Gráfelméleti módszerek stabil párosítások vizsgálatában

Habilitációs tézis

BME VIK, 2013

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Mellékletek: Szakmai önéletrajz
Oklevél másolat
PhD oklevél másolat
Honosító határozat másolata
tudománymetriai táblázatok
oktatott tárgyak listája
egyéb információk
Utolsó frissítés: 2013. május 30.

Tartalomjegyzék

Bevezetés

Jelen disszertáció a szerzőnek a BME Villamosmérnöki és Informatikai karán elindított habilitációs eljárásának kelléke. Célja, hogy a PhD fokozat megszerzését követően a szerző (esetenként társszerzőkkel közösen) elért egyes tudományos eredményeit bemutassa és összefoglalja. A könnyebb áttekinthetőség kedvéért az önálló eredményeket tézispontokban rendszerezzük. Ezt követi a feldolgozott témakör áttekintése és az egyes eredmények bővebb kifejtése angol nyelven. A feltehetőleg csak a témában különösen jártas olvasót érdeklő bizonyítások illetőleg technikai részletek szintén ebben a részben találhatók.

A stabil párosítások elméletét Gale és Shapley úttörő munkája [31] tette ismertté és keltette fel közgazdászok, játékelmélettel, diszkrét matematikával ill. számítástudománnyal folgalkozó kutatók érdeklődését és lett mindezen tudományterületeken számos érdekes eredmény és probléma forrása. Gale és Shapley modelljében n férfi és n nő rendezi sorba a saját preferenciája szerint az ellentétes nem képviselőit, nekünk pedig azt a természetes kérdést kell eldönteni, hogy vajon összeházasíthatók-e egymással tartósan e szereplők, azaz ne létezzen olyan férfi és nő, akik egymást jobban kedvelik, mint a házastársukat. Gale és Shapley munkája nyomán kiderült, hogy egy meglepően ésszerű viselkedést szimuláló, gyors algoritmus segítségével mindig elérhető a fenti cél, azaz egy ú.n. stabil párosítás. Ahogyan ez a játékelméleti modellekben megszokott, nem fontos, hogy férfiakról és nőkről szól a fenti történet, hasonlóan természetes kérdések vethetők fel akkor is, ha a modell szereplői önálló döntésre képes, kizárólag a saját céljukat elérni kívánó „játékosok”. Ilyen megközelítésben a modell számos gyakorlati problémához kapcsolódik, ám a legtöbb ilyen problémához valamiféle általánosításra van szükség. Az egyetemi felvételi problémában az egyetemi szakokra kell az e szakora jelentező diákokat úgy felvenni, hogy ne legyen egy hallgatónak és egy szaknak sem közös érdeke a kapott megoldástól való eltérés. (A modellben a szakok a felvételi alapján rangsorolják a jelentkezőket, utóbbiak pedig a saját preferenciasorrendjük szerint adják be a jelentkezésüket.) Az ilyen módon általánosított „kétoldalú piacon” a megoldás olyan stabil házasításnak felel meg, ahol a piac egyik oldalának szereplői (azaz a szakok) számára megengedett a poligámia, továbbá azt is meg kell engedni, hogy a modell kétféle szereplője között ne legyen minden kapcsolat (házasság) lehetséges. Egyéb munkaerőpiaci vagy ütemezéselméleti alkalmazásokban az is előfordul, hogy a piac mindkét oldalán megengedett a játékosok számára a poligámia. Ezekről a problémákról az derült ki, hogy Gale és Shapley stabil párosításokról szóló tétele továbbra is igaz marad.

A problémakör általánosításának egy másik fontos iránya az „egyoldalú piacok” esete. Itt a részvevő játékosok nem két típusból kerülnek ki, ennek megfelelően nincs megkötés arra nézve, hogy melyik játékospárok között alakulhat ki kapcsolat, más szóval a modellhez tartozó gráf nem feltétlenül páros. Erre példa a stabil szobatárs probléma, aholis úgy kell n játékost csoportokra (szobákba) osztanunk, hogy bármely csoportban legfeljebb két játékos legyen, ám ne találjunk két olyan játékost, akik a megadott beosztás helyett szívesebben alkotnának inkább egy csoportot. Ugyan ebben az esetben nincs mindig stabil párosítás, de Irving hatékony algoritmust adott ennek keresésére [35].

A mára már igen kiterjedt témát immár legalább négy könyv dolgozza fel ([52, 33, 42, 43]), és a terület fontosságát az is jól jelzi, hogy Roth és Shapley 2012-ben az e témakörben elért eredményeikért kaptak közgazdaság Nobel díjat. A jelen munka a stabil párosítások elméletének egy újszerű megközelítését igyekszik bemutatni, melynek kidolgozását a szerző még a [28] PhD disszertációjában ill. a [25] cikkében kezdte meg. E megközelítés egyik fontos eleme, hogy hogy a kétoldalú piacok esetét Tarski egy jól ismert, az analízisben és a halmazelméletben alkalmazott fixponttételének segítségével tárgyalja [61]. E fixponttétellel való kapcsolat teszi világossá, hogy Gale és Shapley algoritmusa miért is természetes, miért működik, de megmagyarázza egyúttal a stabil párosítások jól ismert hálótulajdonságát is. Érdekes, hogy az egyoldalú piacokra vonatkozó eredmények egy másik fixponttétellel, konkrétan a topológiából ismert Kakutani-félével mutatnak váratlan rokonságot. Egy másik fontos összetevője az általunk vázolt megközelítésnek, hogy a diszkrét matematika, és főképpen a gráfelmélet eredményeit és terminológiáját alkalmazza. Ez bár a kombinatorikus szemével nézve nagyon természetesnek tűnik, egyáltalán nem az. Az szakirodalomban (a szerző munkáitól eltekintbve) szinte sosem élnek ezzel a közgazdászok számára talán szokatlan megközelítéssel, jóllehet számos eredmény és bizonyítás sokkal szemléletesebb lesz ezáltal. Végül a bevett módszer helyett, amely szerint az egyes általánosításokat szinte a nulláról bizonyítjuk, az itt vázolt tárgyalásmódban egy kiválasztási függvények segítségével megfogalmazott általános problémát vizsgálunk, és a kiválasztási függvényekről megfogalmazott különféel feltevésekből vezetjük le a stabil megoldás létezését. Eközben előfordul, hogy egy általánosított modellre vonatkozó tételt egy jól ismert, egyszerűbb modellre való visszavezetéssel bizonyítunk be.

A disszertáció az alábbiak szerint épül fel. A következő szakasz a tézisek listája. Ezt követi ezek részletes kifejtése, angol nyelven. Az 1. fejezetben a kétoldalú piacokkal foglalkozunk bemutatjuk a Gale-Shapley tételt és felidézzük a rendszerint helytelenül Tarskinak tulajdonított (ám Knaster által publikált) fixponttételt. Az 1.1 szakaszban azt az általánosítást vizsgáljuk, amikor előírt kvóták szerinti poligámia is megengedett, és kiválasztási függvény segítségével definiáljuk a stabilitást és bizonyítjuk általánosított stabil párosítás létezését pl. matroidokra épülő modellekben. Az 1.2 szakaszban bemutatunk néhány, a stabil megoldások hálótulajdonságára vonatkozó korábbi ismeretet, de egyúttal itt szerepel az egyetemi felvételiben használt ú.n. vonalhúzások korábban ismeretlen hálótulajdonsága is. Az ezt követő az 1.3 szakaszban a hálótulajdonság alkalmazásával igazolható további hasznos tulajdonságra mutatunk rá, bemutatjuk Teo és Sethuraman egy eredményének általánosítását, a stabil megoldások egy fontos partíciós tulajdonságát, valamint a „rural hospital" tétel általánosítását a matroid-modellre. A kétoldalú piacokról szóló fejezetet az 1.4 szakasz zárja. Itt felidézzük a stabil párosítás poliéder lineáris leírására vonatkozó korábbi eredményeket. Ezek általánosításaként olyan poliéderek lineáris leírását mutatjuk be, amelyeket komonoton növekedő kiválasztási függvényekhez tartozó stabil megoldások határoznak meg. Bár ezen leírások jól karakterizálják az adott poliédert, a feltételek implicitek, és sok van belőlük. Azonban egy szintén ebben a szakaszban tárgyalt tétel szerint a stabil b-párosítások esetére jóval könnyebben kezelhető lineáris feltételek is megadják a leírást.

A 2. fejezetben az egyoldalú piacokat vizsgáljuk, aholis tehát a játékosok és a köztük levő lehetséges kapcsolatok gráfja nem feltétlenül páros. A 2.1 szakaszban bemutatjuk Tan karakterizációját, és annak a b-párosításos modellre történő általánosítását, valamint egy olyan konstrukciót írunk le, amivel a stabil b-párosítás probléma a stabil párosítás problémára vezethető vissza. Ugyanitt mutatunk ré a Tan-féle karakterizáció és a Scarf lemma ill. a Kakutani-féle fixponttétel kapcsolatára, és arra, hogy ha egy modellben stabil b-párosítás nincs is, stabil b'-párosítás mindig létezik alkalmas, b-hez elég közeli b' fokszámkorlátokra. A 2.2 szakaszban olyan modelleket vizsgálunk, amelyekben a preferenciasorrendek nem feltétlenül szigorúak, bizonyos kapcsolatok (házasságok) kötelezően megköttetendők, mások pedig tilosak. E modellekben természetesen nem feltétlenül van stabil párosítás, azonban Irving algoritmusának egy érdekes általánosítása segítségével hatékonyan tudunk megoldást találni, ha van.

A 3. fejezetben azt mutatjuk meg, hogy hálózati folyamok esetén is lehet értelme preferenciákról beszélni, és ahogyan egy páros gráfban érdekes probléma mind a maximális nagyságú, mind a stabil párosítás keresése, úgy egy hálózatban sem csak maximális nagyságú (archaikus szóhasználattal élve: maximális értékű) folyamról beszélhetünk, hanem stabilról is, amelyben a hálózat csúcsai reprezentálta játékosok egyike sem szeretne eltérni az adott folyamtól. Baïou és Balinski korábbi eredményének segítségével tudjuk megmutatni, hogy stabil folyam mindig létezik, ill. a 3.1 szakaszban a stabil folyamok hálótulajdonságának felhasználásával magyarázzuk meg, hogy a modellbeli játékosok miért tekinthetők vagy eladó, vagy vevő típusúnak.

A disszertációt a 4. fejezet zárja, amiben a stabil folyamok egy természetes általánosítását mutatjuk be. Itt egy olyan kétoldalú munkaerőpiacot vizsgálunk, amelynek szereplői orvosok és kórházak, a lehetséges kapcsolatok pedig munkaszerződések, ám minden egyes szerződéshez valamiféle intenzitás is tartozik. Ez a modell általánosítja Hatfield és Milgrom ünnepelt modelljét, és az itt szereplő stabil megoldás létezését igazoló tétel olyan esetekben is alkalmazható, ahol a Hatfield-Milgrom modell csődöt mond.

A disszertációt a habilitációs eljáráshoz szükséges további dokumentumok másolatainak csatolt példányai zárják: a diplomáról, PhD oklevélről, ennek honosításáról szóló határozat, tudománymetriai táblázatok, valamint az oktatott tárgyak listája.

Tézispontok

1. Az egyetemi felvételi ponthatárok hálótulajdonságának bizonyítása (Theorem12). (Közös eredmény Jankó Zsuzsannával. Forrás: [37].)

2. A stabil párosítások hálótulajdonságának felhasználásával Teo és Sethuraman tételének általánosítása (Theorem 21). Ennek az eredménynek speciális esete Klaus és Klein későbbi [40] cikkének fő eredménye. (Forrás: [24].)

3. Stabil b-párosítások egy érdekes csillag-partíciós tulajdonságának igazolása (Theorem 23 és Theorem 39). Ettől az eredménytől függetlenül, lineáris programozási eszközökkel Sethuraman, Teo és Qian hasonló eredményt bizonyítottak [56] cikkükben. A csillag-partíciós tulajdonság az egyik kulcsa a stabil b-párosítás poliéder lineáris leírásának. (Forrás: [26, 24].)

4. Többféle, általánosított stabil párosításokkal kapcsolatos poliéder leírása (Theorem 29 és Theorem 31). (Forrás: [26].)

5. A nem feltétlenül páros gráfokon értelmezett stabil b-párosítás problémának a stabil párosítás problémára történő visszavezetése egyszerű konstrukióval (Observation 36 és Observation 37). Irving stabil párosítást kereső algoritmusának általánosítása stabil b-párosítás keresésére. (Közös eredmény Katarína Cechlárovával. Forrás: [15].)

6. Irving algoritmusának általánosítása szuperstabil párosítások keresésére, ha tiltott élek is megengedettek (2.2 szakasz), illetve a szabad (blokkolni képtelen) éleket tartalmazó modellben a stabil párosítás létezésének NP-teljessége (Theorem 44). (Közös eredmény Katarína Cechlárovával. Forrás: [16].)

7. Stabil folyamok létezése, ezek hálótulajdonsága, a stabil folyam probléma visszavezetése stabil allokáció keresésére (Theorem 48). (Forrás: [27].)

8. Kétoldalú piacokon részbenrendezéseken alapuló kiválasztási függvénnyel definiált stabil párosítás létezésének igazolása, ezen stabil párosítások hálótulajdonságának bizonyítása (Theorem 61). (Közös eredmény Rashid Farooq-kal és Akihisa Tamuraval. Forrás: [20].)

1. fejezet

The bipartite case: stable marriages

Let G be a bipartite graph (parallel edges are allowed) with colour classes M and W and let E denote the set of edges of G. (It might be convenient to think that vertices of M and W represent men and women, respectively, and edges are along possible marriages.) For vertex v of G, let $E(v)$ denote the set of edges incident with v. (Note that $E(v)$ might not be finite, even if $V(G)$ is finite). We call $(G, \mathcal{O})$ a *bipartite preference system* if graph G is as above and $\mathcal{O}$ is a set of orders $<_v$ for $v \in M \cup W$, such that $<_v$ is a well-order on $E(v)$. We say that edge e of G *M-dominates* (*W-dominates*) edge f of G if e and f are incident with the same vertex m of M (w of W) and $e <_m f$ ($e <_w f$). Edge e *dominates* edge f if e M-dominates or W-dominates f. Set F of edges M-dominates (W-dominates) edge set H if each edge h of H is M-dominated (W-dominated) by some edge f of F.

Let $(G, \mathcal{O})$ be a bipartite preference system. Subset S of E is a *stable matching* if no edge of S is dominated by S, and S dominates $E \setminus S$. Note that any stable matching is necessarily a matching (i.e. a set of disjoint edges), as if two edges of a stable matching shares a vertex then one of them would dominate the other.

Theorem 1 (Gale and Shapley 1962 [31]) *For any finite bipartite preference system $(G, \mathcal{O})$, there exists a stable matching.*

The original proof of Theorem 1 is a construction of a special stable matching with the so called deferred acceptance algorithm. This is an iterative procedure that alternatively repeat two steps. It starts with a proposal step in which each vertex of M selects its most preferred edge. In the next refusal step, certain edges that have been selected in the proposal step, get deleted. Namely, m's edge e gets deleted, if there is another edge f selected by some other vertex with $f <_w e$ for some vertex w of W. If no edge is deleted in a refusal step then output the edges selected in the last proposal step. Otherwise start the procedure all over for the reduced bipartite preference system that we get after the deletions of the refusal step. Gale and Shapley have proved that this deferred acceptance algorithm constructs a stable matching of $(G, \mathcal{O})$ which is „man-optimal”, that is, no edge that has been deleted in a refusal step can be in any stable matching, or equivalently, each M-vertex gets the best possible partner and each W-vertex receives the worst possible partner that he/she can have in a stable matching.

In the literature, the usual setting and the definition of a stable matching is somewhat different from ours. We describe that terminology as well, so as to have a „dictionary” between the two languages. If G is a graph, a *matching* N of G is a set of disjoint edges of G. Equivalently, we can regard a matching as an involution on the vertices of G, that is, a function $\mu : V(G) \to V(G)$ in such a way that $\mu(\mu(v)) = v$ for each vertex v of G. Indeed, if uv is an edge of matching N then $\mu(u) = v$ and $\mu(v) = u$, and if w is not covered by N then $\mu(w) = w$ for the corresponding involution. From involution μ, we can also reconstruct matching N as $uv \in N$ if and only if $\mu(u) = v$.

The conventional counterpart of a bipartite preference system is the following. We have given disjoint sets M and W. For vertex m of M (w of W), let $\prec_m$ ($\prec_w$) be a linear order on $W \cup \{m\}$ ($M \cup \{w\}$). In this model, matching μ is a stable matching if

$$\mu(v) \leq_v v \text{ for each vertex } v \text{ of } M \cup W \text{ and} \tag{1.1}$$

$$\text{for all } m \in M \text{ and } w \in W,\ w \prec_m \mu(m) \text{ implies } \mu(w) \prec_w m\ . \tag{1.2}$$

That is, a matching is stable, if each person (vertex) is not worse off by his/her partner than by remaining single (or, in other words, matching μ is *individually rational*) and whenever man m (i.e. a vertex in the

M side) prefers woman w to his eventual partner, then w must prefer $\mu(w)$ to the partnership with m. Note that traditionally, the preference orders are on the vertices, and not on the edges. (This makes the traditional model in some sense richer than ours, as it allows that a man m can accept woman w but w prefers to remain single to marry m). For simple graphs however, preference order $\prec_v$ induces a linear order $<_v$ on $E(v)$, so a stable matching in our sense corresponds to a stable matching μ in the latter sense. Our terminology is more general in the sense that it readily allows graphs with parallel edges in a bipartite preference system. This feature is useful if we use a model where the vertices correspond to firms and workers, so parallel edges between a firm and a worker may mean different wages in case of an employment.

