi^\vee = \text{Hom}_{\mathfrak{o}_K}(\pi, K/\mathfrak{o}_K)$ be the Pontryagin dual of π . Pontryagin duality sets up an anti-equivalence between the category of torsion \mathfrak{o}_K -modules and the category of all compact linear-topological \mathfrak{o}_K -modules.

By a smooth \mathfrak{o}_K -torsion representation of G (resp. of $B = \mathbf{B}(F)$) we mean a torsion \mathfrak{o}_K -module π together with a smooth (ie. stabilizers are open) and linear action of the group G (resp. of B). π is admissible if for any $U \leq G$ open subgroup, the vector space $k_K \otimes_{\mathfrak{o}_K} \pi^U$ is finite dimensional.

For example, if $\mathbf{G} = \mathbf{GL}_n$ and $F = \mathbb{Q}_p$, B is the subgroup of upper triangular matrices, N consists of the strictly upper triangular matrices (1 on the diagonal), T is the diagonal subgroup, $N_0 = \mathbf{N}(\mathbb{Z}_p)$, the simple roots are $\alpha_1, \dots, \alpha_{n-1}$ where $\alpha_i(\text{diag}(t_1, \dots, t_n)) = t_i t_{i+1}^{-1}$, x_{α_i} sends a matrix to its $(i, i+1)$ -coefficient, $u_{\alpha_i}(\cdot)$ is the strictly upper triangular matrix, with $(i, i+1)$ -coefficient \cdot and 0 everywhere else.

Let $C^\infty(G)$ (respectively $C_c^\infty(G)$) denote the set of locally constant $G \rightarrow k_K$ functions (respectively locally constant functions with compact support), with the group G acting by left multiplication ($gf : x \mapsto f(g^{-1}x)$ for $f \in C^\infty(G)$ and $g, x \in G$).

Let $G_0 \leq G$ be a compact open subgroup and $\Lambda(G_0)$ denote the completed group ring of the profinite group G_0 over o_K . Any smooth o_K -representation π is the union of its finite G_0 -subrepresentations, therefore π^\vee is a left $\Lambda(G_0)$ -module (through the inversion map on G_0).

Let $\Omega(G_0) = \Lambda(G_0)/\varpi_K \Lambda(G_0)$. $\Omega(N_0)$ is noetherian and has no zero divisors, so it has a fraction (skew) field. If M is a $\Omega(N_0)$ -module, by the rank of M we mean $\dim_{k_K}(\text{Frac}(\Omega(N_0)) \otimes_{\Omega(N_0)} M)$.

Let $\ell : N_0 \rightarrow \mathbb{Z}_p$ (for now) any surjective group homomorphism and denote by $H_0 \triangleleft N_0$ the kernel of ℓ . The ring $\Lambda_\ell(N_0)$, denoted by $\Lambda_{H_0}(N_0)$ in [17], is a generalisation of the ring $\mathcal{O}_\mathcal{E}$, which corresponds to $\Lambda_{\text{id}}(N_0^{(2)})$ where $N_0^{(2)}$ is the \mathbb{Z}_p -points of the unipotent radical of a split Borel subgroup in \mathbf{GL}_2 . We refer the reader to [17] for the proofs of some of the following claims.

The maximal ideal $\mathcal{M}(H_0)$ of the completed group o_K -algebra $\Lambda(H_0) = o_K[[H_0]]$ is generated by ϖ_k and by the kernel of the augmentation map $o_K[[H_0]] \rightarrow o_K$.

The ring $\Lambda_\ell(N_0)$ is the $\mathcal{M}(H_0)$ -adic completion of the localization of $\Lambda(N_0)$ with respect to the Ore subset $S_\ell(N_0)$ of elements which are not in the ideal $\mathcal{M}(H_0)\Lambda(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(H_0)[[X]]$ of skew Taylor series over $\Lambda(H_0)$ in the variable $X = [u] - 1$ where $u \in N_0$ and $\ell(u)$ is a topological generator of $\ell(N_0) = \mathbb{Z}_p$. Then $\Lambda_\ell(N_0)$ is viewed as the ring of infinite skew Laurent series $\sum_{n \in \mathbb{Z}} a_n X^n$ over $\Lambda(H_0)$ in the variable X with $\lim_{n \rightarrow -\infty} a_n = 0$ for the compact topology of $\Lambda(H_0)$. For a different characterization of this ring in terms of a projective limit $\Lambda_\ell(N_0) \cong \varprojlim_{n,k} \Lambda(N_0/H_k)[1/X]/\varpi_K^n$ for $H_k \triangleleft N_0$ normal subgroups contained and open in H_0 satisfying $\bigcap_{k \geq 0} H_k = \{1\}$ see also [23].

For a finite index subgroup \mathcal{G}_2 in a group \mathcal{G}_1 we denote by $J(\mathcal{G}_1/\mathcal{G}_2) \subset \mathcal{G}_1$ a (fixed) set of representatives of the left cosets in $\mathcal{G}_1/\mathcal{G}_2$.

Chapter 2

The Schneider-Vigneras functor for principal series

2.1 Principal series

In this chapter fix $n \in \mathbb{N}$, and let $G = \mathbf{GL}_n(\mathbf{F})$, and $G_0 = \mathbf{GL}_n(o_F)$.

Let B be the set of upper triangular matrices in G , T the set of diagonal matrices, N the set of upper triangular unipotent matrices. Let N^- be the lower unipotent matrices - the opposite of N - and $N_0 = N \cap G_0$ - a totally decomposed compact open subgroup of N - those matrices which has coefficients in o_F .

By the abuse of notation let $w \in W$ denote also the permutation matrices - representatives of W in G (with $w_{ij} = 1$ if $w(j) = i$, and $w_{ij} = 0$ otherwise), and also the corresponding permutation of the set $\{1, 2, \dots, n\}$. For $w \in W$ denote length of w —the length of the shortest word representing w in the terms of the standard generators of W —by $l(w)$.

Let the kernel of the projection $pr : G_0 \rightarrow \mathbf{GL}_n(k_F)$ be $U^{(1)}$. This is a compact open pro- p normal subgroup of G_0 . We have $G = G_0B$ and $U^{(1)} \subset (N^- \cap U^{(1)})B$.

Let

$$\chi = \chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n : T \rightarrow k_K^*$$

be a locally constant character of T with $\chi_i : F^* \rightarrow k_K^*$ multiplicative. Note that for all i we have $\chi_i(1 + \pi_F o_F) = 1$ and $\chi_i(o_F^*) \subset k_F^* \cap k_K^* \leq \overline{\mathbb{F}_p}^*$. Since $T \simeq B/[B, B]$, also denote the corresponding $B \rightarrow k_K^*$ character by χ . Let

$$\pi = \text{Ind}_B^G(\chi) = \{f \in C^\infty(G) \mid \forall g \in G, b \in B : f(gb) = \chi^{-1}(b)f(g)\}$$

π is called a principal series representation of G . π is irreducible exactly when for all i we have $\chi_i \neq \chi_{i+1}$ ([16], theorem 4). For any open right B -invariant subset $X \subset G$ we write $\text{Ind}_B^X = \{f \in \text{Ind}_B^G(\chi) | f|_{G \setminus X} \equiv 0\}$.

We can understand the structure of π better (see [21], section 4.), by the Bruhat decomposition $G = \bigcup_{w \in W} BwB$. Fix a total ordering \prec_T refining the Bruhat ordering \prec of W , and let

$$w_1 = \text{id}_W \prec_T w_2 \prec_T w_3 \prec_T \cdots \prec_T w_n = w_0.$$

Let us denote by $G_m = \bigcup_{1 \leq l \leq m} Bw_l B$ - a closed subset of G . We obtain a descending B -invariant filtration of π by

$$\pi_m = \text{Ind}_B^{G \setminus G_m}(\chi) = \{F \in \text{Ind}_B^G(\chi) | F|_{G_m} \equiv 0\} \quad (0 < m \leq n!),$$

with quotients π_{m-1}/π_m via $f \mapsto f(\cdot w_m)$ isomorphic to $\pi(w_m, \chi) = C_c^\infty(N/N'_{w_m})$ (see [17], section 12), where $N'_{w_m} = N \cap w_m N w_m^{-1}$, with N acting by left translations and T acting via

$$(t\phi)(n) = \chi(w_m^{-1} t w_m) \phi(t^{-1} n t).$$

For any $w \in W$ put

$$N_w = \{n \in N | \forall i < j, w^{-1}(i) < w^{-1}(j) : n_{ij} = 0\} = N \cap w N^- w^{-1} \leq N,$$

and $N_{0,w} = N_0 \cap N_w$. Then we have the following form of the Bruhat decomposition $G = \coprod_{w \in W} N_w w B$.

2.2 The action of B_+ on G

The first goal is to partition G to N_0 -invariant open subsets $\{U_w | w \in W\}$ indexed by the Weyl-group, which are respected by the B_+ -action in the sense that if $x \in U_w$ $b \in B_+$ then there exists $w' \preceq w$ in W such that $b^{-1}x \in U_{w'}$.

Definition Let for any $w \in W$ $r_w : N^- \cap G_0 \rightarrow \mathbf{G}(k_F), n^- \mapsto pr(w n^- w^{-1})$, $R_w = w r_w^{-1}(\mathbf{N}(k_F))$, $R = \cup_{w \in W} R_w$.

We have that

$$R_w = \left\{ (a_{ij}) \in G | \forall i, j : a_{ij} \begin{cases} = 1, & \text{if } w^{-1}(i) = j \\ = 0, & \text{if } w^{-1}(i) < j \\ \in o_F, & \text{if } w^{-1}(i) > j \text{ and } w(j) > i \\ \in \varpi_F o_F, & \text{if } w^{-1}(i) > j \text{ and } w(j) < i \end{cases} \right\}$$

For $n = 3$ in details (with $o = o_F$ and $\varpi = \varpi_F$):

w	R_w		w	R_w
$\text{id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \varpi o & 1 & 0 \\ \varpi o & \varpi o & 1 \end{pmatrix}$	(23) =	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \varpi o & o & 1 \\ \varpi o & 1 & 0 \end{pmatrix}$
(12) = $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} o & 1 & 0 \\ 1 & 0 & 0 \\ \varpi o & \varpi o & 1 \end{pmatrix}$	(123) =	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} o & o & 1 \\ 1 & 0 & 0 \\ \varpi o & 1 & 0 \end{pmatrix}$
(132) = $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} o & 1 & 0 \\ o & \varpi o & 1 \\ 1 & 0 & 0 \end{pmatrix}$	(13) =	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} o & o & 1 \\ o & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Let $\mathbf{N}(k_F)$ be the k_F -points of \mathbf{N} (the upper triangular unipotent matrices with coefficients in k_F). k_F has canonical (multiplicative) injection to $o_F \subset F$, hence any subgroup $\mathbf{H}(k_F) \leq \mathbf{N}(k_F)$ is mapped injectively to N_0 (however this is not a group homomorphism). We denote this subset of N_0 by $\widetilde{\mathbf{H}(k_F)}$.

Proposition 2.2.1 *A set of double coset representatives of $U^{(1)} \backslash G/B$ is $\cup_{w \in W} \widetilde{\mathbf{N}_w(k_F)} w$. Every element of G can be written uniquely in the form rb with $r \in R$ and $b \in B$.*

Proof By the Bruhat decomposition of $\mathbf{G}(k_F)$ a set of double coset representatives of $U^{(1)} \backslash G_0/(B \cap G_0)$ is the set as above. Since $G = G_0 B$, we have the first part of proposition.

Let $g = unwb \in G$ with $u \in U^{(1)}$, $w \in W$, $n \in \widetilde{\mathbf{N}_w(k_F)}$ and $b \in B$. Then $g = w(w^{-1}nw)u'b$ with $u' = w^{-1}n^{-1}unw \in U^{(1)}$. But then there exist $n' \in N^- \cap U^{(1)}$ and $b' \in B$ such that $u' = n'b'$. Then $g = w(w^{-1}nwn')(b'b)$, where $w^{-1}nwn' \in r_w^{-1}(\mathbf{N}(k_F))$ because of the definition of N_w .

For any $w \in W$ we clearly have $U^{(1)} \widetilde{\mathbf{N}_w(k_F)} w B = R_w B$. Hence the uniqueness follows: if $rb = r'b'$ then there exists $w \in W$ such that $r, r' \in R_w$ and $b'b^{-1} = (r'^{-1}w^{-1})(wr) \in B \cap N^- = \{\text{id}\}$. \square

Definition For any $w \in W$ let $U_w = U^{(1)} \widetilde{\mathbf{N}_w(k_F)} w B$. This way we partitioned G into open subsets indexed by the Weyl group. We obviously have $U_w = R_w B$.

Corollary 2.2.2 *For any $w \in W$ we have that U_w is (left) N_0 -invariant.*

Proof Let $n' \in N_0$ and $x = unwb \in U^{(1)}\widetilde{\mathbf{N}_w(k_F)}wB$. We have $N_0 = N_{0,w}(N'_w \cap N_0)$, thus $n'n = mm'$ for some $m \in N_{0,w}$ and $m' \in N'_w \cap N_0$, moreover we can write $m = m_1m_0 \in (N_w \cap U^{(1)})\widetilde{\mathbf{N}_w(k_F)}$. By the definition of N'_w

$$n'x = (n'un'^{-1}m_1)m_0w(w^{-1}m'wb) \in U^{(1)}\widetilde{\mathbf{N}_w(k_F)}wB,$$

meaning that U_w is N_0 -invariant. \square

Proposition 2.2.3 *Let $y \in U_w = R_wB$, $nt \in B_+ = N_0T_+$, and $x = t^{-1}n^{-1}y \in U_{w'} = R_{w'}B$. Then $w' \preceq w$.*

Proof Let $y = rb$ with $r \in R_w$ and $b \in B$. By the previous proposition we may assume that $n = \text{id}$. If $t = \text{diag}(t_1, t_2, \dots, t_n) \in G_0$, then

$$x = w(w^{-1}t^{-1}w(w^{-1}r)w^{-1}tw)(w^{-1}t^{-1}wb),$$

where $w^{-1}t^{-1}w(w^{-1}r)w^{-1}tw \in r_w^{-1}(\mathbf{N}(k_F))$, because it is in N^- and the coefficients under the diagonal have the same valuation as those in $w^{-1}r$. T_+ as a monoid is generated by $T \cap G_0$, the center $Z(G)$ and the elements with the form $(\varpi_F, \varpi_F, \dots, \varpi_F, 1, 1, \dots, 1)$, hence it is enough to prove the proposition for such t -s.

So fix $t = (t_1 = \varpi_F, t_2 = \varpi_F, \dots, t_l = \varpi_F, t_{l+1} = 1, t_{l+2} = 1, \dots, t_n = 1)$, $r = (r_{ij})$ and try to write x in the form as in Proposition 2.2.1. For all $j = 0, 1, 2, \dots, n$ we construct inductively a decomposition $x = (t^{(j)})^{-1}r^{(j)}b^{(j)}$ together with $w^{(j)} \in W$, where

- $w^{(j+1)} \preceq w^{(j)}$ for $j < n$ and such that the first j columns of $w^{(j)}$ are the same as the first j columns of $w^{(j+1)}$,
- $t^{(j)} = \text{diag}(t_i^{(j)}) \in T$ with

$$t_i^{(j)} = \begin{cases} 1, & \text{if } (w^{(j)})^{-1}(i) \leq j \\ t_i, & \text{if } (w^{(j)})^{-1}(i) > j \end{cases},$$

- $r^{(j)} \in R_{w^{(j)}}$, and if we change the first j columns of $r^{(j)}$ to the first j columns of $(t^{(j)})^{-1}r^{(j)}$ it is still in $R_{w^{(j)}}$ (by de definition of $t^{(j)}$ it is enough to verify the condition for $(t^{(j)})^{-1}r^{(j)}$),
- $b^{(j)} \in B$.

Then $w^{(n)} \preceq w^{(n-1)} \preceq w^{(n-2)} \preceq \dots \preceq w^{(1)} = w$. However for $j = n$ we have $t^{(n)} = \text{id}$, hence $w^{(n)} = w'$ by disjointness of the sets $R_v B$ for $v \in W$, so we have the proposition.

For $j = 0$ we have $t^{(0)} = t, r^{(0)} = r, b^{(0)} = b$ and $w^{(0)} = w$. From j to $j + 1$:

- If $w^{(j)}(j + 1) \leq l$, then let $w^{(j+1)} = w^{(j)}$, so $t^{(j+1)} = e_{w^{(j)}(j+1)}^{-1} t^{(j)}$, where for $1 \leq k \leq n$ we denote $e_k = e_k(\varpi_F)$ the diagonal matrix with ϖ_F in the k -th row and 1 everywhere else. We can choose $r^{(j+1)} = e_{w^{(j)}(j+1)}^{-1} r^{(j)} e_{j+1}$, and $b^{(j+1)} = e_{j+1}^{-1} b^{(j)}$.

Then the first j columns of $(t^{(j+1)})^{-1} r^{(j+1)}$ are equal of those of $(t^{(j)})^{-1} r^{(j)}$, and the entries at place $(i, j + 1)$ with $i \neq w^{(j+1)}(j + 1)$ are multiplied by ϖ_F . Because of the conditions for $r^{(j)}$, this is in $R_{w^{(j+1)}}$. The other conditions for $w^{(j+1)}, t^{(j+1)}, r^{(j+1)}$ and $b^{(j+1)}$ obviously hold.

- If $w^{(j)}(j + 1) > l$ and if $\nu_F(r_{i,j+1}^{(j)}) \geq 1$ for all $i \leq l$, then it suffices to choose $w^{(j+1)} = w^{(j)}, t^{(j+1)} = t^{(j)}, r^{(j+1)} = r^{(j)}$ and $b^{(j+1)} = b^{(j)}$.
- Assume that $w^{(j)}(j + 1) > l$ and that there exists $i \leq l$ such that $\nu_F(r_{i,j+1}^{(j)}) = 0$. Let i_0 be the maximal such i . Then choose $w^{(j+1)}(j + 1) = i_0$, and $t^{(j+1)} = e_{i_0}^{-1} t^{(j)}$.

Let $r' = e_{i_0}^{-1} r^{(j)} e_{j+1} ((r_{i_0,j+1}^{(j)})^{-1} \cdot \varpi)$, where $e_j(\alpha)$ is the diagonal matrix with $\alpha \in F$ in the j -th row and 1 everywhere else. Note that $r'_{i_0,j+1} = 1$ and r' differs from $r^{(j)}$ only in the i_0 -th row and the $j+1$ -st column. But $(t^{(j+1)})^{-1} r'$ is not in $\mathbf{GL}_n(o_F)$ - for example $\nu_F(r'_{i_0,(w^{(j)})^{-1}(i_0)}) = -1$, and there might be some other elements of r' in the i_0 -th row and columns between the $j + 2$ -nd and $j' = (w^{(j)})^{-1}(i_0)$ -th.

To see this note first that $w^{(j)}(j + 1) > l \geq i_0$, so $(w^{(j)})^{-1}(i_0) \neq j + 1$. In particular the right multiplication with e_{j+1} does not change the entry at place $(i_0, (w^{(j)})^{-1}(i_0))$. Since $r^{(j)} \in R_{w^{(j)}}$, the defining conditions of $R_{w^{(j)}}$ and that $(w^{(j)})^{-1}(i_0) \neq j + 1$ imply $(w^{(j)})^{-1}(i_0) > j + 1$. Thus $(t_{i_0}^{(j)})^{-1} = (t_{i_0})^{-1} = \varpi_F^{-1}$, since $i_0 \leq l$. By the definition of $R_{w^{(j)}}$ we have $r_{i_0,(w^{(j)})^{-1}(i_0)}^{(j)} = 1$. Therefore $r'_{i_0,(w^{(j)})^{-1}(i_0)} = \varpi_F^{-1}$ which has valuation -1 .

But note, that in the $j + 1$ -st column of r' the i_0 -th element is 1, all the other has valuation at least 1. Thus the first $j + 1$ columns of $(t^{(j+1)})^{-1} r'$

satisfy the condition for the first $j + 1$ columns of $(t^{(j+1)})^{-1}r^{(j+1)}$ - this is meaningful, because we already fixed the first $j + 1$ columns of $w^{(j+1)}$. So we want to find $r^{(j+1)} = r'b'$ with $b' \in B$ such that the first $j + 1$ columns of b' is those of the identity matrix, and $(t^{(j+1)})^{-1}r^{(j+1)} \in R_{w^{(j+1)}}$ for some $w^{(j)} \preceq w^{(j+1)}$.

Let $j_0 = j + 1$, and if $j_i < j'$ then

$$j_{i+1} = \min\{h \mid j + 1 < h, r'_{i_0, h} \notin o_F, w^{(j)}(j_i) > w^{(j)}(h)\}.$$

We claim that the set on the right hand side contains j' if $j_i < j'$. We prove it by induction on i . For $i = 0$ we already verified it. Assume by contradiction that $w^{(j)}(j_i) < i_0 = w^{(j)}(j')$. Since $j' > j_i$ we get $r'_{i_0, j_i} \in \varpi_F o_F$, because $r^{(j)} \in R_{w^{(j)}}$. But then $r'_{i_0, j_i} \in o_F$, because $r' \in e_{i_0}^{-1}r^{(j)} \cdot \text{Mat}(o_F)$, contradicting the defining conditions of j_i . Thus we have $w^{(j)}(j_i) \geq i_0 = w^{(j)}(j')$.

Let s be minimal such that $j_s = j'$ and set $j_{s+1} = n + 1$. We claim that $r^{(j+1)}$ will be in $R_{w^{(j+1)}}$ with $w^{(j+1)} = w^{(j)}(j_{s-1}, j_s)(j_{s-2}, j_{s-1}) \cdots (j_0, j_1)$. Then the condition $w^{(j+1)} \prec w^{(j)}$ holds, because the multiplication from right with each transposition (j_i, j_{i+1}) decreases the inversion number and the length respectively, by the definition of j_{i+1} .

For the existence of a $b' \in B$ such that $r'b' \in R_{w^{(j+1)}}$ we prove the following statements inductively:

Lemma 2.2.4 *For all $j + 1 \leq k \leq n$ there exist*

- $b^{(k)} \in B$ such that the first k column of $r^{(k)} = r'b^{(k)}$ satisfy the defining condition for the first k column in $R_{w^{(j+1)}}$, and if we have $k < n$ then $r^{(k)}$ and $r^{(k+1)}$ differ only in the $k + 1$ -st column.
- a linear combination $s^{(k)}$ of the columns $j + 1, j + 2, \dots, k$ in $r^{(k)}$ for which we have

$$s_i^{(k)} = \begin{cases} 1, & \text{if } i = i_0 \\ 0, & \text{if } (w^{(j+1)})^{-1}(i) \leq k, \text{ and } i \neq i_0 \\ \varpi_F x, & \text{for some } x \in o_F \text{ otherwise} \end{cases}$$

and the maximal i such that $\nu_F(s_i^{(k)}) = 1$ is $w^{(j)}(j_{i'})$, where i' is so, that $j_{i'} \leq k < j_{i'+1}$.

Proof This holds for $k = j + 1$ with $b^{(j+1)} = \text{id}$, $r^{(j+1)} = r'$ and $s^{(j+1)}$ the $j + 1$ -st column of r' . To verify the condition for $s^{(j+1)}$ note that $r'_{(w^{(j)}(j+1), j+1)} = \varpi_F$ and if $i > j + 1$, then by the definition of $R_{w^{(j)}}$ we have that $r'_{i, j+1}$ has valuation at least 1 and $r'_{(i, j+1)} = \varpi_F (r'_{i_0, j+1})^{-1} r'_{i, j+1}$ has valuation at least 2.

Assume that we have $r^{(k)}$, $b^{(k)}$ and $s^{(k)}$. Let i' be so that $j_{i'} \leq k < j_{i'+1}$ and s' be the $k + 1$ -st column of $r^{(k)}$ (which is equal with the $k + 1$ -st column of r' , thus for $i \neq i_0$ we have $s'_i = r'_{i, k+1}$) and $s'' = s' - r'_{(i_0, k+1)} s^{(k)}$. Then by the conditions on s' we can change the $k + 1$ -st column of $r^{(k)}$ to s'' with multiplication from right by an element $b'' \in B$. Moreover $s''_{i_0} = 0$, and the element in s'' with minimal valuation and biggest row index is the $w^{(j+1)}(k + 1)$ -st:

- If $\nu_F(r'_{(i_0, k+1)}) \geq 0$ then for $i \neq i_0$ we have $s'_i \equiv s''_i = s'_i - r'_{(i_0, k+1)} s^{(k)} \pmod{\varpi_F}$, hence the element with minimal valuation is in the row $w^{(j+1)}(k + 1) = w^{(j)}(k + 1)$ (because $r^{(j)} \in R_{w^{(j)}}$ and $j_{i'+1} \neq k + 1$).
- If $\nu_F(r'_{(i_0, k+1)}) < 0$ then it is -1 and for $i \neq i_0$ we have $s''_i = r'_{(i, k+1)} - r'_{(i_0, k+1)} \cdot s^{(k)}$. Where on the right hand side the first term has positive valuation for $i > w^{(j)}(k + 1)$ and 0 valuation for $i = w^{(j)}(k + 1)$ (because $r^{(j)} \in R_{w^{(j)}}$), and the second has valuation $0 = -1 + 1$ for $i = w^{(j)}(j_{i'})$ and at least 1 for $i > w^{(j)}(j_{i'})$ (by the induction hypothesis on $s^{(k)}$). Moreover $j_{i'} \neq k + 1$, because $j_{i'} \leq k$, hence $w^{(j)}(j_{i'}) \neq w^{(j)}(k + 1)$.
If $w^{(j)}(j_{i'}) < w^{(j)}(k + 1)$ then $j_{i'+1} \neq k + 1$ and $w^{(j)}(k + 1) = w^{(j+1)}(k + 1)$. If $w^{(j)}(j_{i'}) > w^{(j)}(k + 1)$ then $j_{i'+1} = k + 1$ and $w^{(j+1)}(k + 1) = w^{(j+1)}(j_{i'+1}) = w^{(j)}(j_{i'})$.

By multiplying this column with $(s''_{w^{(j+1)}(k+1)})^{-1}$ we get the element $r^{(k+1)}$ (we also have to multiply the $k + 1$ -st row of b'' with $s''_{w^{(j+1)}(k+1)}$, this is $b^{(k+1)}$). This satisfies the condition for the $k + 1$ -st row of $R_{w^{(j+1)}}$ because the defining conditions for $r^{(j)} \in R_{w^{(j)}}$, $s^{(k)}$ and the equality

$$\{i | (w^{(j+1)})^{-1}(i) < k + 1\} = \{i | (w^{(j)})^{-1}(i) < k + 1\} \setminus \{w^{(j)}(j_{i'})\} \cup \{i_0\}.$$

The last thing to verify is the existence of an appropriate linear combination $s^{(k+1)}$. Let $s^{(k+1)} = s^{(k)} - s_{w^{(j+1)}(k+1)}^{(k)} (s''_{w^{(j+1)}(k+1)})^{-1} \cdot s''$. Since

$\nu_F(s_{w^{(j+1)}(k+1)}^{(k)}) > 0$, we have $\nu_F(s_i^{(k+1)}) > 0$ if $i \neq i_0$, and by the previous argument also $s_{w^{(j+1)}(j')}^{(k+1)} = 0$ for $j' \leq k+1$ and $j' \neq j+1$.