Let us now return to our terminology. If S is a stable matching then we can partition E into three disjoint parts as $E = S_M \cup S \cup S_W$, in such a way that S_M (and S_W) is M-dominated (W-dominated, respectively) by S. (Note that such a partition is unique only if no edge of $E \setminus S$ is both M- and W-dominated by S.) From here, it is easy to see that sets $S^M := S \cup S_M$ and $S^W := S \cup S_W$ have the following properties.

$$S^M \cup S^W = E \tag{1.3}$$

$$\text{no edge of } S^M \cap S^W \text{ dominates another edge of } S^M \cap S^W \tag{1.4}$$

$$S^M \text{ is } M\text{-dominated by } S^M \cap S^W \text{ and } S^W \text{ is } W\text{-dominated by } S^M \cap S^W \tag{1.5}$$

For convenience, we call (S^M, S^W) a *stable pair* if S^M and S^W have the above properties (1.3,1.4,1.5). Clearly, if (S^M, S^W) is a stable pair then $S := S^M \cap S^W$ is a stable matching. Equivalently, we can say that pair (S^M, S^W) is stable if besides (1.3) we have

$$\mathcal{C}_M(S^M) = S^M \cap S^W = \mathcal{C}_W(S^W) \ , \tag{1.6}$$

where $\mathcal{C}_M(S^M)$ (and $\mathcal{C}_W(S^W)$) denotes those elements of S^M (and S^W) that are not M-dominated (and W-dominated) by other elements of S^M (S^W). This means that (S^M, S^W) is a stable pair if and only if

$$S^W = E \setminus (S^M \setminus \mathcal{C}_M(S^M)) \text{ and} \tag{1.7}$$

$$S^M = E \setminus (S^W \setminus \mathcal{C}_W(S^W)) \tag{1.8}$$

holds. By substituting (1.7) into (1.8), we get that (S^M, S^W) is a stable pair if and only if (1.7) holds with

$$f(S^M) := E \setminus \left[\left(E \setminus [S^M \setminus \mathcal{C}_M(S^M)]\right) \setminus \mathcal{C}_W \left(E \setminus [S^M \setminus \mathcal{C}_M(S^M)]\right) \right] = S^M \ . \tag{1.9}$$

A key observation in our treatment that function f is *monotone*, that is, $f(A) \subseteq f(B)$ whenever $A \subseteq B \subseteq E$. (To see this, it is useful to observe that function $A \mapsto A \setminus \mathcal{C}(A)$ is monotone for $\mathcal{C} = \mathcal{C}_M$ and $\mathcal{C} = \mathcal{C}_W$.) Thus we can invoke the Knaster-Tarski fixed point theorem.

Theorem 2 (Knaster and Tarski 1928 [41]) *If $f : 2^E \to 2^E$ is a monotone function then f has a fixed point.*

As a consequence of Theorem 2, the function f that comes from the bipartite preference model according to (1.9) has a fixed point S^M. Using (1.7) as a definition for S^W, we can construct a stable pair (S^M, S^W) from the above fixed point, and a stable matching S from stable pair (S^M, S^W). This proves the existence part of Theorem 1. What is more interesting, than this single reduction is that for finite ground sets, there is a most simple algorithmic proof of Theorem 2 that can be applied in our construction. (Note that in Theorem 2, E can be an arbitrarily large set.) Namely, $f(\emptyset) \subseteq f(f(\emptyset)) \subseteq f(f(f(\emptyset))) \subseteq \ldots$ by the monotonicity of f, so if E is finite then this increasing chain has to get stabilized at some fixed point A of f. This fixed point A is contained in any fixed point of f by the procedure. Similarly, chain $f(E) \supseteq f(f(E)) \supseteq f(f(f(E))) \supseteq \ldots$ must get stabilized at the inclusionwise maximal fixed point of f. It turns out that the deferred algorithm itself is essentially the iteration of f starting from E. If we exchange the role of M and W then the deferred acceptance algorithm concludes with the woman-optimal stable matching that we can construct by iterating f starting from the $\emptyset$. This observation is enough to prove that the deferred acceptance algorithm outputs the man-optimal stable matching.

1.1 Multiple partner matchings

The stable marriage theorem of Gale and Shapley (Theorem 1) has several extensions and generalizations that nicely fit into our framework. We start with the stable admissions problem described in [31]. We have a bipartite preference system $(G, \mathcal{O})$ and a function $b : M \cup W \to \mathbb{N}$ with $b(w) = 1$ for all $w \in W$. Set F of edges (M, b)*-dominates* ((W, b)*-dominates*) edge e of G if there is a vertex m of M (w of W) and different edges $f_1, f_2, \ldots, f_{b(m)}$ ($f_1, f_2, \ldots, f_{b(w)}$) of F so that $f_i <_m e$ for $i = 1, 2, \ldots, b(m)$ ($f_i <_w e$ for $i = 1, 2, \ldots, b(w)$). Set F of edges b-dominates edge set H if each edge h of H is (M, b)-dominated or (W, b)-dominated by F. Subset S of the edges of G is a *stable admission scheme* if no edge of S dominates another edge of S, and S dominates $E \setminus S$. By definition, each vertex in W is incident with at most one edge and vertex m of M is incident with at most $b(m)$ edges of a stable admission scheme. (The story is, that vertices of M represent colleges that offer admission, W stands for the set of students that look for admission in a college, and b is the quota of the colleges. We look for a stable admission scheme in which no college-student pair mutually prefer each other to their assignments.)

The same argument that we gave for the stable marriage problem goes through for the above admissions case, even in the more general case in which we do not require that $b(w) = 1$ for $w \in W$. (The generalization of a stable admission scheme to this case we call a *stable b-matching.*) The iteration of the corresponding monotone function describes the modified deferred acceptance algorithm that finds the optimal stable admissions. This implies the following theorem.

Theorem 3 *For any bipartite preference system $(G, \mathcal{O})$ and $b : V(G) \to \mathbb{N}$ there exists a stable b-matching. If M and W are the colour classes of G then there is an M-optimal stable b-matching S, in which each vertex m of M is incident with the most preferred $b(m)$ edges of $E(m)$ that can be present a stable b-matching. Simultaneously, each vertex w of W is incident with the least preferred $b(w)$ edges of $E(w)$ that can appear in a stable b-matching.*

We can generalize the notions of stable matchings and stable admissions. Let $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ be set functions. Pair (S^M, S^W) is a $\mathcal{C}_M\mathcal{C}_W$*-stable pair* if (1.3,1.6) holds. Subset S of E is a $\mathcal{C}_M\mathcal{C}_W$*-stable set* if S dominates exactly the elements of $E \setminus S$, that is, if

$$\mathcal{C}_M(S) = \mathcal{C}_W(S) = S \text{ and} \tag{1.10}$$

$$\mathcal{C}_M(S \cup \{e\}) = S \text{ or } \mathcal{C}_W(S \cup \{e\}) = S \text{ for any element } e \text{ of } E \tag{1.11}$$

What are the crucial properties of a dominance function $\mathcal{C}$ that make our argument work? These are that

$$\mathcal{C}(A) \subseteq A \text{ for any } A \tag{1.12}$$

and that function

$$\overline{\mathcal{C}}(A) := A \setminus \mathcal{C}(A) \text{ is monotone.} \tag{1.13}$$

We call function $\mathcal{C}$ *comonotone* if (1.12,1.13) hold. These properties imply that function f defined in (1.9) is monotone and there is a stable pair. Still, it might happen that $\mathcal{C}_M$ and $\mathcal{C}_W$ are comonotone so there is a stable pair (S^M, S^W), but no stable set exists. However, $S := S^M \cap S^W$ is a stable set if $\mathcal{C}_M$ and $\mathcal{C}_W$ have the additional property that

$$\mathcal{C}(A) = \mathcal{C}(B) \text{ whenever } \mathcal{C}(A) \subseteq B \subseteq A\ . \tag{1.14}$$

This is claimed in the following theorem.

Theorem 4 (Fleiner 2000 [28]) *If $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are comonotone set functions then there exists a $\mathcal{C}_M\mathcal{C}_W$-stable pair (S^M, S^W). If, moreover, both $\mathcal{C}_M$ and $\mathcal{C}_W$ have property* (1.14) *then there exists a $\mathcal{C}_M\mathcal{C}_W$-stable set S.*

We give two more interesting examples of comonotone functions with property (1.14), hence extend the stable marriage theorem in two different directions. A *partial well-order* is a partial order so that each subset of the ground set has a minimal element in the induced order.

Observation 5 *If $<$ is a partial order on E and $\mathcal{C}(A)$ denotes the set of $<$-minima of A for subset A of E then $\mathcal{C}$ is comonotone. If $<$ is a partial well-order then $\mathcal{C}$ has property* (1.14)*.*

Theorem 4 and Observation 5 implies the following property of partial orders.

Corollary 6 (see Fleiner [28, 22, 25]) *If $<_1$ and $<_2$ are partial orders on E then there are subsets E_1, E_2 of E such that*

$$E_1 \cup E_2 = E \textit{ and}$$
$$E_1 \cap E_2 \textit{ is the set of } <_1\textit{-minima of } E_1 \textit{ and the set of } <_2\textit{-minima of } E_2$$

If both $<_1$ and $<_2$ are partial well-orders then there is a common antichain S of $<_1$ and $<_2$ (i.e. no two elements of S are comparable in any of the orders) so that for any $e \in E$ there is an $s \in S$ with $s <_1 e$ or $s <_2 e$.

Observation 7 *Let $<$ be a linear order on finite set E, and $\mathcal{M}$ be a matroid on E. Denote by $\mathcal{C}^<_{\mathcal{M}}(A)$ the output of the greedy algorithm on input A, where the algorithm goes through the elements of A in the order given by $<$. Then $\mathcal{C}^<_{\mathcal{M}}$ is comonotone and has property* (1.14)*.*

Theorem 4 and Observation 7 extends the stable matching theorem to matroids.

Corollary 8 (see Fleiner [28, 22, 25]) *Let $<_1$ and $<_2$ be linear orders on finite set E, and $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids on E. Then there is a common independent set S of $\mathcal{M}_1$ and $\mathcal{M}_2$ so that for each element e of E we have*

$$e \in \operatorname{span}_i\{s \in S : s \leq_i e\} \textit{ for some } i \in \{1, 2\}\ .$$

There also is a common spanning set T of $\mathcal{M}_1$ and $\mathcal{M}_2$ such that for any element t of T, subset T is the lexicographically $<_i$-minimal spanning subset of $\mathcal{M}_i$ that contains $T \setminus \{t\}$ for some $i \in \{1, 2\}$.

See [22, 28] for a reduction of the stable b-matching theorem to the first part of Corollary 8.

1.2 The lattice of stable sets and path independent choice functions

The marriage model of Gale and Shapley has attracted some interest in the theory of social choice. There are related results to the ones that we discussed so far. Again, the terminology is different from ours, so in this section we attempt to cover some basics and point out an interesting connection of this approach to the monotone function based framework.

Let $(G, \mathcal{O})$ be a bipartite preference system and let F and W be the two colour classes of G. We will identify vertices of F with different firms and the vertices of W with workers. Edge fw of G represents that firm f considers w as a potential employee and worker w can accept f as an employer. Firms would like to get certain specific jobs to be done, and this is why they have a more complex preference function on workers than plain ranking. Namely, each firm f has a so called choice function $\mathcal{C}_f$ that selects from any set W' of workers a subset $\mathcal{C}_f(W')$ of W' that firm f would hire if on the labour-market only firm f and workers of W' would be present. Set function $\mathcal{C} : 2^W \to 2^W$ is a *choice function* if there is a well-order $<$ on 2^W such that $\mathcal{C}(W')$ is the $<$-minimal subset of W', for any subset W' of W.

Continuing on a paper of Crawford and Knoer [17], Kelso and Crawford [39] extended the admissions model to a model where each firm has a choice function and each worker has an ordinary preference ranking on firms.

An assignment of workers to firms is called *stable* if it is not blocked by a worker-firm pair. Worker-firm pair (w, f) *blocks* an assignment if w prefers f to his/her employer and in the meanwhile firm f would take worker w if w would be available (that is $w \in \mathcal{C}_f(W_f \cup \{w\})$, where W_f is the set of workers assigned to firm f).

Note that in the above fairly general model there might be no stable assignment. However, if each choice function has the so-called substitutability property, then a stable assignment always exists. We say that set function $\mathcal{C}_f : 2^W \to 2^W$ of firm f has the *property of substitutability*, if

$$w \in \mathcal{C}_f(W') \text{ implies } w \in \mathcal{C}_f(W' \setminus \{w'\}) \tag{1.15}$$

for any subset W' of the set W of workers and for any two different workers w, w' of W'. This means that if a firm would like to employ a certain worker, then it still would like to hire him/her if some other worker leaves the labour-market.

Theorem 9 (Kelso-Crawford 1982 [39]) *If each firm's preference is a substitutable choice function in the worker-firm assignment model, then there is a stable assignment of workers to firms.*

The proof of Crawford and Kelso is via the accordingly modified deferred acceptance algorithm. They also observed that firm-proposing results in the firm-optimal assignment, and the worker-proposal based method leads to the worker-optimal situation. In [47, 48], Roth extended Theorem 9 to a many-to-many model. A *stable assignment* is a bipartite assignment graph A with colour classes F and W, such that for any $w \in W$ and $f \in F$ we have that $wf \in E(A)$ if and only if $f \in \mathcal{C}_w(\Gamma_A(w) \cup f)$ and $w \in \mathcal{C}_f(\Gamma_A(f) \cup w)$. (Notation $\Gamma_A(w)$ stands for the neighbours of w in A.)

Theorem 10 (Roth 1984 [47, 48]) *Let F and W be disjoint finite sets, and for each $f \in F$ and $w \in W$ let $\mathcal{C}_w : 2^F \to 2^F$ and $\mathcal{C}_f : 2^W \to 2^W$ be choice functions with substitutability property (1.15). Then there is a stable assignment in the model.*

Clearly, the stable marriage theorem of Gale and Shapley is a special case of Theorem 10, where the choice functions simply select the highest ranked partner from the input. For the college model, the choice function of college c selects the best $b(c)$ choices.

A key observation in this section is that a choice function trivially has properties (1.12) and (1.14), and a little effort shows that for finite ground sets substitutability property (1.15) implies property (1.13), so choice functions in Theorems 9 and 10 are comonotone. A fairly trivial construction shows that the above theorems are special cases of Theorem 4. Thus, our monotone framework is relevant for these results so we can prove that there always exists a stable set, that is, a stable assignment.

In [48], Roth studied three models: the one-to-one, the many-to-one and the many-to-many with substitutable preferences. He showed that for all three models there is a firm-optimal, „worker-pessimal" and a worker-optimal, „firm-pessimal" stable assignment. The name „polarization of interests" refers to this observation. Further on, Roth introduced the notion of the *consensus property*, that means the following. If each agent on one side of the market combines his/her most preferred assignment from a set of stable assignments, then this way another stable assignment is constructed. This is a generalization of the lattice property of stable schemes in the marriage model. (The observation that stable marriages in the marriage model have a natural lattice structure is attributed to John Conway.) Unfortunately, this property does not always hold in Theorem 10. In [48], Roth asked whether some lattice structure can still be defined on stable assignments. Blair answered this question positively in [14]. His idea was that instead of defining lattice operations, he introduced a more or less natural partial order on stable assignments. This partial order turned out to be a lattice order hence one can generalize the lattice property of bipartite stable matchings.

Theorem 11 (Blair 1988 [14]) *Let F be a set of firms and W be a set of workers. Let, for each $f \in F$ ($w \in W$), set function $\mathcal{C}_f : 2^W \to 2^W$ ($\mathcal{C}_w : 2^F \to 2^F$) be given with properties (1.12,1.14,1.15). Let $\mathcal{A}$ be the set of stable assignments of the model, and define for $A_1, A_2 \in \mathcal{A}$*

$$A_1 < A_2 \text{ if } \Gamma_{A_1}(f) = \mathcal{C}_f(\Gamma_{A_1}(f) \cup \Gamma_{A_2}(f))$$

holds for each firm f. (That is, each firm would choose A_1 if all choices provided by A_1 and A_2 would be offered.) Then $\mathcal{A}$ is nonempty and $<$ is a lattice order, that is, any two stable assignments have a $<$-maximal lower bound and a $<$-minimal upper bound.

It seems that Theorem 11 is not general enough for the following example where choice functions do not have property 1.14.

In Hungary, university admissions are determined with the scores of the applicants. Applicants have strict rankings on their applications. Each application belongs to some applicant and describes a certain university and a subject. We call these university-subject pairs *colleges*. Each college assigns to its applicants certain scores based on the applicants' previous grades and entrance exam results. Different applicants may have the same score at the same college and an applicant may have different scores at different colleges.

Each college has a strict quota on the number of its admitted students. After all information (that is rankings and scores) is known, each college has to declare a certain score limit. This score limit has to be *feasible*, that is if each applicant is admitted to the first college in her ranking where her score is above the limit then no college exceeds its quota. We say that a score limit is *stable* if it is feasible but decreasing the quota of any college results in a non-feasible score limit. The *SSL (stable score limit) problem* is described by the applicants, colleges, applicant rankings and scores and the task is the determination a stable score limit. For the details see Biró [12].

We can define choice functions for applicants and for colleges in the above model. Namely, the choice function C_a of applicant a selects from any set of colleges the first college in a's ranking. If we fix a college c and a subset A of the applicants then choice function of college c selects subset $C_c(A)$ of A the following way. College c calculates the lowest score s such that the number of applicants in A with score above s is not exceeding the quota of c. Then the choice set $C_c(A)$ is the set of applicants of A with score more than s. It is not difficult to see that a stable assignment with these choice functions is exactly an admission scheme that is determined by a stable score limit.

Clearly, both the applicants' and colleges' choice functions have properties (1.12,1.15), and applicants choice function has the (1.14) property as well that does not hold for the colleges' function. A recent result of Jankó [37] shows that in spite of this, stable core limits form a lattice for the following partial order. We say that a score limit S is higher than score limit S' if for each college c, S assigns a greater or equal score limit to c than S' does.

Theorem 12 (Jankó 2009 [37]) *If S_1 and S_2 are stable score limits in the SSL problem then there is unique lowest stable score limit $S_1 \vee S_2$ that is higher than S_1 and S_2 and there is a unique highest score limit $S_1 \wedge S_2$ that is lower than S_1 and S_2.*

Theorem 12 implies that there is a unique highest stable score limit that describes the college-optimal admission scheme and a unique lowest stable score limit that is student-optimal and admits the maximum set of students.

We have mentioned earlier that (1.15) implies (1.13) for functions on finite ground sets, so the functions that describe the choice of the agents of the market are comonotone with property (1.14). To prove the lattice property in our framework, we have to go back to the roots, that is to the fixed point theorem. Recall that we have cited the Knaster-Tarski fixed point theorem on monotone set functions. This theorem is often attributed to Tarski for the reason that 27 years after Knaster's French paper in a polish journal, he published a lattice theoretical generalization of the result in English in a much easier reachable journal [61]. (Actually, Tarski has also formulated a corollary of the fixed point theorem there in terms of Boolean algebras that is more general than Theorem 4.) Note that the proof of the set function theorem is just as difficult as the proof of the lattice generalization, but in the latter paper Tarski has explicitly stated the lattice property of fixed points that we need now.

If L is a lattice on E with partial order $<$ and lattice operations $\wedge, \vee$ then function $f : E \to E$ is *monotone* if $a < b$ implies $f(a) < f(b)$. Lattice L' is a *sublattice* of L if its ground set E' is a subset of E and the lattice operations on L' are the original operations $\wedge$ and $\vee$ restricted to E'. Lattice L' is a *lattice subset* of L if its ground set E' is a subset of E and the lattice order on L' is the restriction of $<$ to E'. Lattice L is complete if any subset E' of its ground set has a meet (greatest lower bound) $\bigwedge E'$ and a join (least upper bound) $\bigvee E'$. (In particular, these lattices have a minimal and maximal element.) Note that any finite lattice is complete.

Theorem 13 (Tarski 1955 [61]) *Let L be a complete lattice on ground set E and $f : E \to E$ be a monotone function. Then the fixed points of f form a nonempty complete lattice subset of L.*

It turns out that Theorem 13 is relevant in the setting of Theorem 11 and it implies that stable assignments exist and form a lattice as described. For the details, see [22]. In the next section, we discuss a property that implies that the lattice subset in Theorem 13 is a sublattice. If that is the case then stable pairs have a very rich structure that allows us to prove further results.

We finish this section with pointing out a connection between properties of set functions we have used so far. Set function $\mathcal{C} : 2^E \to 2^E$ is *path independent* if

$$\mathcal{C}(A \cup B) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B)) \text{ for any subsets } A, B \text{ of } E \tag{1.16}$$

The central notion of path independence has been introduced by Plott in 1973 [45] into the theory of social choice. Our observation is the following.

Theorem 14 *Set function $\mathcal{C} : 2^E \to 2^E$ is path independent with property* (1.12) *if and only if it is comonotone and has property* (1.14).

Lemma 15 (see [28, 22, 25]) *Set function $\mathcal{C} : 2^E \to 2^E$ is comonotone if and only if $\mathcal{C}$ has property* (1.12) *and*

$$\mathcal{C}(B) \cap A \subseteq \mathcal{C}(A) \textit{ whenever } A \subseteq B \subseteq E \tag{1.17}$$

We give an example indicating that comonotonicity and (1.14) is necessary in Theorem 14. That is, there is a function with properties (1.12) and (1.14) that is not path independent.

Example 16 *Let $|E| > k \geq 1$, fix element x of E and define $\mathcal{C} : 2^E \to 2^E$ by*

$$\mathcal{C}(A) = \begin{cases} A & \text{if } |A| > k \\ A \setminus \{x\} & \text{if } |A| \leq k \ . \end{cases}$$

Then $\mathcal{C}$ has properties (1.12,1.14) but it is neither path independent nor comonotone.

Example 16 and Theorem 14 together imply that there exists a comonotone function that is not path independent.

1.3 The stable matching lattice

In this section, we discuss the situation that the lattice subset of stable pairs in Theorem 4 and the lattice subset of fixed points in Theorem 13 are both sublattices. For a finite ground set E, we call function $f : 2^E \to 2^E$ *strongly monotone* if f is monotone with property

$$|f(A \cup \{e\})| \leq |f(A)| + 1 \text{ for any subset } A \text{ and element } e \text{ of } E \ . \tag{1.18}$$

Set function $\mathcal{C} : 2^E \to 2^E$ is *increasing*[1] if

$$A \subseteq B \subseteq E \text{ implies } |\mathcal{C}(A)| \leq |\mathcal{C}(B)| \ . \tag{1.19}$$

Note that $|\mathcal{C}^<_{\mathcal{M}}(A)| = \text{rank}(A)$ for comonotone function $\mathcal{C}^<_{\mathcal{M}}$ described in Observation 7, hence $\mathcal{C}^<_{\mathcal{M}}$ is increasing.

First we give a sufficient condition for a monotone function on subset lattices so that the lattice subset of its fixed points is a sublattice. The interested reader may find the details of an even more general treatment in [25].

Theorem 17 (see [28, 22, 25]) *If $f : 2^E \to 2^E$ is a strongly monotone function on finite set E, then fixed points of f form a nonempty sublattice of* $(2^E, \cap, \cup)$.

The following Lemma is a link between strongly monotone and increasing comonotone functions.

Lemma 18 (see [28, 22, 25]) *If function $\mathcal{C} : 2^E \to 2^E$ is increasing and comonotone then $\overline{\mathcal{C}}$ is strongly monotone.*

Based on Lemma 18, we can give a sufficient condition for the property that stable pairs in Theorem 4 form a sublattice. Note that independently from our work, Alkan [8] has also found Theorem 19 (see also [32]). He used the name *cardinal monotonicity* for our increasing notion.

Theorem 19 (Alkan 2000 [8], Fleiner 2000[28]) *If E is finite and $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are increasing, comonotone functions then $\mathcal{C}_M\mathcal{C}_W$-stable sets have the same cardinality and form a nonempty, complete lattice with lattice operations $S_1 \vee S_2 := \mathcal{C}_M(S_1 \cup S_2)$ and $S_1 \wedge S_2 := \mathcal{C}_W(S_1 \cup S_2)$. Moreover, $S_1 \cap S_2 = (S_1 \wedge S_2) \cap (S_1 \vee S_2)$ and $S_1 \cup S_2 = (S_1 \wedge S_2) \cup (S_1 \vee S_2)$, or equivalently,*

$$\chi(S_1) + \chi(S_2) = \chi(S_1 \wedge S_2) + \chi(S_1 \vee S_2) \tag{1.20}$$

holds for any two $\mathcal{C}_M\mathcal{C}_W$-stable sets S_1, S_2

Theorem 19 can be regarded as a generalization of the „consensus property" observed by Roth in [48]. Namely, from Theorem 19, it follows that if $\mathcal{S}$ is a set of $\mathcal{C}_M\mathcal{C}_W$-stable sets then $\mathcal{C}_M(\bigcup \mathcal{S})$ (the *first choice of $\mathcal{C}_M$ from $\mathcal{S}$*) is a $\mathcal{C}_M\mathcal{C}_W$-stable set.

But more is true. Let us denote by $\mathcal{S}_i^M$ the *ith choice of $\mathcal{C}_M$ from $\mathcal{S}$* defined as the first choice of $\mathcal{C}_M$ from the support of

$$\sum_{S \in \mathcal{S}} \chi(S) - \sum_{j=1}^{i-1} \chi(\mathcal{S}_j^M) \ .$$

If $\mathcal{S}$ is a chain of k $\mathcal{C}_M\mathcal{C}_W$-stable sets then $\mathcal{S}_i^M \in \mathcal{S}$ for $i = 1, 2, \ldots, k$. Otherwise, there are two uncomparable stable sets of $\mathcal{S}$, say S_1 and S_2 of $\mathcal{S}$ that we can exchange into $S_1 \wedge S_2$ and $S_1 \vee S_2$.

[1] Independently of us, Hatfield and Milgrom also observed that this property is key to prove the sublattice property of stable sets [34]. They call the same notion „law of aggregate demand".

By (1.20), this uncrossing operation does not change $\sum_{S\in\mathcal{S}}\chi(S)$. If we apply a sequence of uncrossing operations on $\mathcal{S}$ then we can transform collection $\mathcal{S}$ into a chain of $\mathcal{C}_M\mathcal{C}_W$-stable sets in finite number of steps (see [28] for the details). As the uncrossing steps do not change $\sum_{S\in\mathcal{S}}\chi(S)$, this chain must be the chain of the ith choices of $\mathcal{C}_M$. The above argument also holds for $\mathcal{C}_W$ and gives the following.

Theorem 20 *If E is a finite ground set, $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are increasing comonotone functions and $\mathcal{S}$ is a set of n (not necessarily different) $\mathcal{C}_M\mathcal{C}_W$-stable sets then the ith choice of $\mathcal{C}_M$ from $\mathcal{S}$ is a $\mathcal{C}_M\mathcal{C}_W$-stable set and coincides with the $(n+1-i)$th choice of $\mathcal{C}_W$ from $\mathcal{S}$.*

Theorem 20 generalizes the following nice structural result of Teo and Sethuraman on stable matchings. The original proof used linear programming tools.

Theorem 21 (Teo and Sethuraman 1998 [62]) *Let $(G,\mathcal{O})$ be a bipartite preference system with colour classes M and W of G and let $S_i, S_2, \dots, S_n$ be (not necessarily different) stable matchings. For each vertex v of $M\cup W$ order edges of $\bigcup_{i=1}^n S_i\cap E(v)$ as $e_v^1 \leq_v e_v^2 \leq_v \dots$ in such a way that each edge is listed as many times as it appears in an S_i. (So the length of this chain is the number of S_i's that cover v.) Then*

$$S_M^k := \{e_m^k : m\in M\} \text{ and } S_W^k := \{e_w^k : w \in W\}$$

are stable matchings for $k\in\{1,2,\dots,n\}$ and $S_M^k = S_W^{n+1-k}$.

We sketch a short direct proof of this result. We need the consequence of Theorem 19 that if each man chooses his best partner from a set of stable marriages then this induces a stable matching scheme in which each woman receives the worst husband from the given set.

Proof: For vertex v of G and stable matchings S and S' let $S \preceq_v S'$ denote that v prefers S to S'. List $S_1, S_2, \dots, S_n$ as $S_v^1 \preceq_v S_v^2 \preceq_v \dots \preceq_v S_v^n$ for each vertex v. Observe that $S_M^k = \bigwedge_{m\in M}\bigvee_{i=1}^k S_m^i$ and $S_W^k = \bigwedge_{w\in W}\bigvee_{i=1}^k S_w^i$, Chains $S_M^1, S_M^2, \dots, S_M^n$ and $S_W^1, S_W^2, \dots, S_W^n$ are opposite, hence $S_M^k = S_W^{n+1-k}$. □

With the help of Theorem 22, it is straightforward to generalize the above direct proof of Theorem 21 to stable b-matchings. Independently of us, after our result, this was done by Klaus and Klijn. The main result in their paper [40] is the special case of Theorem 21 in the many-to-one case, i.e. when $b\equiv 1$ on one colour class of the underlying graph. Klaus and Klijn have a little different proof: they simply join the meets of all k-subsets of the given set of stable matchings.

In what follows, we point out the so called splitting property of stable b-matchings. To this end, we use the generalization of the Comparability Theorem of Roth and Sotomayor [51] to the many-to-many model by Baïou and Balinski. The Comparability Theorem states that in a fixed bipartite preference system, if two stable b-matchings are different for some agent a, then a strictly prefers one b-matching to the other (that is, a would choose one of the b-matchings if all options of the two b-matchings were offered). For a short and direct proof see [26].

Theorem 22 (Baïou and Balinski 2000 [9]) *Let S and S' be two stable b-matchings for bipartite preference system $(G,\mathcal{O})$, let v be a vertex of graph G and $S_v := S\cap E(v)$ and $S'_v := S'\cap E(v)$. If $S_v \neq S'_v$ then $|S_v| = |S'_v| = b(v)$ and the $b(v)$ $<_v$-best edges of $S_v\cup S'_v$ are either S_v or S'_v.*

A consequence of Theorem 22 that is interesting in itself is that in the polygamous stable marriage problem each participating person p can partition the members of the other gender into as many groups as p's quota in such a way that in any polygamous stable marriage scheme p receives at most one partner from each group. This result turns out to be useful for the linear characterization of the stable b-matching polytope. See Section 1.4 for the details.

Corollary 23 (Fleiner 2002 [26]) *Let $(G,\mathcal{O})$ be a bipartite preference system and $b : V(G)\to\mathbb{N}$ be a quota function. Then for each vertex v of G, there is a partition of $E(v)$ into $b(v)$ parts $E_1(v), E_2(v), \dots, E_{b(v)}(v)$ so that $|S\cap E_i(v)|\leq 1$ for any stable b-matching S and any integer i with $1\leq i\leq b(v)$.*

Note that Corollary 23 has also been observed by Sethuraman et al. in [56] and turned out to be crucial in giving an alternative proof for Theorem 28, the characterization of the stable admissions polytope.

An interesting corollary of Theorem 22 and Theorem 20 is the following „middle choice” property of stable b-matchings.

Corollary 24 *If $(G,\mathcal{O})$ is a finite bipartite preference system, $b: V(G) \to \mathbb{N}$ is a quota function and $M_1, M_2, \dots, M_{2k-1}$ are stable b-matchings then there is a stable b-matching M of $(G,\mathcal{O})$ that assigns each vertex v of G with the edges of the the kth best assignment of v out of $M_1, M_2, \dots, M_{2k-1}$.*

The last result in this section generalizes a well-known fact in the stable admissions model (that is also valid in the many-to-many case). In that model, those colleges that cannot fill up their quota in some stable admission scheme receive the very same set of students in any stable assignment. A special case of this property is that in any bipartite preference system always the same persons get married in each stable marriage scheme. Theorem 25 is a direct corollary of Theorem 19.

Theorem 25 (Fleiner 2000 [28]) *If $\mathcal{M}_1, \mathcal{M}_2$ are matroids on a common ground set and S_1, S_2 have the property of S in Corollary 8 for linear orders $<_1, <_2$, then $\mathrm{span}_{\mathcal{M}_i}(S_1) = \mathrm{span}_{\mathcal{M}_i}(S_2)$ for $i \in \{1, 2\}$.*

1.4 Stable matching polyhedra

Recently, linear programming become a popular tool to study bipartite and nonbipartite stable matchings, see Abeledo, Blum, Roth, Rothblum, Sethuraman, Teo, Vande Vate and others in [3, 4, 1, 2, 50, 62, 56]. In this section, we survey linear descriptions of bipartite stable matching polyhedra. The earliest such result is that of Vande Vate.

We denote by $P^b(G,\mathcal{O})$ the convex hull of characteristic vectors in $\mathbb{R}^E$ of stable b-matchings of bipartite preference system $(G,\mathcal{O})$. (So $P^{\mathbf{1}}(G,\mathcal{O})$ is the polytope of ordinary stable matchings.) As usual in linear programming, we define $x(S) := \sum\{x(e) : e \in S\}$ for a vector $x \in \mathbb{R}^E$ and subset S of E.

Theorem 26 (Vande Vate 1989 [63]) *Let $(G,\mathcal{O})$ be a bipartite preference system with colour classes M and W, $|M| = |W|$ and $E = M \times W$. Then*

$$P^{\mathbf{1}}(G,\mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(E(v)) = 1\ \forall\ v \in M \cup W,\ x(\psi(e)) \leq 1\ \forall\ e \in E\}$$

where $\psi(mw) := \{f \in E : f \geq_m mv \text{ or } f \geq_w mw\}$.

Rothblum gave a shorter proof of a modified description for a more general problem in [53], and his proof was further simplified by Roth *et al.* in [50].

Theorem 27 (Rothblum 1992 [53]) *Let $(G,\mathcal{O})$ be a bipartite preference system with colour classes M and W. Then*

$$P^{\mathbf{1}}(G,\mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(E(v)) \leq 1\ \forall\ v \in M \cup W,\ x(\phi(e)) \geq 1\ \forall\ e \in E\}$$

where $\phi(mw) := \{f \in E : f \leq_m mw \text{ or } f \leq_w mw\}$.

Based on these results, standard tools of linear programming allow us to find a maximum weight stable matching in polynomial time.

But these results handle only the stable matching problem and do not say much about stable b-matchings. The following theorem of Baïou and Balinski [10] gives a linear description of the stable admissions polytope and generalizes Theorem 27. Note that Sethuraman et al. gave an alternative proof for the following Theorem 28 based on Corollary 23.