If $w^{(j+1)}(k+1) > w^{(j)}(j')$, then $s_{w^{(j+1)}(k+1)}^{(k)} > 1$ and $s^{(k+1)} \equiv s^{(k)} \pmod{\varpi_F^2}$. If $w^{(j+1)}(k+1) < w^{(j)}(j')$ then by the definition of $R_{w^{(j+1)}}$ for all $i > w^{(j+1)}(k+1)$ we have $\nu(s_i'') > 1$ and again $s_i^{(k+1)} \equiv s_i^{(k)} \pmod{\varpi_F^2}$. If $w^{(j+1)}(k+1) = w^{(j)}(j')$, then by the definition of $R_{w^{(j)}}$ we have $s'_{w^{(j)}(j')} = r'_{(w^{(j)}(j'), k+1)} = 0$, $s''_{w^{(j+1)}(k+1)} = 0 - r'_{(i_0, k+1)} s_{w^{(j)}(j')}^{(k)}$ and $s^{(k+1)} =$

$$= s^{(k)} - s_{w^{(j)}(j')}^{(k)} (-r'_{(i_0, k+1)} s_{w^{(j)}(j')}^{(k)})^{-1} \cdot (s' - r'_{(i_0, k+1)} s^{(k)}) = (r'_{(i_0, k+1)})^{-1} s',$$

which satisfies the condition because s' is the $j'_{+1} = k+1$ -st column of $r'^{(k)}$ and because of the definition of $R_{w^{(j)}}$. \square

To finish the proof we set $b' = b'^{(n)}$, $r^{(j+1)} = r' b'^{(n)} \in R_{w^{(j+1)}}$ and $b^{(j+1)} = (b'^{(n)})^{-1} (r_{i_0, j+1}^{(j)} \cdot e_{j+1}^{-1}) b^{(j)} \in B$.

\square

Corollary 2.2.5 *For any $w \in W$ we have $BwB = N_w wB \subset \cup_{w' \preceq w} U_{w'}$. In particular for any $0 < m_0 \leq n!$ we have that*

$$\bigcup_{m \geq m_0} U_{w_m} \subset G \setminus G_{m_0-1} = \bigcup_{m \geq m_0} Bw_m B.$$

Proof Let $x = n_w w b \in N_w wB$. Then there exists $t \in T_+$ such that $n' = t n_w t^{-1} \in N_0$. Thus $x = t^{-1} n' w (w^{-1} t w) b = t^{-1} n' w b''$ with $b'' \in B$. By the previous proposition for $w = w \cdot \text{id} \in R_w B$ and $(n')^{-1} t \in B_+$, there exist $w' \prec w$, $r_{w'} \in R_{w'}$ and $b' \in B$ such that $t^{-1} n' w = r_{w'} b'$, hence $x = r_{w'} (b' b'') \in U_{w'}$. The second assertion follows from that:

$$\bigcup_{m \geq m_0} U_{w_m} = G \setminus \bigcup_{1 \leq m < m_0} U_{w_m} \subset G \setminus \bigcup_{1 \leq m < m_0} Bw_m B = G \setminus G_{m_0-1}.$$

\square

Remark We can achieve the results of this section not only for \mathbf{GL}_n , but different groups: let $G' = \mathbf{G}'(F)$ be such that

- G' is isomorphic to a closed subgroup in G which we also denote by G' ,
- In G' a maximal torus is $T' = T \cap G'$, a Borel subgroup $B' = B \cap G'$ with unipotent radical $N' = N \cap G'$, such that $N_{G'}(T') = N_G(T) \cap G'$ and hence $W' \leq W$ with $w_0 \in W'$, with representatives w' of W' in $G'_0 \leq G_0$ such that the representatives w of W in G can be written in the form $w = w't$ such that $t \in T \cap G_0$.
- $G'_0 = G_0 \cap G'$ with $G' = G'_0 B'$ and
- $U'^{(1)} = U^{(1)} \cap G'$ such that $U'^{(1)} \subset (N'^{-} \cap U'^{(1)})B'$ for $N'^{-} = w_0 N' w_0$.

For example these conditions are satisfied for the group \mathbf{SL}_n .

The proof of the first proposition works for such G' , and from a decomposition $x = r'b' \in R'_w B' \subset G'$ we get some $r \in R_w$ and $b \in B$ such that $x = rb \in G$. Hence the B'_+ -action on G' respects the restriction of \prec to W' in the sense that if $x \in R_{w'} B'$ and $b' \in B'$ then there exists $w'' \preceq w'$ in W' such that $b'^{-1}x \in R_{w''} B'$.

2.3 Generating B_+ -subrepresentations

For any torsion \mathfrak{o}_K -module X with \mathfrak{o}_K -linear B -action denote the (partially ordered) set of generating B_+ -subrepresentations of X (those B_+ -submodules M of X for which $BM = X$) by $\mathcal{B}_+(X)$.

For example $\mathrm{Ind}_B^{U_{w_0}}(\chi) \simeq C^\infty(N_0)$ is the minimal generating B_+ -subrepresentation of the Steinberg representation $\pi_{n!-1} = \mathrm{Ind}_B^{Bw_0B}(\chi) \simeq C_c^\infty(N)$. (cf [17], Lemma 2.6)

Proposition 2.3.1 *Let X be a smooth admissible and irreducible torsion \mathfrak{o}_K -representation of G . Then $M_0 = B_+ X^{U^{(1)}}$ is a generating B_+ -subrepresentation of X . For any $M \in \mathcal{B}_+(X)$ there exists a $t_+ \in T_+$ such that $t_+ M_0 \subset M$.*

Proof X is a ϖ_K vectorspace as well, because $\varpi_K X \leq X$, hence by the irreducibility it is either 0 or X , and since X is torsion $\varpi_K X = X$ gives $X = 0$.

BM_0 is a B -subrepresentation, and also a G_0 -subrepresentation (because $U^{(1)} \triangleleft G_0$). $G_0 B = B G_0 = G$, so BM_0 is a G -subrepresentation of X . M_0 is

not $\{0\}$, since $U^{(1)}$ is pro- p and since X is irreducible $BM_0 = X$, hence M_0 is generating. And M_0 is clearly a B_+ -submodule of X .

X is admissible, hence $X^{U^{(1)}}$ has a finite generating set, say R . Let M be as in the proposition. For any $r \in R$ there exists an element $t_r \in T_+$ such that $t_r r \in M$ ([17], Lemma 2.1). The cardinality of R is finite, hence for $t_+ = \prod_{r \in R} t_r$ we have $t_r^{-1} t_+ \in T_+$ for all $r \in R$, and then $t_+ M_0 \subset M$. \square

From now on let $\pi = \text{Ind}_B^G(\chi)$ as before and $M_0 = B_+ \pi^{U^{(1)}}$. Then $\pi^{U^{(1)}}$ (as a vector space) is generated by

$$f_r : \begin{cases} urb & \mapsto \chi^{-1}(b) \\ y \neq urb & \mapsto 0 \end{cases} \quad \left(r \in U^{(1)} \setminus G/B = \bigcup_{w \in W} \widetilde{N_w(k_F)w} \right).$$

If we denote the coset $U^{(1)}wB$ also with w , then $\pi^{U^{(1)}}$ is generated by $\{f_w | w \in W\}$ as an N_0 -module. Hence any $f \in M_0$ can be written in the form $\sum_{i=1}^s \lambda_i n_i t_i f_{w_i}$ for some $\lambda_i \in k_K, n_i \in N_0, t_i \in T_+$ and $w_i \in W$.

Proposition 2.3.2 M_0 is minimal in $\mathcal{B}_+(\pi)$.

Remark In [17] section 12 Schneider and Vigneras treated the case of the subquotients π_{m-1}/π_m . Unfortunately M_0 does not generally give the minimal generating B_+ -subrepresentation of π_{m-1}/π_m on this subquotient, since that their method does not work on the whole π . It is not true even for $\mathbf{GL}_3(\mathbb{Q}_p)$: an explicit example is shown in Corollary 2.5.2.

Proof By the previous proposition, it is enough to show, that for any $t' \in T_+$ we have $M_0 \subset B_+ t' M_0$.

If $t' \in G_0$, then $t'^{-1} \in T_+$ thus we have $B_+ t' = B_+$, and $B_+ t' M_0 = B_+ M_0 = M_0$. The same is true for central elements $t' \in Z(G)$. So it is enough to prove for $t' = (\varpi_F, \varpi_F, \dots, \varpi_F, 1, 1, \dots, 1)$ that $M_0 \subset B_+ t' M_0$.

Let $j_0 \in \mathbb{N}$ be such that $t'_{j_0} = \varpi_F$ and $t'_{j_0+1} = 1$. We need to show, that for all $w \in W$ we have $f_w \in B_+ t' M_0$. We prove it by descending induction on w with respect to \prec .

Let us denote $N_{j_0}^{(1)} = \{n \in N \cap U^{(1)} | \forall i < j, (j_0 - i)(j - j_0) < 0 : n_{ij} = 0\}$, $N_{w, j_0} = N_w \cap N_{j_0}^{(1)}$ and

$$J_{w, j_0} = J(N_{w, j_0} / t' N_{w, j_0} t'^{-1}) \subset N_0 \cap U^{(1)}.$$

It is enough to prove the following:

Lemma 2.3.3 *Let $g = \sum_{m \in J_{w,j_0}} mt'f_w$. Then $\chi(w^{-1}t'w)f_w - g$ is in $\sum_{w':w \prec w'} N_0 f_{w'}$.*

We claim that for $r \in R_w$ we have

$$t'f_w(r) = \begin{cases} \chi(w^{-1}t'w), & \text{if } \forall i \leq j_0 < j, w^{-1}(i) > w^{-1}(j) : r_{ij} \in \varpi_F^2 o_F, \\ 0, & \text{otherwise.} \end{cases}$$

$t'f_w(r) = f(t'^{-1}r)$ is nonzero if and only if $t'^{-1}r \in U^{(1)}wB$. Following the proof of Proposition 2.2.3, it is equivalent to that for all $1 \leq j \leq n$ we have $w = w^{(j)}$ and that the first j column of $(t^{(j)})^{-1}r^{(j)}$ is as the first j column of $U^{(1)}w$. This holds if and only if $r_{ij} \in \varpi_F^2 o_F$ for all i and j as above. Then we have $r^{(n)} = t'^{-1}rw^{-1}t'w$ and $b^{(n)} = w^{-1}(t')^{-1}w$, hence our claim.

Therefore $\chi(w^{-1}t'w)f_w|_{U_w} = \sum_{m \in J_{w,j_0}} mt'f_w|_{U_w}$. Hence by the induction hypothesis and Proposition 2.2.3 it suffices to prove that g is $U^{(1)}$ -invariant.

To do that, first notice that since f_w is $U^{(1)}$ -invariant, we have that $t'f_w$ is $t'U^{(1)}t'^{-1}$ -invariant. Moreover, since for all $m \in J_{w,j_0}$ we have $m \in N_0 \cap U^{(1)} \subseteq t'N_0t'^{-1}$, m normalizes $t'U^{(1)}t'^{-1}$, $mt'f_w$ is also $t'U^{(1)}t'^{-1}$ -invariant, and so is g .

On the other hand, we can write

$$g = \sum_{m \in J_{w,j_0}} mt'f_w = \sum_{m \in J_{w,j_0}} t'(t'^{-1}mt')f_w = t' \left(\sum_{n \in t'^{-1}N_{w,j_0}t'/N_{w,j_0}} nf_w \right),$$

where the sum in the bracket on the right hand side is obviously $t'^{-1}N_{w,j_0}t'$ -invariant, hence g is N_{w,j_0} -invariant.

Denote $N'_{w,j_0} = N'_w \cap N_{j_0}^{(1)}$. Then N_{w,j_0} centralizes $t'^{-1}N'_{w,j_0}t'$: let $n_0 = \text{id} + m_0 \in t'^{-1}N'_{w,j_0}t'$, $n \in N_{w,j_0}$,

$$(n^{-1}n_0n - n_0)_{xy} = (n^{-1}m_0n - m_0)_{xy} = \sum_{x \leq s \leq t \leq y} (n^{-1})_{xs}(m_0)_{st}n_{ty} - (m_0)_{xy},$$

and by the definition $N_{j_0}^{(1)}$, $(m_0)_{st}$ is 0, unless $s \leq j_0 \leq t$ and hence $(n^{-1})_{xs}m_{st}n_{ty} = 0$, unless $x = s$ and $y = t$.

By the definition of N'_w we have $w^{-1}N'_{w,j_0}w \subset B$, so for any $u \in U^{(1)}$ and $n_0 \in t'^{-1}N'_{w,j_0}t' \subset G_0$ we have $n_0uw = (n_0un_0^{-1})w(w^{-1}n_0w) \in U^{(1)}wB$, and hence f_w is $t'^{-1}N'_{w,j_0}t'$ -invariant.

Altogether for any representative $n \in J_{w,j_0}$

$$nf_w(n_0x) = f_w(n^{-1}n_0x) = f_w(n_0n^{-1}x) = f_w(n^{-1}x) = nf_w(x),$$

meaning that nf_w is $t'^{-1}N'_{w,j_0}t'$ -invariant, and $t'nf_w$ is N'_{w,j_0} -invariant. So g is also N'_{w,j_0} -invariant.

$U^{(1)}$ is contained in $\langle t'U^{(1)}t'^{-1}, N_{w,j_0}, N'_{w,j_0} \rangle$, so g is $U^{(1)}$ -invariant, and we are done. \square

Corollary 2.3.4 *For any $f \in M_0$ there exists $t \in T_+$ such that f can be written in form $\sum_{i=1}^s \lambda_i n_i t f_{w_i}$ for some $\lambda_i \in k_K, n_i \in N_0$ and $w_i \in W$.*

Define the $k_K[B_+]$ -submodules $M_{0,m} = \sum_{m' > m} B_+ f_{w_{m'}} \leq \text{Ind}_B^{G_m}(\chi)$. We obtain a descending filtration $M_0 = M_{0,0} \geq M_{0,1} \geq \dots \geq M_{0,n!} = 0$. Then $M_{0,n!-1} = \text{Ind}_B^{U_{w_0}}(\chi)$ is the minimal generating subrepresentation of $\pi_{n!-1}$.

Proposition 2.3.5 *Let $1 < m \leq n!$, $w = w_{m-1}$ and $n' \in N'_{0,w} = N'_w \cap N_0$ and $t \in T_+$. Then $g = n' t f_w - t f_w \in M_{0,m}$.*

Proof For $w' \prec w$ we have $t f_w|_{U_{w'}} = n' t f_w|_{U_{w'}} = 0$ and following the proof of Proposition 2.2.3 we get $n' t f_w|_{U_w} = t f_w|_{U_w}$. Moreover g is $tU^{(1)}t^{-1}$ -invariant, thus it is contained in $\sum_{m' > m-1} t f_{w_{m'}} \subset M_{0,m}$. \square

Corollary 2.3.6 *For any $f \in M_0$ there exists $t \in T_+$ such that f can be written in form $\sum_{i=1}^s \lambda_i n_i t f_{w_i}$ for some $\lambda_i \in k_K, w_i \in W$ and $n_i \in N_{0,w_i}$.*

Remarks 1. π is the modulo ϖ_K reduction of the p -adic principal series representation. This can be done with any $l \in \mathbb{N}$ for the modulo ϖ_K^l reduction. Then the ϖ_K -torsion part of the minimal generating B_+ -representation is exactly M_0 .

2. This can be carried out in the same way for groups $G' = \mathbf{G}'(F)$ as in the previous section satisfying moreover $N_0 \subset G'$. For example $\mathbf{G}' = \mathbf{SL}_n$ has this property (but its center is not connected), or $G' = P$ for arbitrary $P \leq G$ parabolic subgroup has also (but these are not reductive).

2.4 The Schneider-Vigneras functor

Following Schneider and Vigneras ([17], section 2) we introduce the functor D from torsion \mathfrak{o}_K -modules to modules over the Iwasawa algebra of N_0 .

Let us denote the completed group ring of N_0 over \mathfrak{o}_K by $\Lambda(N_0)$, and define

$$D_{SV}(\rho) = \varinjlim_{M \in \mathcal{B}_+(\rho)} M^\vee,$$

as an $\Lambda(N_0)$ -module, equipped with a natural T_+^{-1} -action ψ .

On $D_{SV}(\pi)$ the action of ϖ_K is 0, hence we can view it as a $\Omega(N_0) = \Lambda(N_0)/\varpi_K\Lambda(N_0)$ -module.

By Proposition 2.3.2 we have

Proposition 2.4.1 *The $\Omega(N_0)$ -module $D_{SV}(\pi)$ is equal to M_0^\vee .*

Remarks 1. We do not know whether $D_{SV}(\pi)$ is finitely generated or it has rank 1 as an $\Omega(N_0)$ -module.

2. On M_0 we have an action of $U^{(1)}$: if $x \in U^{(1)}$, $n \in N_0$, $t \in T_+$ and $w \in W$ then we can write $n^{-1}xn = n_1n_2 \in U^{(1)}$ with $n_1 \in N_0$ and $n_2 \in B^-T \cap U^{(1)}$ (with $B^- = N^-T$), thus

$$xntf_w = n(n^{-1}xn)tf_w = (nn_1)t(t^{-1}n_2t)f_w = (nn_1)tf_w \in M_0,$$

since $t^{-1}n_2t \in U^{(1)}$ and f_w is $U^{(1)}$ -invariant. Thus on $D_{SV}(\pi)$ there is an action of $\Lambda(U^{(1)})$, therefore an action of $\Lambda(I)$ (with I denoting the Iwahori subgroup).

Till this point we considered only the $\Lambda(N_0)$ -module structure of $D_{SV}(\pi)$. Now we shall examine the ψ -action as well. We need to get an étale module from $D_{SV}(\pi)$, thus we examine the ψ -invariant images of $D_{SV}(\pi)$ in an étale module.

Let D be a topologically étale (see [18] the first lines of Section 4) (φ, Γ) -module over $\Omega(N_0)$, with the following properties:

- D is torsion-free as an $\Omega(N_0)$ -module,
- on D the topology is Hausdorff,
- D has a basis of neighborhoods of 0, containing φ -invariant $\Omega(N_0)$ -submodules ($O \leq D$ open such that $\varphi_t(O) \subseteq O$ for all $t \in T_+$).

Theorem 2.4.2 *If D is as above and $F : D_{SV}(\pi) \rightarrow D$ is a continuous ψ -invariant map (where ψ is the canonical left inverse of φ on D), then F factors through the natural map $F_0 : D_{SV}(\pi) \rightarrow D_{SV}(\pi_{n!-1})$: there exists a continuous ψ -invariant map $G : D_{SV}(\pi_{n!-1}) \rightarrow D$ such that $F = F_0 \circ G$.*

Proof $\overline{D_{SV}(\pi) - tors}$ is in the kernel of F (the torsion submodules exist, because the rings are Ore rings).

In $M_0/(M_0 \cap \pi_{n!-1})$ there are no nontrivial $k_K[N_0]$ -divisible elements, because if $f \in M_0$ the image of it in $M_0/(M_0 \cap \pi_{n!-1})$ is $f' = f|_{G \setminus Bw_0B}$. Assume by contradiction that f' is $k_K[N_0]$ -divisible. If it is nontrivial, then there exists $bw_mb \in G$ such that $f(bw_mb) \neq 0$ with some $m < n!$. Let $n' \in N'_{0,w_m} = N_0 \cap w_m N_0 w_m^{-1}$ with $n' \neq \text{id}$, and $[n'] - [\text{id}] \in k_K[N_0]$. Then for any $g \in M_0$ we have

$$([n'] - [\text{id}])g(w_m) = g(n'^{-1}w_m) - g(w_m) = g(w_m(w_m^{-1}n'^{-1}w_m)) - g(w_m) = 0,$$

because $w_m^{-1}n'^{-1}w_m \in N$. Thus f' is not divisible by $[n'] - [\text{id}]$.

It follows that F factors through $(M_0 \cap \pi_{n!-1})^\vee$: The fact that there are no nontrivial divisible submodules in $M_0/(M_0 \cap \pi_{n!-1})$ implies that for any (closed) submodule the maps $f \mapsto \lambda f$ are not surjective for all $\lambda \in k_K[N_0]^\vee$. Hence dual maps are not injective for all λ - it has no torsionfree quotient arising as a dual of a submodule of $M_0/(M_0 \cap \pi_{n!-1})$, thus $(M_0/(M_0 \cap \pi_{n!-1}))^\vee \leq \overline{D_{SV}(\pi) - tors}$. Now consider the exact sequence

$$0 \rightarrow M_0 \cap \pi_{n!-1} \rightarrow M_0 \rightarrow M_0/(M_0 \cap \pi_{n!-1}) \rightarrow 0.$$

We claim that F factors through $M_{0,n!-1}^\vee$ as well. If $f \in (M_0 \cap \pi_{n!-1})^\vee$ such that $f|_{M_{0,n!-1}} \equiv 0$, then $\psi_t(u^{-1}f)|_{t^{-1}M_{0,n!-1}} \equiv 0$ for all $u \in N_0$:

The ψ -action on $D_{SV}(\pi)$ comes from the T_+ -action on π , hence $\psi_t(u^{-1}f)(t^{-1}x) = (u^{-1}f)(tt^{-1}x) = f(ux) = 0$ if $x \in M_{0,n!-1}$.

For all $O \subseteq D$ open subset there exists $t \in T_+$ such that $\text{Ker}(f \mapsto f|_{t^{-1}M_{0,n!-1}}) \subset F^{-1}(O)$, since F is continuous and $\bigcup_{t \in T_+} t^{-1}M_{0,n!-1} = \pi_{0,n!-1}$. If O is φ and N_0 -invariant as well, then

$$F(f) = \sum_{u \in N_0/tN_0t^{-1}} u\varphi_t(F(\psi_t(u^{-1}f))) \subseteq O.$$

Then $F(f) = 0$ by the Hausdorff property.

By [17], Proposition 12.1, we have $D_{SV}(\pi_{n!-1}) = M_{0,n!-1}^\vee$, which completes the proof. \square

- Remarks**
1. For this we do not need the Γ -action of D , the statement is true for D étale φ -modules with continuous N_0 and φ -action.
 2. Let D' be the maximal quotient of $D_{SV}(\pi)$, which is torsionfree, Hausdorff and on which the action of ψ is nondegenerate in the following sense: for all $d \in D' \setminus \{0\}$ and $t \in T_+$ there exists $u \in N_0$ such that $\psi_t(ud) \neq 0$. Then the natural map from D' to $D_{SV}(\pi_{n!-1})$ is bijective.
 3. By [22] section 4 if $F = \mathbb{Q}_p$, we have that $D^0(\pi_{n!-1}) = D_{SV}(\pi_{n!-1})$ and $D^i(\pi_{n!-1}) = 0$ for $i > 0$.

Following [17] we choose a surjective homomorphism $\ell : N_0 \rightarrow \mathbb{Q}_p$. Then we can get (φ, Γ) -modules from $D_{SV}(\pi)$: Let $\Lambda_\ell(N_0)$ denote the ring $\Lambda_{N_1}(N_0)$ of [17] with $N_1 = \text{Ker}(\ell)$, with maximal ideal $\mathcal{M}_\ell(N_0)$, $\Omega_\ell(N_0) = \Lambda_\ell(N_0)/\varpi_K \Lambda_\ell(N_0)$ and $D_\ell(\pi) = \Omega_\ell(N_0) \otimes_{\Omega(N_0)} D_{SV}(\pi)$.

Corollary 2.4.3 *Let D be a finitely generated topologically étale (φ, Γ) -module over $\Omega_\ell(N_0)$, and $F' : D_\ell(\pi) \rightarrow D$ a continuous map. Then F' factors through the natural map $F'_0 : D_\ell(\pi) \rightarrow D_\ell(\pi_{n!-1})$.*

Proof If D is a finitely generated topologically étale (φ, Γ) -module over $\Omega_\ell(N_0)$, then it automatically satisfies the conditions above:

D is étale, hence $\Omega_\ell(N_0)$ -torsion free (Theorem 8.20 in [18]), thus $\Omega(N_0)$ -torsion free as well. It is Hausdorff, since finitely generated and the weak topology is Hausdorff on $\Omega_\ell(N_0)$ (Lemma 8.2.iii in [17]).

We only need to verify the condition for the neighborhoods. The sets $\mathcal{M}_\ell(N_0)^k D + \Omega(N_0) \otimes_{k[[X]]} X^n \ell(D)^{++}$ (where $\ell(D)$ is the étale (φ, Γ) -module attached to D at the category equivalence [18] Theorem 8.20) are open φ -invariant $\Omega(N_0)$ submodules and form a basis of neighborhoods of 0 in the weak topology of D .

Thus $D_{SV}(\pi) \rightarrow D_\ell(\pi) \rightarrow D$ factors through $D_{SV}(\pi) \rightarrow D_{SV}(\pi_{n!-1})$, hence the corollary. \square

2.5 Some properties of M_0

In this section we point out some properties of M_0 , which make the picture more difficult than the known case of subquotients π_{m-1}/π_m . Recall ([17]

section 12) that $\pi_{m-1}/\pi_m \simeq \pi(w_m, \chi)$, which has a minimal generating B_+ -subrepresentation

$$M(w_m, \chi) = C^\infty(N_0/N'_{w_m} \cap N_0) \in \mathcal{B}_+(\pi(w_m, \chi)).$$

Proposition 2.5.1 *Let $n = 3$, $F = \mathbb{Q}_p$, then $M_0 \cap \pi_{n!-1} \supsetneq M_{0,n!-1}$.*

Corollary 2.5.2 *Thus $M_0 \cap \pi_{n!-1}$ is not equal to the minimal generating B_+ -subrepresentation of $\pi_{n!-1}$, which is $C^\infty(N_0) = M_{0,n!-1}$ ([17] section 12).*

Proof Assume that $\chi = \chi_1 \otimes \chi_2 \otimes \chi_3 : T \rightarrow k_K^*$ is a character, such that neither χ_1/χ_2 , nor χ_2/χ_3 is trivial on o_K^* . Similar construction can be carried out in the other cases.