Theorem 28 (Baïou and Balinski 1999 [10], see also Sethuraman et al. [56]) *Let $(G,\mathcal{O})$ be a bipartite preference system and $b: M \cup W \to \mathbb{N}$ be a quota function so that $b(w) = 1$ for all nodes w of W. Then*

$$\begin{aligned} P^b(G,\mathcal{O}) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0}, \\ & x(E(w)) \leq 1\ \forall\ w \in W,\ x(E(m)) \leq b(m)\ \forall\ m \in M, \\ & x(C(m, w_1, w_2, \dots, w_{b(m)})) \geq b(m) \\ & \textit{for all combs } C(m, w_1, w_2, \dots, w_{b(m)})\}\ , \end{aligned}$$

where a comb is defined for $m \in M$ and $mw_1 >_m mw_2 >_m \dots >_m mw_{b(m)}$ as

$$\begin{aligned} C(m, w_1, w_2, \dots, w_{b(m)}) = & \{mw \in E : mw \leq_m mw_1\} \cup \\ & \{mw_i' \in E : m'w_i \leq_{w_i} mw_i \textit{ for some } i = 1, 2, \dots, b(m)\} \end{aligned}$$

Because of the comb constraints, the above characterization can consist of $\Omega(n^B)$ linear inequalities, where n is the number of „colleges" and B is the maximum of all quotas. But in spite of the exponential number of constraints, it is still possible to find an optimum weight stable admission by the ellipsoid method, using the separation algorithm of Baïou and Balinski.

In [28, 25], with the help of the theory of blocking polyhedra and lattice polyhedra, Fleiner gave a linear description of certain polyhedra that are related to $\mathcal{C}_M\mathcal{C}_W$-stable sets. Fix functions $\mathcal{C}_M\mathcal{C}_W : 2^E \to 2^E$ and let us denote by

$$\begin{aligned}
\mathcal{S} &:= \{S \subseteq W : S \text{ is an } \mathcal{C}_M\mathcal{C}_W\text{-stable set}\}\\
\mathcal{B} &:= \{B \subseteq E : B \cap S \neq \emptyset \text{ for any } S \in \mathcal{S}\}\\
\mathcal{A} &:= \{A \subseteq E : |A \cap S| \leq 1 \text{ for any member } S \text{ of } \mathcal{S}\}\\
K &:= E \setminus \bigcup \mathcal{S}
\end{aligned}$$

family of $\mathcal{C}_M\mathcal{C}_W$-stable sets, the *blocker*, the *antiblocker* of $\mathcal{S}$, and the set of nonstable elements, respectively. Define further the $\mathcal{C}_M\mathcal{C}_W$-stable set polytope, its dominant and submissive polyhedra by

$$P(\mathcal{C}_M, \mathcal{C}_W) := \text{conv}\{\chi^S : S \in \mathcal{S}\} \tag{1.21}$$

$$P(\mathcal{C}_M, \mathcal{C}_W)^\uparrow := P(\mathcal{C}_M, \mathcal{C}_W) + \mathbb{R}^E_+ = \{x + y : x \in P(\mathcal{C}_M, \mathcal{C}_W), y \geq 0\} \tag{1.22}$$

$$P(\mathcal{C}_M, \mathcal{C}_W)^\downarrow := \{x - y : x \in P(\mathcal{C}_M, \mathcal{C}_W), y \geq 0\} \cap \mathbb{R}^E_+ . \tag{1.23}$$

Theorem 29 (Fleiner 2000 [28, 25]) *If $\mathcal{C}_M, \mathcal{C}_W : 2^E \to 2^E$ are increasing comonotone functions then*

$$P(\mathcal{C}_M, \mathcal{C}_W)^\uparrow = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(B) \geq 1 \ \text{ for } B \in \mathcal{B}\} , \tag{1.24}$$

$$P(\mathcal{C}_M, \mathcal{C}_W)^\downarrow = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(K) = 0 \text{ and } x(A) \leq 1 \text{ for any } A \in \mathcal{A}\} , \tag{1.25}$$

$$P(\mathcal{C}_M, \mathcal{C}_W) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(B) \geq 1 \ \text{ for } B \in \mathcal{B} \text{ and } x(A) \leq 1 \ \text{ for } A \in \mathcal{A}\} . \tag{1.26}$$

If Theorem 29 is applied to the bipartite stable b-matching problem then it gives the following linear description of the stable b-matching polytope.

Theorem 30 (Fleiner 2000 [28, 23, 25]) *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : V(G) \to \mathbb{N}$ be a quota function. Then*

$$P^b(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(A) \leq 1\ \forall A \in \mathcal{A},\ x(B) \geq 1\ \forall\ B \in \mathcal{B}\}$$

where

$$\begin{aligned}
\mathcal{A} &:= \{A \subseteq E : |A \cap S| \leq 1 \text{ for any stable } b\text{-matching } S\} \text{ and}\\
\mathcal{B} &:= \{B \subseteq E : B \cap S \neq \emptyset \text{ for any stable } b\text{-matching } S\} .
\end{aligned}$$

Note that the constraints in Theorem 27 are special cases of the ones in Theorem 30. However, there are two important differences between Theorem 30 and the above earlier results. A shortage of Theorem 30 is that it uses implicit constraints, hence if it is specialized to the stable marriage problem, it might require more constraints than Rothblum's explicit description. (This is why Theorem 30 is rather an extension than a generalization of Theorem 27.) A positive feature of Theorem 30 is that unlike Theorem 28, both the matrix and the right hand side vector in the description contains only 0 and 1 entries.

The following result is a strengthening of Theorem 30 for the stable b-matching polytope and it is a genuine generalization of Theorem 27.

Theorem 31 (Fleiner 2002 [26]) *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : M \cup W \to \mathbb{N}$ be a quota function. Then the star-partitions $E_1(v), E_2(v), \ldots, E_{b(v)}(v)$ of $E(v)$ described in Corollary 23 satisfy*

$$\begin{aligned}
P^b(G, \mathcal{O}) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0},\\
& x(E_i(v)) \leq 1\ \forall\ v \in M \cup W, 1 \leq i \leq b(v),\\
& x(\phi_{i,j}(mw)) \geq 1\ \forall mw \in E, 1 \leq i \leq b(m), 1 \leq j \leq b(w)\} ,\\
\textit{where} \quad & \phi_{i,j}(mw) := \{mw\} \cup [\phi(mw) \cap (E_i(m) \cup E_j(w))]
\end{aligned}$$

Note that star-partitions in Corollary 23 can be found with m deferred acceptance algorithms, where m is the number of edges of G (see [26]).

It is interesting to compare the linear descriptions of the stable matching polytopes, that is Rothblum's Theorem 27, Theorem 28 of Baïou and Balinski and Theorem 31 by Fleiner. Rothblum's result gives a linear description of the stable matching polytope for the one-to-one case with $O(n+m)$ constraints, the one of Baïou and Balinski does it for many-to-one markets with $O(n+mD^B)$ constraints and Fleiner characterized the many-to-many polytope by $O(n+mB^2)$ linear inequalities. (Here n denotes the number of agents, m is the number of possible relationships, B is the maximum quota, and maximum degree D is the maximum number of possible relationships that an agent can have.) An advantage of the first two characterizations is that the linear constraints are explicit, while in Theorem 31 we need some preprocessing to write down the inequalities. An advantage of Theorem 31 is that it handles the most general problem.

However both Theorem 27 and 28 can handle more general situations. Due to Observation 36, we can optimize over the stable admissions polytope by applying Theorem 27 on b-splitting $(G^b, \mathcal{O}^b)$. If we apply the b-splitting only on one side of the market, then Theorem 28 allows us to optimize over the stable b-matching polytope. The first construction gives us a linear programming problem in $O(mB)$ dimensions with $O((n+m)B)$ constraints, while the second creates an LP problem in $O(mB)$ dimensions with $O(nB + Bm(BD)^B)$ constraints. But this is not the end of the story. If we apply Observation 36 and 37 as in the proof of Theorem 35, then we can readily use Rothblum's characterization (Theorem 27) to optimize over the stable b-matching polytope, and for this we solve an LP problem with $O((n+m)B)$ constraints in $O(mB)$ dimensions. Hence in selecting the algorithm for optimizing stable matchings, we have some tradeoff between the number of constraints, the dimension of the problem and implicitness of the linear inequalities.

It is a natural question whether there is a similar polyhedral description of the stable matching polytope for the nonbipartite (one-sided) case. Feder proved in [21] that there is no hope for such a characterization: finding a minimum weight stable matching in the stable roommates problem is NP-complete.

2. fejezet

The nonbipartite case: the stable roommates problem

We have seen that the bipartite nature of the stable marriage theorem was crucial for the application of the Knaster-Tarski fixed point theorem. There is however a natural nonbipartite model in which similar questions can be asked. We discuss certain nonbipartite versions in this section. The interested reader is referred to [5] for further details.

A *graphic preference system* is a pair $(G, \mathcal{O})$ where G is a graph and $\mathcal{O} = \{<_v : v \in V(G)\}$ so that $<_v$ is a a linear order on $E(v)$. Let $b : V(G) \to \mathbb{N}$. Set F of edges of G *b-dominates* edge e of G if there is a vertex v of e and different edges $f_1, f_2, \ldots, f_{b(v)}$ of F so that $f_i <_v e$ for $i = 1, 2, \ldots, b(v)$. Set F of edges b-dominates edge set H if each edge h of H is b-dominated by F. A *stable b-matching* is a subset S of the edges of G such that no edge of S is dominated by S, and S dominates $E \setminus S$. By definition, each vertex v of G is incident with at most $b(v)$ edges of a stable b-matching. A *stable matching* is the short name of a stable **1**-matching.

An important difference from the bipartite model is that in nonbipartite graphic preference systems there might exist no stable matching whatsoever. An example is a 3-cycle where preferences are also cyclic. The first efficient algorithm to decide the existence of a stable matching for this case is due to Irving [35]. Although, after Irving's result, several different algorithms were designed for the same (or for a more general) problem (see [21, 58, 60, 36]), Irving's algorithm still plays an important role in studying stable matchings: its appropriate generalization is the main tool for proving results on generalized models.

2.1 Tan's characterization

Based on Irving's proof, Tan in [59] gave a compact characterization of those models that contain a stable matching. In this section, we study Tan's result.

Recall that a function w assigning non-negative weights to edges in G is called a *fractional matching* if $\sum_{v \in e} w(e) \leq 1$ for every vertex v. A fractional matching w is called *stable* if every edge e contains a vertex v such that $\sum_{v \in f, f \leq_v e} w(f) = 1$.

Theorem 32 (Tan, 1991 [59]) *In any graphic preference system, there exists a half-integral fractional stable matching. In other words, there exists a set S of edges whose connected components are single edges and cycles, such that every edge e of the graph contains a vertex v of $V(S)$ such that $e \leq_v s$ for each $s \in S$ containing v.*

Tan's original proof is based on Irving's algorithm [35]. Tan observed that if we do not stop the algorithm when we find a rotation with identical 1-arc and 2-arc sets, but instead we conclude that the edges of these rotations will have weight half, then we have a polynomial time algorithm for finding a stable fractional matching in a graphic preference systems. (Note that Tan's terminology was pretty different from the above one that comes from Aharoni [6].)

Observe that if the support of a half-integral fractional stable matching contains only even cycles then there obviously exists a stable matching: we only have to throw away each second edge of the cycle components of the support. Tan also proved the following curious fact. (For a short and direct proof that is independent from Irving's algorithm, see [6].)

Theorem 33 (Tan 1991 [59]) *Let $(G, \mathcal{O})$ be a graphic preference system. If an odd cycle appears in the support of some fractional stable matching of $(G, \mathcal{O})$, then this very cycle appears in the support of any fractional stable matching of $(G, \mathcal{O})$.*

As the characteristic vector of a stable matching is a half-integral fractional stable matching, the presence of an odd cycle in the support of a half-integral fractional stable matching is equivalent to the non-existence of a stable matching.

Theorem 32 follows directly from a well-known game theoretical lemma of Scarf. An advantage of this reduction is that Scarf's lemma is very flexible and it allows us to deduce a generalization of Theorem 32 to stable b-matchings in nonbipartite graphs. We call function $w : E(G) \to \mathbb{N}$ a *fractional b-matching* if $\sum_{v \in e} w(e) \leq b(v)$ for every vertex v of G. A fractional b-matching w is called *stable* if for every edge e either $w(e) = 1$ or e contains a vertex v such that $\sum_{v \in f, f \leq_v e} w(f) = b(v)$.

Theorem 34 *If $(G, \mathcal{O})$ is a graphic preference system and $b : V(G) \to \mathbb{N}$ then there exists a half-integral fractional stable b-matching. In other words, there exist disjoint subsets S and S^{half} of edges such that*

- *the components of S^{half} are cycles,*
- *for any vertex v of G,*
$$|E(v) \cap S| + \frac{1}{2}|E(v) \cap S^{half}| \leq b(v) \tag{2.1}$$
- *if S^{half} covers some vertex v then (2.1) holds with equality and the $<_v$-maximal edge of $E(v) \cap (S \cup S^{half})$ belongs to S^{half}, and at last*
- *each edge e of $E \setminus S$ has a vertex v such that (2.1) holds with equality and $s <_v e$ for any $s \in E(v) \cap (S \cup S^{half})$.*

Obviously, if all components of S^{half} are even in Theorem 34 then S together with each second edge of S^{half} is a stable b-matching. Just like in case of nonbipartite stable matchings, a half-integral fractional stable b-matching characterizes the existence of a stable b-matching.

Theorem 35 *Let $(G, \mathcal{O})$ be a graphic preference system, $b : V(G) \to \mathbb{N}$ and C be an odd cycle of G. If w is a half-integral stable b-matching for $(G, \mathcal{O})$ and $w(e) = \frac{1}{2}$ for each edge e of C then the edges of C receive weight $\frac{1}{2}$ in any half-integral fractional stable b-matching.*

That is, if S^{half} contains an odd cycle then this very odd cycle is contained in the half-support of any half-integral stable b-matching, hence no (integral) stable b-matching can exist. We prove Theorem 35 with the help of two constructions that reduce the stable b-matching problem to the stable matching problem. Let graphic preference system $(G, \mathcal{O})$ and quota function $b : V(G) \to \mathbb{N}$ be given. Applying a *b-splitting* to preference system $(G, \mathcal{O})$ results in another preference system $(G^b, \mathcal{O}^b)$ such that

$$\begin{aligned}
V(G^b) &:= \{v(i) : v \in V(G) \text{ and } i = 1, 2, \ldots, b(v)\} \\
E(G^b) &:= \{u(i)v(j) : uv \in E(G) \text{ and } u(i), v(j) \in V(G^b)\} \\
\mathcal{O}^b &:= \{<_{v(i)} : v(i) \in V(G^b)\} \\
u(i)v(j) <_{u(i)} u(i)w(k) &\iff \begin{cases} uv <_u uw \text{ or} \\ v = w \text{ and } j < k \end{cases}
\end{aligned}$$

That is, we substitute each vertex v of G by $b(v)$ different copies, and two vertices of G^b are joined by an edge if the corresponding vertices of G are different and adjacent. Preferences are inherited from $(G, \mathcal{O})$ we only have to take extra care of that if the two edges come from the same edge.

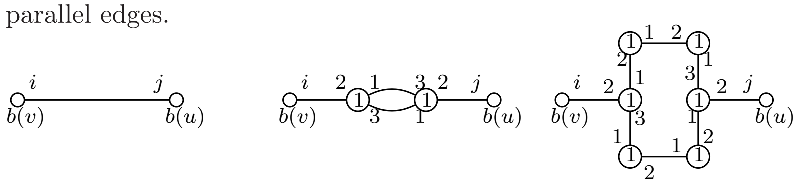
Observation 36 *Let $(G, \mathcal{O})$ be a graphic preference system and $b : V(G) \to \mathbb{N}$ in such a way that $b(u) = 1$ or $b(v) = 1$ for any edge uv of G. Then there is a one-to-one correspondence of half-integral fractional stable b-matchings of $(G, \mathcal{O})$ to half-integral fractional stable matchings of $(G^b, \mathcal{O}^b)$ in such a way that an odd cycle of weight half of the fractional stable b-matching corresponds to an odd cycle of weight half in the fractional stable matching and vice versa.*

That is, there exists a stable b-matching of $(G, \mathcal{O})$ if and only if there is a stable matching of $(G^b, \mathcal{O}^b)$. The problem is that the condition on b in Observation 36 does not hold in general. The next construction solves this difficulty. The *subdivided preference system of* $(G, \mathcal{O})$ is preference system $(G^s, \mathcal{O}^s)$ such that

$$\begin{aligned} V(G^s) &:= V(G) \cup \{v(e) : v \in V(G) \text{ and } e \in E(v)\} \\ E(G^s) &:= \{vv(e) : v \in V(G) \text{ and } e \in E(v)\} \cup \{e_{uv} : uv = e \in E(G)\}, \\ &\quad \text{where } e_{uv} \text{ joins } u(e) \text{ to } v(e) \\ \mathcal{O}^s &:= \{\prec_v : v \in V(G)\} \cup \{<_{v(e)} : v(e) \in V(G^s)\}, \text{ such that} \end{aligned}$$

$$uu(e) \prec_u uu(f) \text{ if } e <_u f, \text{ and } e_{uv} <_{v(e)} vv(e) <_{v(e)} e_{vu} \text{ for } uv = e \in E(G)\ .$$

That is, we subdivide each edge of G by two vertices and introduce a new edge between the subdividing vertices. (In other words, we substitute each edge with a path on four vertices so that the middle edge has a parallel copy.) Note that in [15] a similar construction was applied that used a 6-cycle to avoid parallel edges.



The preference order of the old vertices come from the original preference order and the preference order of a subdividing vertex is such that each of the two parallel edges is the best at one of its ends and the worst at the other end. Define $b^s : V(G^s) \to \mathbb{N}$ by $b^s(v) := b(v)$ if $v \in V(G)$ and $b^s(v(e)) := 1$ for $v \in V, e \in E_G(v)$.

Observation 37 *Let $(G, \mathcal{O})$ be a graphic preference system and $b : V(G) \to \mathbb{N}$. Preference system $(G^s, \mathcal{O}^s)$ and quota function b^s has the property needed in Observation 36, that is, $b^s(x) = 1$ or $b^s(y) = 1$ for any edge xy of G^s.*

Moreover, any half-integral fractional stable b-matching of $(G, \mathcal{O})$ induces a half-integral fractional stable b^s-matching of $(G^s, \mathcal{O}^s)$ and vice versa. In both constructions, an odd cycle of weight half induces another odd cycle of weight half.

In particular, there is a stable b-matching of $(G, \mathcal{O})$ if and only if there is a stable matching of $(G^s, \mathcal{O}^s)$. Moreover, G^s is 3-chromatic, hence any stable (b-)matching problem can be reduced to one on a 3-chromatic graph. If G is bipartite then G^s is also bipartite, and we have already used this fact in Section 1.4. Proof:[Sketch of the proof of Theorem 35.] Let w and C be as in Theorem 35. By Observations 36 and 37, there is a fractional stable matching w' of $((G^s)^b, (\mathcal{O}^s)^b)$ that corresponds to w, hence C induces an odd cycle C' of $(G^s)^b$ with w'-weight $\frac{1}{2}$. By Theorem 33, any half-integral fractional stable matching of $((G^s)^b, (\mathcal{O}^s)^b)$ assigns weight $\frac{1}{2}$ to each edge of C'. This means that any half-integral fractional stable b-matching of $(G, \mathcal{O})$ must induce a half-integral stable matching of $((G^s)^b, (\mathcal{O}^s)^b)$ that assigns weight $\frac{1}{2}$ to C', hence any half-integral stable b-matching of $(G, \mathcal{O})$ must assign weight $\frac{1}{2}$ to each edge of C. □

There is another interesting consequence of Theorem 34 on approximate stable b-matchings.

Theorem 38 *If $(G, \mathcal{O})$ is a graphic preference system and $b : V(G) \to \mathbb{N}$ then there is a subset U of $V(G)$ with $|U| \le \frac{1}{3}|V(G)|$ such that for any $b' : V(G) \to \mathbb{N}$ of the form*

$$b'(v) := \begin{cases} b(v) & \text{if } v \notin U \\ b(v) \pm 1 & \text{if } v \in U \end{cases}$$

there is a stable b'-matching of $(G, \mathcal{O})$.

Proof: Let S, S^{half} be as in Theorem 34 and construct U by choosing one vertex from each odd cycle of S^{half}. As each odd cycle is of length at least 3, the size of U is at most the third of $|V(G)|$. Construct subset T of S^{half} by throwing away each second edge of each even cycle of S^{half} and by selecting each second edge of each odd component with the exception of points of U, where we select both or none of the edges depending on whether $b'(u) = b(u)+1$ or $b'(u) = b(u)-1$. Clearly, $S \cup T$ is a stable b'-matching of $(G, \mathcal{O})$. □

Our next observation is that Corollary 23 has a direct generalization to nonbipartite models.

Theorem 39 *Let $(G, \mathcal{O})$ be a preference system and $b : V(G) \to \mathbb{N}$ be a quota function. Then for each vertex v of G, there is a partition of $E(v)$ into $b(v)$ parts $E_1(v), E_2(v), \ldots, E_{b(v)}(v)$ so that $|S \cap E_i(v)| \le 1$ for any stable b-matching S and any integer i with $1 \le i \le b(v)$.*

The proof of Theorem 34 is an application of the following lemma of Scarf to vector b, the extended incidence matrix and the extended domination matrix of the preference system. Notation $[n]$ stands for the set of the first n positive integers.

Theorem 40 (Scarf 1967 [54]) *Let $n < m$ be positive integers, b be a vector in $\mathbb{R}^n_+$ and $B = (b_{i,j})$, $C = (c_{i,j})$ be matrices of dimensions $n \times m$, satisfying the following three properties: the first n columns of B form an $n \times n$ identity matrix (i.e. $b_{i,j} = \delta_{i,j}$ for $i, j \in [n]$), the set $\{x \in \mathbb{R}^n_+ : Bx = b\}$ is bounded, and $c_{i,i} \le c_{i,k} \le c_{i,j}$ for any $i \in [n]$, $i \ne j \in [n]$ and $k \in [m] \setminus [n]$.*

Then there is a nonnegative vector x of $\mathbb{R}^m_+$ such that $Bx = b$ and the columns of C that correspond to supp (x) *form a dominating set, that is, for any column $i \in [m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \ge c_{k,j}$ for any $j \in$* supp (x).

Theorem 40 can be interpreted such that in any weighted hypergraphic preference system there always exists a fractional stable b-matching. It turned out that Theorem 40 is a close relative of the topological fixed point theorem of Kakutani (see also [7, 6]). A *simplicial complex* is a non-empty family $\mathcal{C}$ of subsets of a finite ground set such that $A \subset B \in \mathcal{C}$ implies $A \in \mathcal{C}$. Members of $\mathcal{C}$ are called *simplices* or *faces*. Let us call simplicial complex $\mathcal{C}$ *manifold-like* if, denoting its rank by n (that is, the maximum cardinality of a simplex in it is $n+1$), every face of cardinality n of $\mathcal{C}$ is contained in two faces of cardinality $n+1$. The *dual* $\mathcal{C}^*$ of a complex $\mathcal{C}$ is the set of complements of its simplices. Just like in the case of complexes, members of a dual complex are also called *faces*.

Lemma 41 (Aharoni 2001 [6]) *If $\mathcal{C}$ and $\mathcal{C}'$ are two manifold-like complexes on the same ground set, then the number of maximum cardinality faces of $\mathcal{C}$ that are also minimum cardinality faces of ${\mathcal{C}'}^*$ is even.*

What examples are there of manifold-like complexes? Of course, a triangulation of a closed manifold is of this sort. (We call this complex a *manifold-complex*.) Another well known example of a dual manifold-like complex is the *cone complex*: let X be a set of vectors in $\mathbb{R}^n$, and b a vector not lying in the positive cone spanned by any $n-1$ elements of X. A third example of a manifold-like complex is the *domination complex*. Let C be a matrix as in Theorem 40 with the additional property that in each row of C all entries are different. Then the family of dominating column sets together with the extra member $[n]$ is a manifold-like complex. (For the details, see [7].)

If we plug the cone complex and the domination complex in Lemma 41 and apply a general position argument then we can deduce Theorem 40. The application of Lemma 41 to the cone complex and the manifold complex yields the following discrete version of Kakutani's fixed point theorem.

Theorem 42 (see [5]) *Let us label the vertices of an n-dimensional simplex S by the unit vectors of $\mathbb{R}^{n+1}$, and label the other vertices of a triangulation T of S by vectors of $\mathbb{R}^{n+1}$ in such a way that the label of any vertex v of T is in the positive hull of the labels of those vertices of S that lie on the minimal face of S that contain v. For any vector $b \in \mathbb{R}^{n+1}_+$, there is a elementary simplex in the triangulation of S whose vertex labels contain b in their cone.*

Theorem 42 and a continuity argument implies Kakutani's fixed point theorem.

Theorem 43 (Kakutani 1941 [38]) *Let $S \subseteq \mathbb{R}^n$ be an n-dimensional simplex and $K(S)$ be the family of closed convex subsets of S. Let $\Phi : S \to K(S)$ be upper semicontinuous, that is, whenever $y_i \to y$, $x_i \to x$ and $y_i \in \Phi(x_i)$ then $y \in \Phi(x)$. Then there is a point x of S so that $x \in \Phi(x)$.*

Brouwer's fixed point theorem is the special case of Theorem 43 where $\Phi(x)$ is one point for all x. The discrete version of Brouwer's fixed point theorem is Sperner's lemma and Theorem 42, the discrete version of Kakutani's theorem is a genuine generalization of Sperner's lemma. Note that a result of Shapley in [57] that generalize Sperner's lemma is formally a special case of Theorem 42. Still, Shapley's method in [57] proves Theorem 42.

It is interesting to see the close relationship of the bipartite stable matching theorem to the lattice theoretical fixed point theorem of Tarski and of the nonbipartite version to the topological fixed point theorem of Kakutani. One might ask himself whether there is some fixed-point theorem that implies a generalization of Tan's results on nonbipartite preference systems.

2.2 The stable roommates problem with free and forbidden edges

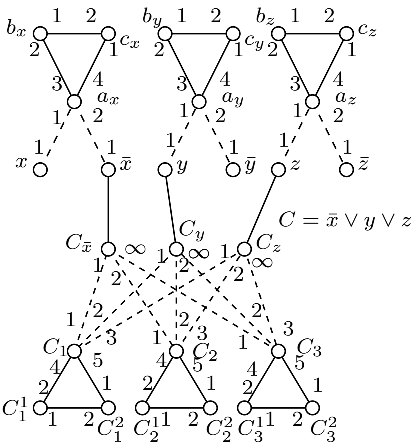
In this section, we study three generalizations of the one sided stable matching problem. We consider the problem with special edges, look at preference orders that are not necessarily strict, and we look for stable partnerships, where instead of linear orders, agents' preferences are described by some personal choice functions.

In the stable roommates problem, ordinary edges between two agents have two different features: on one hand, an edge can be present in a matching and dominate other edges, and on the other hand it may block a matching. A possible way to generalize the stable roommates problem is to allow edges that have only one of those properties. That is, we call an edge *forbidden* if it may block a matching, but it cannot be present in the stable matching that we look for. So the presence of a forbidden edge makes it more difficult to find a stable matching. A *free* edge is the opposite: we may use it in our matching, but it never blocks. The stable roommates problem with forbidden and free edges is given by preference system $(G, \mathcal{O})$, disjoint subsets $E_{forbidden}$ and E_{free} of E and we ask if there exists a matching S of $E \setminus E_{forbidden}$ that is not blocked by an edge of $E \setminus E_{free}$.

By the definition, if we declare an ordinary edge free then all stable matchings remain stable and some new may emerge, hence it becomes easier to find one. Forbidding an ordinary edge may kill some stable matchings but never creates a new one, so it makes it more difficult to find one. We shall show that for the decision problem the opposite holds: the problem with forbidden edges is tractable and the presence of free edges makes it hard.

Theorem 44 *[Cechlárová and Fleiner 2008 [16]] For a preference system $(G, \mathcal{O})$ and subset $F \subseteq E(G)$ of free edges it is NP-complete to decide the existence of a stable matching. The problem is NP-complete already if F consists of disjoint edges.*

Proof:[Sketch of the proof.] We show a polynomial reduction of the 3-SAT problem to the the stable roommates problem with free edges. For this reason we have to construct in polynomial time for each 3-CNF boolean formula Φ a preference system $(G, \mathcal{O})$ and set of free edges such that Φ is satisfiable if and only if there is a matching in $(G, \mathcal{O})$ that can be blocked only by free edges.

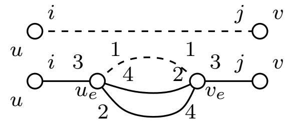


For each variable x of Φ, let us define vertices a_x, b_x, c_x, x and $\bar{x}$, edges a_xb_x, b_xc_x, c_xa_x free edges a_xx and $a_x\bar{x}$. These latter edges are first choices of x and $\bar{x}$ respectively, the preference order of a_x is $a_xx, a_x\bar{x}, a_xb_x, a_xc_x$, b_x prefers b_xc_x to b_xa_x and c_x prefers c_xa_x to c_xb_x. For each clause C we have vertices C_i and C_i^j for $i = 1, 2, 3$, $j = 1, 2$ and C_l, for each literal l in C. Construct free edges C_lC_i that are the ith choices of C_l for $i = 1, 2$ and C_lC_3 is the very last choice of C_l. For each C_i, the edges C_iC_l are the first three choices in an arbitrary order.

For $i = 1, 2, 3$, add all edges between C_i, C_i^1 and C_i^2 such that these edges are ordinary and preferences along these triangles are cyclic. To finish the construction of G, for each literal l in some clause C, connect vertex l with C_l. All nonspecified preferences are arbitrary. The figure shows the part of G corresponding to clause $C = \bar{x} \vee y \vee z$ and variables x, y and z. It is easy to see that if there is a truth assignment to Φ then there is a matching in G that is not blocked by ordinary edges. For this, we choose edges xv_x for each true variable and edges $\bar{x}v_x$ for each false one, all edges b_xc_x and $C_i^1C_i^2$. It is straightforward to complete this matching it to one we need.

If there is a matching S that is not blocked by an ordinary edge then for each variable x, exactly one of a_xx and $a_x\bar{x}$ is present in S, as otherwise S would contain a stable matching of triangle $a_xb_xc_x$ that does not exist. Similarly, each vertex C_i is covered by a free edge of S as otherwise S would contain a stable matching of triangle $C_iC_i^1C_i^2$, which is impossible. This means that S contains no edge lC_l.

If $xv_x \in S$ then declare x true, otherwise let x be false. We prove that this is a truth assignment of Φ, that is, each clause has a true literal. If C is a clause then there is an edge C_3C_l of S. As lC_l cannot block this means that l is covered by a free edge, hence l is a true literal in C. This follows that Φ has a truth assignment.



To show the second part of the Theorem, we construct for each stable roommates problem with free edges an equivalent problem one where free edges are disjoint.

To do this it is enough to substitute each free edge with a little graph similar to the construction we had in the subdivided preference system. The difference is that for an edge $e = uv$, we add a free edge $u_e v_e$ that is first choice for both u_e and v_e. (See the figure.) After this change all free edges are disjoint and there is a matching that is not blocked by ordinary edges in $(G, \mathcal{O})$ if and only if there is one after the construction. □

Péter Biró [13] asked the complexity of the stable roommates problem with free edges in the special case where vertices of the underlying graph G are partitioned into disjoint sets and free edges are exactly those ones that connect different partition sets. Note that the above construction for the reduction of 3-SAT has this property, so this special case is NP-complete. Rob Irving asked the same question for the case where free edges are the ones that connect two vertices of the same partition set. As a graph where free edges are disjoint has this property, our second construction in the above proof shows the NP-completeness of this problem as well.

Now we turn our attention to forbidden edges. The first result on this is probably the following.

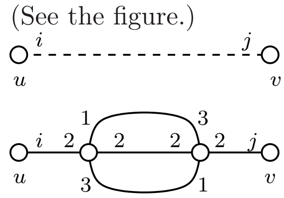
Theorem 45 (Dias et al. [19]) *If $(G, \mathcal{O})$ is a bipartite preference system and subset F of edges is forbidden then there is a linear time algorithm to find a stable matching that does not contain an edge of F, if such exists.*

Fleiner et al. in [29] extended Theorem 45 to nonbipartite preference systems, where one may allow indifferences in the preferences. Our goal here is to describe an extension of Irving's algorithm to solve an even more general problem. To define that problem, we focus on choice functions that come from not necessarily linear preference orders.

In both the stable marriage and the stable roommates problems, strict (linear) preferences of the participating agents play a crucial role. However, in many practical situations, one has to deal with indifferences in the preference orders. A natural model for this is that preference orders are partial (rather than linear) orders. One can extend the notion of a stable matching to this model in at least three different ways. One possibility is that a matching is *weakly stable* if no pair of agents a, b exists such that they mutually strictly prefer one another to their eventual partner. Ronn proved that deciding the existence of a weakly stable matching is NP-complete [46]. Based on Theorem 44, there is an alternative proof.

Theorem 46 (see Ronn [46]) *It is NP-complete to decide the existence of a weakly stable matching in the stable roommates problem where each vertex has a weak linear preference order on the incident edges with at most two edges in tie.*

Proof:[Sketch of the proof.] Construct graph G' the following way. Substitute each free edge of G with the gadget we used for the subdivided preference system. That is, instead of each free edge create a 3-path with two parallel edges in the middle. Further, add a third parallel edge to each gadget such that this third edge is tied with the nonparallel edges at both endvertices. (See the figure.)



It is easy to check that there there is a matching of G not blocked by any ordinary edge if and only if there is a weakly stable matching in $(G', \mathcal{O}')$, where preferences of $\mathcal{O}'$ are the same as of $\mathcal{O}$ at vertices of G and given by the figure at the new vertices.

Hence for each instance of the stable roommates problem with free edges, one can construct in polynomial time an equivalent instance of the problem described in Theorem 46, that is we have a polynomial reduction of the former problem to the latter. As the former problem is NP-complete by Theorem 44, the latter one has this property as well. □

A more restrictive notion than weak stability is the following. A matching is *strongly stable* if there are no agents a and b such that a strictly prefers b to his eventual partner and b does not prefer his eventual partner to a. Scott gave an algorithm that finds a strongly stable matching or reports if none exists in $O(m^2)$ time [55]. The most restrictive notion is that of super-stability. A matching is *super-stable* if there exist no two agents a and b such that neither of them prefers his eventual situation to being a partner of the other. In other words, a matching is super-stable, if it is stable for any linear extensions of the preference orders of the agents.

For the case where indifference is transitive (preferences are weak linear orders), Irving and Manlove gave an $O(m)$ algorithm to find a super-stable matching, if exists [36]. Interestingly, the algorithm has two phases, just like Irving's [35], but its second phase is completely different. The authors remark in [36] that the algorithm works without modification for the more general case when preference orders are partial orders.

Through indifferences of the agents in a preference system with weak linear orders, we may create special type of edges. Namely, it is possible that there are two parallel edges (say e and e') between two vertices and both vertices are indifferent between the parallel copies. This means that both e and e' may block a matching but nor e, neither e' can be present in a super-stable matching, as its parallel copy would block. It is easy to see that the super-stable roommates problem is equivalent with the one where we delete e' and forbid e.