Let \prec_T be the following total ordering of the Weyl group of G refining the Bruhat ordering:

$$\begin{aligned} w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \prec_T w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \prec_T w_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \prec_T \\ \prec_T w_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \prec_T w_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \prec_T w_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = w_0. \end{aligned}$$

And let

$$\begin{aligned} h = \sum_{a=0}^{p^2-1} \sum_{b=0}^{p^2-1} \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f_{w_2} \in M_0, \\ f = h - \frac{1}{\chi_3(p^2)} \sum_{a=0}^{p^3-1} \sum_{b=0}^{p^3-1} h \left(\begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f_{w_5}. \end{aligned}$$

Then it is easy to verify that $f \in M_0 \cap \pi_5$, and that $f(z) \neq 0$ for

$$z = \begin{pmatrix} p^2 & 0 & 1 \\ 1 & 0 & 0 \\ p & 1 & 0 \end{pmatrix} \in Bw_0B \setminus N_0w_0B.$$

Thus $f \notin M_{0,5} = B_+f_6 \subseteq \{f \in \pi | \text{supp}(f) \leq N_0w_0B\}$. □

However, if $f \in M_0 \cap \pi_5$ then $\text{supp}(f)$ is contained in $Bw_0B \cap \bigcup_{i>3} R_iB$: A straightforward computation shows that for any $n \in N_0$, $t \in T_+$, $w \in W$ and

- for any $r \in R_{w_1}$ we have $ntf_w(r) = ntf_w(w_1)$. Let $r' = w_1 \in G_5$,
- for any $r \in R_{w_2}$ we have $ntf_w(r) = ntf_w(r')$ for

$$r' = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & 0 \\ \beta' & 0 & 1 \end{pmatrix} \in G_5, \text{ where } r = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & 0 \\ \beta' & \gamma' & 1 \end{pmatrix},$$

- for any $r \in R_{w_3}$ we have $ntf_w(r) = ntf_w(r')$ for

$$r' = \begin{pmatrix} 1 & 0 & 0 \\ \alpha' - \beta\gamma' & \gamma & 1 \\ 0 & 1 & 0 \end{pmatrix} \in G_5, \text{ where } r = \begin{pmatrix} 1 & 0 & 0 \\ \alpha' & \gamma & 1 \\ \beta' & 1 & 0 \end{pmatrix}.$$

Thus if $i < 4$ and $r \in R_{w_i}$, then since $r' \notin Bw_0B$ we have $f(r) = f(r') = 0$.

Proposition 2.5.3 *The quotients $M_{0,m-1}/M_{0,m-1} \cap \pi_m$ via $f \mapsto f(\cdot w_m)$ are isomorphic to $M(w_m, \chi)$.*

Proof It is obvious, that $f(\cdot w_m) \equiv 0$ implies $f|_{G_m \setminus G_{m-1}} \equiv 0$ and $f \in M_{0,m-1} \cap \pi_m$. Hence the map $M_{0,m-1}/M_{0,m-1} \cap \pi_m \rightarrow M(w_m, \chi)$, $f \mapsto f(\cdot w_m)$ is injective.

Let $t_0 = \text{diag}(\varpi_F^{n-1}, \varpi_F^{n-2}, \dots, \varpi_F, 1) \in T_+$, and for any $l \in \mathbb{N}$ let $U^{(l)} = \text{Ker}(G_0 \rightarrow \mathbf{G}(o_F/\varpi_F^l o_F))$. For $x = rb \in R_{w_m}B$ we have

$$\sum_{n \in (N_0 \cap U^{(l)})/t_0^l N_0 t_0^{-l}} nt_0^l f_{w_m}(rb) = \begin{cases} \chi^{-1}(b), & \text{if } r \in U^{(l)}w_m, \\ 0, & \text{if not.} \end{cases}$$

The image of these generate $M(w_m, \chi)$ as an N_0 -module, so $f \mapsto f(\cdot w_m)$ is surjective. \square

Since $M_{0,m} \leq \pi_m$, $M(w_m, \chi)$ is naturally a quotient of $M_{0,m-1}/M_{0,m}$, we have $D_{SV}(\pi_{m-1}/\pi_m) \leq (M_{0,m-1}/M_{0,m})^\vee$.

Proposition 2.5.4 *For $m = 1$ and $m = n! - n + 1, n! - n + 2, \dots, n!$ $(M_{0,m-1}/M_{0,m})^\vee = D_{SV}(\pi_{m-1}/\pi_m)$. For other m -s it is not true, for example if $n = 3$, $F = \mathbb{Q}_p$ and $m = 2, 3$.*

Proof By the previous proposition it is enough to show that $M_{0,m} = M_{0,m-1} \cap \pi_m$ for $m = 1$ and $m > n! - n$.

For $m = 1$ the quotient is obviously k_K , for $m > n! - n$ we have $w \prec w_m$ implies $w = w_{n!}$, so if $f \in B_+ f_{w_m} \cap \pi_{m-1} = B_+ f_{w_m} \cap \pi_{n!-1}$, then $\text{supp}(f) \subset U^{(1)} R_{w_{n!-1}}^{(1)} B$. But

$$M_{0,n!-1} \simeq C^\infty(N_0) \simeq \{f \in \pi_{n!-1} | \text{supp}(f) \subset U^{(1)} R_{w_{n!-1}} B\}.$$

The function f constructed in the beginning of this section is in $M_{0,1} \cap \pi_2 \setminus M_{0,2}$. The same can be done for $m = 3$. \square

Chapter 3

Comparison of functors

3.1 A $\Lambda_\ell(N_0)$ -variant of Breuil's functor

Our first goal is to associate a (φ, Γ) -module over $\Lambda_\ell(N_0)$ (not just over $\mathcal{O}_\mathcal{E}$) to a smooth \mathfrak{o} -torsion representation π of G in the spirit of [3] that corresponds to $D_\xi^\vee(\pi)$ via the equivalence of categories of [18] between (φ, Γ) -modules over $\mathcal{O}_\mathcal{E}$ and over $\Lambda_\ell(N_0)$.

From now on let $\mathfrak{o} = \mathfrak{o}_K, \varpi = \varpi_K$. Let H_k be the normal subgroup of N_0 generated by $s^k H_0 s^{-k}$, ie. we put

$$H_k = \langle n_0 s^k H_0 s^{-k} n_0^{-1} \mid n_0 \in N_0 \rangle .$$

H_k is an open subgroup of H_0 normal in N_0 and we have $\bigcap_{k \geq 0} H_k = \{1\}$. Denote by F_k the operator $\mathrm{Tr}_{H_k/sH_k s^{-1}} \circ (s \cdot)$ on π and consider the skew polynomial ring $\Lambda(N_0/H_k)/\varpi^h[F_k]$ where $F_k \lambda = (s \lambda s^{-1}) F_k$ for any $\lambda \in \Lambda(N_0/H_k)/\varpi^h$. The set of finitely generated $\Lambda(N_0/H_k)[F_k]$ -submodules of π^{H_k} that are stable under the action of Γ and admissible as a representation of N_0/H_k is denoted by $\mathcal{M}_k(\pi^{H_k})$.

Lemma 3.1.1 *We have $F = F_0$ and $F_k \circ \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \cdot) = \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \cdot) \circ F_0$ as maps on π^{H_0} .*

Proof We compute

$$\begin{aligned}
F_k \circ \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \cdot) &= \mathrm{Tr}_{H_k/s H_k s^{-1}} \circ (s \cdot) \circ \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \cdot) = \\
&= \mathrm{Tr}_{H_k/s H_k s^{-1}} \circ \mathrm{Tr}_{s H_k s^{-1}/s^{k+1} H_0 s^{-k-1}} \circ (s^{k+1} \cdot) = \\
&= \mathrm{Tr}_{H_k/s^{k+1} H_0 s^{-k-1}} \circ (s^{k+1} \cdot) = \\
&= \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ \mathrm{Tr}_{s^k H_0 s^{-k}/s^{k+1} H_0 s^{-k-1}} \circ (s^{k+1} \cdot) = \\
&= \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \cdot) \circ \mathrm{Tr}_{H_0/s H_0 s^{-1}} \circ (s \cdot) = \\
&= \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \cdot) \circ F_0 .
\end{aligned}$$

□

Note that if $M \in \mathcal{M}(\pi^{H_0})$ then $\mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k M)$ is a $s^k N_0 s^{-k} H_k$ -subrepresentation of π^{H_k} . So in view of the above Lemma we define M_k to be the N_0 -subrepresentation of π^{H_k} generated by $\mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k M)$, ie. $M_k = N_0 \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k M)$. By Lemma 3.1.1 M_k is a $\Lambda(N_0/H_k)/\varpi^h[F_k]$ -submodule of π^{H_k} .

Lemma 3.1.2 *For any $M \in \mathcal{M}(\pi^{H_0})$ the N_0 -subrepresentation M_k lies in $\mathcal{M}_k(\pi^{H_k})$.*

Proof Let $\{m_1, \dots, m_r\}$ be a set of generators of M as a $\Lambda(N_0/H_0)/\varpi^h[F]$ -module. We claim that the elements $\mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k m_i)$ ($i = 1, \dots, r$) generate M_k as a module over $\Lambda(N_0/H_k)/\varpi^h[F_k]$. Since both H_k and $s^k H_0 s^{-k}$ are normalized by $s^k N_0 s^{-k}$, for any $u \in N_0$ we have

$$\mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k u s^{-k} \cdot) = (s^k u s^{-k} \cdot) \circ \mathrm{Tr}_{H_k/s^k H_0 s^{-k}} . \quad (3.1)$$

Therefore by continuity we also have

$$\mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \lambda s^{-k} \cdot) = (s^k \lambda s^{-k} \cdot) \circ \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}$$

for any $\lambda \in \Lambda(N_0/H_0)/\varpi^h$. Now writing any $m \in M$ as $m = \sum_{j=1}^r \lambda_j F^{i_j} m_j$ we compute

$$\begin{aligned}
\mathrm{Tr}_{H_k/s^k H_0 s^{-k}} \circ (s^k \sum_{j=1}^r \lambda_j F^{i_j} m_j) &= \sum_{j=1}^r (s^k \lambda s^{-k}) F_k^{i_j} \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k m_j) \in \\
&\in \sum_{j=1}^r \Lambda(N_0/H_k)/\varpi^h[F_k] \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k m_j) .
\end{aligned}$$

For the stability under the action of Γ note that Γ normalizes both H_k and $s^k H_0 s^{-k}$ and the elements in Γ commute with s .

Since M is admissible as an N_0 -representation, $s^k M$ is admissible as a representation of $s^k N_0 s^{-k}$. Further by (3.1) the map $\text{Tr}_{H_k/s^k H_0 s^{-k}}$ is $s^k N_0 s^{-k}$ -equivariant therefore its image is also admissible. Finally, M_k can be written as a finite sum

$$\sum_{u \in J(N_0/s^k N_0 s^{-k} H_k)} u \text{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M)$$

of admissible representations of $s^k N_0 s^{-k}$ therefore the statement. \square

Lemma 3.1.3 *Fix a simple root $\alpha \in \Delta$ such that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$. Then for any $M \in \mathcal{M}(\pi^{H_0})$ the kernel of the trace map*

$$\text{Tr}_{H_0/H_k} : Y_k = \sum_{u \in J(N_{\alpha,0}/s^k N_{\alpha,0} s^{-k})} u \text{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M) \rightarrow N_0 F^k(M) \quad (3.2)$$

is finitely generated over o . In particular, the length of $Y_k^\vee[1/X]$ as a module over $o/\varpi^h((X))$ equals the length of $M^\vee[1/X]$.

Proof Since any $u \in N_{\alpha,0} \leq N_0$ normalizes both H_0 and H_k and we have $N_{\alpha,0} H_0 = N_0$ by the assumption that $\ell(N_{\alpha,0}) = \mathbb{Z}_p$, the image of the map (3.2) is indeed $N_0 F^k(M)$. Moreover, by the proof of Lemma 2.6 in [3] the quotient $M/N_0 F^k(M)$ is finitely generated over o . Therefore we have $M^\vee[1/X] \cong (N_0 F^k(M))^\vee[1/X]$ as a module over $o/\varpi^h((X))$. In particular, their length are equal:

$$l = \text{length}_{o/\varpi^h((X))} M^\vee[1/X] = \text{length}_{o/\varpi^h((X))} (N_0 F^k(M))^\vee[1/X] .$$

We compute

$$\begin{aligned} l &= \text{length}_{o/\varpi^h((X))} M^\vee[1/X] = \text{length}_{o/\varpi^h((\varphi^k(X)))} (s^k M)^\vee[1/X] \geq \\ &\geq \text{length}_{o/\varpi^h((\varphi^k(X)))} (\text{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M))^\vee[1/X] = \\ &= \text{length}_{o/\varpi^h((X))} (o/\varpi^h[[X]] \otimes_{o/\varpi^h[[\varphi^k(X)]]} \text{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M))^\vee[1/X] \geq \\ &\geq \text{length}_{o/\varpi^h((X))} Y_k^\vee[1/X] . \end{aligned}$$

By the existence of a surjective map (3.2) we must have equality in the above inequality everywhere. Therefore we have $\text{Ker}(\text{Tr}_{H_0/H_k})^\vee[1/X] = 0$, which shows that $\text{Ker}(\text{Tr}_{H_0/H_k})$ is finitely generated over o , because M is admissible, and so is $\text{Ker}(\text{Tr}_{H_0/H_k}) \leq M$. \square

The kernel of the natural homomorphism

$$\Lambda(N_0/H_k)/\varpi^h \rightarrow \Lambda(N_0/H_0)/\varpi \cong k[[X]]$$

is a nilpotent prime ideal in the ring $\Lambda(N_0/H_k)/\varpi^h$. We denote the localization at this ideal by $\Lambda(N_0/H_k)/\varpi^h[1/X]$. For the justification of this notation note that any element in $\Lambda(N_0/H_k)/\varpi^h[1/X]$ can uniquely be written as a formal Laurent-series $\sum_{n \gg -\infty} a_n X^n$ with coefficients a_n in the finite group ring $o/\varpi^h[H_0/H_k]$. Here X —by an abuse of notation—denotes the element $[u_0] - 1$ for an element $u_0 \in N_{\alpha,0} \leq N_0$ with $\ell(u_0) = 1 \in \mathbb{Z}_p$. The ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ admits a conjugation action of the group Γ that commutes with the operator φ defined by $\varphi(\lambda) = s\lambda s^{-1}$ (for $\lambda \in \Lambda(N_0/H_k)/\varpi^h[1/X]$). A (φ, Γ) -module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a finitely generated module over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ together with a semilinear commuting action of φ and Γ . Note that φ is no longer injective on the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ for $k \geq 1$, in particular it is not flat either. However, we still call a (φ, Γ) -module D_k over $\Lambda(N_0/H_k)/\varpi^h[1/X]$ étale if the natural map

$$1 \otimes \varphi: \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h[1/X]} D_k \rightarrow D_k$$

is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$ -modules. For any $M \in \mathcal{M}(\pi^{H_0})$ we put

$$M_k^\vee[1/X] = \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\Lambda(N_0/H_k)/\varpi^h} M_k^\vee$$

where $(\cdot)^\vee$ denotes the Pontryagin dual $\text{Hom}_o(\cdot, K/o)$.

The group N_0/H_k acts by conjugation on the finite $H_0/H_k \triangleleft N_0/H_k$. Therefore the kernel of this action has finite index. In particular, there exists a positive integer r such that $s^r N_{\alpha,0} s^{-r} \leq N_0/H_k$ commutes with H_0/H_k . Therefore the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ is contained as a subring in $\Lambda(N_0/H_k)/\varpi^h[1/X]$.

Lemma 3.1.4 *As modules over the group ring $o/\varpi^h((\varphi^r(X)))[H_0/H_k]$ we have an isomorphism*

$$M_k^\vee[1/X] \rightarrow o/\varpi^h((\varphi^r(X)))[H_0/H_k] \otimes_{o/\varpi^h((\varphi^r(X)))} Y_k^\vee[1/X] .$$

In particular, $M_k^\vee[1/X]$ is induced as a representation of the finite group H_0/H_k , so the reduced (Tate-) cohomology groups $\tilde{H}^i(H', M_k^\vee[1/X])$ vanish for all subgroups $H' \leq H_0/H_k$ and $i \in \mathbb{Z}$.

Proof By the definition of M_k we have a surjective $o/\varpi^h[[\varphi^r(X)]] [H_0/H_k]$ -linear map

$$f: o/\varpi^h[[\varphi^r(X)]] [H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k \rightarrow M_k$$

sending $\lambda \otimes y$ to λy for $\lambda \in o/\varpi^h[[\varphi^r(X)]] [H_0/H_k]$ and $y \in Y_k$. Further, by Lemma 3.1.3 the kernel of the restriction of f to the H_0/H_k -invariants

$$(o/\varpi^h[[\varphi^r(X)]] [H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k)^{H_0/H_k} = \left(\sum_{h \in H_0/H_k} h \right) \otimes Y_k$$

is finitely generated over o . By taking the Pontryagin dual of f and inverting X we obtain an injective $o/\varpi^h((\varphi^r(X))) [H_0/H_k]$ -homomorphism

$$\begin{aligned} f^\vee[1/X]: M_k^\vee[1/X] &\rightarrow (o/\varpi^h[[\varphi^r(X)]] [H_0/H_k] \otimes_{o/\varpi^h[[\varphi^r(X)]]} Y_k)^\vee[1/X] \cong \\ &\cong o/\varpi^h((\varphi^r(X))) [H_0/H_k] \otimes_{o/\varpi^h((\varphi^r(X)))} (Y_k^\vee[1/X]) \end{aligned}$$

that becomes surjective after taking H_0/H_k -coinvariants. Since $M_k^\vee[1/X]$ is a finite dimensional representation of the finite p -group H_0/H_k over the local artinian ring $o/\varpi^h((X))$ with residual characteristic p , the map $f^\vee[1/X]$ is in fact an isomorphism as its cokernel has trivial H_0/H_k -coinvariants. \square

Denote by $H_{k,-}/H_k$ the kernel of the group homomorphism

$$s(\cdot)s^{-1}: N_0/H_k \rightarrow N_0/H_k .$$

It is a finite normal subgroup contained in $H_0/H_k \leq N_0/H_k$. If k is big enough so that H_k is contained in sH_0s^{-1} then we have $H_{k,-} = s^{-1}H_k s$, otherwise we always have $H_{k,-} = H_0 \cap s^{-1}H_k s$. The ring homomorphism

$$\varphi: \Lambda(N_0/H_k)/\varpi^h \rightarrow \Lambda(N_0/H_k)/\varpi^h$$

factors through the quotient map $\Lambda(N_0/H_k)/\varpi^h \twoheadrightarrow \Lambda(N_0/H_{k,-})/\varpi^h$. We denote by $\tilde{\varphi}$ the induced ring homomorphism

$$\tilde{\varphi}: \Lambda(N_0/H_{k,-})/\varpi^h \rightarrow \Lambda(N_0/H_k)/\varpi^h .$$

Note that $\tilde{\varphi}$ is injective and makes $\Lambda(N_0/H_k)/\varpi^h$ a free module of rank

$$\begin{aligned} \nu &= |\text{Coker}(s(\cdot)s^{-1}: N_0/H_k \rightarrow N_0/H_k)| = \\ &= p|\text{Coker}(s(\cdot)s^{-1}: H_0/H_k \rightarrow H_0/H_k)| = \\ &= p|\text{Ker}(s(\cdot)s^{-1}: H_0/H_k \rightarrow H_0/H_k)| = p|H_{k,-}/H_k| \end{aligned}$$

over $\Lambda(N_0/H_{k,-})/\varpi^h$.

Lemma 3.1.5 *We have a series of isomorphisms of $\Lambda(N_0/H_k)/\varpi^h[1/X]$ -modules*

$$\begin{aligned}
\mathrm{Tr}^{-1} &= \mathrm{Tr}_{H_{k,-}/H_k}^{-1} : (\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] \xrightarrow{(1)} \\
&\xrightarrow{(1)} \mathrm{Hom}_{\Lambda(N_0/H_k), \varphi}(\Lambda(N_0/H_k), M_k^\vee[1/X]) \xrightarrow{(2)} \\
&\xrightarrow{(2)} \mathrm{Hom}_{\Lambda(N_0/H_{k,-}), \tilde{\varphi}}(\Lambda(N_0/H_k), (M_k^\vee[1/X])^{H_{k,-}}) \xrightarrow{(3)} \\
&\xrightarrow{(3)} \Lambda(N_0/H_k) \otimes_{\Lambda(N_0/H_{k,-}), \tilde{\varphi}} M_k^\vee[1/X]^{H_{k,-}} \xrightarrow{(4)} \\
&\xrightarrow{(4)} \Lambda(N_0/H_k) \otimes_{\Lambda(N_0/H_{k,-}), \tilde{\varphi}} (M_k^\vee[1/X])_{H_{k,-}} \xrightarrow{(5)} \\
&\xrightarrow{(5)} \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h, \varphi} M_k^\vee[1/X] .
\end{aligned}$$

Proof (1) follows from the adjoint property of \otimes and Hom . The second isomorphism follows from noting that the action of the ring $\Lambda(N_0/H_k)$ over itself via φ factors through the quotient $\Lambda(N_0/H_{k,-})$ therefore $H_{k,-}$ acts trivially on $\Lambda(N_0/H_k)$ via this map. So any module-homomorphism $\Lambda(N_0/H_k) \rightarrow M_k^\vee[1/X]$ lands in the $H_{k,-}$ -invariant part $M_k^\vee[1/X]^{H_{k,-}}$ of $M_k^\vee[1/X]$. The third isomorphism follows from the fact that $\Lambda(N_0/H_k)$ is a free module over $\Lambda(N_0/H_{k,-})$ via $\tilde{\varphi}$. The fourth isomorphism is given by (the inverse of) the trace map $\mathrm{Tr}_{H_{k,-}/H_k} : (M_k^\vee[1/X])_{H_{k,-}} \rightarrow M_k^\vee[1/X]^{H_{k,-}}$ which is an isomorphism by Lemma 3.1.4. The last isomorphism follows from the isomorphism $(M_k^\vee[1/X])_{H_{k,-}} \cong \Lambda(N_0/H_{k,-}) \otimes_{\Lambda(N_0/H_k)} M_k^\vee[1/X]$. \square

Remark Here φ always acted only on the ring $\Lambda(N_0/H_k)$, hence denoting φ_t the action $n \mapsto tnt^{-1}$ for a fixed $t \in T_+$ and choosing k large enough such that $tH_0t^{-1} \geq H_k$ we get analogously an isomorphism

$$\begin{aligned}
\mathrm{Tr}_{t^{-1}H_k t/H_k}^{-1} &: (\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi_t, \Lambda(N_0/H_k)/\varpi^h} M_k)^\vee[1/X] \rightarrow \\
&\rightarrow \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h, \varphi_t} M_k^\vee[1/X] .
\end{aligned}$$

We denote the composite of the five isomorphisms in Lemma 3.1.5 by Tr^{-1} emphasising that all but (4) are tautologies. Our main result in this section is the following generalization of Lemma 2.6 in [3].

Proposition 3.1.6 *The map*

$$\begin{aligned}
&\mathrm{Tr}^{-1} \circ (1 \otimes F_k)^\vee[1/X] : \quad (3.3) \\
M_k^\vee[1/X] &\rightarrow \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X]
\end{aligned}$$

is an isomorphism of $\Lambda(N_0/H_k)/\varpi^h[1/X]$ -modules. Therefore the natural action of Γ and the operator

$$\begin{aligned}\varphi: M_k^\vee[1/X] &\rightarrow M_k^\vee[1/X] \\ f &\mapsto (\mathrm{Tr}^{-1} \circ (1 \otimes F_k)^\vee[1/X])^{-1}(1 \otimes f)\end{aligned}$$

make $M_k^\vee[1/X]$ into an étale (φ, Γ) -module over the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$.

Proof Since M_k is finitely generated over $\Lambda(N_0/H_k)/\varpi^h[F_k]$ by Lemma 3.1.2, the cokernel C of the map

$$1 \otimes F_k: \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h} M_k \rightarrow M_k \quad (3.4)$$

is finitely generated as a module over $\Lambda(N_0/H_k)/\varpi^h$. Further, it is admissible as a representation of N_0 (again by Lemma 3.1.2), therefore C is finitely generated over o . In particular, we have $C^\vee[1/X] = 0$ showing that (3.3) is injective.

For the surjectivity put $Y_k = \sum_{u \in J(N_{\alpha,0}/s^k N_{\alpha,0} s^{-k})} u \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M)$. This is an $o/\varpi^h[[X]]$ -submodule of M_k . By Lemma 3.1.3 we have

$$\begin{aligned}\mathrm{length}_{o/\varpi^h((\varphi^r(X)))}(Y_k^\vee[1/X]) &= \\ = |N_{\alpha,0} : s^r N_{\alpha,0} s^{-r}| \mathrm{length}_{o/\varpi^h((X))}(Y_k^\vee[1/X]) &= p^r l.\end{aligned}$$

By Lemma 3.1.4 we obtain

$$\begin{aligned}\mathrm{length}_{o/\varpi^h((\varphi^r(X)))} M_k^\vee[1/X] &= \\ = |H_0 : H_k| \cdot \mathrm{length}_{o/\varpi^h((\varphi^r(X)))} Y_k^\vee[1/X] &= |H_0 : H_k| p^r l.\end{aligned}$$

Consider the ring homomorphism

$$\varphi: \Lambda(N_0/H_k)/\varpi^h[1/X] \rightarrow \Lambda(N_0/H_k)/\varpi^h[1/X]. \quad (3.5)$$

Its image is the subring $\Lambda(sN_0 s^{-1} H_k/H_k)/\varpi^h[1/\varphi(X)]$ over which the ring $\Lambda(N_0/H_k)/\varpi^h[1/X]$ is a free module of rank $\nu = |N_0 : sN_0 s^{-1} H_k| = p |H_{k,-} : H_k|$. So we obtain

$$\begin{aligned}p \mathrm{length}_{o((\varphi^r(X)))} \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] &= \\ = \mathrm{length}_{o((\varphi^{r+1}(X)))} \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h[1/X]} M_k^\vee[1/X] &= \\ = \nu \mathrm{length}_{o((\varphi^{r+1}(X)))} \Lambda(sN_0 s^{-1} H_k/H_k)/\varpi^h[1/\varphi(X)] \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h[1/X]} & \\ \otimes M_k^\vee[1/X] \stackrel{(*)}{=} \nu \mathrm{length}_{o((\varphi^r(X)))} M_k^\vee[1/X]_{H_{k,-}} &= \\ = \nu \mathrm{length}_{o((\varphi^r(X)))} (o/\varpi^h[H_0/H_{k,-}] \otimes_{o/\varpi^h} Y_k^\vee[1/X]) &= \\ = \nu |H_0 : H_{k,-}| p^r l = p |H_0 : H_k| p^r l = p \mathrm{length}_{o/\varpi^h((\varphi^r(X)))} M_k^\vee[1/X]. &\end{aligned}$$

Here the equality (*) follows from the fact that the map φ induces an isomorphism between $\Lambda(N_0/H_{k,-})/\varpi^h[1/X]$ and $\Lambda(sN_0s^{-1}H_k/H_k)/\varpi^h[1/\varphi(X)]$ sending the subring $o((\varphi^r(X)))$ isomorphically onto $o((\varphi^{r+1}(X)))$.

This shows that (3.3) is an isomorphism as it is injective and the two sides have equal length as modules over the artinian ring $o/\varpi^h((X))$. \square

Remark We also obtain in particular that the map (3.4) has finite kernel and cokernel. Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$ -submodule $M_{k,*}$ of M_k such that the kernel of $1 \otimes F_k$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi} M_{k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi} M_k$. We denote by M_k^* the image of $1 \otimes F_k$.