The stable matching problem with forbidden edges is given by a preference system $(G, \mathcal{O})$ and a subset F of E, the set of *forbidden edges*. The problem is to find a stable matching S of $(G, \mathcal{O})$ that does not contain any forbidden edge of F. Clearly, this problem is a special case of the super-stable matching problem. Fleiner *et al.* exhibited a reduction of the super-stable matching problem with forbidden edges this problem to 2-SAT [29] and in [30], the same authors extended Irving's algorithm to this case. Here we describe this latter result, and hence solve the stable roommates problem with forbidden pairs and the super-stable roommates problem, as well.

We have seen that a forbidden edge in the stable roommates problem can be regarded as special edge in the super-stable roommates problem. Interestingly, in the extension of Irving's algorithm we explicitly use forbidden edges: while Irving's original algorithm had only edge deletions, in the extension we eventually need to forbid an edge. Recall that if Irving's algorithm deletes an edge then no new stable matching is created by this and if the deletion kills some stable matching then some other has to survive. Our algorithm deletes an edge in two steps. First it is forbidding edge e if either there exists no super-stable matching in the current instance or if there is a super-stable matching avoiding e. Hence if we find a super-stable matching after forbidding e then it is super-stable before that, and if we conclude that no super-stable matching exists then the conclusion is valid before that step. Our algorithm eventually deletes a forbidden edge e if no super-stable matching exists that is blocked only by e, that is, if no new super-stable matching is created by the deletion.

There is a hierarchy of the different type steps of the following algorithm. Every time we try to execute the first possible of them. To describe these step types, we say that edge $e = E(v)$ of G (forbidden or not) is a *first choice edge of* v, if there is no edge $f \in E(v) \setminus F$ with $f <_v e$ (i.e., if no free edge can dominate e at vertex v). Note that there can be more than one 1st choices of v present.

Proposal step. If $e = vw$ is a 1st choice of v then orient e from v to w. Just like in Irving's algorithm, arcs created in a proposal step are called *1-arcs*. Note that it is possible that a vertex sends more than one 1-arc and a 1-arc can also be bioriented. After we found all 1-arcs, the algorithm looks for a

Mild rejection step. If 1-arc e of G_i points to v and $E(v) \ni f \not<_v e$ (that is, f is not better than e according to v in G) then forbid f.

After all such steps are done we move on to find a

Firm rejection step. If some free 1-arc e of G_i points to v and $e <_v f \in E(v)$ (e is better than f according to v) then we delete f.

Note that the above f is already forbidden by a mild rejection step.

As soon as no more proposal and (mild or firm) rejection step can be executed, the following generalization of the first-last property holds: each vertex v that is incident with a free edge sends and receives exactly one 1-arc, and these represents the unique best and worse choices of v, respectively. If some vertex v is not incident with a free edge then v still may be incident with forbidden edges. If this happens then no super-stable matching exist. Otherwise each vertex is either a singleton of G or it sends and receives exactly one 1-arc. The algorithm works further with these latter vertices.

Assume that in G no proposal or rejection step can be executed. An edge $e \in E(v)$ is a *second choice of* v if $e >_v f \notin F$ implies that f is the unique 1st choice of v. In other words, e is a second choice, if the only free edge that dominates e at v is the unique 1-arc leaving v. Every vertex v of G with degree at least two is incident with at least one free second choice edge: if not other, then the unique 1-arc pointing to v. Now the algorithm can make a

2nd choice step. If $e = vw$ is a second choice of v then (counterintuitively) orient e from w to v. Arcs created at this step are called *2-arcs*.

What is the meaning of a 2-arc? It can be interpreted as an implication: if wv is a 2-arc, a is the 1-arc entering w and a' is the 1-arc leaving v then for any super-stable matching S, $a \in S$ implies $a' \in S$. This observation allows us to build an implication structure on the set of 1-arcs. In this implication structure 1-arc e *sm-implies* 1-arc f if there is a directed path starting with e ending with f and using 1-arcs and 2-arcs in an alternating manner. Clearly, if e sm-implies f and S is a super-stable matching then $f \in S$ whenever $e \in S$. We say that 1-arcs e and f are called *sm-equivalent*, if e sm-implies f and f sm-implies e, or, in other words if there is a directed cycle D formed by 1-arcs and 2-arcs in an alternating manner such that D contains both e and f. (Note that D may use the same vertex more than once.) Sm-equivalence is clearly an equivalence relation and if C is an sm-class and S is a super-stable matching then either C is disjoint from S or C is contained in S.

Beyond determining sm-equivalence classes, 2-arcs yield further implications between sm-classes: if uu' is a 1-arc of sm-class C and vv' is a 1-arc of sm-class C' and $u'v$ is a 2-arc, then sm-class C „implies" sm-class C' in such a way that if C is not disjoint from super-stable matching S then S contains both classes C and C'. Assume that sm-class C is on the top of this implication structure, i.e. C is not implied by any other sm-class (but C may imply certain other classes). We can summarize this in the formula that C has the property that

$$\text{if } vv' \text{ is a 1-arc of } C \text{ and } w'v \text{ is a 2-arc} \\ \text{then (the unique) 1-arc } ww' \text{ is sm-equivalent to } vv'. \tag{2.2}$$

To find a top sm-class C, introduce an auxiliary digraph on the vertices of G, such that if uu' is a 1-arc and $u'v$ is a 2-arc, then we introduce an arc uv of the auxiliary graph. We can find a source strong component of the auxiliary graph in linear time by depth first search. If it contains vertices $u_1, u_2, \ldots, u_k$ then it determines a top sm-class $C = \{u_1u_1', u_2u_2', \ldots, u_ku_k'\}$ formed by 1-arcs. Note that it is possible here that $u_i = u_j'$ for different i and j.

2nd choice elimination step. If for 1-arcs $u_iu_i', u_ju_j' \in C$ there are 2-arcs vu_i and vu_j with $vu_i \not<_v vu_j$ then forbid vu_i.

Note that a 2nd choice elimination step might create some new 2-arcs so we might have to take further 2nd choice steps. After these steps even the top sm-class C may change. Sooner or later we arrive to a situation where none of the above steps are possible. At that moment, if uu' is a 1-arc of C then there is unique 2-arc leaving u' and there is a unique 2-arc pointing to u, that is, u has a unique 2nd choice. This is exactly the same situation that we have in the ordinary stable roommates problem in case of a rotation. That is, if $C \subseteq S$ for a super-stable matching S then matching $S \setminus C \cup C'$ is also super-stable, where C' denotes the set of 2-arcs within C. So if none of the above steps can be executed then we try our last option.

Rotation elimination step. Forbid all edges of C in G.

After a rotation elimination step, the 2-arcs within C become 1-arcs in the opposite direction. This may involve certain mild and firm rejection steps (e.g. we shall delete the forbidden 1-arcs of C), 2nd choice steps, 2nd choice elimination steps and further rotation eliminations. None of these steps kill all super-stable matchings and none of them creates a super-stable matching. Furthermore, if none of the above steps are possible then we are left with a matching on the vertices that are adjacent with ordinary (non-forbidden) edges. So if no vertex is adjacent to a forbidden edge in this situation then what we left with is a super-stable matching. If there are some forbidden edges as well then all these edges must be bidirected 1-arcs and no super-stable matching exists in the particular problem.

3. fejezet

Stable flows

A well-known extensions of the stable marriage problem is proved by Baïou and Balinski [11]. They showed that if each edge of the underlying bipartite graph has a nonnegative capacity and each vertex has a nonnegative quota then the accordingly modified deferred acceptance algorithm always finds a so called stable allocation. An allocation is an assignment of nonnegative values to the edges that do not exceed the corresponding capacities such that the total allocation of no vertex exceeds its quota. In other words, an allocation is a marriage scheme where a bipartite marriage can be formed with an „intensity" different from 0 and 1 and each participant has an individual upper bound on his/her total „marriage intensity". An allocation is stable if any unsaturated edge e has a saturated end vertex v such that each edge e' incident with v has an assigned value 0 whenever v prefers e to e'. That is, if the intensity of a marriage is not maximum then one of the spouses has maximum total marriage intensity and none of his/her marriages is worse than the particular marriage. Beyond proving the existence of stable assignments, Baïou and Balinski used flow-type arguments to speed up the deferred acceptance algorithm in [11]. Later, Dean and Munshi came up with an even faster network flow based algorithm for the same problem [18].

It is fairly well-known that the bipartite matching problem can be formulated in the more general network flow model, and the alternating path algorithm for maximum bipartite matchings is a special case of the augmenting path algorithm of Ford and Fulkerson for maximum flows. However, it seems that the question whether there exists a flow generalization of the stable marriage theorem has not been addressed so far. This very problem is in the focus in this section. It turned out that our model is closely related to so-called „supply chains" well-known in the Economics literature. Prior to our work, Ostrovsky had a related result in [44]. There, he considered only acyclic networks, but instead of the Kirchhoff law, he required a less restrictive property called „same side substitutability" and „cross side complementarity". Ostrovsky proved the existence of a „chain stable network" and justified that these „chain stable networks" form a lattice under a natural partial order. These results are very close to ours and cry for a common generalization. This will be subject of future work.

Recall that by a *network* we mean a quadruple (D, s, t, c), where $D = (V, A)$ is a digraph, s and t are different nodes[1] of D and $c : A \to \mathbb{R}_+$ is a function that determines the capacity $c(a)$ of each arc a of A. Vertices s and t are called *terminals*, other vertices of G are *nonterminals*. A *flow* of network (D, s, t, c) is a function $f : A \to \mathbb{R}$ such that capacity condition $0 \leq f(a) \leq c(a)$ holds for each arc a of A and each nonterminal vertex v of D satisfies the Kirchhoff law: $\sum_{u:uv\in A} f(uv) = \sum_{u:vu\in A} f(vu)$, that is, the amount of the incoming flow equals the amount of the outgoing flow for v. Note that there is no conceptual difference between terminals s and t: both are ordinary vertices that are exempt from the Kirchhoff law[2].

A *network with preferences* is a network (D, s, t, c) along with a preference order $\leq_v$ for each vertex v, such that $\leq_v$ is a linear order on the arcs that are incident to v and we say that v prefers a to a' if $a \leq_v a'$ holds. (Note that preference orders $\leq_s$ and $\leq_t$ of the terminals do not play a role in the notion of stability as we shall never compare an incoming and an outgoing arc of the same vertex. So

[1] Sometimes it is assumed that no arc enters vertex s and no arc leaves vertex t. We do not require this assumption for the reason that this way we prove a more general result. Still, if the reader finds it difficult to follow the argument, it might be convenient to consider the source-sink case and skip the irrelevant parts.

[2] It seems that this fact is not completely clear for many. Perhaps the reason is that when network flows are taught, it is usually emphasized that the role of s and t are different: the former is „the source" and the latter one is a „the sink". To convince the sceptic, it is illuminative to give them a task to find a formula for the minimum value of an st flow in a network. (It is not 0 in general.)

we may think that for each nonterminal vertex v there are two independent preference orders: one is on the incoming arcs and the other one on the outgoing ones.) For a given network with preferences, it is convenient to think that vertices of D are „players" that trade with a certain good. An arc uv of D from player u to player v with capacity $c(uv)$ represents the possibility that player u can supply at most $c(uv)$ units of good to player v. A „trading scheme" is described by a flow f of the network, as for any two players u and v, flow $f(uv)$ determines the amount of goods that u sells to v. Everybody in the market would like to trade as much as possible, that is, each player v strives to maximize the amount of flow through v. In particular, if flow f allows player v to receive some more flow (that is, there are goods on the market offered to v and v is happy to buy them) and v can also send some more flow (i.e. some other player would be happy to buy more goods from v) then flow f does not correspond to a stable market situation.

Another instability occurs when $vw \leq_v vu$ (player v prefers to sell to w rather than to u) and flow f is such that w would be happy to buy more goods from v (that is $f(vw) < c(vw)$ and w has some extra selling opportunity), moreover $f(vu) > 0$ (v sells a positive amount of goods to u). In this situation, v would send flow rather to w than to u, hence this cannot occur in a stable market situation. A similar instability can be described if we talk about entering arcs instead of outgoing ones, that is, if we exchange the roles of buying and selling.

To formalize our concept of stability, we need a few definitions. For a network (D, s, t, c) and flow f we say that arc a is *f-unsaturated* if $f(a) < c(a)$, that is, if it is possible to send some extra flow through a. A *blocking walk of flow f* is a directed walk $P = (v_1, a_1, v_2, a_2, \ldots, a_{k-1}, v_k)$ such that $a_i \in A$ and $v_i \in V$ $(\forall i)$ and the following properties hold.

$$\text{arc } a_i \text{ points from } v_i \text{ to } v_{i+1} \text{ for } i = 1, 2, \ldots, k-1 \text{ (i.e. } P \text{ is a walk) and} \tag{3.1}$$

$$v_2, v_3, \ldots, v_{k-1} \text{ are nonterminal vertices and} \tag{3.2}$$

$$\text{each arc } a_i \text{ is } f\text{-unsaturated and} \tag{3.3}$$

$$v_1 \text{ is terminal or there is an arc } a' = v_1 u \text{ such that} f(a') > 0 \text{ and } a_1 <_{v_1} a' \text{ and} \tag{3.4}$$

$$v_k \text{ is terminal or there is an arc } a'' = w v_k \text{ such that} f(a'') > 0 \text{ and } a_{k-1} <_{v_k} a'' . \tag{3.5}$$

So directed walk P is blocking if each player that corresponds to an inner vertex of P is happy and capable to increase the flow along P, moreover v_1 can send extra flow either because v_1 is a terminal node or because v_1 may decrease the flow toward some vertex u that v_1 prefers less than v_2, and at last, v_k can receive some extra flow either because either v_k is a terminal node or v_k can refuse flow from w whom v_k ranks below v_{k-1}. We say that an f-unsaturated walk $P = (v_1, v_2, \ldots, v_k)$ is *f-dominated at* v_1 if (3.4) does not hold, and P is *f-dominated at* v_k if (3.5) does not hold. A flow f of a network with preferences is *stable* if no blocking walk of f exists. In the *stable flow problem*, we have given a network with preferences and our task is to find a stable flow if such exists.

A special case of the stable flow problem is the stable allocation problem of Baïou and Balinski [11]. The *stable allocation problem* is defined by finite disjoint sets W and F of workers and firms, a map $q : W \cup F \to \mathbb{R}$, a set E of edges between W and F along with a map $p : E \to \mathbb{R}$ and for each worker or firm $v \in W \cup F$ a linear order $<_v$ on those pairs of E that contain v. We shall refer to pairs of E as „edges" and hopefully it will not cause ambiguity. Quota $q(v)$ denotes the maximum of total assignment that worker or firm v can accept and capacity $p(wf)$ of edge $e = wf$ means the maximum allocation that worker w can be assigned to firm f along e. An *allocation* is a nonnegative map $g : E \to \mathbb{R}$ such that $g(e) \leq p(e)$ holds for each $e \in E$ and for any $v \in W \cup F$ we have

$$g(v) := \sum_{x : vx \in E} g(vx) \leq q(v) \ , \tag{3.6}$$

that is, the total assignment $g(v)$ of player v cannot exceed quota $q(v)$ of v. If (3.6) holds with equality then we say that player v is *g-saturated.* An allocation is *stable* if for any edge wf of E at least one of the following properties hold:

$$\text{either } g(wf) = p(wf) \quad \text{(the particular employment is realized with full capacity)} \tag{3.7}$$

$$\text{or } \textstyle\sum_{wf' \leq_w wf} g(wf') = q(w), \text{ that is worker } w \text{ is } g\text{-saturated and } w \text{ does not prefer } f \text{ to any of his employers (we say that } wf \text{ is } g\text{-dominated at } w) \tag{3.8}$$

$$\text{or } \textstyle\sum_{w'f \leq_f wf} g(w'f) = q(f), \text{ that is firm } f \text{ is } g\text{-saturated and } f \text{ does not prefer } w \text{ to any of its employees (we say that } wf \text{ is } g\text{-dominated at } f). \tag{3.9}$$

Note that (3.9) and (3.8) implies that if g is a stable allocation, then for each firm f and each worker w

$$\text{there is at most one edge } e \text{ dominated at } f \text{ with } g(e) > 0 \text{ and} \tag{3.10}$$

$$\text{there is at most one edge } e \text{ dominated at } w \text{ with } g(e) > 0\ . \tag{3.11}$$

If g_1 and g_2 are allocations and $w \in W$ is a worker then we say that *allocation* g_1 *dominates allocation* g_2 *for worker* w (in notation $g_1 \leq_w g_2$) if one of the following properties is true:

$$\text{either } g_1(wf) = g_2(wf) \text{ for each } f \in F \text{ or} \tag{3.12}$$

$$\begin{array}{c} \sum_{f'\in F} g_1(wf') = \sum_{f'\in F} g_2(wf') = q(w), \text{ and} \\ g_1(wf) < g_2(wf) \text{ and } g_1(wf') > 0 \text{ implies that } wf' <_w wf. \end{array} \tag{3.13}$$

That is, if w can freely choose his allocation from $\max(g_1, g_2)$ then w would choose g_1 either because g_1 and g_2 are identical for w or because w is saturated in both allocations and g_1 represents w's choice out of $\max(g_1, g_2)$. By exchanging the roles of workers and firms, one can define domination relation $\leq_f$ for any firm f, as well.