Note that for $k = 0$ we have $M_0 = M$. Let now $0 \leq j \leq k$ be two integers. By Lemma 3.1.4 the space of H_j -invariants of M_k is equal to $\text{Tr}_{H_j/H_k}(M_k)$ upto finitely generated modules over o . On the other hand, we compute

$$\begin{aligned} N_0F_j^{k-j}(M_j) &= N_0\text{Tr}_{H_j/s^{k-j}H_0s^{-k}}(s^{k-j}\cdot) \circ \text{Tr}_{H_j/s^jH_0s^{-j}}(s^jM) = \\ &= N_0\text{Tr}_{H_j/s^kH_0s^{-k}}(s^kM) = N_0\text{Tr}_{H_j/H_k} \circ \text{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) = \\ &= \text{Tr}_{H_j/H_k}(N_0\text{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)) = \text{Tr}_{H_j/H_k}(M_k) \end{aligned}$$

since both H_k and H_j are normal in N_0 whence we have $(u\cdot) \circ \text{Tr}_{H_j/H_k} = \text{Tr}_{H_j/H_k} \circ (u\cdot)$ for all $u \in N_0$. So taking H_j/H_k -coinvariants of $M_k^{\vee}[1/X]$, we have a natural identification

$$\begin{aligned} M_k^{\vee}[1/X]_{H_j/H_k} &\cong (M_k^{H_j/H_k})^{\vee}[1/X] \cong \\ &\cong (\text{Tr}_{H_j/H_k}(M_k))^{\vee}[1/X] = (N_0F_j^{k-j}(M_j))^{\vee}[1/X] \cong M_j^{\vee}[1/X] \end{aligned} \quad (3.6)$$

induced by the inclusion $N_0F_j^{k-j}(M_j) \subseteq M_k^{H_j} \subseteq M_k$.

Lemma 3.1.7 *We have $\text{Tr}_{H_j/H_k} \circ F_k = F_j \circ \text{Tr}_{H_j/H_k}$.*

Proof We compute

$$\begin{aligned} \text{Tr}_{H_j/H_k} \circ F_k &= \text{Tr}_{H_j/H_k} \circ \text{Tr}_{H_k/sH_ks^{-1}} \circ (s\cdot) = \\ \text{Tr}_{H_j/sH_ks^{-1}} \circ (s\cdot) &= \text{Tr}_{H_j/sH_ks^{-1}} \circ \text{Tr}_{sH_ks^{-1}/sH_ks^{-1}}(s\cdot) = \\ \text{Tr}_{H_j/sH_ks^{-1}} \circ (s\cdot)\text{Tr}_{H_j/H_k} &= F_j \circ \text{Tr}_{H_j/H_k} . \end{aligned}$$

\square

Proposition 3.1.8 *The identification (3.6) is φ and Γ -equivariant.*

Proof It suffices to treat the case when k is large enough so that we have $H_{k,-} = s^{-1}H_k s$. So from now on we assume $H_k \leq sH_0 s^{-1} \leq sN_0 s^{-1}$. As Γ acts both on M_k and M_j by multiplication coming from the action of Γ on π , the map (3.6) is clearly Γ -equivariant. In order to avoid confusion we are going to denote the map φ on $M_k^\vee[1/X]$ (resp. on $M_j^\vee[1/X]$) temporarily by φ_k (resp. by φ_j). Let f be in M_k^\vee such that its restriction to $M_{k,*}$ is zero (see the Remark after Prop. 3.1.6).

We regard f as an element in $(M_k^*/M_{k,*})^\vee \leq (M_k^*)^\vee$. We are going to compute $\varphi_k(f)$ and $\varphi_j(f|_{\text{Tr}_{H_j/H_k}(M_k^*)})$ explicitly and find that the restriction of $\varphi_k(f)$ to $\text{Tr}_{H_j/H_k}(M_k^*)$ is equal to $\varphi_j(f|_{\text{Tr}_{H_j/H_k}(M_k^*)})$. Note that we have an isomorphism $M_k^\vee[1/X] \cong M_k^{*\vee}[1/X] \cong (M_k^*/M_{k,*})^\vee[1/X]$ (resp. $M_j^\vee[1/X] \cong \text{Tr}_{H_j/H_k}(M_k^*)^\vee[1/X]$).

Let $m \in M_k^* \leq M_k$ be in the form

$$m = \sum_{u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)$$

with elements $m_u \in M_k$ for $u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})$. By the remark after Proposition 3.1.6 M_k^* is a finite index submodule of M_k . Note that the elements m_u are unique upto $M_{k,*} + \text{Ker}(F_k)$. Therefore $\varphi_k(f) \in (M_k^*)^\vee$ is well-defined by our assumption that $f|_{M_{k,*}} = 0$ noting that the kernel of F_k equals the kernel of $\text{Tr}_{H_{k,-}/H_k}$ since the multiplication by s is injective and we have $F_k = s \circ \text{Tr}_{H_{k,-}/H_k}$. So we compute

$$\begin{aligned} \varphi_k(f)(m) &= ((1 \otimes F_k)^\vee)^{-1}(\text{Tr}_{H_{k,-}/H_k}(1 \otimes f))(m) = \\ &= ((1 \otimes F_k)^\vee)^{-1}(1 \otimes \text{Tr}_{H_{k,-}/H_k}(f))\left(\sum_{u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)\right) = \\ &= \text{Tr}_{H_{k,-}/H_k}(f)(F_k^{-1}(u_0 F_k(m_{u_0}))) = f(\text{Tr}_{H_{k,-}/H_k}((s^{-1}u_0 s)m_{u_0})) \end{aligned} \tag{3.7}$$

where u_0 is the single element in $J(N_0/sN_0s^{-1})$ corresponding to the coset of 1. In order to simplify notation put f_* for the restriction of f to $\text{Tr}_{H_j/H_k}(M_k)$ and

$$U = J(N_0/sN_0s^{-1}) \cap H_j s N_0 s^{-1} .$$

Note that we have $0 = \varphi_j(f_*)(uF_j(m'))$ for all $m' \in M_j$ and

$$u \in J(N_0/sN_0s^{-1}) \setminus U .$$

Therefore using Lemma 3.1.7 we obtain

$$\begin{aligned}
\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m) &= \varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\
&= \varphi_j(f_*)(\sum_{u \in J(N_0/sN_0s^{-1})} uF_j \circ \mathrm{Tr}_{H_j/H_k}(m_u)) = \\
&= \sum_{u \in U} f(\mathrm{Tr}_{H_j,-/H_j}(s^{-1}\bar{u}s\mathrm{Tr}_{H_j/H_k}(m_u))) = \\
&= \sum_{u \in U} f(s^{-1}\bar{u}s\mathrm{Tr}_{H_j,-/H_k}(m_u)) \quad (3.8)
\end{aligned}$$

where for each $u \in U$ we choose a fixed \bar{u} in $sN_0s^{-1} \cap H_ju$. Note that $f(s^{-1}\bar{u}s\mathrm{Tr}_{H_j,-/H_k}(m_u))$ does not depend on this choice: If $\bar{u}_1 \in sN_0s^{-1} \cap H_ju$ is another choice then we have $(\bar{u}_1)^{-1}\bar{u} \in sN_0s^{-1} \cap H_j$ whence $s^{-1}(\bar{u}_1)^{-1}\bar{u}s$ lies in $H_{j,-} = N_0 \cap s^{-1}H_js$ so we have

$$\begin{aligned}
s^{-1}\bar{u}s\mathrm{Tr}_{H_j,-/H_k}(m_u) &= s^{-1}\bar{u}_1ss^{-1}(\bar{u}_1)^{-1}\bar{u}s\mathrm{Tr}_{H_j,-/H_k}(m_u) = \\
&= s^{-1}\bar{u}_1s\mathrm{Tr}_{H_j,-/H_k}(m_u) .
\end{aligned}$$

Moreover, the equation (3.8) also shows that $\varphi_j(f_*)$ is a well-defined element in $(\mathrm{Tr}_{H_j/H_k}(M_k^*))^\vee$. On the other hand, for the restriction of $\varphi_k(f)$ to $\mathrm{Tr}_{H_j/H_k}(M_k)$ we compute

$$\begin{aligned}
\varphi_k(f)(\mathrm{Tr}_{H_j/H_k} m) &= \varphi_k(f)(\sum_{w \in J(H_j/H_k)} w \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u)) = \\
&= \sum_{w \in J(H_j/H_k)} \sum_{u \in J(N_0/sN_0s^{-1})} \varphi_k(f)(wuF_k(m_u)) = \\
&= \sum_{\substack{u \in U \\ w \in J(H_j/H_k) \cap (sN_0s^{-1}u^{-1})}} f(\mathrm{Tr}_{H_k,-/H_k}((s^{-1}wus)m_u)) = \\
&= f(\sum_{v = s^{-1}w\bar{u}^{-1}s \in J(H_j,-/H_k,-)} \mathrm{Tr}_{H_k,-/H_k} \sum_{u \in U} vs^{-1}\bar{u}sm_u) = \\
&= \sum_{u \in U} f(s^{-1}\bar{u}s\mathrm{Tr}_{H_j,-/H_k}(m_u))
\end{aligned}$$

that equals $\varphi_j(f_*)(\mathrm{Tr}_{H_j/H_k} m)$ by (3.8). Finally, let now $f \in M_k^\vee$ be arbitrary. Since $M_{k,*}$ is finite, there exists an integer $r \geq 0$ such that $X^r f$

vanishes on $M_{k,*}$. By the above discussion we have $\varphi_k(X^r f)(\text{Tr}_{H_j/H_k} m) = \varphi_j(X^r f_*)(\text{Tr}_{H_j/H_k} m)$. The statement follows noting that $\varphi(X^r)$ is invertible in the ring $\Lambda(N_0/H_j)/\varpi^h[1/X]$. \square

So we may take the projective limit $M_\infty^\vee[1/X] = \varprojlim_k M_k^\vee[1/X]$ with respect to these quotient maps. The resulting object is an étale (φ, Γ) -module over the ring

$$\varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X] \cong \Lambda_\ell(N_0)/\varpi^h.$$

Moreover, by taking the projective limit of (3.6) with respect to k we obtain a φ - and Γ -equivariant isomorphism $(M_\infty^\vee[1/X])_{H_j} \cong M_j^\vee[1/X]$. So we just proved

Corollary 3.1.9 *For any object $M \in \mathcal{M}(\pi^{H_0})$ the (φ, Γ) -module $M^\vee[1/X]$ over $\mathfrak{o}/\varpi^h((X))$ corresponds to $M_\infty^\vee[1/X]$ via the equivalence of categories in Theorem 8.20 in [18].*

Note that whenever $M \subset M'$ are two objects in $\mathcal{M}(\pi^{H_0})$ then we have a natural surjective map $M'^\vee_\infty[1/X] \twoheadrightarrow M^\vee_\infty[1/X]$. So in view of the above corollary we define

$$D_{\xi, \ell, \infty}^\vee(\pi) = \varprojlim_{k \geq 0, M \in \mathcal{M}(\pi^{H_0})} M_k^\vee[1/X] = \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} M^\vee_\infty[1/X].$$

We call two elements $M, M' \in \mathcal{M}(\pi^{H_0})$ equivalent ($M \sim M'$) if the inclusions $M \subseteq M + M'$ and $M' \subseteq M + M'$ induce isomorphisms $M^\vee[1/X] \cong (M + M')^\vee[1/X] \cong M'^\vee[1/X]$. This is equivalent to the condition that M equals M' upto finitely generated \mathfrak{o} -modules. In particular, this is an equivalence relation on the set $\mathcal{M}(\pi^{H_0})$. Similarly, we say that $M_k, M'_k \in \mathcal{M}_k(\pi^{H_k})$ are equivalent if the inclusions $M_k \subseteq M_k + M'_k$ and $M'_k \subseteq M_k + M'_k$ induce isomorphisms

$$M_k^\vee[1/X] \cong (M_k + M'_k)^\vee[1/X] \cong M'^\vee_k[1/X].$$

Proposition 3.1.10 *The maps*

$$\begin{aligned} M &\mapsto N_0 \text{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M) \\ \text{Tr}_{H_0/H_k}(M_k) &\leftarrow M_k \end{aligned}$$

induce a bijection between the sets $\mathcal{M}(\pi^{H_0})/\sim$ and $\mathcal{M}_k(\pi^{H_k})/\sim$. In particular, we have

$$D_{\xi,\ell,\infty}^\vee(\pi) = \varprojlim_{k \geq 0} \varprojlim_{M_k \in \mathcal{M}_k(\pi^{H_k})} M_k^\vee[1/X].$$

Proof We have $\mathrm{Tr}_{H_0/H_k}(N_0 \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M)) = N_0 \mathrm{Tr}_{H_0/s^k H_0 s^{-k}}(s^k M) = N_0 F^k(M)$ which is equivalent to M . Conversely,

$$N_0 \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k \mathrm{Tr}_{H_0/H_k}(M_k)) = N_0 \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M_k) = N_0 F_k^k(M_k)$$

is equivalent to M_k as it is the image of the map

$$1 \otimes F_k^k : \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi^k, \Lambda(N_0/H_k)/\varpi^h} \rightarrow M_k$$

having finite cokernel. □

We equip the pseudocompact $\Lambda_\ell(N_0)$ -module $D_{\xi,\ell,\infty}^\vee(\pi)$ with the weak topology, ie. with the projective limit topology of the weak topologies of $M_\infty^\vee[1/X]$. (The weak topology on $\Lambda_\ell(N_0)$ is defined in section 8 of [17].) Recall that the sets

$$O(M, l, l') = f_{M,l}^{-1}(\Lambda(N_0/H_l) \otimes_{u_\alpha} X' M^\vee[1/X]^{++}) \quad (3.9)$$

for $l, l' \geq 0$ and $M \in \mathcal{M}(\pi^{H_0})$ form a system of neighbourhoods of 0 in the weak topology of $D_{\xi,\ell,\infty}^\vee(\pi)$. Here $f_{M,l}$ is the natural projection map $f_{M,l}: D_{\xi,\ell,\infty}^\vee(\pi) \twoheadrightarrow M_l^\vee[1/X]$ and $M^\vee[1/X]^{++}$ denotes the set of elements $d \in M^\vee[1/X]$ with $\varphi^n(d) \rightarrow 0$ in the weak topology of $M^\vee[1/X]$ as $n \rightarrow \infty$.

3.2 A natural transformation from D_{SV} to $D_{\xi,\ell,\infty}^\vee$

Lemma 3.2.1 *Let W be in $\mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$. There exists a positive integer $k_0 > 0$ such that for all $k \geq k_0$ we have $s^k M \subseteq W$. In particular, both $M_k = N_0 \mathrm{Tr}_{H_k/s^k H_0 s^{-k}}(s^k M)$ and $N_0 F^k(M)$ are contained in W for all $k \geq k_0$.*

Proof By the assumption that M is finitely generated over $\Lambda(N_0/H_0)/\varpi^h[F]$ and W is a B_+ -subrepresentation it suffices to find an integer s^{k_0} such that we have $s^{k_0} m_i$ lies in W for all the generators m_1, \dots, m_r of M . This, however, follows from Lemma 2.1 in [17] noting that the powers of s are cofinal in T_+ . □

In particular, we have a homomorphism $W^\vee \rightarrow M_k^\vee$ of $\Lambda(N_0)$ -modules induced by this inclusion. We compose this with the localization map $M_k^\vee \rightarrow M_k^\vee[1/X]$ and take projective limits with respect to k in order to obtain a $\Lambda(N_0)$ -homomorphism

$$\text{pr}_{W,M}: W^\vee \rightarrow M_\infty^\vee[1/X].$$

Lemma 3.2.2 *The map $\text{pr}_{W,M}$ is ψ_s - and Γ -equivariant.*

Proof The Γ -equivariance is clear as it is given by the multiplication by elements of Γ on both sides. For the ψ_s -equivariance let $k > 0$ be large enough so that H_k is contained in $sH_0s^{-1} \leq sN_0s^{-1}$ (ie. $H_{k,-} = s^{-1}H_k s$) and M_k is contained in W . Let f be in $W^\vee = \text{Hom}_o(W, o/\varpi^h)$ such that $f|_{N_0sM_{k,*}} = 0$. By definition we have $\psi_s(f)(w) = f(sw)$ for any $w \in W$. Denote the restriction of f to M_k by $f|_{M_k}$ and choose an element $m \in M_k^* \leq M_k$ written in the form

$$m = \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u) = \sum_{u \in J(N_0/sN_0s^{-1})} us\text{Tr}_{H_{k,-}/H_k}(m_u).$$

Then we compute

$$\begin{aligned} f|_{M_k}(m) &= \sum_{u \in J(N_0/sN_0s^{-1})} f(us\text{Tr}_{H_{k,-}/H_k}(m_u)) = \\ &= \sum_{u \in J(N_0/sN_0s^{-1})} (u^{-1}f)(s\text{Tr}_{H_{k,-}/H_k}(m_u)) = \\ &= \sum_{u \in J(N_0/sN_0s^{-1})} \psi_s(u^{-1}f)(\text{Tr}_{H_{k,-}/H_k}(m_u)) = \\ &\stackrel{(3.7)}{=} \sum_{u \in J(N_0/sN_0s^{-1})} \varphi(\psi_s(u^{-1}f)|_{M_k})(F_k(m_u)) = \\ &= \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)|_{M_k})(uF_k(m_u)) = \\ &= \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)|_{M_k})(m) \end{aligned}$$

as for distinct $u, v \in J(N_0/sN_0s^{-1})$ we have $u\varphi(f_0)(vF_k(m_v)) = 0$ for any $f_0 \in (M_k^*)^\vee$. So by inverting X and taking projective limits with respect to

k we obtain

$$\mathrm{pr}_{W,M}(f) = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\mathrm{pr}_{W,M}(\psi_s(u^{-1}f)))$$

as we have $(M_k^*)^\vee[1/X] \cong M_k^\vee[1/X]$. However, since $M_\infty^\vee[1/X]$ is an étale (φ, Γ) -module over $\Lambda_\ell(N_0)/\varpi^h$ we have a unique decomposition of $\mathrm{pr}_{W,M}(f)$ as

$$\mathrm{pr}_{W,M}(f) = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi(u^{-1}\mathrm{pr}_{W,M}(f)))$$

so we must have $\psi(\mathrm{pr}_{W,M}(f)) = \mathrm{pr}_{W,M}(\psi_s(f))$. For general $f \in W^\vee$ note that $N_0sM_{k,*}$ is killed by $\varphi(X^r)$ for $r \geq 0$ big enough, so we have

$$\begin{aligned} X^r\psi(\mathrm{pr}_{W,M}(f)) &= \psi(\mathrm{pr}_{W,M}(\varphi(X^r)f)) = \\ &= \mathrm{pr}_{W,M}(\psi_s(\varphi(X^r)f)) = X^r\mathrm{pr}_{W,M}(\psi_s(f)). \end{aligned}$$

The statement follows since X^r is invertible in $\Lambda_\ell(N_0)$. \square

By taking the projective limit with respect to $M \in \mathcal{M}(\pi^{H_0})$ and the injective limit with respect to $W \in \mathcal{B}_+(\pi)$ we obtain a ψ_s - and Γ -equivariant $\Lambda(N_0)$ -homomorphism

$$\mathrm{pr} = \varinjlim_W \varprojlim_M \mathrm{pr}_{W,M} : D_{SV}(\pi) \rightarrow D_{\xi,\ell,\infty}^\vee(\pi).$$

Remarks 1. The natural maps $\pi^\vee \rightarrow D_\xi^\vee(\pi)$ and $\pi^\vee \rightarrow D_{\xi,\ell,\infty}^\vee(\pi)$ both factor through the map $\pi^\vee \rightarrow D_{SV}(\pi)$.

2. The natural topology on D_{SV} obtained as the quotient topology from the compact topology on π^\vee via the surjective map $\pi^\vee \twoheadrightarrow D_{SV}(\pi)$ is compact, but may not be Hausdorff in general. However, if $\mathcal{B}_+(\pi)$ contains a minimal element (as in the case of the principal series see Proposition 2.3.2) then it is also Hausdorff. However, the map pr factors through the maximal Hausdorff quotient of $D_{SV}(\pi)$, namely $\overline{D}_{SV}(\pi) = (\bigcap_{W \in \mathcal{B}_+(\pi)} W)^\vee$. Indeed, pr is continuous and $D_{\xi,\ell,\infty}^\vee(\pi)$ is Hausdorff, so the kernel of pr is closed in $D_{SV}(\pi)$ (and contains 0).

3. Assume that $h = 1$, ie. π is a smooth representation in characteristic p . Then $D_{\xi,\ell,\infty}^\vee(\pi)$ has no nonzero $\Lambda(N_0)/\varpi$ -torsion. Hence the $\Lambda(N_0)/\varpi$ -torsion part of $D_{SV}(\pi)$ is contained in the kernel of pr .

4. If $D_{SV}(\pi)$ has finite rank and its torsion free part is étale over $\Lambda(N_0)$ then $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi)$ is also étale and of finite rank r over $\Lambda_\ell(N_0)$. Moreover, the map $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \text{pr} : \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \rightarrow D_{\xi, \ell, \infty}(\pi)$ has dense image by Lemma 3.2.1. Thus $D_{\xi, \ell, \infty}^\vee(\pi)$ has rank at most r over $\Lambda_\ell(N_0)$.

One can show the above Remark 2 algebraically, too. Let $M \in \mathcal{M}(\pi^{H_0})$ be arbitrary. Then the map $1 \otimes \text{id}_{M^\vee} : M^\vee \rightarrow M^\vee[1/X]$ has finite kernel, so the image $(1 \otimes \text{id}_{M^\vee})(M^\vee)$ is isomorphic to M_0^\vee for some finite index submodule $M_0 \leq M$. Moreover, M_0^\vee is a ψ - and Γ -invariant treillis in $D = M^\vee[1/X] = M_0^\vee[1/X]$. Therefore the map $(1 \otimes F)^\vee$ is injective on M_0^\vee since it is injective after inverting X and M_0^\vee has no X -torsion. This means that $1 \otimes F : o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]], \varphi} M_0 \rightarrow M_0$ is surjective, ie. we have $M_0 = N_0 F^k(M_0)$ for all $k \geq 0$. However, for any $W \in \mathcal{B}_+(\pi)$ and k large enough (depending a priori on W) we have $N_0 F^k(M_0) \subseteq W$, so we deduce $M_0 \subset \bigcap_{W \in \mathcal{B}_+} W$.

Corollary 3.2.3 *If $\pi = \text{Ind}_{B_0}^B \pi_0$ is a compactly induced representation of B for some smooth o/ϖ^h -representation π_0 of B_0 then we have $D_\xi^\vee(\pi) = 0$. In particular, D_ξ^\vee is not exact on the category of smooth o/ϖ^h -representations of B . (However, it may still be exact on a smaller subcategory with additional finiteness conditions.)*

Proof By the 2nd remark above the map $\pi^\vee \rightarrow D_\xi^\vee(\pi)$ factors through the maximal Hausdorff quotient $\overline{D}_{SV}(\pi)$ of $D_{SV}(\pi)$. By Lemma 3.2 in [17], we have $\overline{D}_{SV}(\pi) = (\bigcap_\sigma W_\sigma)^\vee$ where the B_+ -subrepresentations W_σ are indexed by order-preserving maps $\sigma : T_+/T_0 \rightarrow \text{Sub}(\pi_0)$ where $\text{Sub}(\pi_0)$ is the partially order set of B_0 -subrepresentations of π_0 . The explicit description of the B_+ -subrepresentations W_σ (there denoted by M_σ) before Lemma 3.2 in [17] shows that we have in fact $\bigcap_\sigma W_\sigma = \{0\}$ whence the natural map $\pi^\vee \rightarrow D_\xi^\vee(\pi)$ is zero. However, by the construction of this map this can only be zero if $D_\xi^\vee(\pi) = 0$.

Since the principal series arises as a quotient of a compactly induced representation, the exactness of D_ξ^\vee would imply the vanishing of D_ξ^\vee on the principal series, too—which is not the case by Ex. 7.6 in [3]. \square

Proposition 3.2.4 *Let D be an étale (φ, Γ) -module over $\Lambda_\ell(N_0)/\varpi^h$, and $f : D_{SV}(\pi) \rightarrow D$ be a continuous ψ_s and Γ -equivariant $\Lambda(N_0)$ -homomorphism.*

Then f factors uniquely through pr , ie. there exists a unique ψ - and Γ -equivariant $\Lambda(N_0)$ -homomorphism $\hat{f} : D_{\xi, \ell, \infty}^\vee(\pi) \rightarrow D$ such that $f = \hat{f} \circ \text{pr}$.

Proof Note that the uniqueness of \hat{f} follows from Lemma 3.2.1 since any continuous $\Lambda_\ell(N_0)$ -homomorphism of $D_{\xi, \ell, \infty}^\vee(\pi)$ factors through $M_\infty^\vee[1/X]$ for some $M \in \mathcal{M}(\pi^{H_0})$. Indeed, if \hat{f}' is another lift then the image of pr is contained in the kernel of $\hat{f} - \hat{f}'$.

At first we construct a homomorphism $\hat{f}_{H_0} : D_\xi^\vee = (D_{\xi, \ell, \infty}^\vee)_{H_0} \rightarrow D_{H_0}$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 D_{SV}(\pi) & \xrightarrow{\text{pr}} & D_{\xi, \ell, \infty}^\vee(\pi) & \xrightarrow{(\cdot)_{H_0}} & D_\xi^\vee(\pi) \\
 & \searrow f & & & \downarrow \hat{f}_{H_0} \\
 & & D & \xrightarrow{(\cdot)_{H_0}} & D_{H_0}
 \end{array}$$

Consider the composite map $f' : \pi^\vee \rightarrow D_{SV}(\pi) \xrightarrow{f} D \rightarrow D_{H_0}$. Note that f' is continuous and D_{H_0} is Hausdorff, so $\text{Ker}(f')$ is closed in π^\vee . Therefore $M_0 = (\pi^\vee / \text{Ker}(f'))^\vee$ is naturally a subspace in π . We claim that M_0 lies in $\mathcal{M}(\pi^{H_0})$. Indeed, M_0^\vee is a quotient of $\pi_{H_0}^\vee$, hence $M_0 \leq \pi^{H_0}$ and it is Γ -invariant since f' is Γ -equivariant. M_0 is admissible because it is discrete, hence M_0^\vee is compact, equivalently finitely generated over $o/\varpi^h[[X]]$, because M_0^\vee can be identified with a $o/\varpi^h[[X]]$ -submodule of D_{H_0} which is finitely generated over $o/\varpi^h((X))$. The last thing to verify is that M is finitely generated over $o/\varpi^h[[X]][F]$, which follows from the following

Lemma 3.2.5 *Let D be an étale (φ, Γ) -module over $o/\varpi^h((X))$ and $D_0 \subset D$ be a ψ and Γ -invariant compact (or, equivalently, finitely generated) $o/\varpi^h[[X]]$ submodule. Then D_0^\vee is finitely generated as a module over $o/\varpi^h[[X]][F]$ where for any $m \in D_0^\vee = \text{Hom}_o(D_0, o/\varpi^h)$ we put $F(m)(f) = m(\psi(f))$ (for all $f \in D_0$).*

Proof As the extension of finitely generated modules over a ring is again finitely generated, we may assume without loss of generality that $h = 1$ and D is irreducible, ie. D has no nontrivial étale (φ, Γ) -submodule over $o/\varpi((X))$.

If $D_0 = \{0\}$ then there is nothing to prove. Otherwise D_0 contains the smallest ψ and Γ stable $o[[X]]$ -submodule D^\natural of D . So let $0 \neq m \in D_0^\vee$

be arbitrary such that the restriction of m to D^{\natural} is nonzero and consider the $o/\varpi[[X]][F]$ -submodule $M = o/\varpi[[X]][F]m$ of D_0^{\vee} generated by m . We claim that M is not finitely generated over o . Suppose for contradiction that the elements $F^r m$ are not linearly independent over o/ϖ . Then we have a polynomial $P(x) = \sum_{i=0}^n a_i x^i \in o/\varpi[x]$ such that $0 = P(F)m(f) = m(\sum a_i \psi^i(f)) = m(P(\psi)f)$ for any $f \in D^{\natural} \subset D_0$. However, $P(\psi): D^{\natural} \rightarrow D^{\natural}$ is surjective by Prop. II.5.15. in [5], so we obtain $m|_{D^{\natural}} = 0$ which is a contradiction. In particular, we obtain that $M^{\vee}[1/X] \neq 0$. However, note that $M^{\vee}[1/X]$ has the structure of an étale (φ, Γ) -module over $o/\varpi((X))$ by Lemma 2.6 in [3]. Indeed, M is admissible, Γ -invariant, and finitely generated over $o/\varpi[[X]][F]$ by construction. Moreover, we have a natural surjective homomorphism $D = D_0[1/X] = (D_0^{\vee})^{\vee}[1/X] \rightarrow M^{\vee}[1/X]$ which is an isomorphism as D is assumed to be irreducible. Therefore we have $(D_0^{\vee}/M)^{\vee}[1/X] = 0$ showing that D_0^{\vee}/M is finitely generated over o . In particular, both M and D_0^{\vee}/M are finitely generated over $o/\varpi[[X]][F]$ therefore so is D_0^{\vee} . \square

Now $D_0 = M_0^{\vee}$ is a ψ - and Γ -invariant $o/\varpi^h[[X]]$ -submodule of D therefore we have an injection $f_0: M_0^{\vee}[1/X] \hookrightarrow D$ of étale (φ, Γ) -modules. The map $\hat{f}_{H_0}: D_{\xi}^{\vee} \rightarrow D_{H_0}$ is the composite map $D_{\xi}^{\vee} \twoheadrightarrow M_0^{\vee}[1/X] \hookrightarrow D$. It is well defined and makes the above diagram commutative, because the map

$$\pi^{\vee} \rightarrow D_{SV}(\pi) \xrightarrow{\text{pr}} D_{\xi, \ell, \infty}^{\vee}(\pi) \xrightarrow{(\cdot)^{H_0}} D_{\xi}^{\vee}(\pi) \rightarrow M_0^{\vee}[1/X]$$

is the same as $\pi^{\vee} \rightarrow M_0^{\vee} \rightarrow M_0^{\vee}[1/X]$.

Finally, by Corollary 3.1.9 $M^{\vee}[1/X]$ (resp. D_{H_0}) corresponds to $M_{\infty}^{\vee}[1/X]$ (resp. to D) via the equivalence of categories in Theorem 8.20 in [18] therefore f_0 can uniquely be lifted to a φ - and Γ -equivariant $\Lambda_{\ell}(N_0)$ -homomorphism $f_{\infty}: M_{\infty}^{\vee}[1/X] \hookrightarrow D$. The map \hat{f} is defined as the composite $D_{\xi, \ell, \infty}^{\vee} \twoheadrightarrow M_{\infty}^{\vee}[1/X] \hookrightarrow D$. Now the image of $f - \hat{f} \circ \text{pr}$ is a ψ_s -invariant $\Lambda(N_0)$ -submodule in $(H_0 - 1)D$ therefore it is zero by Lemma 8.17 and the proof of Lemma 8.18 in [18]. Indeed, for any $x \in D_{SV}(\pi)$ and $k \geq 0$ we may write $(f - \hat{f} \circ \text{pr})(x)$ in the form $\sum_{u \in J(N_0/s^k N_0 s^{-k})} u \varphi^k((f - \hat{f} \circ \text{pr})(\psi^k(u^{-1}x)))$ that lies in $(H_k - 1)D$. \square

3.3 Étale hull

In this section we construct the étale hull of $D_{SV}(\pi)$: an étale T_+ -module $\widetilde{D}_{SV}(\pi)$ over $\Lambda(N_0)$ with an injection $\iota : D_{SV}(\pi) \rightarrow \widetilde{D}_{SV}(\pi)$ with the following universal property: For any étale (φ, Γ) -module D' over $\Lambda(N_0)$, and ψ_s - and Γ -equivariant map $f : D_{SV}(\pi) \rightarrow D'$, f factors through $\widetilde{D}_{SV}(\pi)$, ie. there exists a unique ψ - and Γ -equivariant $\Lambda(N_0)$ -homomorphism $\tilde{f} : \widetilde{D}_{SV}(\pi) \rightarrow D'$ making the diagram

$$\begin{array}{ccc} D_{SV}(\pi) & \xrightarrow{\iota} & \widetilde{D}_{SV}(\pi) \\ f \downarrow & \swarrow \tilde{f} & \\ D' & & \end{array}$$

commutative. Moreover, if we assume further that D' is an étale T_+ -module over $\Lambda(N_0)$ and the map f is ψ_t -equivariant for all $t \in T_+$ then the map \tilde{f} is T_+ -equivariant.

Definition Let D be a $\Lambda(N_0)$ -module and $T_* \leq T_+$ be a submonoid. Assume moreover that the monoid T_* (or in the case of ψ -actions the inverse monoid T_*^{-1}) acts \mathcal{o} -linearly on D , as well.

We call the action of T_* a φ -action (relative to the $\Lambda(N_0)$ -action) and denote the action of t by $d \mapsto \varphi_t(d)$, if for any $\lambda \in \Lambda(N_0)$, $t \in T_*$ and $d \in D$ we have $\varphi_t(\lambda d) = \varphi_t(\lambda)\varphi_t(d)$. Moreover, we say that the φ -action is *injective* if for all $t \in T_*$ the map φ_t is injective. The φ -action of T_* is *nondegenerate* if for all $t \in T_*$ we have

$$D = \sum_{u \in J(N_0/tN_0t^{-1})} \text{Im}(u \circ \varphi_t) = \sum_{u \in J(N_0/tN_0t^{-1})} u(\varphi_t(D)) .$$

We call the action of T_*^{-1} a ψ -action of T_* (relative to the $\Lambda(N_0)$ -action) and denote the action of $t^{-1} \in T_*^{-1}$ by $d \mapsto \psi_t(d)$, if for any $\lambda \in \Lambda(N_0)$, $t \in T_*$ and $d \in D$ we have $\psi_t(\varphi_t(\lambda)d) = \lambda\psi_t(d)$. Moreover, we say that the ψ -action of T_* is *surjective* if for all $t \in T_*$ the map ψ_t is surjective. The ψ -action of T_* is *nondegenerate* if for all $t \in T_*$ we have

$$\{0\} = \bigcap_{u \in J(N_0/tN_0t^{-1})} \text{Ker}(\psi_t \circ u^{-1}) .$$

The nondegeneracy is equivalent to the condition that for any $t \in T_*$ $\text{Ker}(\psi_t)$ does not contain any nonzero $\Lambda(N_0)$ -submodule of D .

We say that a φ - and a ψ -action of T_* are *compatible* on D , if

$(\varphi\psi)$ for any $t \in T_*$, $\lambda \in \Lambda(N_0)$, and $d \in D$ we have $\psi_t(\lambda\varphi_t(d)) = \psi_t(\lambda)d$.

Note that with $\lambda = 1$ we also have $\psi_t \circ \varphi_t = \text{id}_D$ for any $t \in T_*$ assuming $(\varphi\psi)$.

We also consider φ - and ψ -actions of the monoid $\mathbb{Z}_p \setminus \{0\}$ on $\Lambda(N_0)$ -modules via the embedding $\xi: \mathbb{Z}_p \setminus \{0\} \rightarrow T_+$. Modules with a φ -action (resp. ψ -action) of $\mathbb{Z}_p \setminus \{0\}$ are called (φ, Γ) -modules (resp. (ψ, Γ) -modules).

For example, the natural φ - and ψ -actions of T_+ on $\Lambda(N_0)$ are compatible.

Remarks 1. Note that the ψ -action of the monoid T_* is in fact an action of the inverse monoid T_*^{-1} . However, we assume T_+ to be commutative so it may also be viewed as an action of T_* .

2. Pontryagin duality provides an equivalence of categories between compact $\Lambda(N_0)$ -modules with a continuous ψ -action of T_* and discrete $\Lambda(N_0)$ -modules with a continuous φ -action of T_* . The surjectivity of the ψ -action corresponds to the injectivity of φ -action. Moreover, the ψ -action is nondegenerate if and only if so is the corresponding φ -action on the Pontryagin dual.

If D is a $\Lambda(N_0)$ -module with a φ -action of T_* then there exists a homomorphism

$$\Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_t} D \rightarrow D, \lambda \otimes d \mapsto \lambda\varphi_t(d) \quad (3.10)$$

of $\Lambda(N_0)$ -modules. We say that the T_* -action on D is *étale* if the above map is an isomorphism. The φ -action of T_* on D is étale if and only if it is injective and for any $t \in T_*$ we have

$$D = \bigoplus_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(D). \quad (3.11)$$

Similarly, we call a $\Lambda(N_0)$ -module together with a φ -action of the monoid $\mathbb{Z}_p \setminus \{0\}$ an étale (φ, Γ) -module over $\Lambda(N_0)$ if the action of $\varphi = \varphi_s$ is étale.

If D is an étale T_* -module over $\Lambda(N_0)$ then there exists a ψ -action of T_* compatible with the étale φ -action (see [17] Section 6).

Dually, if D is a $\Lambda(N_0)$ -module with a ψ -action of T_* then there exists a map

$$\begin{aligned} \iota_t: D &\rightarrow \Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_t} D \\ d &\mapsto \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(u^{-1}d). \end{aligned}$$

Lemma 3.3.1 *For any $t \in T_*$ the map ι_t is a homomorphism of $\Lambda(N_0)$ -modules. It is injective for all $t \in T_*$ if and only if the ψ -action of T_* on D is nondegenerate.*

Proof Fix $t \in T_*$. For any $\lambda \in \Lambda(N_0)$ and $u, v \in N_0$ we put $\lambda_{u,v} = \psi_t(u^{-1}\lambda v)$. Note that for any fixed $v \in N_0$ we have

$$\lambda v = \sum_{u \in J(N_0/tN_0t^{-1})} u \varphi_t(\lambda_{u,v})$$

and for any fixed $u \in N_0$ we have

$$u^{-1}\lambda = \sum_{v \in J(N_0/tN_0t^{-1})} \varphi_t(\lambda_{u,v})v^{-1}.$$

So we compute

$$\begin{aligned} \iota_t(\lambda x) &= \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(u^{-1}\lambda x) = \\ &= \sum_{u,v \in J(N_0/tN_0t^{-1})} u \otimes \psi_t(\varphi_t(\lambda_{u,v})v^{-1}x) = \\ &= \sum_{u,v \in J(N_0/tN_0t^{-1})} u \otimes \lambda_{u,v} \psi_t(v^{-1}x) = \\ &= \sum_{u,v \in J(N_0/tN_0t^{-1})} u \varphi_t(\lambda_{u,v}) \otimes \psi_t(v^{-1}x) = \\ &= \sum_{v \in J(N_0/tN_0t^{-1})} \lambda v \otimes \psi_t(v^{-1}x) = \lambda \iota_t(x). \end{aligned}$$

The second statement follows from noting that $\Lambda(N_0)$ is a free right module over itself via the map φ_t with free generators $u \in J(N_0/tN_0t^{-1})$. \square

Lemma 3.3.2 *Let D be a $\Lambda(N_0)$ -module with a ψ -action of T_* and $t \in T_*$. Then there exists a ψ -action of T_* on $\varphi_t^*D = \Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_t} D$ making the homomorphism ι_t ψ -equivariant. Moreover, if we assume in addition that the ψ -action on D is nondegenerate then so is the ψ -action on φ_t^*D .*

Proof Let $t' \in T_*$ be arbitrary and define the action of $\psi_{t'}$ on φ_t^*D by putting

$$\psi_{t'}(\lambda \otimes d) = \sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda \varphi_t(u')) \otimes \psi_{t'}(u'^{-1}d) \text{ for } \lambda \in \Lambda(N_0), d \in D,$$

and extending $\psi_{t'}$ to $\varphi_t^* D$ α -linearly. Note that we have

$$\begin{aligned} & \psi_{t'}(\varphi_{t'}(\mu)\lambda \otimes d) = \\ = & \sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\varphi_{t'}(\mu)\lambda\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}d) = \mu\psi_{t'}(\lambda \otimes d) . \end{aligned}$$

Moreover, the map $\psi_{t'}$ is well-defined since we have

$$\begin{aligned} \psi_{t'}(\lambda\varphi_t(\mu) \otimes d) &= \sum_{v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu)\varphi_t(v')) \otimes \psi_{t'}(v'^{-1}d) = \\ &= \sum_{v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(\mu v')) \otimes \psi_{t'}(v'^{-1}d) = \\ &= \sum_{u', v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(\mu_{u', v'}))) \otimes \psi_{t'}(v'^{-1}d) = \\ &= \sum_{u', v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u'))\varphi_t(\mu_{u', v'}) \otimes \psi_{t'}(v'^{-1}d) = \\ &= \sum_{u', v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \mu_{u', v'}\psi_{t'}(v'^{-1}d) = \\ &= \sum_{u', v' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \psi_{t'}(\varphi_{t'}(\mu_{u', v'})v'^{-1}d) = \\ &= \sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}\mu d) = \psi_{t'}(\lambda \otimes \mu d) , \end{aligned}$$

where $\mu_{u', v'} = \psi_{t'}(u'^{-1}\mu v')$. Introducing the notation $J' = J(N_0/t'N_0t'^{-1})$ and $J'' = J(N_0/t''N_0t''^{-1})$ we further compute

$$\begin{aligned} \psi_{t''}(\psi_{t'}(\lambda \otimes d)) &= \psi_{t''}\left(\sum_{u' \in J(N_0/t'N_0t'^{-1})} \psi_{t'}(\lambda\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}d)\right) = \\ &= \sum_{u'' \in J''} \sum_{u' \in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'))\varphi_t(u'')) \otimes \psi_{t''}(u''^{-1}\psi_{t'}(u'^{-1}d)) = \\ &= \sum_{u'' \in J''} \sum_{u' \in J'} \psi_{t''}(\psi_{t'}(\lambda\varphi_t(u'\varphi_{t'}(u'')))) \otimes \psi_{t''}(\psi_{t'}(\varphi_{t'}(u'')^{-1}u'^{-1}d)) = \\ &= \psi_{t''t'}(\lambda \otimes d) \end{aligned}$$

showing that it is indeed a ψ -action of the monoid T_* .

For the second statement of the Lemma we compute

$$\begin{aligned}
& \psi_{t'}(\iota_t(x)) = \\
&= \sum_{u' \in J(N_0/t'N_0t'^{-1})} \sum_{u \in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}\psi_t(u^{-1}x)) = \\
&= \sum_{u' \in J(N_0/t'N_0t'^{-1})} \sum_{u \in J(N_0/tN_0t^{-1})} \psi_{t'}(u\varphi_t(u')) \otimes \psi_{t'}(\psi_t(\varphi_t(u')^{-1}u^{-1}x)) .
\end{aligned}$$

Note that in the above sum $u\varphi_t(u')$ runs through a set of representatives for the cosets $N_0/tt'N_0t'^{-1}t^{-1}$. Moreover, $v = \psi_{t'}(u\varphi_t(u'))$ is nonzero if and only if $u\varphi_t(u')$ lies in $t'N_0t'^{-1}$ and the nonzero values of v run through a set $J'(N_0/tN_0t^{-1})$ of representatives of the cosets N_0/tN_0t^{-1} . In case $v \neq 0$ we have $\psi_{t'}(\varphi_t(u')^{-1}u^{-1}x) = \psi_{t'}(\varphi_t(u')^{-1}u^{-1})\psi_{t'}(x)$. So we obtain

$$\begin{aligned}
\psi_{t'}(\iota_t(x)) &= \sum_{v \in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(\psi_{t'}(\varphi_{t'}(v)x)) = \\
&= \sum_{v \in J'(N_0/tN_0t^{-1})} v \otimes \psi_t(v^{-1}\psi_{t'}(x)) = \iota_t(\psi_{t'}(x)) .
\end{aligned}$$

Assume now that the ψ -action of T_* on D is nondegenerate. Any element in $x \in \varphi_t^*D$ can be uniquely written in the form $\sum_{u \in J(N_0/tN_0t^{-1})} u \otimes x_u$. Assume that for a fixed $t' \in T_*$ we have $\psi_{t'}(u_0'^{-1}x) = 0$ for all $u_0' \in N_0$. Then we compute

$$\begin{aligned}
0 &= \psi_{t'}(u_0'^{-1}x) = \\
&= \sum_{u' \in J(N_0/t'N_0t'^{-1})} \sum_{u \in J(N_0/tN_0t^{-1})} \psi_{t'}(u_0'^{-1}u\varphi_t(u')) \otimes \psi_{t'}(u'^{-1}x_u) .
\end{aligned}$$

Put $y = u_0'^{-1}u\varphi_t(u')$. For any fixed u_0' the set

$$\{y \mid u \in J(N_0/tN_0t^{-1}), u' \in J(N_0/t'N_0t'^{-1})\}$$

forms a set of representatives of $N_0/tt'N_0(tt')^{-1}$, and we have $\psi_{t'}(y) \neq 0$ if and only if y lies in $t'N_0t'^{-1}$ in which case we have $\psi_{t'}(y) = t'^{-1}yt'$. So the nonzero values of $\psi_{t'}(y)$ run through a set of representatives of N_0/tN_0t^{-1} . Since we have the direct sum decomposition $\varphi_t^*D = \bigoplus_{v \in J(N_0/tN_0t^{-1})} v \otimes D$ we obtain $\psi_{t'}(u'^{-1}x_u) = 0$ for all $u' \in J(N_0/t'N_0t'^{-1})$ and $u \in J(N_0/tN_0t^{-1})$ such that $y = u_0'^{-1}u\varphi_t(u')$ is in $t'N_0t'^{-1}$. However, for any choice of u' and u there exists such a u_0' , so we deduce $x = 0$. \square

Proposition 3.3.3 *Let D be a $\Lambda(N_0)$ -module with a ψ -action of T_* . The following are equivalent:*

1. *There exists a unique φ -action on D , which is compatible with ψ and which makes D an étale T_* -module.*
2. *The ψ -action is surjective and for any $t \in T_*$ we have*

$$D = \bigoplus_{u_0 \in J(N_0/tN_0t^{-1})} \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq u_0}} \text{Ker}(\psi_t \circ u^{-1}) . \quad (3.12)$$

In particular, the action of ψ is nondegenerate.

3. *The map ι_t is bijective for all $t \in T_*$.*

Proof 1 \implies 3 In this case the map ι_t is the inverse of the isomorphism (3.10) so it is bijective by the étale property.

3 \implies 2: The injectivity of ι_t shows the nondegeneracy of the ψ -action. Further if $1 \otimes d = \iota_t(x)$ then we have $\psi_t(x) = d$ so the ψ -action is surjective. Moreover, $\iota_t^{-1}(u_0 \otimes D)$ equals $\bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \text{Ker}(\psi_t \circ u^{-1})$ therefore D can be written as a direct sum (3.12).

2 \implies 1: Fix $t \in T_*$. For any $d \in D$ we have to choose $\varphi_t(d)$ such that $\psi_t(\varphi_t(d)) = d$. By the surjectivity of ψ_t we can choose $x \in D$ such that $\psi_t(x) = d$. Using the assumption we can write $x = \sum_{u_0 \in J(N_0/tN_0t^{-1})} x_{u_0}$, with

$$x_{u_0} \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq u_0}} \text{Ker}(\psi_t \circ u^{-1}) .$$

By the compatibility $(\varphi\psi)$ we should have

$$\varphi_t(d) \in \bigcap_{\substack{u \in J(N_0/tN_0t^{-1}) \\ u \neq 1}} \text{Ker}(\psi_t \circ u^{-1})$$

as we have $\psi_t(u) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$.

A convenient choice is $\varphi_t(d) = x_1$, and there exists exactly one such element in D : if x' would be an other, then

$$x_1 - x' \in \bigcap_{u \in J(N_0/tN_0t^{-1})} \text{Ker}(\psi_t \circ u^{-1}) = \{0\} .$$

This shows the uniqueness of the φ -action. Further, $x_1 = \varphi_t(d) = 0$ would mean that x lies in $\text{Ker}(\psi_t)$ whence $d = \psi_t(x) = 0$ —therefore the injectivity. Similarly, by definition we also have $x_{u_0} = u_0\varphi_t \circ \psi_t(u_0^{-1}x)$ for all $u_0 \in J(N_0/tN_0t^{-1})$. By the surjectivity of the ψ -action any element in D can be written of the form $\psi_t(u_0^{-1}x)$ for any fixed $u_0 \in J(N_0/tN_0t^{-1})$ so we obtain

$$u_0\varphi_t(D) = \bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \text{Ker}(\psi_t \circ u^{-1}) .$$

The étale property (3.11) follows from this using our assumption 2. Moreover, this also shows $\psi_t(u\varphi_t(d)) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ which implies $(\varphi\psi)$ using that $\psi_t \circ \varphi_t = \text{id}_D$ by construction. Finally, $\varphi_t(\lambda)\varphi_t(d) - \varphi_t(\lambda d)$ lies in the kernel of $\psi_t \circ u_0^{-1}$ for any $u_0 \in J(N_0/tN_0t^{-1})$, $\lambda \in \Lambda(N_0)$ and $d \in D$, so it is zero. \square

From now on if we have an étale T_* -module over $\Lambda(N_0)$ we a priori equip it with the compatible ψ -action, and if we have a $\Lambda(N_0)$ -module with a ψ -action, which satisfies the above property 2, we equip it with the compatible φ -action, which makes it étale. The construction of the étale hull and its universal property is given in the following

Proposition 3.3.4 *For any $\Lambda(N_0)$ -module D , with a ψ -action of T_* there exists an étale T_* -module \tilde{D} over $\Lambda(N_0)$ and a ψ -equivariant $\Lambda(N_0)$ -homomorphism $\iota : D \rightarrow \tilde{D}$ with the following universal property: For any ψ -equivariant $\Lambda(N_0)$ -homomorphism $f : D \rightarrow D'$ into an étale T_* -module D' we have a unique morphism $\tilde{f} : \tilde{D} \rightarrow D'$ of étale T_* -modules over $\Lambda(N_0)$ making the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\iota} & \tilde{D} \\ f \downarrow & \swarrow \tilde{f} & \\ D' & & \end{array}$$

commutative. \tilde{D} is unique upto a unique isomorphism. If we assume the ψ -action on D to be nondegenerate then ι is injective.

Proof We will construct \tilde{D} as the injective limit of φ_t^*D for $t \in T_*$. Consider the following partial order on the set T_* : we put $t_1 \leq t_2$ whenever we have $t_2t_1^{-1} \in T_*$. Note that by Lemma 3.3.2 we obtain a ψ -equivariant isomorphism

$\varphi_{t_2 t_1^{-1}}^* \varphi_{t_1}^* D \cong \varphi_{t_2}^* D$ for any pair $t_1 \leq t_2$ in T_* . In particular, we obtain a ψ -equivariant map $\iota_{t_1, t_2}: \varphi_{t_1}^* D \rightarrow \varphi_{t_2}^* D$. Applying this observation to $\varphi_{t_1}^* D$ for a sequence $t_1 \leq t_2 \leq t_3$ we see that the $\Lambda(N_0)$ -modules $\varphi_t^* D$ ($t \in T_*$) with the ψ -action of T_* form a direct system with respect to the connecting maps ι_{t_1, t_2} . We put

$$\tilde{D} = \varinjlim_{t \in T_*} \varphi_t^* D$$

as a $\Lambda(N_0)$ -module with a ψ -action of T_* . For any fixed $t' \in T_*$ we have

$$\begin{aligned} \varphi_{t'}^* \tilde{D} &= \Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_{t'}} \varinjlim_{t \in T_*} \varphi_t^* D \cong \\ &\cong \varinjlim_{t \in T_*} \Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_{t'}} \varphi_t^* D \cong \varinjlim_{t' t \in T_*} \varphi_{t' t}^* D \cong \tilde{D} \end{aligned}$$

showing that there exists a unique φ -action of T_* on \tilde{D} making \tilde{D} an étale T_* -module over $\Lambda(N_0)$ by Proposition 3.3.3.

For the universal property, let $f: D \rightarrow D'$ be an ψ -equivariant map into an étale T_* -module D' over $\Lambda(N_0)$. By construction of the map φ_t on \tilde{D} ($t \in T_*$) we have $\varphi_t(\iota(x)) = (1 \otimes x)_t$ where $(1 \otimes x)_t$ denotes the image of $1 \otimes x \in \varphi_t^* D$ in \tilde{D} . So we put

$$\tilde{f}((\lambda \otimes x)_t) = \lambda \varphi_t(f(x)) \in D'$$

and extend it \mathcal{o} -linearly to \tilde{D} . Note right away that \tilde{f} is unique as it is φ_t -equivariant. The map $\tilde{f}: \tilde{D} \rightarrow D'$ is well-defined as we have

$$\begin{aligned} \tilde{f}(\iota_{t, t'}(1 \otimes x)) &= \tilde{f}\left(\sum_{u' \in N_0/t'N_0t'^{-1}} u' \otimes_{t'} \psi_{t'}(u'^{-1} \otimes_t x)\right) = \\ &= \sum_{u', v' \in N_0/t'N_0t'^{-1}} \tilde{f}(u' \otimes_{t'} \psi_{t'}(u'^{-1} \varphi_t(v')) \otimes_t \psi_{t'}(v'^{-1} x)) = \\ &= \sum_{u', v' \in N_0/t'N_0t'^{-1}} \tilde{f}(u' \varphi_{t'} \circ \psi_{t'}(u'^{-1} \varphi_t(v')) \otimes_{t'} \psi_{t'}(v'^{-1} x)) = \\ &= \sum_{v' \in N_0/t'N_0t'^{-1}} \tilde{f}(\varphi_t(v') \otimes_{t'} \psi_{t'}(v'^{-1} x)) = \\ &= \sum_{v' \in N_0/t'N_0t'^{-1}} \varphi_t(v') \varphi_{t'}(f(\psi_{t'}(v'^{-1} x))) = \\ &= \sum_{v' \in N_0/t'N_0t'^{-1}} \varphi_t(v' \varphi_{t'} \circ \psi_{t'}(v'^{-1} f(x))) = \varphi_t(f(x)) = \tilde{f}(1 \otimes_t x) \end{aligned}$$

noting that $\iota_{t,tt'}$ is a $\Lambda(N_0)$ -homomorphism. Here the notation \otimes_t indicates that the tensor product is via the map φ_t . By construction \tilde{f} is a homomorphism of étale T_* -modules over $\Lambda(N_0)$ satisfying $\tilde{f} \circ \iota = f$.

The injectivity of ι in case the ψ -action on D is nondegenerate follows from Lemmata 3.3.1 and 3.3.2. \square

Example If D itself is étale then we have $\tilde{D} = D$.