For any stable allocation problem, one can design a network (D, s, t, c) such that $V(D) = \{s, t\} \cup W \cup F$, $A(D) = \{sw : w \in W\} \cup \{ft : f \in F\} \cup \{wf : wf \in E\}$ and $c(sw) = q(w)$, $c(ft) = q(f)$ and $c(wf) = p(wf)$ for any worker w and firm f. That is, we consider the underlying bipartite graph, orient its edges from W to F, add new vertices s and t, with an arc from s to each worker-node and an arc from each firm-node to t, and capacities are given by the original edge-capacities and the corresponding quotas. Preference orders $<_v$ on the arcs incident to v are induced by the preference order on the corresponding edges incident to v, or, if there is no such edge, then it is a trivial linear order. It is straightforward to see from the definitions that g is a stable allocation if and only if there exists a stable flow f such that $g(e) = f(\vec{e})$ holds for each edge $e \in E$, where $\vec{e}$ is the arc that corresponds to edge e. The stable allocation problem was introduced by Baïou and Balinski as a certain „continuous" version of the stable marriage problem in [11]. It turned out that a natural extension of the deferred acceptance algorithm of Gale and Shapley [31] works for the stable allocation problem and the structure of stable allocations is similar to that of stable marriages. Beyond stating the existence of stable allocations, the theorem below describes some structural properties of them.

Theorem 47 (See Baïou and Balinski [11]) *1. If a stable allocation problem is described by* W, F, E, p *and* q *then there always exists a stable allocation* g*. Moreover, if* p *and* q *are integral, then there exists an integral stable allocation* g*.*

2. If g_1 *and* g_2 *are stable allocations and* $v \in W \cup F$ *then* $g_1 \leq_v g_2$ *or* $g_2 \leq_v g_1$ *holds.*

3. Stable allocations have a natural lattice structure. Namely, if g_1 *and* g_2 *are stable allocations then* $g_1 \vee g_2$ *and* $g_1 \wedge g_2$ *are stable allocations, where*

$$(g_1 \vee g_2)(wf) = \begin{cases} g_1(wf) & \textit{if } g_1 \leq_w g_2 \\ g_2(wf) & \textit{if } g_2 \leq_w g_1 \end{cases} \tag{3.14}$$

and

$$(g_1 \wedge g_2)(wf) = \begin{cases} g_1(wf) & \textit{if } g_1 \leq_f g_2 \\ g_2(wf) & \textit{if } g_2 \leq_f g_1 \end{cases} \tag{3.15}$$

In other words, if workers choose from two stable allocations then we get another stable allocation, and this is also true for the firms' choices. Moreover, it is true that

$$(g_1 \vee g_2)(wf) = \begin{cases} g_1(wf) & \textit{if } g_1 \geq_f g_2 \\ g_2(wf) & \textit{if } g_2 \geq_f g_1 \end{cases} \tag{3.16}$$

and

$$(g_1 \wedge g_2)(wf) = \begin{cases} g_1(wf) & \textit{if } g_1 \geq_w g_2 \\ g_2(wf) & \textit{if } g_2 \geq_w g_1 \end{cases} \tag{3.17}$$

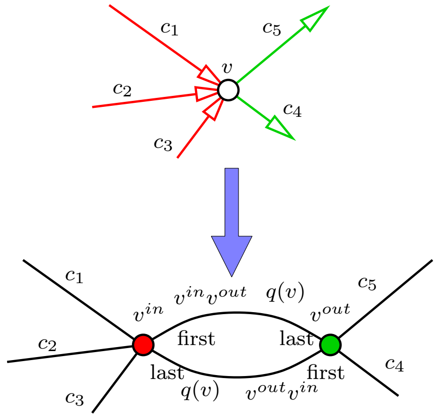
That is, in stable allocation $g_1 \vee g_2$ *where each worker picks his better assignment, each firm receives the worse out of the two. Similarly, in* $g_1 \wedge g_2$ *the choice of the firms means the less preferred situation to the workers.*

Our goal here is to exhibit a generalization of Theorem 47. The "natural" approach to achieve this would be an appropriate generalization of the deferred acceptance algorithm of Gale and Shapley. The difficulty is that though the Gale-Shapley algorithm can handle quota function q, somehow it has problems with ensuring the Kirchhoff law.

Theorem 48 *If network (D, s, t, c) and preference orders $<_v$ describe a stable flow problem then there always exists a stable flow f. If capacity function c is integral then there exists an integral stable flow.*

With the help of the given stable flow problem, we shall define a particular stable allocation problem. For each nonterminal vertex v of D calculate

$$M(v) := \max\left(\sum_{x:xv\in A(D)} c(xv), \sum_{x:vx\in A(D)} c(vx)\right),$$



that is, $M(v)$ is the maximum of total capacity of those arcs of D that enter and leave v. So $M(v)$ is an upper bound on the amount of flow that can flow through vertex v. Choose $q(v) := M(v) + 1$. Construct graph G_D as follows. Split each vertex v of D into two distinct vertices v^{in} and v^{out}, and for each arc uv of D add edge $u^{out}v^{in}$ to G_D.

For each nonterminal vertex v of D, add two parallel edges between v^{in} and v^{out}: to distinguish between them, we will refer them as $v^{in}v^{out}$ and $v^{out}v^{in}$. Let $p(v^{in}v^{out}) = p(v^{out}v^{in}) := q(v)$, $p(u^{out}v^{in}) := c(uv)$ and $q(v^{in}) = q(v^{out}) := q(v)$. To finish the construction of the stable allocation problem, we need to fix a linear preference order for each vertex of G_D. For vertex v^{in}, let $v^{in}v^{out}$ be the most preferred and $v^{out}v^{in}$ be the least preferred edge (if these edges are present), and the order of the other edges incident to v^{in} are coming from the preference order of v on the corresponding arcs. For vertex v^{out}, the most preferred edge is $v^{out}v^{in}$ and the least preferred one is $v^{in}v^{out}$ (if it makes sense), and the other preferences are coming from $<_v$.

The proof of Theorem 48 is a consequence of Theorem 47 and the following Lemma that describes a close relationship between stable flows and stable allocations.

Lemma 49 *If network (D, s, t, c) and preference orders $<_v$ describe a stable flow problem then $f : A(D) \to \mathbb{R}$ is a stable flow if and only if there is a stable allocation g of G_D such that $f(uv) = g(u^{out}v^{in})$ holds for each arc uv of D.*

3.1 The structure of stable flows

It is well-known about the stable marriage problem that in each stable marriage scheme, the same set of participants get married. That is, if someone does not get a marriage partner in some stable scheme then this very person remains single in all stable marriage schemes. A generalization of this property is the rural hospital theorem by Roth [49] (see also Theorem 5.13 in [52]). It is about the college model, where instead of men we work with colleges, women correspond to students and each college has a quota on the maximum number of admissible students. In the college admission problem, it is true that if a certain college c cannot fill up its quota in a stable admission scheme then c receives the same set of students in any stable admission scheme. (The phenomenon is named after the assignment problem of medical interns to hospitals.)

It seems that the rural hospital theorem cannot be generalized to the stable flow problem. (For more explanation, see [27].) There is however a consequence of the rural hospital theorem that can be generalized, namely, that the size of a stable matching is always the same. We have seen that the stable allocation problem is a special case of the stable flow problem, and from the construction it is apparent that the size of a stable matching (more precisely the total amount of assignments in a stable allocation) equals the value of the corresponding flow.

Theorem 50 *If network (D, s, t, c) and preference orders $<_v$ describe a stable flow problem and f_1 and f_2 are stable flows then the value of f_1 and f_2 are the same. Moreover, $f_1(a) = f_2(a)$ for any arc of D that is incident to terminal vertex s or t.*

As we have seen in Theorem 47, stable allocations have a lattice structure. Based on the connection of stable allocations and stable flows described in Lemma 49, we can prove that stable flows of a network with preferences also form a natural lattice. So assume that f is a stable flow in network $(D, s, t, c,)$

with preferences and let stable allocation g_f of G_D be the canonical representation of f as in the proof of Lemma 49.

Observe that any nonterminal vertex v of D, exactly one of $g_f(v^{in}v^{out})$ and $g_f(v^{out}v^{in})$ is positive by the choice of q and g_f. For stable flow f, we can classify the vertices of D different from s and t: v is an f*-vendor* if $g_f(v^{in}v^{out}) > 0$ and v is an f*-customer* if $g_f(v^{out}v^{in}) > 0$. If v is an f-vendor then no edge $v^{out}u^{in}$ can be g_f-dominated at v^{out} (as $g_f(v^{in}v^{out}) > 0$), hence player v sends as much flow to other vertices as much they accept. Similarly, if v is an f-customer then no edge $u^{out}v^{in}$ can be g_f-dominated at v^{out}, that is, player v receives as much flow as the others can supply her.

To explore the promised lattice structure of stable flows, let f_1 and f_2 two stable flows with canonical representations g_{f_1} and g_{f_2}, respectively. From Theorem 47 we know that stable allocations form a lattice, so $g_{f_1} \vee g_{f_2}$ and $g_{f_1} \wedge g_{f_2}$ are also stable allocations of G_D, and by Theorem 48, these stable allocations define stable flows $f_1 \vee f_2$ and $f_1 \wedge f_2$, respectively. How can we determine these latter flows directly, without the canonical representations? To answer this, we translate the lattice property of stable allocations on G_D to stable flows of D.

Theorem 50 shows that stable flows cannot differ on arcs incident to terminal vertex s or t, so on these arcs $f_1 \vee f_2$ and $f_1 \wedge f_2$ are determined. However, vertices different from s and t may have completely different situations in stable flows f_1 and f_2. The two colour classes of graph G_D are formed by the v^{in} and v^{out} type vertices, respectively. So, by Theorem 47, $g_{f_1} \vee g_{f_2}$ can be determined such that (say) each vertex v^{out} selects the better allocation and each vertex v^{in} receives the worse allocation out of the ones that g_{f_1} and g_{f_2} provides them. Similarly, for stable allocation $g_{f_1} \wedge g_{f_2}$ the „in"-type vertices choose according to their preferences and the „out"-type ones are left with the less preferred allocations. This means the following in the language of flows. If we want to construct $f_1 \vee f_2$ and v is a vertex different from s and t then either all arcs entering v will have the same flow in $f_1 \vee f_2$ as in f_1, or for all arcs a entering v we have $(f_1 \vee f_2)(a) = f_2(a)$ holds. A similar statement is true for the arcs leaving v. To determine which of the two alternatives is the right one, the following rules apply:

- If v is an f_1-vendor and an f_2-customer then v chooses f_2. If v is an f_2-vendor and an f_1-customer then v chooses f_1. That is, each vertex strives to be a customer.
- If v is an f_1-vendor and an f_2-vendor and v transmits more flow in f_1 than in f_2 (i.e. $0 < g_{f_1}(v^{in}v^{out}) < g_{f_2}(v^{in}v^{out})$) then v chooses f_1. That is, vendors prefer to sell more.
- If v is an f_1-customer and an f_2-customer and v transmits more flow in f_1 than in f_2 (i.e. $0 < g_{f_1}(v^{out}v^{in}) < g_{f_2}(v^{out}v^{in})$) then v chooses f_2. That is, customers prefer to buy less.
- Otherwise v is a customer in both f_1 and f_2 or v is a vendor in both flows and v transmits the same amount in both flows (i.e. $g_{f_1}(v^{out}v^{in}) = g_{f_2}(v^{out}v^{in})$ and $g_{f_1}(v^{in}v^{out}) = g_{f_2}(v^{in}v^{out})$). In this situation, v chooses the better „selling position" and gets the worse „buying position" out of stable flows f_1 and f_2.

Clearly, for the construction of $f_1 \wedge f_2$, one always has to choose the „other" options than the one that the above rules describe.

The lattice structure of stable flows defines a partial order on stable flows: $f_1 \preceq f_2$ if and only if $f_1 \vee f_2 = f_2$ holds, or equivalently, if $f_1 \wedge f_2 = f_1$ is true. By to the above rules, this means that each f_1-customer v is an f_2-customer, such that v buys at least as much in f_1 as in f_2. Each f_2-vendor u is an f_1-vendor and u sells at most as much in f_1 as in f_2. If w plays the same role (vendor or customer) in both flows and transmits the same amount then v prefers the selling position of f_2 and the buying position of f_1.

4. fejezet

Matchings with partially ordered contracts

In this section, we describe a mathematical model that extends both the stable flow model and that of Hatfield and Milgrom described in [34].

Let D and H be two disjoint sets of agents. We regard D as the set of doctors and H as the set of hospitals. By a contract x, we always mean an agreement between doctor $D(x) \in D$ and hospital $H(x) \in H$. Let X denote the set of all possible contracts in the model. For any subset X' of X, doctor d of D and hospital h of H, $X'(d) = \{x \in X' : D(x) = d\}$ and $X'(h) = \{x \in X' : H(x) = h\}$ denotes all the contracts that involve doctor d and hospital h, respectively.

The main difference between our model and that of Hatfield and Milgrom in [34] is that in our model we allow certain implications between contracts. An example is that if x is a contract that assigns doctor $D(x)$ to hospital $H(x)$ for some i days a week then it is always possible to choose contract x' between $D(x)$ and $H(x)$ that describes the same job as x does except for the total weakly workload is j days for $j < i$. Or, instead of contract x doctor $D(x)$ and hospital $H(x)$ may agree on signing a contract x' for a job that needs a lower qualification than x needs. In these examples, the possibility contract x implies the possibility of contract x' and we denote this fact by $x' \preceq x$. We assume that $P = (X, \preceq)$ is a partially ordered set on the set X of possible contracts[1]. It is easy to check that if there is no implication between contracts whatsoever (that is, if any two contracts are incomparable in poset P, i.e. if P is trivial) then our model reduces to that of Hatfield and Milgrom.

Just like in the Hatfield-Milgrom model, hospitals and doctors have certain preferences on the contracts they participate in. This is described by choice functions as follows. Assume that $X' \subseteq X$ is an lower ideal of P. Then $C_d(X')$ denotes those contracts of $X'(d)$ that doctor d would pick from $X'(d)$ if she is allowed to choose freely. Note that though in the Hatfield-Milgrom model, choice function C_d always selects at most one contract (hence it is a so-called one-to-many matching market), we do not assume this property. For any hospital h, we have a similar choice function C_h that selects the favourite contracts of hospital h from $X'(h)$. We assume that C_d and C_h always select an antichain of P. (That is, if d can work t or t' hours for h according to contracts x and x' then d never wants to sign both contracts x and x' and the same is true for h.) As each agent in our two-sided market has a choice function, we can define two joint choice functions: one for the doctors and one for the hospitals. Formally,

$$C_D(X') = \bigcup\{C_d(X') : d \in D\} \qquad \text{and} \qquad C_H(X') = \bigcup\{C_h(X') : h \in H\}$$

denote the doctors' and hospitals' choice function, respectively. Clearly, each choice function C we defined so far is mapping lower ideals of P into antichains of P such that $C(L) \subseteq L$ holds for any lower ideal L of P. For such a choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$, we define another choice function $C^* : \mathcal{L}(P) \to \mathcal{L}(P)$ by $C^*(L) := \mathrm{Li}(C(L))$. As there is a bijection between antichains and lower ideals of P, not only C determines C^*, but we can calculate C from C^* by $C(L) = \mathrm{Max}(C^*(L))$. Obviously, if P is trivial then $C = C^*$. We can also talk about choice functions in a more general sense. If $\mathcal{L}$ is a subset of 2^X then a choice function on $\mathcal{L}$ is a mapping $C : \mathcal{L} \to \mathcal{L}$ such that $C(L) \subseteq L$ holds for any element L of $\mathcal{L}$. Note that choice functions C^*_D and C^*_H are choice functions in this latter sense, as well.

[1] Later we shall see that all our results are true in the more general setting where we do not assume any acyclicity about implications between contracts. One can define „lower ideals" on the transitive closure of the implication digraph and these „lower ideals" form a complete sublattice of 2^X.

There are two important properties of choice functions that we shall assume in our model. Recall that choice function $C : \mathcal{L} \to \mathcal{L}$ on subset $\mathcal{L}$ of 2^X is path independent if

$$C(L) \subseteq L' \subseteq L \Rightarrow C(L) = C(L') \tag{4.1}$$

holds for any two members L and L' of $\mathcal{L}$. Note that in the Hatfield-Milgrom model, choice functions are defined by a strict linear order on the subsets X such that $C(Y)$ is that subset of Y that comes first in this linear order. (We shall see an example of such a choice function in Example 57.) Clearly, such choice functions are path independent by definition. Note that in the „traditional" definition of path independence is different from ours. Actually, (4.1) is weaker than that as shown below.

Observation 51 *If for choice function $C : \mathcal{L} \to \mathcal{L}$ identity $C(A \cup B) = C(C(A) \cup C(B))$ holds for any members A, B of $\mathcal{L}$ then (4.1) is also true for C.*

In Lemma 55 we shall see that assuming substitutability (that we define a bit later) of C then "traditional" path independence is equivalent to (4.1). The following statement is easy to check.

Observation 52 *If P is a partial order on X then choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ is path independent if and only if choice function $C^* : \mathcal{L}(P) \to \mathcal{L}(P)$ is path independent.*

From now on, $\mathcal{L}$ denotes a complete sublattice of 2^X. To get some intuition, the reader might simply think that $\mathcal{L} = \mathcal{L}(P)$, but our results that we claim for general complete sublattices are more general than the ones with this restriction. We do think that general complete sublattices still capture some interesting Economics models that do not fit in the poset-framework.

If $C : \mathcal{L} \to \mathcal{L}$ is a choice function then we can compare certain members of $\mathcal{L}$ with the help of C the following way. We say that member L is *C-better* than member L' (denoted by $L' \preceq_C L$) if $C(L \cup L') = L$. We can extend this notion for antichains if choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ maps lower ideals to antichains. This way, antichain A of P is *C-better* than $A' \in \mathcal{A}(P)$ (denoted by $A' \preceq_C A$) if $C(\mathrm{Li}(A \cup A')) = A$. Note that the same notation for lower ideals and antichains does not cause ambiguity as the range of C determines which one we talk about. Note further that $\preceq_C$ is not necessarily a partial order.