Corollary 3.3.5 *The functor $D \mapsto \tilde{D}$ from the category of $\Lambda(N_0)$ -modules with a ψ -action of T_* to the category of étale T_* -modules over $\Lambda(N_0)$ is exact.*

Proof $\Lambda(N_0)$ is a free $\varphi_t(\Lambda(N_0))$ -module, so $\Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_t} -$ is exact, and so is the direct limit functor. \square

Corollary 3.3.6 *Assume that D is a $\Lambda(N_0)$ -module with a nondegenerate ψ -action of T_* and $f: D \rightarrow D'$ is an injective ψ -equivariant $\Lambda(N_0)$ -homomorphism into the étale T_* -module D' over $\Lambda(N_0)$. Then \tilde{f} is also injective.*

Proof Since D is nondegenerate we may identify $\varphi_t^* D$ with a $\Lambda(N_0)$ -submodule of \tilde{D} . Assume that $x = \sum_{u \in J(N_0/tN_0t^{-1})} u \otimes_t x_u \in \varphi_t^* D$ lies in the kernel of \tilde{f} . Then $x_u = \psi_t(u^{-1}x) \in D \subseteq \varphi_t^* D \subseteq \tilde{D}$ ($u \in J(N_0/tN_0t^{-1})$) also lies in the kernel of \tilde{f} . However, we have $\tilde{f}(x_u) = f(x_u)$ showing that $x_u = 0$ for all $u \in J(N_0/tN_0t^{-1})$ as f is injective. \square

Example Let D be a (classical) irreducible étale (φ, Γ) -module over $k((X))$ and $D_0 \subset D$ a ψ - and Γ -invariant treillis in D . Then we have $\tilde{D}_0 \cong D$ unless D is 1-dimensional and $D_0 = D^\natural$ in which case we have $\tilde{D}_0 = D_0$.

Proof If D is 1-dimensional then $D^\natural = D^+$ is an étale (φ, Γ) -module over $k[[X]]$ (Prop. II.5.14 in [5]) therefore it is equal to its étale hull. If $\dim D > 1$ then we have $D^\natural = D^\# \subseteq D_0$ by Cor. II.5.12 and II.5.21 in [5]. By Corollary 3.3.6 $\tilde{D}^\# \subseteq \tilde{D}_0$ injects into D and it is φ - and ψ -invariant. Since $D^\#$ is not φ -invariant (Prop. II.5.14 in [5]) and it is the maximal compact $o[[X]]$ -submodule of D on which ψ acts surjectively (Prop. II.4.2 in [5]) we obtain that \tilde{D}_0 is not compact. In particular, its X -divisible part is nonzero therefore equals D as the X -divisible part of \tilde{D}_0 is an étale (φ, Γ) -submodule of the irreducible D . \square

Proposition 3.3.7 *The T_+^{-1} action on $D_{SV}(\pi)$ is a surjective nondegenerate ψ -action of T_+ .*

Proof Let $d \in D_{SV}(\pi)$ and $t \in T_+$. Since the action of both t and $\Lambda(N_0)$ on $D_{SV}(\pi)$ comes from that on π^\vee we have $t^{-1}\varphi_t(\lambda)d = t^{-1}t\lambda t^{-1}d = \lambda t^{-1}d$, so this is indeed a ψ -action. The surjectivity of each ψ_t follows from the injectivity of the multiplication by t on each $W \in \mathcal{B}_+(\pi)$. Finally, if W is in $\mathcal{B}_+(\pi)$ then so is $t^*W = \sum_{u \in J(N_0/tN_0t^{-1})} utW$ for any $t \in T_+$. Take an element $d \in D_{SV}(\pi)$ lying in the kernel of $\psi_t(u^{-1}\cdot)$ for all $u \in J(N_0/tN_0t^{-1})$. Then there exists a generating B_+ -subrepresentation W of π such that the restriction of $t^{-1}u^{-1}d$ to W is zero for all $u \in J(N_0/tN_0t^{-1})$. Then the restriction of d to t^*W is zero showing that d is zero in $D_{SV}(\pi)$ therefore the nondegeneracy. Alternatively, the nondegeneracy of the ψ -action also follows from the existence of a ψ -equivariant injective map $D_{SV}(\pi) \hookrightarrow D_{SV}^0(\pi)$ into an étale T_+ -module $D_{SV}^0(\pi)$ ([17] Proposition 3.5 and Remark 6.1). \square

Question Let $D_{SV}^{(0)}(\pi)$ as in [17]. We have that $D_{SV}^{(0)}(\pi)$ is an étale T_* -module over $\Lambda(N_0)$ ([17] Proposition 3.5) and $f : D_{SV}(\pi) \hookrightarrow D_{SV}^{(0)}(\pi)$ is a ψ -equivariant map ([17] Remark 6.1). By the universal property of the étale hull and Corollary 3.3.6 $\widetilde{D}_{SV}(\pi)$ also injects into $D_{SV}^{(0)}(\pi)$. Whether or not this injection is always an isomorphism is an open question. In case of the Steinberg representation this is true by Proposition 11 in [22].

We call the submonoid $T'_* \leq T_* \leq T_+$ cofinal in T_* if for any $t \in T_*$ there exists a $t' \in T'_*$ such that $t \leq t'$. For example $\xi(\mathbb{Z}_p \setminus \{0\})$ is cofinal in T_+ .

Corollary 3.3.8 *Let D be a $\Lambda(N_0)$ -module with a ψ -action of T_* and denote by \widetilde{D} (resp. by \widetilde{D}') the étale hull of D for the ψ -action of T_* (resp. of T'_*). Then we have a natural isomorphism $\widetilde{D}' \xrightarrow{\sim} \widetilde{D}$ of étale T'_* -modules over $\Lambda(N_0)$. More precisely, if $f : D \rightarrow D_1$ is a ψ -equivariant $\Lambda(N_0)$ -homomorphism into an étale T'_* -module D_1 then f factors uniquely through $\iota : D \rightarrow \widetilde{D}$.*

Proof Since $T'_* \leq T_*$ is cofinal in T_* we have

$$\varinjlim_{t' \in T'_*} \varphi_{t'}^* D \cong \varinjlim_{t \in T_*} \varphi_t^* D = \widetilde{D}.$$

\square

By Corollary 3.3.8 there exists a homomorphism $\tilde{\text{pr}} : \widetilde{D_{SV}}(\pi) \rightarrow D_{\xi, \ell, \infty}^{\vee}(\pi)$ of étale (φ, Γ) -modules over $\Lambda(N_0)$ such that $\text{pr} = \tilde{\text{pr}} \circ \iota$. Our main result in this section is the following

Theorem 3.3.9 $D_{\xi, \ell, \infty}^{\vee}(\pi)$ is the pseudocompact completion of $\Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ in the category of étale (φ, Γ) -modules over $\Lambda_{\ell}(N_0)$, ie. we have

$$D_{\xi, \ell, \infty}^{\vee}(\pi) \cong \varprojlim_D D$$

where D runs through the finitely generated étale (φ, Γ) -modules over $\Lambda_{\ell}(N_0)$ arising as a quotient of $\Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by a closed submodule. This holds in any topology on $\Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ making both the maps $1 \otimes \iota : D_{SV}(\pi) \rightarrow \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$, $d \mapsto 1 \otimes \iota(d)$ and $1 \otimes \tilde{\text{pr}} : \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \rightarrow D_{\xi, \ell, \infty}^{\vee}(\pi)$ continuous.

Remark Since the map $\text{pr} : D_{SV}(\pi) \rightarrow D_{\xi, \ell, \infty}^{\vee}(\pi)$ is continuous, there exists such a topology on $\Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$. For instance we could take either the final topology of the map $D_{SV}(\pi) \rightarrow \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ or the initial topology of the map $\Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \rightarrow D_{\xi, \ell, \infty}^{\vee}(\pi)$.

Proof The homomorphism $\tilde{\text{pr}}$ factors through the map $1 \otimes \text{id} : \widetilde{D_{SV}}(\pi) \rightarrow \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ since $D_{\xi, \ell, \infty}^{\vee}(\pi)$ is a module over $\Lambda_{\ell}(N_0)$, so we obtain a homomorphism

$$1 \otimes \tilde{\text{pr}} : \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \rightarrow D_{\xi, \ell, \infty}^{\vee}(\pi)$$

of étale (φ, Γ) -modules over $\Lambda_{\ell}(N_0)$. At first we claim that $1 \otimes \tilde{\text{pr}}$ has dense image. Let $M \in \mathcal{M}(\pi^{H_0})$ and $W \in \mathcal{B}_+(\pi)$ be arbitrary. Then by Lemma 3.2.1 the map $\text{pr}_{W, M, k} : W^{\vee} \rightarrow M_k^{\vee}$ is surjective for $k \geq 0$ large enough. This shows that the natural map

$$1 \otimes \text{pr}_{W, M, k} : \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} W^{\vee} \rightarrow \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} M_k^{\vee} \cong M_k^{\vee}[1/X]$$

is surjective. However, $1 \otimes \text{pr}_{W, M, k}$ factors through $\Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi)$ by the Remarks after Lemma 3.2.2. In particular, the natural map

$$1 \otimes \text{pr}_{M, k} : \Lambda_{\ell}(N_0) \otimes_{\Lambda(N_0)} \widetilde{D_{SV}}(\pi) \rightarrow M_k^{\vee}[1/X]$$

is surjective for all $M \in \mathcal{M}(\pi^{H_0})$ and $k \geq 0$ large enough (whence in fact for all $k \geq 0$). This shows that the image of the map

$$1 \otimes \text{pr}: \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \rightarrow D_{\xi, \ell, \infty}^\vee(\pi)$$

is dense whence so is the image of $1 \otimes \widetilde{\text{pr}}$. By the assumption that $1 \otimes \widetilde{\text{pr}}$ is continuous we obtain a surjective homomorphism

$$\widehat{1 \otimes \widetilde{\text{pr}}}: \varprojlim_D D \rightarrow D_{\xi, \ell, \infty}^\vee(\pi)$$

of pseudocompact (φ, Γ) -modules over $\Lambda_\ell(N_0)$ where D runs through the finitely generated étale (φ, Γ) -modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi)$.

Let $0 \neq (x_D)_D$ be in the kernel of $\widehat{1 \otimes \widetilde{\text{pr}}}$. Then there exists a finitely generated étale (φ, Γ) -module D over $\Lambda_\ell(N_0)$ with a surjective continuous homomorphism $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi) \twoheadrightarrow D$ such that $x_D \neq 0$. By Proposition 3.2.4 this map factors through $D_{\xi, \ell, \infty}^\vee(\pi)$ contradicting to the assumption $\widehat{1 \otimes \widetilde{\text{pr}}}((x_D)_D) = 0$. \square

Remark Breuil's functor D_ξ^\vee can therefore be computed from D_{SV} the following way: For a smooth \mathfrak{o}/ϖ^h -representation π we have

$$D_\xi^\vee(\pi) \cong (\varprojlim_D D)_{H_0} \cong \varprojlim_D D_{H_0}$$

where D runs through the finitely generated étale (φ, Γ) -modules over $\Lambda_\ell(N_0)$ arising as a quotient of $\Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi)$ by a closed submodule.

Chapter 4

Nongeneric ℓ

Assume from now on that $\ell = \ell_\alpha$ is a nongeneric Whittaker functional defined by the projection of N_0 onto $N_{\alpha,0} \cong \mathbb{Z}_p$ for some simple root $\alpha \in \Delta$.

4.1 The action of T_+

Our goal in this section is to define a φ -action of T_+ on $D_{\xi,\ell,\infty}^\vee(\pi)$ or equivalently, on $D_\xi^\vee(\pi)$ extending the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and making $D_{\xi,\ell,\infty}^\vee(\pi)$ an étale T_+ -module over $\Lambda_\ell(N_0)$. Let $t \in T_+$ be arbitrary. Note that by the choice of this ℓ we have $tH_0t^{-1} \subseteq H_0$. In particular, T_+ acts via conjugation on the ring $\Lambda(N_0/H_0) \cong o[[X]]$; we denote the action of $t \in T_+$ by φ_t . This action is via the character α mapping T_+ onto $\mathbb{Z}_p \setminus \{0\}$. In particular, $o[[X]]$ is a free module of finite rank over itself via φ_t . Moreover, we define the Hecke action of $t \in T_+$ on π^{H_0} by the formula $F_t(m) := \text{Tr}_{H_0/tH_0t^{-1}}(tm)$ for any $m \in \pi^{H_0}$. For $t, t' \in T_+$ we have

$$\begin{aligned} F_{t'} \circ F_t &= \text{Tr}_{H_0/t'H_0t'^{-1}} \circ (t' \cdot) \circ \text{Tr}_{H_0/tH_0t^{-1}} \circ (t \cdot) = \\ &= \text{Tr}_{H_0/t'H_0t'^{-1}} \circ \text{Tr}_{t'H_0t'^{-1}/t'H_0t^{-1}t'^{-1}} \circ (t't \cdot) = F_{t't} . \end{aligned}$$

For any $M \in \mathcal{M}(\pi^{H_0})$ we put $F_t^* M := N_0 F_t(M)$.

Lemma 4.1.1 *For any $M \in \mathcal{M}(\pi^{H_0})$ we have $F_t^* M \in \mathcal{M}(\pi^{H_0})$.*

Proof We have

$$\begin{aligned} F(F_t^* M) &= F(N_0 F_t(M)) \subset N_0 F F_t(M) = \\ &= N_0 F_{st}(M) = N_0 F_t(F(M)) \subseteq F_t^* M . \end{aligned}$$

So F_t^*M is a module over $\Lambda(N_0/H_0)/\varpi^h[F]$. Moreover, if m_1, \dots, m_r generates M , then the elements $F_t(m_i)$ ($1 \leq i \leq r$) generate F_t^*M , so it is finitely generated. The admissibility is clear as $F_t^*M = \sum_{u \in J(N_0/tN_0t^{-1})} uF_t(M)$ is the sum of finitely many admissible submodules. Finally, F_t^*M is stable under the action of Γ as F_t commutes with the action of Γ . \square

By the definition of F_t^*M we have a surjective $o/\varpi^h[[X]]$ -homomorphism

$$1 \otimes F_t: o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]], \varphi_t} M \rightarrow F_t^*M$$

which gives rise to an injective $o/\varpi^h((X))$ -homomorphism

$$(1 \otimes F_t)^\vee[1/X]: (F_t^*M)^\vee[1/X] \hookrightarrow o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} M^\vee[1/X]. \quad (4.1)$$

Moreover, there is a structure of an $o/\varpi^h[[X]][F]$ -module on

$$o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]], \varphi_t} M$$

by putting $F(\lambda \otimes m) := \varphi_t(\lambda) \otimes F(m)$. Similarly, the group Γ also acts on $o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]], \varphi_t} M$ semilinearly. The map $1 \otimes F_t$ is F and Γ -equivariant as F_t, F , and the action of Γ all commute. We deduce that $(1 \otimes F_t)^\vee[1/X]$ is a φ - and Γ -equivariant map of étalae (φ, Γ) -modules.

Note that for any $t \in T_+$ there exists a positive integer $k \geq 0$ such that $t \leq s^k$, ie. $t' := t^{-1}s^k$ lies in T_+ . So we have $F_t^*(F_{t'}^*M) = F_{s^k}^*M = N_0F^k(M) \subseteq M$. So we obtain an isomorphism $M^\vee[1/X] \cong (F_{s^k}^*M)^\vee[1/X] = (F_t^*(F_{t'}^*M))^\vee[1/X]$ as $M/N_0F^k(M)$ is finitely generated over o .

Lemma 4.1.2 *The map (4.1) is an isomorphism of étale (φ, Γ) -modules for any $M \in \mathcal{M}(\pi^{H_0})$ and $t \in T_+$.*

Proof The composite $(1 \otimes F_{t'})^\vee[1/X] \circ (1 \otimes F_t)^\vee[1/X] = (1 \otimes F^k)^\vee[1/X]$ is an isomorphism by Lemma 2.6 in [3]. So $(1 \otimes F_t)^\vee[1/X]$ is also an isomorphism as both $(1 \otimes F_t)^\vee[1/X]$ and $(1 \otimes F_{t'})^\vee[1/X]$ are injective. \square

Now taking projective limits we obtain an isomorphism of pseudocompact étalae (φ, Γ) -modules

$$\begin{aligned} (1 \otimes F_t)^\vee[1/X]: D_\xi^\vee(\pi) &\rightarrow \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} M^\vee[1/X]) \\ (m)_{(F_t^*M)^\vee[1/X]} &\mapsto ((1 \otimes F_t)^\vee[1/X](m))_{M^\vee[1/X]}. \end{aligned}$$

Moreover, since $o((X))$ is finite free over itself via φ_t , we have an identification

$$\begin{aligned} \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} M^\vee[1/X]) &\cong \\ &\cong o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} D_\xi^\vee(\pi). \end{aligned}$$

Using the maps $(1 \otimes F_t)^\vee[1/X]$ we define a φ -action of T_+ on $D_\xi^\vee(\pi)$ by putting $\varphi_t(d) := ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d)$ for $d \in D_\xi^\vee(\pi)$.

Proposition 4.1.3 *The above action of T_+ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D_\xi^\vee(\pi)$ into an étale T_+ -module over $o/\varpi^h[[X]]$.*

Proof By the definition of the T_+ -action it is indeed an extension of the action of the monoid $\mathbb{Z}_p \setminus \{0\}$. For $t, t' \in T_+$ we compute

$$\begin{aligned} \varphi_{t'} \circ \varphi_t(d) &= ((1 \otimes F_{t'})^\vee[1/X])^{-1} \circ ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = \\ &= ((1 \otimes F_t)^\vee[1/X] \circ (1 \otimes F_{t'})^\vee[1/X])^{-1}(1 \otimes d) = \\ &= ((1 \otimes F_{t't})^\vee[1/X])^{-1}(1 \otimes d) = \varphi_{t't}(d) = \varphi_{t't}(d). \end{aligned}$$

Further, we have

$$\begin{aligned} \varphi_t(\lambda d) &= ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes \lambda d) = ((1 \otimes F_t)^\vee[1/X])^{-1}(\varphi_t(\lambda) \otimes d) = \\ &= \varphi_t(\lambda)((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = \varphi_t(\lambda)\varphi_t(d) \end{aligned}$$

showing that this is indeed a φ -action of T_+ . The étale property follows from the fact that $(1 \otimes F_t)^\vee[1/X]$ is an isomorphism for each $t \in T_+$. \square

The inclusion $u_\alpha: \mathbb{Z}_p \rightarrow N_{\alpha,0} \leq N_0$ induces an injective ring homomorphism—still denoted by u_α by a certain abuse of notation— $u_\alpha: \widehat{o((X))}^p \hookrightarrow \Lambda_\ell(N_0)$ where $\widehat{o((X))}^p$ denotes the p -adic completion of the Laurent-series ring $o((X))$. For each $t \in T_+$ this gives rise to a commutative diagram

$$\begin{array}{ccc} \widehat{o((X))}^p & \xrightarrow{u_\alpha} & \Lambda_\ell(N_0) \\ \varphi_t \downarrow & & \downarrow \varphi_t \\ \widehat{o((X))}^p & \xrightarrow{u_\alpha} & \Lambda_\ell(N_0) \end{array}$$

with injective ring homomorphisms. On the other hand, by the equivalence of categories in Thm. 8.20 in [18] we have a φ - and Γ -equivariant identification $M_\infty^\vee[1/X] \cong \Lambda_\ell(N_0) \otimes_{o(\widehat{(X)})^p, u_\alpha} M^\vee[1/X]$. Therefore tensoring the isomorphism (4.1) with $\Lambda_\ell(N_0)$ via u_α we obtain an isomorphism

$$\begin{aligned} (1 \otimes F_t)_\infty^\vee[1/X]: (F_t^* M)_\infty^\vee[1/X] &\cong \Lambda_\ell(N_0) \otimes_{u_\alpha} (F_t^* M)^\vee[1/X] \rightarrow \\ &\rightarrow \Lambda_\ell(N_0) \otimes_{u_\alpha} o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} M^\vee[1/X] \cong \\ &\cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} \Lambda_\ell(N_0) \otimes_{u_\alpha} M^\vee[1/X] \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} M_\infty^\vee[1/X]. \end{aligned} \quad (4.2)$$

Taking projective limits again we deduce an isomorphism

$$\begin{aligned} (1 \otimes F_t)_\infty^\vee[1/X]: D_{\xi, \ell, \infty}^\vee(\pi) &\rightarrow \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} D_{\xi, \ell, \infty}^\vee(\pi) \\ (m)_{(F_t^* M)_\infty^\vee[1/X]} &\mapsto ((1 \otimes F_t)_\infty^\vee[1/X](m))_{M_\infty^\vee[1/X]} \end{aligned}$$

for all $t \in T_+$ using the identification

$$\varprojlim_{M \in \mathcal{M}(\pi^{H_0})} (\Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} M_\infty^\vee[1/X]) \cong \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0), \varphi_t} D_{\xi, \ell, \infty}^\vee(\pi).$$

Using the maps $(1 \otimes F_t)_\infty^\vee[1/X]$ we define a φ -action of T_+ on $D_{\xi, \ell, \infty}^\vee(\pi)$ by putting $\varphi_t(d) := ((1 \otimes F_t)_\infty^\vee[1/X])^{-1}(1 \otimes d)$ for $d \in D_{\xi, \ell, \infty}^\vee(\pi)$.

Corollary 4.1.4 *The above action of T_+ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D_{\xi, \ell, \infty}^\vee(\pi)$ into an étale T_+ -module over $\Lambda_\ell(N_0)$. The reduction map $D_{\xi, \ell, \infty}^\vee(\pi) \rightarrow D_\xi^\vee(\pi)$ is T_+ -equivariant for the φ -action.*

We can view this φ -action of T_+ in a different way: Let us define $F_{t,k} := \text{Tr}_{H_k/tH_k t^{-1}} \circ (t \cdot)$. Then we have a map

$$1 \otimes F_{t,k}: \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h, \varphi_t} M_k \rightarrow F_{t,k}^* M_k := N_0 F_{t,k}(M_k), \quad (4.3)$$

where we have $F_{t,k}^* M \in \mathcal{M}_k(\pi^{H_k})$. Let k be large enough such that we have $tH_0 t^{-1} \geq H_k$. After taking Pontryagin duals, inverting X , taking projective limit and using the remark after Lemma 3.1.5 we obtain a homomorphism of étale (φ, Γ) -modules

$$\varprojlim_k \text{Tr}_{t^{-1}H_k t}^{-1} \circ (1 \otimes F_{t,k})^\vee[1/X]: (F_t^* M)_\infty^\vee[1/X] \rightarrow \Lambda_\ell(N_0) \otimes_{\varphi_t} M_\infty^\vee[1/X]. \quad (4.4)$$

This map is indeed Γ - and φ -equivariant because we compute

$$\begin{aligned} F_k \circ F_{t,k} &= \mathrm{Tr}_{H_k/sH_k s^{-1}} \circ (s \cdot) \circ \mathrm{Tr}_{H_k/tH_k t^{-1}} \circ (t \cdot) = \\ &= \mathrm{Tr}_{H_k/s^k t H_k t^{-1} s^{-k}} \circ (s^k t \cdot) = \\ &= \mathrm{Tr}_{H_k/tH_k t^{-1}} \circ (t \cdot) \circ \mathrm{Tr}_{H_k/sH_k s^{-1}} \circ (s \cdot) = F_{t,k} \circ F_k . \end{aligned}$$

Now we have two maps (4.2) and (4.4) between $(F_t^* M)_\infty^\vee[1/X]$ and $\Lambda_\ell(N_0) \otimes_{\varphi_t} M_\infty^\vee[1/X]$ that agree after taking H_0 -coinvariants by definition. Hence they are equal by the equivalence of categories in Thm. 8.20 in [18].

We obtain in particular that the map (4.3) has finite kernel and cokernel as it becomes an isomorphism after taking Pontryagin duals and inverting X . Hence there exists a finite $\Lambda(N_0/H_k)/\varpi^h$ -submodule $M_{t,k,*}$ of M_k such that the kernel of $1 \otimes F_{t,k}$ is contained in the image of $\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi} M_{t,k,*}$ in $\Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi} M_k$. We denote by $M_{t,k}^* \leq F_{t,k}^* M_k$ the image of $1 \otimes F_{t,k}$. We conclude that as in Proposition 3.1.6, we can describe the φ_t -action in the following way:

$$\begin{aligned} \varphi_t: M_k^\vee[1/X] &\rightarrow (F_{t,k}^* M_k)^\vee[1/X] \\ f &\mapsto (\mathrm{Tr}_{t^{-1}H_k t/H_k}^{-1} \circ (1 \otimes F_{t,k}))^\vee[1/X]^{-1}(1 \otimes f) \quad (4.5) \end{aligned}$$

Being an étale T_+ -module over $\Lambda_\ell(N_0)$ we equip $D_{\xi,\ell,\infty}^\vee(\pi)$ with the ψ -action of T_+ : ψ_t is the canonical left inverse of φ_t for all $t \in T_+$.

Proposition 4.1.5 *The map $\mathrm{pr}: D_{SV}(\pi) \rightarrow D_{\xi,\ell,\infty}^\vee(\pi)$ is ψ -equivariant for the ψ -actions of T_+ on both sides.*

Proof We proceed as in the proofs of Proposition 3.1.8 and Lemma 3.2.2. We fix $t \in T_+$, $W \in \mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$ and show that $\mathrm{pr}_{W,M}$ is ψ_t -equivariant. Fix k such that $F_{t,k}^* M_k \leq W$ and $tH_0 t^{-1} \geq H_k$.

At first we compute the formula analogous to (3.7). Let f be in M_k^\vee such that its restriction to $M_{t,k,*}$ is zero and $m \in M_{t,k}^* \leq F_{t,k}^* M_k$ be in the form

$$m = \sum_{u \in J(N_0/tN_0 t^{-1})} u F_{t,k}(m_u)$$

with elements $m_u \in M_k$ for $u \in J(N_0/tN_0 t^{-1})$. $M_{t,k}^*$ is a finite index submodule of $F_{t,k}^* M_k$. Note that the elements m_u are unique upto $M_{t,k,*} + \mathrm{Ker}(F_{t,k})$. Therefore $\varphi_t(f) \in (M_{t,k}^*)^\vee$ is well-defined by our assumption that $f|_{M_{t,k,*}} = 0$

noting that the kernel of $F_{t,k}$ equals the kernel of $\mathrm{Tr}_{t^{-1}H_k t/H_k}$ since the multiplication by t is injective and we have $F_{t,k} = t \circ \mathrm{Tr}_{t^{-1}H_k t/H_k}$. So we compute

$$\begin{aligned} \varphi_t(f)(m) &= ((1 \otimes F_{t,k})^\vee)^{-1}(\mathrm{Tr}_{t^{-1}H_k t/H_k}(1 \otimes f))(m) = \\ &= ((1 \otimes F_{t,k})^\vee)^{-1}(1 \otimes \mathrm{Tr}_{t^{-1}H_k t/H_k}(f))\left(\sum_{u \in J((N_0/H_k)/t(N_0/H_k)t^{-1})} u F_{t,k}(m_u)\right) = \\ &= \mathrm{Tr}_{t^{-1}H_k t/H_k}(f)(F_{t,k}^{-1}(u_0 F_{t,k}(m_{u_0}))) = f(\mathrm{Tr}_{t^{-1}H_k t/H_k}((t^{-1}u_0 t)m_{u_0})) \end{aligned} \quad (4.6)$$

where u_0 is the single element in $J(N_0/tN_0t^{-1})$ corresponding to the coset of 1.

Now let f be in W^\vee such that the restriction $f|_{N_0 t M_{t,k,*}} = 0$. By definition we have $\psi_t(f)(w) = f(tw)$ for any $w \in W$. Choose an element $m \in M_{t,k}^* \subset F_{t,k}^* M_k$ written in the form

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} u F_{t,k}(m_u) = \sum_{u \in J(N_0/tN_0t^{-1})} u t \mathrm{Tr}_{t^{-1}H_k t/H_k}(m_u).$$

Then we compute

$$\begin{aligned} f|_{F_{t,k}^* M_k}(m) &= \sum_{u \in J(N_0/tN_0t^{-1})} f(ut \mathrm{Tr}_{t^{-1}H_k t/H_k}(m_u)) = \\ &= \sum_{u \in J(N_0/tN_0t^{-1})} \psi_t(u^{-1}f)(\mathrm{Tr}_{t^{-1}H_k t/H_k}(m_u)) = \\ &\stackrel{(4.6)}{=} \sum_{u \in J(N_0/tN_0t^{-1})} \varphi_t(\psi_t(u^{-1}f)|_{F_{t,k}^* M_k})(F_{t,k}(m_u)) = \\ &= \sum_{u \in J(N_0/tN_0t^{-1})} u \varphi_t(\psi_t(u^{-1}f)|_{M_k})(u F_{t,k}(m_u)) = \\ &= \sum_{u \in J(N_0/tN_0t^{-1})} u \varphi_t(\psi_t(u^{-1}f)|_{M_k})(m) \end{aligned}$$

as for distinct $u, v \in J(N_0/tN_0t^{-1})$ we have $u \varphi_t(f_0)(v F_{t,k}(m_v)) = 0$ for any $f_0 \in (M_{t,k}^*)^\vee$. So by inverting X and taking projective limits with respect to k we obtain

$$\mathrm{pr}_{W, F_t^* M}(f) = \sum_{u \in J(N_0/tN_0t^{-1})} u \varphi_t(\mathrm{pr}_{W, M}(\psi_t(u^{-1}f)))$$

as we have $(M_{t,k}^*)^\vee[1/X] \cong (F_{t,k}^*M)^\vee[1/X]$. Since the map (4.2) is an isomorphism we may decompose $\mathrm{pr}_{W,F_t^*M}(f)$ uniquely as

$$\mathrm{pr}_{W,F_t^*M}(f) = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\psi_t(u^{-1}\mathrm{pr}_{W,F_t^*M}(f)))$$

so we must have $\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$. For general $f \in W^\vee$ note that $N_0sM_{t,k,*}$ is killed by $\varphi_t(X^r)$ for $r \geq 0$ big enough, so we have

$$\begin{aligned} X^r\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) &= \psi_t(\mathrm{pr}_{W,F_t^*M}(\varphi_t(X^r)f)) = \\ &= \mathrm{pr}_{W,M}(\psi_t(\varphi_t(X^r)f)) = X^r\mathrm{pr}_{W,M}(\psi_t(f)) . \end{aligned}$$

Since X^r is invertible in $\Lambda_\ell(N_0)$, we obtain

$$\psi_t(\mathrm{pr}_{W,F_t^*M}(f)) = \mathrm{pr}_{W,M}(\psi_t(f))$$

for any $f \in W^\vee$. The statement follows taking the projective limit with respect to $M \in \mathcal{M}(\pi^{H_0})$ and the inductive limit with respect to $W \in \mathcal{B}_+(\pi)$. \square

4.2 Compatibility with parabolic induction

Let $P = L_P N_P$ be a parabolic subgroup of G containing B with Levi component L_P and unipotent radical N_P and let π_P be a smooth o/ϖ^h -representation of L_P that we view as a representation of P^- via the quotient map $P^- \twoheadrightarrow L_P$ where $P^- = L_P N_{P^-}$ is the parabolic subgroup opposite to P . Since T is contained in L_P , we may consider the same cocharacter $\xi: \mathbb{Q}_p^* \rightarrow T$ for the group L_P instead of G . Further, we put $N_{L_P} = N \cap L_P$ and $N_{L_P,0} = N_0 \cap L_P$.

As in [3] denote by $W = N_G(T)/T$ (resp. by $W_P = (N_G(T) \cap L_P)/T$) the Weyl group of G (resp. of L_P) and by $w_0 \in W$ the element of maximal length. We have a canonical system

$$K_P = \{w \in W \mid w^{-1}(\Phi_P^+) \subseteq \Phi_+\}$$

of representatives (the Kostant representatives) of the right cosets $W_P \backslash W$ where Φ_P^+ denotes the set of positive roots of L_P with respect to the Borel subgroup $L_P \cap B$. We have a generalized Bruhat decomposition

$$G = \coprod_{w \in K_P} P^- w B = \coprod_{w \in K_P} P^- w N .$$

Now let π_P be a smooth representation of L_P over o/ϖ^h . We regard π_P as a representation of P^- via the quotient map $P^- \twoheadrightarrow L_P$. Then the parabolically induced representation $\text{Ind}_{P^-}^G \pi_P$ admits [21] (see also [7] §4.3) a filtration by B -subrepresentations whose graded pieces are contained in

$$\mathcal{C}_w(\pi_P) = c - \text{Ind}_{P^-}^{P^-wN} \pi_P$$

for $w \in K_P$ where $c - \text{Ind}_{P^-}^*$ stands for the space of locally constant functions on $* \supseteq P^-$ with compact support modulo P^- . B acts on $\mathcal{C}_w(\pi_P)$ by right translations. Moreover, the first graded piece equals $\mathcal{C}_1(\pi_P)$.

Lemma 4.2.1 *Let $\pi' \leq \mathcal{C}_w(\pi_P)$ be any B -subrepresentation for some $w \in K_P \setminus \{1\}$. Then we have $D_\xi^\vee(\pi') = 0$.*

Proof By the right exactness of D_ξ^\vee (Prop. 2.7(ii) in [3]) it suffices to treat the case $\pi' = \mathcal{C}_w(\pi_P)$. For this the same argument works as in Prop. 6.2 [3] with the following modification:

The particular shape of ℓ is only used in Lemma 6.5 in [3] (note that the subgroup $H_0 = \text{Ker}(\ell: N_0 \rightarrow \mathbb{Z}_p)$ is denoted by N_1 therein). For an element $w \neq 1$ in the Weyl group we have $(w^{-1}N_{P^-w} \cap N_0) \backslash N_0 / H_0 = \{1\}$ if and only if H_0 does not contain $w^{-1}N_{P^-w} \cap N_0$. Whenever $w^{-1}N_{P^-w} \cap N_0 \not\subseteq H_0$, the statement of Lemma 6.5 in [3] is true and there is nothing to prove.

In case we have $\{1\} \neq w^{-1}N_{P^-w} \cap N_0 \subseteq H_0$, the statement of Lemma 6.5 is not true for $\ell = \ell_\alpha$. However, the argument using it in the proof of Prop. 6.2 can be replaced by the following: the operator F acts on the space $\mathcal{C}((w^{-1}N_{P^-w} \cap N_0) \backslash N_0, \pi_P^w)^{H_0}$ nilpotently. Indeed, the trace map $\text{Tr}_{H_0/sH_0s^{-1}}$

$$\mathcal{C}((w^{-1}N_{P^-w} \cap N_0) \backslash N_0, \pi_P^w)^{sH_0s^{-1}} \rightarrow \mathcal{C}((w^{-1}N_{P^-w} \cap N_0) \backslash N_0, \pi_P^w)^{H_0}$$

is zero as each double coset $(w^{-1}N_{P^-w} \cap H_0) \backslash H_0 / sH_0s^{-1}$ has size divisible by p and any function in $\mathcal{C}((w^{-1}N_{P^-w} \cap N_0) \backslash N_0, \pi_P^w)^{sH_0s^{-1}}$ is constant on these double cosets. The statement follows from Prop. 2.7(iii) in [3]. \square

In order to extend Thm. 6.1 in [3] (the compatibility with parabolic induction) to our situation ($\ell = \ell_\alpha$) we need to distinguish two cases: whether the root subgroup N_α is contained in L_P or in N_P . Similarly to [7] we define the $s^\mathbb{Z}N_{L_P}$ -ordinary part $\text{Ord}_{s^\mathbb{Z}N_{L_P}}(\pi_P)$ of a smooth representation π_P of L_P as follows. We equip $\pi_P^{N_{L_P,0}}$ with the Hecke action $F_P = \text{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}} \circ (s \cdot)$ of s making $\pi_P^{N_{L_P,0}}$ a module over the polynomial ring $o/\varpi^h[F_P]$ and put

$$\text{Ord}_{s^\mathbb{Z}N_{L_P}}(\pi_P) = \text{Hom}_{o/\varpi^h[F_P]}(o/\varpi^h[F_P, F_P^{-1}], \pi_P^{N_{L_P,0}})_{F_P\text{-fin}}$$

where $F_P - fin$ stands for those elements in the Hom-space whose orbit under the action of F_P is finite. By Lemmata 3.1.5 and 3.1.6 in [7] we may identify $\text{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ with an $o/\varpi^h[F_P]$ -submodule in $\pi_P^{N_{L_P},0}$ by sending a map $f \in \text{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)$ to its value $f(1) \in \pi_P^{N_{L_P},0}$ at $1 \in o/\varpi^h[F_P, F_P^{-1}]$.

Proposition 4.2.2 *Let π_P be a smooth locally admissible representation of L_P over o/ϖ^h which we view by inflation as a representation of P^- . We have an isomorphism*

$$D_\xi^\vee(\text{Ind}_{P^-}^G \pi_P) \cong \begin{cases} D_\xi^\vee(\pi_P) & \text{if } N_\alpha \subseteq L_P \\ o/\varpi^h((X)) \widehat{\otimes}_{o/\varpi^h} \text{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee & \text{if } N_\alpha \subseteq N_P \end{cases}$$

as étale (φ, Γ) -modules. In particular, for $P = B$ we have $D_\xi^\vee(\text{Ind}_B^G \pi_B) \cong o/\varpi^h((X)) \widehat{\otimes}_{o/\varpi^h} \pi_B^\vee$, ie. the value of D_ξ^\vee at the principal series is the same (φ, Γ) -module of rank 1 regardless of the choice of ℓ (generic or not).

Proof By Lemma 4.2.1 and the right exactness of D_ξ^\vee (Prop. 2.7(ii) in [3]) it suffices to show that $D_\xi^\vee(\mathcal{C}_1(\pi_P))$ is isomorphic either to $D_\xi^\vee(\pi_P)$ or $o/\varpi^h((X)) \widehat{\otimes}_{o/\varpi^h} \text{Ord}_{s^{\mathbb{Z}}N_{L_P}}(\pi_P)^\vee$. Moreover, the proof of Prop. 6.7 in [3] goes through without modification so we have an isomorphism $D_\xi^\vee(\mathcal{C}_1(\pi_P)) \cong D^\vee((\text{Ind}_{P^- \cap N_0}^{N_0} \pi_P)^{H_0})$. Hence we are reduced to computing $D^\vee((\text{Ind}_{P^- \cap N_0}^{N_0} \pi_P)^{H_0})$ in terms of π_P . We further have an identification

$$\text{Ind}_{P^- \cap N_0}^{N_0} \pi_P \cong \mathcal{C}(N_{P,0}, \pi_P) \cong \mathcal{C}(N_{P,0}, o/\varpi^h) \otimes_{o/\varpi^h} \pi_P$$

by equation (40) in [3]. We need to distinguish two cases.

Case 1: $N_\alpha \subseteq L_P$. In this case we have $N_{P,0} \subseteq H_0$. Hence we deduce $(\mathcal{C}(N_{P,0}, o/\varpi^h) \otimes_{o/\varpi^h} \pi_P)^{H_0} = \pi_P^{H_0/N_{P,0}} = \pi_P^{H_{P,0}}$. So we have

$$D_\xi^\vee(\text{Ind}_{P^-}^G \pi_P) \cong D^\vee((\text{Ind}_{P^- \cap N_0}^{N_0} \pi_P)^{H_0}) \cong D^\vee(\pi_P^{H_{P,0}}) \cong D_\xi^\vee(\pi_P)$$

in this case as claimed.

Case 2: $N_\alpha \subseteq N_P$. In this case we have $N_{L_P,0} \subseteq H_0$ and $N_{P,0}/(N_{P,0} \cap H_0) \cong \mathbb{Z}_p$. So we have an identification

$$\mathcal{C}(N_{P,0}, \pi_P)^{H_0} \cong \mathcal{C}(N_{P,0}/(N_{P,0} \cap H_0), \pi_P^{N_{L_P},0}) \cong \mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{L_P},0}).$$

Here the Hecke action $F = F_G = \text{Tr}_{H_0/sH_0s^{-1}} \circ (s \cdot)$ of s on the right hand side is given by the formula

$$F_G(f)(a) = \begin{cases} F_P(f(a/p)) & \text{if } a \in p\mathbb{Z}_p \\ 0 & \text{if } a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p \end{cases},$$

where $F_P = \text{Tr}_{N_{L_P,0}/sN_{L_P,0}s^{-1}} \circ (s \cdot)$ denotes the Hecke action of s on $\pi_P^{N_{L_P,0}}$.

Now let M be a finitely generated $o/\varpi^h[[X]][F]$ submodule of $\mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{L_P,0}})$ that is stable under the action of Γ and is admissible as a representation of \mathbb{Z}_p . By possibly passing to a finite index submodule of M we may assume without loss of generality that the natural map $M^\vee \rightarrow M^\vee[1/X]$ is injective whence the map $\text{id} \otimes F: o/\varpi^h[[X]] \otimes_{o/\varpi^h[[X]], F} M \rightarrow M$ is surjective.

Let $f \in M$ be arbitrary. By continuity of f there exists an integer $n \geq 0$ such that f is constant on the cosets of $p^n\mathbb{Z}_p$. Writing $f = \sum_{i=0}^{p^n-1} [i] \cdot F^n(f_i)$ (where $[i] \cdot$ denotes the multiplication by the group element $i \in \mathbb{Z}_p$) by the surjectivity of $\text{id} \otimes F$ we find that each f_i is necessarily constant as a function on \mathbb{Z}_p satisfying $F_P^n(f_i(0)) = f_i(0)$.

Put $M_* = \{f(0) \mid f \in M\} \subseteq \pi_P^{N_{L_P,0}}$. By the previous discussion F_P acts surjectively on M_* and is generated by the values of elements in $M^{\mathbb{Z}_p}$ (ie. constant functions) as a module over $o/\varpi^h[F_P]$. By the admissibility of M we deduce that $M^{\mathbb{Z}_p}$ hence M_* is finite (or, equivalently, finitely generated over o/ϖ^h). We deduce that in fact we have $M = \mathcal{C}(\mathbb{Z}_p, M_*)$, ie. $M^\vee \cong o/\varpi^h[[X]] \otimes_{o/\varpi^h} M_*^\vee$.

Conversely, whenever we have a $o/\varpi^h[F_P]$ -submodule $M' \leq \pi_P^{N_{L_P,0}}$ that is finitely generated over o/ϖ^h and on which F_P acts surjectively (hence bijectively as the cardinality of o/ϖ^h is finite) then for $M = \mathcal{C}(\mathbb{Z}_p, M')$ we have $M' = M_*$, $M \in \mathcal{M}(\mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{L_P,0}}))$, and $M^\vee \cong o/\varpi^h[[X]] \otimes_{o/\varpi^h} (M')^\vee$ is X -torsion free. In particular, we compute

$$\begin{aligned}
D_\xi^\vee(\mathcal{C}_1(\pi_P)) &\cong \varprojlim_{M \in \mathcal{M}(\mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{LP}, 0}))} M^\vee[1/X] \cong \\
&\cong \varprojlim_{\substack{M \in \mathcal{M}(\mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{LP}, 0})), \\ M^\vee \hookrightarrow M^\vee[1/X]}} o/\varpi^h((X)) \otimes_{o/\varpi^h} M_*^\vee \cong \\
&o/\varpi^h((X)) \widehat{\otimes}_{o/\varpi^h} \left(\varinjlim_{\substack{M \in \mathcal{M}(\mathcal{C}(\mathbb{Z}_p, \pi_P^{N_{LP}, 0})), \\ M^\vee \hookrightarrow M^\vee[1/X]}} M_* \right)^\vee = \\
&= o/\varpi^h((X)) \widehat{\otimes}_{o/\varpi^h} \text{Ord}_{s^{\mathbb{Z}} N_{LP}}(\pi_P)^\vee
\end{aligned}$$

as claimed. \square

Remark For $N_\alpha \subseteq N_P$ we have the equivalent description $D_\xi^\vee(\text{Ind}_P^G \pi_P) \cong \varprojlim_{M \in \mathcal{M}(\pi'_P)} o/\varpi^h[[X]][1/X] \otimes_{o/\varpi^h} M^\vee$, where

$$\pi'_P = (\pi_P^{H_0})_{F_P^\infty=0} = \pi_P^{H_0} / \langle x \in \pi_P^{H_0} \mid \exists n \in \mathbb{N} : F_P^n x = 0 \rangle,$$

and the action of φ (resp. Γ) on $o/\varpi^h[[X]][1/X] \otimes M^\vee$ is the unique $o/\varpi^h[[X]][1/X]$ -semilinear action such that $\varphi(f)(m) = f(\xi(p^{-1})m)$ for $f \in M^\vee$ and $m \in M$ (resp. $x(f)(m) = f(\xi(x^{-1})m)$ for $x \in \mathbb{Z}_p^* \simeq \Gamma$, $f \in M^\vee$ and $m \in M$).

4.3 Compatibility with a reverse functor

In this section the results of [10], section 4 are presented without proofs.

In [18] the functor $D \mapsto \mathfrak{Y}$ is generalized to arbitrary \mathbb{Q}_p -split reductive groups G with connected centre. Let D be an étale (φ, Γ) -module finitely generated over $\mathcal{O}_\mathcal{E}$ and choose a character $\delta: \text{Ker}(\alpha) \rightarrow o^*$. Then we may let the monoid $\xi(\mathbb{Z}_p \setminus \{0\})\text{Ker}(\alpha) \leq T$ (containing T_+) act on D via the character δ of $\text{Ker}(\alpha)$ and via the natural action of $\mathbb{Z}_p \setminus \{0\} \cong \varphi^{\mathbb{N}_0} \times \Gamma$ on D . This way we also obtain a T_+ -action on $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ making $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ an étale T_+ -module over $\Lambda_\ell(N_0)$. In [18] a G -equivariant sheaf \mathfrak{Y} on G/B is attached to D such that its sections on $\mathcal{C}_0 = N_0 w_0 B/B \subset G/B$ is B_+ -equivariantly

isomorphic to the étale T_+ -module $(\Lambda_\ell(N_0) \otimes_{u_\alpha} D)^{bd}$ over $\Lambda(N_0)$ consisting of bounded elements in $\Lambda_\ell(N_0) \otimes_{u_\alpha} D$ (see [18] section 9).

The construction of a G -equivariant sheaf on G/B with sections on $\mathcal{C}_0 = N_0 w_0 B/B \subset G/B$ isomorphic to a dense B_+ -stable $\Lambda(N_0)$ -submodule $D_{\xi,\ell,\infty}^\vee(\pi)^{bd}$ of $D_{\xi,\ell,\infty}^\vee(\pi)$ is not immediate from the work [18] as only the case of finitely generated modules over $\Lambda_\ell(N_0)$ is treated in there. However, the most natural definition of bounded elements in $D_{\xi,\ell,\infty}^\vee(\pi)$ works: The $\Lambda(N_0)$ -submodule $D_{\xi,\ell,\infty}^\vee(\pi)^{bd}$ is defined as the union of ψ -invariant compact $\Lambda(N_0)$ -submodules of $D_{\xi,\ell,\infty}^\vee(\pi)$. The image of $\widetilde{\text{pr}}: \widetilde{D}_{SV}(\pi) \rightarrow D_{\xi,\ell,\infty}^\vee(\pi)$ is contained in $D_{\xi,\ell,\infty}^\vee(\pi)^{bd}$ and the constructions of [18] can be carried over to this situation. The resulting G -equivariant sheaf on G/B is denoted by $\mathfrak{Y} = \mathfrak{Y}_{\alpha,\pi}$.

Now consider the functors $(\cdot)^\vee: \pi \mapsto \pi^\vee$ and the composite

$$\mathfrak{Y}_{\alpha,\cdot}(G/B): \pi \mapsto D_{\xi,\ell,\infty}^\vee(\pi) \mapsto \mathfrak{Y}_{\alpha,\pi}(G/B)$$

both sending smooth, admissible o/ϖ^h -representations of G of finite length to topological representations of G over o/ϖ^h . There exists a natural transformation $\beta_{G/B}$ from $(\cdot)^\vee$ to $\mathfrak{Y}_{\alpha,\cdot}$. This generalizes Thm. IV.4.7 in [4]. The proof of this relies on the observation that the maps $\mathcal{H}_g: D_{\xi,\ell,\infty}^\vee(\pi)^{bd} \rightarrow D_{\xi,\ell,\infty}^\vee(\pi)^{bd}$ in fact come from the G -action on π^\vee . More precisely, for any $g \in G$ and $W \in \mathcal{B}_+(\pi)$ we have maps

$$(g\cdot): (g^{-1}W \cap W)^\vee \rightarrow (W \cap gW)^\vee$$

where both $(g^{-1}W \cap W)^\vee$ and $(W \cap gW)^\vee$ are naturally quotients of W^\vee . These maps fit into a commutative diagram

$$\begin{array}{ccccc} W^\vee & \xrightarrow{\quad} & (g^{-1}W \cap W)^\vee & \xrightarrow{g\cdot} & (W \cap gW)^\vee \\ \downarrow \text{pr}_W & & \downarrow & & \downarrow \\ D_{\xi,\ell,\infty}^\vee(\pi)^{bd} & \xrightarrow{\quad} & \text{res}_{g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}^{\mathcal{C}_0} (D_{\xi,\ell,\infty}^\vee(\pi)^{bd}) & \xrightarrow{g\cdot} & \text{res}_{\mathcal{C}_0 \cap g\mathcal{C}_0}^{\mathcal{C}_0} (D_{\xi,\ell,\infty}^\vee(\pi)^{bd}) \end{array}$$

allowing us to construct the map $\beta_{G/B}$. The proof of this is similar to that of Thm. IV.4.7 in [4]. However, unlike that proof we do not need the full machinery of “standard presentations” in Ch. III.1 of [4] which is not available at the moment for groups other than $\mathbf{GL}_2(\mathbb{Q}_p)$.

4.4 Counterexamples

In [3] the Whittaker functional ℓ is assumed to be generic. However, even if ℓ is not generic, the functor D_ξ^\vee (hence also $D_{\xi,\ell,\infty}^\vee$) is right exact. Here we show that in this case D_ξ^\vee is not faithful and the restriction of D_ξ^\vee to the category SP_{o/ϖ^h} is not exact in general.

From now on let $h = 1$, thus we are over $k = o/\varpi$, and $G = \mathbf{GL}_3(\mathbb{Q}_p)$. Then $|\Delta| = 2$, say $\Delta = \{\alpha, \beta\}$, fix the parabolic subgroup P such that $L_P \cong \mathbf{GL}_2(\mathbb{Q}_p) \times T'$ where T' is a torus and $\ell = \ell_\alpha$. Let the superscript (2) denote the analogous construction of the subgroups B, T, N, T_0 and element s of G in case $G = \mathbf{GL}_2(\mathbb{Q}_p)$. Let $\pi_P \cong \pi^{(2)} \otimes_k \chi$ with $\pi^{(2)}$ a smooth admissible representation of $\mathbf{GL}_2(\mathbb{Q}_p)$.

Proposition 4.4.1 *Let $\pi_P \cong \pi^{(2)} \otimes \chi$ be the twist of a supercuspidal modulo p representation $\pi^{(2)}$ of $\mathbf{GL}_2(\mathbb{Q}_p)$ by a character χ of the torus. Then we have*

$$\dim_{k((X))} D_\xi^\vee (\text{Ind}_{P^-}^G \pi_P) = \begin{cases} 0 & \text{if } N_\beta \subset L_P \\ 2 & \text{if } N_\alpha \subset L_P \end{cases}.$$

Proof We use the compatibility with parabolic induction (Proposition 4.2.2). Note that the torus $T^{(2)}$ is generated by $s^{(2)}$ and $T_0^{(2)}$. So in the case when $N_\beta \subset L_P$ we have an isomorphism

$$\text{Ord}_{s^{\mathbb{Z}} N_{L_P}}(\pi_P) \cong (\text{Ord}_{B^{(2)}}(\pi_2) \otimes \chi)_{|k[F_P]} = 0$$

by the adjunction formula of Emerton’s ordinary parts (Thm. 4.4.6 in [7]). In the other case we apply Thm. 0.10 in [4]. \square

Now let $\chi = \text{id}$ and $\pi_P = \pi^{(2)} \otimes \text{id}$ be a representation of $L_P \cong \mathbf{GL}_2(\mathbb{Q}_p) \times T'$ such that $N_\beta \subset L_P$.

By definition ([3], section 3) the $k[[X]]$ -module structure of $\pi_P^{H_0}$ is isomorphic to those of $\pi^{(2)}$, the \mathbb{Z}_p^* -actions are the same, and

$$F_P m = \sum_{i=0}^{p-1} (1+X)^i F^{(2)} m \quad \text{for } m \in \pi_P^{H_0} = \pi_P^{N_0^{(2)}}.$$

Let $M^{(2)} \in \mathcal{M}(\pi^{(2)})$ and consider the k -vectorspace $(M^{(2)})^\vee / X(M^{(2)})^\vee = (M^{H_0})^\vee$. M^{H_0} is F_P -invariant thus we have an action of F_P on the dual. We describe it with the ψ coming from the étale (φ, Γ) -module structure of $(M^{(2)})^\vee[1/X]$ (cf. Lemma 2.6 and the part after Lemma 3.1 in [3]):

$$F_P(d + X(M^{(2)})^\vee) = \psi \left(\sum_{i=0}^{p-1} (1 + X)^i d \right) + X(M^{(2)})^\vee \quad (d \in (M^{(2)})^\vee).$$

Proposition 4.4.2 *Let $\pi^{(2)}$ be an extension of principal series:*

$$0 \rightarrow \pi_1^{(2)} = \text{Ind}_{B^{(2)-}}^{\mathbf{GL}_2(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2) \xrightarrow{i} \pi^{(2)} \xrightarrow{j} \pi_2^{(2)} = \text{Ind}_{B^{(2)-}}^{\mathbf{GL}_2(\mathbb{Q}_p)}(\chi'_1 \otimes \chi'_2) \rightarrow 0,$$

and $D(\pi^{(2)})$ be the (φ, Γ) -module attached to $\pi^{(2)}$ by the classical Montréal functor D . Then $\text{Ord}_{s^z N_{LP}}(\pi_P)^\vee$ is a quotient of

$$(\Lambda/X\Lambda)_{F^\infty=0} = (\Lambda/X\Lambda)/\langle d \in \Lambda/X\Lambda \mid \exists n \in \mathbb{N} : F^n d = 0 \rangle$$

for a certain lattice Λ containing the smallest ψ -invariant lattice $D^\natural(\pi^{(2)}) \subset D(\pi^{(2)})$.

Proof As before, we have $\text{Ord}_{s^z N_{LP}}(\pi_P) \cong \text{Ord}_{B^{(2)}}(\pi^{(2)}) \otimes \text{id} \cong \text{Ord}_{B^{(2)}}(\pi^{(2)})$. Let us denote it with $\text{Ord}^{(2)}$.

We have $\dim_k(\text{Ord}^{(2)}) \leq 2$, because the ordinary parts of the principal series are 1 dimensional over k (Theorem 4.2.12 in [8]), and the functor $\pi \mapsto \text{Ord}(\pi)$ is left exact (Proposition 3.2.4 in [7]).

For a principal series representation $\pi_0^{(2)}$, if $M \in \mathcal{M}(\pi_0^{(2)})$ such that $M^\vee[1/X]$ is nontrivial, then we have $\text{Ord}_{B^{(2)}}(\pi_0^{(2)}) \leq M^{N_0^{(2)}}$. The minimal generating B_+ -subrepresentation $M_0 \in \mathcal{M}(\pi_0^{(2)})$ of the Steinberg representation is of that kind. Assume indirectly that $M^{N_0^{(2)}}$ does not contain the ordinary part for some $M \in \mathcal{M}(\pi_0^{(2)})$. We have $\dim_{k((X))}(M^\vee[1/X]) \leq 1$ for all $M' \in \mathcal{M}(\pi_0^{(2)})$. But then by Lemma 2.1 in [3] we would have $M' = M + M_0 \in \mathcal{M}(\pi_0^{(2)})$ and $\dim_{k((X))}(M'^\vee[1/X]) \geq 2$.

We show, that there exists $M' \in \mathcal{M}(\pi^{(2)})$ such that $\text{Ord}^{(2)} \leq M'$.

If $\dim_k(\text{Ord}^{(2)}) = 1$, then $\text{Ord}^{(2)} \cong \text{Ord}_{B^{(2)}}(\pi_1^{(2)})$ which is contained in the Steinberg representation $M_1 \leq \pi_1^{(2)}$. Thus $\text{Ord}^{(2)} \leq M' = i(M_1) \in \mathcal{M}(\pi^{(2)})$.

If $\dim_k(\text{Ord}^{(2)}) = 2$, we use the fact that $\text{Ord}_{B^{(2)}}$ is the right adjoint of $\text{Ind}_{B^{(2)-}}^{\text{GL}_2(\mathbb{Q}_p)}$ ([7] Theorem 4.4.6). We have

$$0 \rightarrow \chi_1 \otimes \chi_2 \rightarrow U \cong \text{Ord}^{(2)} \rightarrow \chi'_1 \otimes \chi'_2 \rightarrow 0.$$

Thus the isomorphism $U \rightarrow \text{Ord}^{(2)}$ gives an isomorphism $\text{Ind}_{B^{(2)-}}^{\text{GL}_2(\mathbb{Q}_p)}(U) \rightarrow \pi^{(2)}$.

Let M' be the $k[[X]][F]$ -representation generated by $\text{Ord}^{(2)}$. $M' \in \mathcal{M}(\pi^{(2)})$, because any $f \in M$ viewed as a function $G \rightarrow U$ has support in $N_0^{(2)}B^{(2)-}$, thus M'^V is admissible.

Moreover we can choose M such that $M^V[1/X] \cong D(\pi^{(2)})$: let $M'' \in \mathcal{M}(\pi^{(2)})$ be such that $M''^V[1/X] \cong D(\pi^{(2)})$. Then we also have $M = M' + M'' \in \mathcal{M}(\pi^{(2)})$ (cf Lemma 2.1 in [3]).

Set $\Lambda = M^V \leq M^V[1/X]$. This is ψ -invariant and generates $D(\pi^{(2)})$, thus it contains $D^{\natural}(\pi^{(2)})$. We got that $\text{Ord}_{s^{\mathbb{Z}}N_{LP}}(\pi_P)^V$ is a quotient of $\Lambda/X\Lambda$. Moreover since F_P acts surjectively on $\text{Ord}_{s^{\mathbb{Z}}N_{LP}}(\pi_P)$, the dual is a quotient of $(\Lambda/X\Lambda)_{F_P^{\infty}=0}$. \square

Corollary 4.4.3 *Let $\chi_1 \neq \chi_2$, $\chi'_1 = \chi_2\bar{\varepsilon}^{-1}$ and $\chi'_2 = \chi_1\bar{\varepsilon}$ with $\chi_1 \neq \chi'_1$ and $\bar{\varepsilon} : \mathbb{Q}_p^* \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p^* \rightarrow \mathbb{F}_p^*$ denoting the modulo p cyclotomic character. Then we have an exact sequence*

$$0 \rightarrow \text{Ind}_{P^-}^G(\pi_1^{(2)} \otimes \text{id}) \rightarrow \pi = \text{Ind}_{P^-}^G(\pi^{(2)} \otimes \text{id}) \rightarrow \text{Ind}_{P^-}^G(\pi_2^{(2)} \otimes \text{id}) \rightarrow 0,$$

but the natural map $D_{\xi}^V(\text{Ind}_{P^-}^G(\pi_2^{(2)} \otimes \text{id})) \rightarrow D_{\xi}^V(\text{Ind}_{P^-}^G(\pi^{(2)} \otimes \text{id}))$ is not injective.

Proof The above sequence is exact, because both $- \otimes \text{id}$ and $\text{Ind}_{P^-}^G(-)$ are exact.

By Proposition 4.2.2 we have $D_{\xi}^V(\text{Ind}_{P^-}^G(\pi_2^{(2)} \otimes \text{id})) \cong k((X)) \otimes \text{Ord}_{B^{(2)}}(\pi_2^{(2)})$ and $D_{\xi}^V(\text{Ind}_{P^-}^G(\pi^{(2)} \otimes \text{id})) \cong k((X)) \otimes \text{Ord}_{B^{(2)}}(\pi^{(2)})$ (here we also used that $\text{Ord}_{s^{\mathbb{Z}}N_{LP}}(\pi) \cong \text{Ord}_{B^{(2)}}(\pi^{(2)})$ as before).

For any extension D of the (φ, Γ) -modules $D(\pi_1^{(2)})$ and $D(\pi_2^{(2)})$ there exists an extension $\pi^{(2)}$ of the two principal series with $D(\pi^{(2)}) = D$, since the functor D is essentially surjective (see Thm 0.17(iii) in [4]) and we have $\dim_{\mathbb{F}_p}(\text{Ext}(\pi_2^{(2)}, \pi_1^{(2)})) = 1$ (see [8] Prop. 4.3.15(2)).

Thus it suffices to prove, that there exists a nontrivial extension D and that for any lattice $\Lambda \supseteq D^\natural$ the action F_P on $\Lambda/X\Lambda$ has nontrivial kernel. This is done in the following section. \square

4.5 Extensions of 1 dimensional (φ, Γ) -modules

The most part of the following is folklore, however I could not find it anywhere, so I wrote it down. Let p be an odd prime and $\gamma \in \Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ be a topological generator. Let $\chi : \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \rightarrow \mathbb{Z}_p^*$ be the cyclotomic character.

For $f(X) = \sum_n \lambda_n X^n \in \mathbb{F}_p((X))^*$, write $\deg(f(X)) = \min\{n | \lambda_n \neq 0\}$.

Proposition 4.5.1 *Let D be a one dimensional (φ, Γ) -module over $\mathbb{F}_p((X))$. Then there exists a basis $\{e\}$ of D and $\lambda, \mu \in \mathbb{F}_p^*$ such that the $\varphi(e) = \lambda e$ and $\gamma(e) = \mu e$.*

Proof Let e_0 be any generator of D . Then $\varphi(e_0) = f(X)e$ for some $f \in \mathbb{F}_p((X))$. We can write $f(X) = \lambda_0 X^n f'(X)$ with $\lambda_0 \in \mathbb{F}_p^*$, $n \in \mathbb{Z}$ and $f'(X) \in 1 + X\mathbb{F}_p[[X]]$.

If we change the basis to $e = h(X)e_0$ for any $h(X) \in \mathbb{F}_p((X))^*$, we have $\varphi(h(X)e) = h(X^p)\varphi(e) = (h(X^p)/h(X) \cdot \lambda_0 X^n f'(X))(h(X)e)$. After choosing $h(X) = X^{\lfloor n/p \rfloor} \prod_{j=0}^{\infty} f'(X^{p^j})$ (which is convergent in $\mathbb{F}_p((X))$, since $f'(0) = 1$), we have that $\varphi(e) = \lambda_0 X^m e$, where $0 \leq m < p$ and $p | n - m$.

Let $\gamma(e) = g(X)e = \mu_0 X^l g'(X)e$ with $\mu_0 \in \mathbb{F}_p^*$, $l \in \mathbb{Z}$ and $g'(X) \in 1 + X\mathbb{F}_p[[X]]$. Then we have $\varphi(\gamma(e)) = \gamma(\varphi(e))$, where on the left hand side we have:

$$\varphi(\gamma(e)) = \varphi(\mu_0 X^l g'(X)e) = \lambda_0 \mu_0 X^{pl} g'(X^p) X^m e.$$

On the right hand side

$$\gamma(\varphi(e)) = \gamma(\lambda_0 X^m e) = \lambda_0 \mu_0 ((1 + X)^{X(\gamma)} - 1)^m X^l g'(X)e.$$

Thus we have $X^{pl+m} g'(X^p) = ((1 + X)^{X(\gamma)} - 1)^m X^l g'(X)$, comparing the degrees and the leading coefficients gives $l = m = 0$, $g'(X) = 1$ and we have the proposition. \square

Recall the following definitions of Colmez (cf [5]): For a (φ, Γ) -module D

- we define $D^{nr} = \bigcap_{n \in \mathbb{N}} \varphi^n(D) \leq D$,
- $D^{\natural} \leq D$ to be the smallest ψ -invariant lattice and
- $D^{\#} \leq D$ to be the biggest ψ -invariant lattice on which ψ acts surjectively.

Corollary 4.5.2 *If D is one dimensional with a basis e as above, we have $D^{nr} = \mathbb{F}_p e$, $D^{\natural} = k[[X]]e$ and $D^{\#} = X^{-1}k[[X]]e$.*

Proof The first two statements are clear, the last comes from the facts that $\psi(X^{-1}e) = \psi(\sum_{i=0}^{p-1} (1+X)^i \varphi(X^{-1}e)) = X^{-1}e$ and that $\psi(X^m e) \in X^{m+1}k[[X]]e$ if $m < -1$. \square

Remark For any $\lambda_0, \mu_0 \in \mathbb{F}_p^*$ there exists a one dimensional (φ, Γ) -module, such that the matrix of φ (respectively γ) is λ_0 (respectively μ_0). It is easy to see that in this case the action of φ is étale and the action of γ extends continuously to Γ .

Altogether there are $(p-1)^2$ one dimensional (φ, Γ) -modules over $\mathbb{F}_p((X))$.

Now let D_1 and D_2 be one dimensional (φ, Γ) -modules over $\mathbb{F}_p((X))$. We determine the extensions of D_2 by D_1 . By the previous proposition we might choose a basis $\{e'_i\}$ in D_i such that $\varphi(e'_i) = \lambda_i e'_i$ and $\gamma(e'_i) = \mu_i e'_i$ for $i = 1, 2$ and $\lambda_i, \mu_i \in \mathbb{F}_p^*$.

Proposition 4.5.3

- *If D is an extension of D_2 by D_1 , then in an appropriate basis $\{e_1, e_2\} \subset D$ we have $\varphi(e_1) = \lambda_1 e_1$, $\varphi(e_2) = f(X)e_1 + \lambda_2 e_2$, $\gamma(e_1) = \mu_1 e_1$, $\gamma(e_2) = g(X)e_1 + \mu_2 e_2$, with $f(X) = \sum_i \alpha_i X^i$ and $g(X) \in \mathbb{F}_p((X))$, such that $\alpha_i = 0$ if a) $i > 0$ or b) $i < 0$ and $p|i$, and*

$$\mu_1 f((1+X)^{\chi(\gamma)} - 1) - \mu_2 f(X) = \lambda_1 g(X^p) - \lambda_2 g(X).$$

If $\lambda_1 \neq \lambda_2$ we can also have $\alpha_0 = 0$.

- *Let $f(X), g(X) \in \mathbb{F}_p((X))$ as above. Then there exists a 2 dimensional (φ, Γ) -module D , for which the above statements hold. If $f'(X) \neq \alpha f(X)$ for any $\alpha \in \mathbb{F}_p^*$ and $g'(X)$ are as above with a (φ, Γ) -module D' , then $D \not\cong D'$.*

Proof

- We may choose a basis $\{e_1, e_2\}$ in D such that e_1 is the image of e'_1 and e_2 is a preimage of e'_2 . Then there exist $f(X), g(X) \in \mathbb{F}_p((X))$ such that $\varphi(e_2) = f(X)e_1 + \lambda_2 e_2$ and $\gamma(e_2) = g(X)e_1 + \mu_2 e_2$.

We have $\varphi(\gamma(e_2)) = \varphi(g(X)e_1 + \mu_2 e_2) = (\lambda_1 g(X^p) + \mu_2 f(X))e_1 + \lambda_2 \mu_2 e_2$ and $\gamma(\varphi(e_2)) = \gamma(f(X)e_1 + \lambda_2 e_2) = (\mu_1 f((1+X)^{x(\gamma)} - 1) + \lambda_2 g(x))e_1 + \lambda_2 \mu_2 e_2$, thus

$$\mu_1 f((1+X)^{x(\gamma)} - 1) - \mu_2 f(X) = \lambda_1 g(X^p) - \lambda_2 g(X).$$

Now we look at the basis $\{e_1, e_2 + h(X)e_1\}$ for $h(X) \in \mathbb{F}_p((X))^*$. We have $\varphi(e_2 + h(X)e_1) = (f(X) + \lambda_1 h(X^p) - \lambda_2 h(X))e_1 + \lambda_2 (e_2 + h(X)e_1)$ and $\gamma(e_2 + h(X)e_1) = (g(X) + \mu_1 h((1+X)^{x(\gamma)} - 1) - \mu_2 h(X))e_1 + \mu_2 (e_2 + h(X)e_1)$.

Let $i_0 = pj_0 < 0$ minimal such that $\alpha_{i_0} \neq 0$. Then setting $h(X) = -\lambda_1^{-1} \alpha_{i_0} X^{j_0}$ and $e_2 = e_2 + h(X)e_1$ we can change $\lambda_{i_0} = 0$. Thus we may assume, that $\alpha_{pj_0} = 0$ for $j_0 < 0$.

If $\lambda_1 \neq \lambda_2$, then change e_2 to $e_2 - \alpha_0 (\lambda_1 - \lambda_2)^{-1}$, then $\lambda_0 = 0$. For $i > 0$ we can set $\alpha_i = 0$ inductively.

- It is clear, that the action of φ is étale. (the matrix of φ is upper triangular)

We need that the action of γ extends continuously to Γ . We claim that it is always true if γ has matrix $\begin{pmatrix} \mu_1 & g(X) \\ 0 & \mu_2 \end{pmatrix}$. Let $k_n \in \mathbb{N}$ such that γ^{k_n} converges in Γ . It suffices to verify, that for all $j \in \mathbb{Z}$ there exists $N(j)$ such that for $n, m > N(j)$ in $\gamma^{k_n}(e_2) - \gamma^{k_m}(e_2)$ the coefficient of $X^{j'}$ for $j' \leq j$ is 0. We have

$$\gamma^k(e_2) = \left(\sum_{i=0}^{k-1} \mu_1^i \mu_2^{k-1-i} g((1+X)^{x(\gamma)^i} - 1) \right) e_1 + \mu_2^k e_2,$$

Let $d = \deg(g)$ and $l = \max\{j - d, j + 1\}$. The convergence of γ^{k_n} yields that there exists $N'(j)$ such that for all $n, m > N'(j)$ we have $(p-1)p^l | k_n - k_m$. If $n, m > N'(j)$ then for any $i \in \mathbb{N}$ we have $\mu_2^{k_n-i} = \mu_2^{k_m-i}$ and

$$X^j | g((1+X)^{x(\gamma)^i} - 1) - g((1+X)^{x(\gamma)^{k_n-k_m+i}} - 1).$$

Suppose that $k_n \geq k_m$. Then for $q = (p-1)p^j$ and for some $h(X), h'(X) \in \mathbb{F}_p[[X]]$ we have

$$\begin{aligned} & \gamma^{k_n}(e_2) - \gamma^{k_m}(e_2) = \\ &= \left(\sum_{i=0}^{k_n-k_m-1} \mu_1^i \mu_2^{k_n-1-i} g((1+X)^{\chi(\gamma)^i} - 1) + X^j h(X) \right) e_1 = \\ &= \left(\frac{k_n - k_m}{q} \left(\sum_{i=0}^{q-1} \mu_1^i \mu_2^{k_n-1-i} g((1+X)^{\chi(\gamma)^i} - 1) \right) + X^j h'(X) \right) e_1 = \\ &= X^j h'(X) e_1, \end{aligned}$$

since $pq | k_n - k_m$. Thus $N(j) = N'(j)$ is a convenient choice.

To see that for different choices of $f(X)$ we get different modules let $\{d_1, d_2\}$ be an other basis in D , such that the matrix of φ (and γ) is upper triangular. We will show, that then $d_1 = \alpha e_1$ with $\alpha \in \mathbb{F}_p^*$, unless $f(X) = 0$, which is sufficient for the proposition.

Let $d_1 = a(X)e_1 + b(X)e_2$. $\lambda d_1 = \varphi(d_1) = (\lambda_1 a(X^p) + f(X)b(X^p))e_1 + \lambda_2 b(X^p)e_2$, thus we have $\lambda_2 b(X^p) = \lambda b(X)$, meaning either $\lambda = \lambda_2$ and $b(X) = \beta \in \mathbb{F}_p^*$ or $b(X) = \beta = 0$. We also have $\lambda_1 a(X^p) + f(X)\beta = \lambda a(X)$. Then by the properties of $f(X)$ we have that the coefficients of X^i in $a(X)$ with $i > 0$ is 0, and $\deg(a) = 0$, because otherwise the coefficient of $X^{p \deg(a)}$ is nonzero on the left hand side and 0 on the right. Thus $a(X) = \alpha$ and $f(X) = \delta$ with $\alpha, \delta \in \mathbb{F}_p$. If $\lambda_1 \neq \lambda_2$, then $f(X) = 0$ (see the last statement in the first part of the proposition). If $\lambda_1 = \lambda_2$, then $\lambda_1 \alpha + \delta \beta = \lambda_1 a(X^p) + f(X)\beta = \lambda a(X) = \lambda_1 \alpha$, thus either $\delta = f(X) = 0$ or $\beta = 0$ hence $d = \alpha e_1$.

□

Corollary 4.5.4 *If $\lambda_1 \neq \lambda_2$, then there exists a nontrivial extension of D_2 by D_1 .*

Proof Let $(1+X)^{\chi(\gamma)} - 1 = X(\rho + Xh(X))$, and n with $1-p \leq n < 0$ such that $\mu_1 \rho^n = \mu_2$. We can choose $f(X) = \sum_{i=n}^{-1} \alpha_i X^i$ such that $\mu_1 f((1+X)^{\chi(\gamma)} - 1) - \mu_2 f(X) \in \mathbb{F}_p[[X]]$, because for $i > n$ we have $\mu_1 \rho^i \neq \mu_2$, and we can choose the α_i -s inductively in increasing order. Thus there exists $g(X)$ such that the condition for $f(X)$ and $g(X)$ is satisfied. □

Remark By the modulo p Langlands-correspondence for $\mathbf{GL}_2(\mathbb{Q}_p)$ these 2-dimensional (φ, Γ) -modules (which are the extension of two 1-dimensional ones) correspond to extension of principal series representations of $\mathbf{GL}_2(\mathbb{Q}_p)$.

Let $\pi = \text{Ind}_B^G(\chi_1 \otimes \chi_2)$ and $\pi' = \text{Ind}_B^G(\chi'_1 \otimes \chi'_2)$ (with $\chi_i, \chi'_j : \mathbb{Q}_p^* \rightarrow \mathbb{F}_p^*$ characters) be principal series of $\mathbf{GL}_2(\mathbb{Q}_p)$. By [6], Proposition 4.3.15. there exists nontrivial extension of π' by π if and only if either $\chi_1 = \chi'_1$ and $\chi'_2 = \chi_2$, or $\chi_1 = \chi'_2 \bar{\chi}^{-1}$ and $\chi_2 = \chi'_1 \bar{\chi}$ (where $\bar{\chi}$ is the modulo p reduction of the cyclotomic character).

The (φ, Γ) -module $D(\pi)$ attached to π is not D_1 or D_2 . D_i contains information only of χ_i . However from D we can recover π and π' (and the other way around): $\chi_i|_{1+p\mathbb{Z}_p} = \chi'_j|_{1+p\mathbb{Z}_p} = 1$ we have $\chi'_1(p) = \lambda_1$, $\chi'_1(\gamma) = \mu_1$, $\chi_1(p) = \lambda_2$ and $\chi_1(\gamma) = \mu_2$ (cf. the part before Théorème 0.9 in [4]). If $\chi'_1 \neq \chi_1$, then $\chi_2 = \chi'_1 \bar{\chi}$ and $\chi'_2 = \chi_1 \bar{\chi}$.

Proposition 4.5.5 *Let D be as in the previous proposition. Then*

$$\dim_{\mathbb{F}_p}(D^{nr}) = \begin{cases} 2, & \text{if } f(X) \in \mathbb{F}_p \subset \mathbb{F}_p((X)), \\ 1, & \text{otherwise.} \end{cases}$$

Proof We have

$$\begin{aligned} & \varphi^n(a(X)e_1 + b(X)e_2) = \\ & \left(\lambda_1^n a(X^{p^n}) + \sum_{i=0}^{n-1} \lambda_1^i \lambda_2^{n-1-i} f(X^{p^i}) b(X^{p^n}) \right) e_1 + \lambda_2^n b(X^{p^n}) e_2. \end{aligned}$$

If $d = a_0(X)e_1 + b_0(X)e_2 \in D^{nr}$, and $\text{pr} : D \rightarrow D_2$, then $\text{pr}(d) \in D_2^{nr} = \mathbb{F}_p e'_2$, hence if $d = \varphi^n(a(X)e_1 + b(X)e_2) \in D^{nr}$, then $b(X) = \beta \in \mathbb{F}_p$.

In f the coefficients of X^{pj} with $j < 0$ are 0, hence in the above sum the coefficient of $X^{p^{n-1} \deg(f)}$ is not 0. Thus if $d \in \varphi^n(D)$, then either $\deg(a_0) \leq p^{n-1} \deg(f)$ or $\deg(a_0) \geq 0$. Hence if $d \in D^{nr}$, we have $\deg(a_0) = 0$, and $a(X) = \alpha \in \mathbb{F}_p$.

If $\deg(f) < 0$, then we must have $\beta = 0$. □

Proposition 4.5.6 *Let D be as in Lemma 4.5.3 such that $-p < \deg(f) < 0$. Then $D^\natural = X^{-1}\mathbb{F}_p[[X]]e_1 + \mathbb{F}_p[[X]]e_2$.*

Proof Let $\Lambda = X^{-1}\mathbb{F}_p[[X]]e_1 + \mathbb{F}_p[[X]]e_2$. It is a $k((X))$ -generating submodule, we show that it is ψ -invariant as well. Let $d \in \Lambda$. We can write it in

the form $d = \sum_{i=0}^{p-1} (1+X)^i \varphi(\alpha_i(X)e_1 + \beta_i(X)e_2)$, and a simple computation shows that $\alpha_i(X) \in X^{-1}\mathbb{F}_p[[X]]$ and $\beta_i(X) \in \mathbb{F}_p[[X]]$ for all i . Then $\psi(d) = \alpha_0(X)e_1 + \beta_0(X)e_2 \in \Lambda$. Thus $D^\natural \subseteq \Lambda$.

$\mathbb{F}_p[[X]]e_1 \subset D^\natural$, because if $D' \rightarrow D$ is injective, then so is $D'^\natural \rightarrow D^\natural$ (cf [5] Prop. II.5.17(ii).), and $\mathbb{F}_p((X))e_1 \hookrightarrow D$ as a (φ, Γ) -module, with $D^\natural(\mathbb{F}_p((X))e_1) = \mathbb{F}_p[[X]]e_1$.

We also have that if $D \rightarrow D'$ is surjective, then so is $D^\natural \rightarrow D'^\natural$ (cf [5] Prop. II.5.17(iii).), thus we have an element in the form $d = \lambda X^{-1}e_1 + \lambda_2 e_2$ in D^\natural with some $\lambda \in \mathbb{F}_p$ because $\mathbb{F}_p[[X]]e_1 \leq D^\natural$. Then we have

$$d = \varphi(e_2) + (\lambda X^{-1} - f(X))e_1 = \varphi(e_2) + \sum_{i=0}^{p-1} (1+X)^i \varphi(\alpha_i(X)e_1)$$

with $\alpha_i(X) \in X^{-1}\mathbb{F}_p[[X]]$. We have $\alpha_i(X) \in \mathbb{F}_p[[X]]$ for $i < p + \deg(f)$.

If $\lambda X^{-1} \neq f(X)$, then we also have $\alpha_{p+\deg(f)}(X) \notin \mathbb{F}_p[[X]]$, thus $\psi((1+X)^{-(p+\deg(f))}d) = \alpha_{p+\deg(f)}e_1$, meaning $\Lambda \subseteq D^\natural$.

If $\lambda X^{-1} = f(X)$, then $\psi(d) = e_2 \in D^\natural$ and also $\lambda^{-1}(d - \lambda_2 e_2) = X^{-1}e_1 \in D^\natural$, and we again have $\Lambda \subseteq D^\natural$. □

Corollary 4.5.7 *If D is as above, then the action F_P defined in the previous section has a nontrivial kernel for any $\Lambda \supseteq D^\natural$.*

Proof Recall that $F_P : d + X\Lambda = \psi(\sum_{i=0}^{p-1} (1+X)^i d) + X\Lambda$.

Let $d = X^m e_1 \in \Lambda \cap D_1$ such that $m = \min\{m | m \in \mathbb{Z}, X^m e_1 \in \Lambda\}$. By the Proposition 4.5.6 we have $m \leq -1$. Then $d + X\Lambda \notin X\Lambda$, hence it is enough to prove that $\psi(\sum_{i=0}^{p-1} (1+X)^i d) \in X^{m+1}\mathbb{F}_p[[X]]e_1 \subset X\Lambda$.

If $m < -1$, then it is clear, because then $\Lambda \cap D_1 \supsetneq D_1^\#$, hence ψ is not surjective on it, meaning $\psi(d') \in X^{m+1}\mathbb{F}_p[[X]]e_1$ for any $d' \in \Lambda \cap D_1$, especially for $d' = \sum_{i=0}^{p-1} (1+X)^i d$.

If $m = -1$, then

$$\begin{aligned} \psi\left(\sum_{i=0}^{p-1} (1+X)^i \frac{1}{X} e_1\right) &= \psi\left(\sum_{i=0}^{p-1} (1+X)^i \left(\sum_{j=0}^{p-1} (1+X)^j \varphi\left(\frac{1}{X}\right)\right) e_1\right) = \\ &= \psi\left(\sum_{i,j=0}^{p-1} (1+X)^{i+j} \varphi\left(\frac{1}{X}\right) e_1\right) = \lambda_1(1+(p-1)(1+X))\frac{1}{X} e_1 = \\ &= \lambda_1(p-1)e_1 \in \mathbb{F}_p[[X]]e_1. \end{aligned}$$

□

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