The second important property of a choice function is substitutability (or comonotonicity, as called by Fleiner in [25]) that we define here in a somewhat unusual way. A mapping $U : \mathcal{L} \to \mathcal{L}$ is called *antitone* if $U(L') \subseteq U(L)$ holds whenever $L \subseteq L'$ holds for elements L and L' of $\mathcal{L}$. Choice function $C^* : \mathcal{L} \to \mathcal{L}$ is *substitutable* if there exists an antitone mapping $U : \mathcal{L} \to \mathcal{L}$ such that $C^*(L) = L \cap U(L)$ holds for each member L of $\mathcal{L}$. A choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ that selects an antichain is called substitutable if C^* is substitutable. A choice function C in a traditional two-sided market model selects $C(Y)$ from a set Y of alternatives such that $C(Y)$ is the set of all those choices that are undominated by set Y of alternatives. The substitutability property captures the fact that a broader set of alternatives leaves less undominated choices. Or, equivalently, if the choice set is growing then the set of dominated (hence unselected) alternatives is also growing. This phenomenon is used in the definition of substitutability by Hatfield and Milgrom: elements of X are *substitutes* for choice function $C : 2^X \to 2^X$ if the set of rejected elements is a monotone mapping, that is $R(Y) := Y \setminus C(Y) \subseteq Y' \setminus C(Y') = R(Y')$ whenever $Y \subseteq Y' \subseteq X$.

Observation 53 *If elements of X are substitutes for choice function $C : 2^X \to 2^X$ then C is substitutable.*

Example 54 *Assume that hospital h has a linear preference order on $X(h)$ and $C_h(X')$ is the q_h best elements of $X'(h)$. (Here, there is no partial order on X, or if we insist on having one then it is trivial.) It is easy to check that C_h is path independent and contracts in $X(h)$ are substitutes. To see that C_h is substitutable, we define $U : 2^X \to 2^X$ by $U(X')$ denoting the set of those contracts x of $X(h)$ such that $X'(h)$ contains at most $q_h - 1$ contracts that are better than x according to the preference order of h. Clearly, if $X' \subseteq X''$ then $U(X') \supseteq U(X'')$, so U is antitone. It is also clear by the definition of U that $C_h(X') = X' \cap U(X')$, that is, C_h is indeed substitutable.*

It is well-known that our definition of path-independence is equivalent to the „traditional" one for substitutable choice functions.

Lemma 55 *If choice function $C : \mathcal{L} \to \mathcal{L}$ is substitutable and path-independent then identity $C(A \cup B) = C(C(A) \cup C(B))$ holds for any members A, B of $\mathcal{L}$.*

The following theorem points out an interesting property of substitutable choice functions.

Theorem 56 *If choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ is path-independent and substitutable for some partial order P on a finite ground set X then $\prec_C$ is a partial order on $\{C(L) : L \in \mathcal{L}\}$, that is, on those antichains of P that are in the range of C.*

Next we show that our poset-based model is more general than that of Hatfield and Milgrom in [34].

Example 57 *Assume that we have one hospital h and two doctors d and d'. Contracts x_3, x_4, x_5 and x_3', x_4' and x_5' represent a $3, 4$ and 5 days job for d and d' respectively. Assume that h has the following preference order on feasible contract sets (starting from the best):*

$$\{x_4, x_4'\}, \{x_5, x_3'\}, \{x_3, x_5'\}, \{x_4, x_3'\}, \{x_3, x_4'\}, \{x_3, x_3'\}, \{x_5\}, \{x_5'\}, \{x_4\}, \{x_4'\}, \{x_3\}, \{x_3'\} \ .$$

So $C_h(Y)$ is that subset of Y which is the first in the above order. In particular, we have that $C_h(x_5, x_4, x_3, x_5', x_3') = \{x_5, x_3'\}$, so $x_4 \in R(x_5, x_4, x_3, x_5', x_3')$. Hence, if contracts were substitutes for C_h then R is monotone thus $x_4 \in R(x_5, x_4, x_3, x_5', x_4', x_3') = R(X)$. This means that $x_4 \notin C_h(X)$ contradicting $C_h(X) = \{x_4, x_4'\}$.

However, the above C_h easily fits in our framework if we define poset P by $x_3 \preceq x_4 \preceq x_5$ and $x_3' \preceq x_4' \preceq x_5'$. For any lower ideal L, let $U(L) := \{x_3, x_4, x_3', x_4'\} \cup u(L) \cup u'(L)$ where $u(L) = \emptyset$ if $x_4 \in L$ and $u(L) = \{x_3', x_4', x_5'\}$ if $x_4 \notin L$, and similarly $u'(L) = \emptyset$ if $x_4' \in L$ and $u'(L) = \{x_3, x_4, x_5\}$ if $x_4' \notin L$. As both u and u' are antitone, U is also such. Hence choice function C^* defined by $C^*(L) = L \cap U(L)$ is substitutable and one can easily check that $C^* = C_h$ on lower ideals of P. As C_h is path independent by definition, our model is indeed a genuine generalization of Hatfield and Milgrom's.

Note that for a substitutable choice function $C^* : \mathcal{L} \to \mathcal{L}$ there might be several antitone functions $U : \mathcal{L} \to \mathcal{L}$ such that $C^*(L) = L \cap U(L)$ holds for any member L of $\mathcal{L}$. The next statement shows that there is a canonical one among these antitone functions and this is in fact the minimal of those. (Actually, there is a maximal such U as well, but we do not need this fact.) For a choice function $C^* : \mathcal{L} \to \mathcal{L}$ define $U^* : \mathcal{L} \to \mathcal{L}$ by

$$U^*(L) := \bigcup \{Y \in \mathcal{L} : Y \subseteq C^*(L \cup Y)\} = \bigcup \{Y \in \mathcal{L} : Y \subseteq U(L \cup Y)\} \ . \tag{4.2}$$

Note that the second equality in (4.2) holds by the definition of C^*, and this means that the right hand side defines the same U^* no matter which U (that defines C^*) we use.

Observation 58 *If choice function $C^* : \mathcal{L} \to \mathcal{L}$ is substitutable then U^* in (4.2) is antitone and for any member L of $\mathcal{L}$ we have $C^*(L) = L \cap U^*(L)$.*

There is another useful fact about the antitone function U^* that defines a path-independent substitutable choice function.

Observation 59 *If choice function $C^* : \mathcal{L} \to \mathcal{L}$ is path-independent and substitutable then $U^*(L) = U^*(C^*(L))$ holds for any member L of $\mathcal{L}$.*

At this point, we can generalize the notion of stability to our framework. Let D and H be the sets of doctors and hospitals, respectively and let X denote the set of possible contracts between doctors and hospitals. Assume that we have given a (complete) sublattice $\mathcal{L}$ of 2^X (for example as the set of lower ideals of a partial order P on X), and let $C_D^* = (C_D)^*$ and $C_H^* = (C_H)^*$ denote the joint choice functions of the doctors and of the hospitals, respectively. For members L_1 and L_2 of $\mathcal{L}$ pair (L_1, L_2) is called a *stable pair* if

$$U_D^*(L_1) = L_2 \qquad \text{and} \qquad U_H^*(L_2) = L_1 \tag{4.3}$$

holds. If $\mathcal{L} = \mathcal{L}(P)$ for some poset P on X then antichain A of P is called *stable* if

$$U_D^*(\mathrm{Li}(A)) \cap U_H^*(\mathrm{Li}(A)) = \mathrm{Li}(A) \tag{4.4}$$

Later we shall see that stable pairs are closely related to stable antichains. These latter represent the solution concept of two-sided market situations in our model. What does it mean that an antichain is stable? The first requirement is that if both doctors and hospitals select freely from those contracts that antichain A represents or implies then doctors select $C_D(A) = \mathrm{Max}(\mathrm{Li}(A) \cap U_D^*(\mathrm{Li}(A)) = \mathrm{Max}(\mathrm{Li}(A)) = A$, as $\mathrm{Li}(A) \subseteq U_D^*(\mathrm{Li}(A))$. Similarly follows that $C_H(A) = A$, so hospitals also pick the same antichain A of contracts. Moreover, if there are some further choices available that are represented by antichain Y and both the doctors and the hospitals are happy to pick those (formally, if $Y \subseteq C_D(\mathrm{Li}(A) \cup \mathrm{Li}(Y))$ and $Y \subseteq C_H(\mathrm{Li}(A) \cup \mathrm{Li}(Y)))$ then

$$\mathrm{Li}(Y) \subseteq C_D^*(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \cap C_H^*(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \subseteq$$
$$\subseteq U_D^*(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \cap U_D^*(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \subseteq U_D^*(\mathrm{Li}(A)) \cap U_D^*(\mathrm{Li}(A)) = \mathrm{Li}(A) \ .$$

So $Y \subseteq C_D(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) = C_D(\mathrm{Li}(A)) = A$. This means that we cannot add further choices to A such that both the doctors and the hospitals will select them.

In the Hatfield-Milgrom model, $A \subseteq X$ is a *stable allocation* if $C_D(A) = C_H(A) = A$ and there exists no hospital h and set of contracts $X'' \neq C_h(X')$ with $X'' = C_h(A \cup X'') \subseteq C_D(A \cup X'')$. Assume that A is a feasible allocation, that is, $C_D(A) = C_H(A) = A$ and A' is a blocking set: $A' \subseteq C_D(A \cup A')$ and $A' \subseteq C_H(A \cup A')$. This means that there is a hospital h that picks a different assignment from A and from $A \cup A'$. Let $X'' = C_h(A \cup A')$ denote the choice of this hospital h. Since $X'' = C_h(A \cup A') \subseteq A \cup X'' \subseteq A \cup A'$, we have $X'' = C_h(A \cup X'')$. Because of $X'' \subseteq C_D(A \cup A')$ and $A = C_D(A)$, each doctor in $\cup_{x \in X''} D(x)$ has the same choice as in $A \cup X''$, that is, $X'' \subseteq C_D(A \cup X'')$. So X'' blocks A in the Hatfield-Milgrom sense. This proves that a stable allocation of contracts in the Hatfield-Milgrom framework is a stable antichain. It is not difficult to see that the other direction is also true: any stable antichain is a stable allocation of contracts for the same model with a trivial underlying partial order.

Example 60 *Assume we have two hospitals h and $\bar{h}$ and two doctors d and d'. Contract x_i' represents an i-days job of d' at h, and $\bar{x}_j$ stands for a j-days occupation for d at $\bar{h}$, etc. Poset P is defined by relations of type $z_i \preceq z_j$ for $i \leq j$ and $z \in \{x, x', \bar{x}, \bar{x}'\}$. Let $X := \{z_i : 1 \leq i \leq 5,\ z \in \{x, x', \bar{x}, \bar{x}'\}\}$. Assume that d is a famous doctor with a high salary expectation, so each hospital wants to employ her but for a minimum amount of time. Doctor d' can do the same job equally well but she is young and hence costs less to the employer. Assume that each hospital needs 5 days of work and from a given set of options it selects 1 day of work of doctor d, the maximum amount of work for doctor d' up to 5 days altogether and for the missing days it selects d if she is still available. In particular $C_h = C_{\bar{h}}$ and for example, $C_h(x_1, x_2, x_3, x_1', x_2', x_3') = \{x_2, x_3'\}$, $C_h(x_1, x_2, x_3, x_1', x_2', x_3', x_4') = \{x_1, x_4'\}$ and $C_{\bar{h}}(\bar{x}_1, \bar{x}_2, \bar{x}_1', \bar{x}_2') = \{\bar{x}_2, \bar{x}_2'\}$. Assume moreover that $C_d = C_{d'}$ and both doctors d and d' look for 5 days of work, and both of them prefer hospital h to h': $C_d(x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3) = \{x_3, \bar{x}_2\}$.*

It is easy to check that all four choice functions are substitutable and path-independent on $\mathcal{L}(P)$ and moreover $U_H^*(\mathrm{Li}(A)) = U_H^*(L) = L_1$ and $U_D^*(\mathrm{Li}(A)) = U_D^*(L) = L_2$ holds for

$$A = \{x_1, \bar{x}_4, x_4', \bar{x}_1'\}\ , \qquad L = \mathrm{Li}(A) = \{x_1, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, x_1', x_2', x_3', x_4', \bar{x}_1'\},$$
$$L_1 = \{x_1, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, x_1', x_2', x_3', x_4', \bar{x}_1', \bar{x}_2', \bar{x}_3', \bar{x}_4'\},\ \text{and}$$
$$L_2 = \{x_1, x_2, x_3, x_4, x_5, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, x_1', x_2', x_3', x_4', x_5', \bar{x}_1'\}\ .$$

As $L_1 \cap L_2 = L$, it follows that (L_1, L_2) is a stable pair and A is a stable antichain.

In what follows, we describe the main tools for the proof of our results. Let X be a ground set and define partial order $\sqsubseteq$ on pairs of subsets of X by $(A, B) \sqsubseteq (A', B')$ if $A \subseteq A'$ and $B \supseteq B'$ holds. It is clear that for any sublattice $\mathcal{L}$ of 2^X, $\sqsubseteq$ defines a lattice on $\mathcal{L} \times \mathcal{L}$ with lattice operations $(A, B) \sqcap (A', B') = (A \cap A', B \cup B')$ and $(A, B) \sqcup (A', B') = (A \cup A', B \cap B')$. The following theorem generalizes some results by Hatfield and Milgrom in [34].

Theorem 61 *Let X be a set of possible contracts between set D of doctors and H of hospitals and let $\mathcal{L}$ be a complete sublattice of 2^X. Assume that joint choice functions C_D^* of doctors and C_H^* of hospitals are substitutable. Then stable pairs form a nonempty complete lattice subset of $(\mathcal{L} \times \mathcal{L}, \sqsubseteq)$. In particular, there does exist a stable pair and there is a greatest and a lowest such pair.*

Moreover, if $\mathcal{L} = \mathcal{L}(P)$ is the lattice of lower ideals of some partial order $P = (X, \preceq)$ and both joint choice functions C_D and C_H are substitutatble and path independent then $\preceq_{C_D}$ and $\preceq_{C_H}$ are opposite partial orders on stable antichains and both of them define a lattice.

Note that the 2nd part of Theorem 61 generalizes Theoerm 11 of Blair on the lattice structure of many-to-many stable matchings [14]. Our proof of Theorem 61 leans on ideas of [37], the reader may check the details in [20].

Irodalomjegyzék

[1] Hernán Abeledo and Yosef Blum. Stable matchings and linear programming. *Linear Algebra Appl.*, 245:321–333, 1996.

[2] Hernán G. Abeledo, Yosef Blum, and Uriel G. Rothblum. Canonical monotone decompositions of fractional stable matchings. *Internat. J. Game Theory*, 25(2):161–176, 1996.

[3] Hernán G. Abeledo and Uriel G. Rothblum. Stable matchings and linear inequalities. *Discrete Appl. Math.*, 54(1):1–27, 1994.

[4] Hernán G. Abeledo and Uriel G. Rothblum. Courtship and linear programming. *Linear Algebra Appl.*, 216:111–124, 1995.

[5] Ron Aharoni and Tamás Fleiner. On a lemma of scarf. Technical report, EGRES report TR-2001-15, ISSN 1587-4451, September 2001. `http://www.cs.elte.hu/egres`.

[6] Ron Aharoni and Tamás Fleiner. On a lemma of scarf. *J. Combin. Theory Ser. B*, 87(1):72–80, 2003.

[7] Ron Aharoni and Ron Holzman. Fractional kernels in digraphs. *J. Combin. Theory Ser. B*, 73(1):1–6, 1998.

[8] Ahmet Alkan. A class of multipartner matching models with a strong lattice structre. Forthcoming in *Economic Theory*, 2000.

[9] Mourad Baïou and Michel Balinski. Many-to-many matching: stable polyandrous polygamy (or polygamous polyandry). *Discrete Appl. Math.*, 101(1-3):1–12, 2000.

[10] Mourad Baïou and Michel Balinski. The stable admissions polytope. *Math. Program.*, 87(3, Ser. A):427–439, 2000.

[11] Mourad Baïou and Michel Balinski. The stable allocation (or ordinal transportation) problem. *Math. Oper. Res.*, 27(3):485–503, 2002.

[12] P. Biró. Student admissions in hungary as gale and shapley envisaged. Technical report, University of Glasgow, Department of Computing Science, 2008.

[13] Péter Biró. Personal communication.

[14] Charles Blair. The lattice structure of the set of stable matchings with multiple partners. *Math. Oper. Res.*, 13(4):619–628, 1988.

[15] Katarína Cechlárová and Tamás Fleiner. On a generalization of the stable roommates problem. *ACM Trans. Algorithms*, 1(1):143–156, 2005.

[16] Katarína Cechlárová and Tamás and Fleiner. Stable roommates with free edges. Technical report, EGRES report TR-2009-1, ISSN 1587-4451, February 2009. `http://www.cs.elte.hu/egres`.

[17] Vincent P. Crawford and Elsie Marie Knoer. Job matching with heterogenous firms and workers. *Econometrica*, 49:437–450, 1981.

[18] Brian C. Dean and Siddharth Munshi. Faster algorithms for stable allocation problems. In *Proceedings of the MATCH-UP (Matching Under Preferences) workshop at ICALP 2008 (Reykjavik, 2008)*, pages 133–144, 2008.

[19] Vânia M. F. Dias, Guilherme D. da Fonseca, Celina M. H. de Figueiredo, and Jayme L. Szwarcfiter. The stable marriage problem with restricted pairs. *Theoret. Comput. Sci.*, 306(1-3):391–405, 2003.

[20] Rashid Farooq, Tamás Fleiner, and Akihisa Tamura. Matching with partially ordered contracts. *Jpn. J. Ind. Appl. Math.*, 29(3):401–417, 2012.

[21] Tomás Feder. A new fixed point approach for stable networks and stable marriages. *J. Comput. System Sci.*, 45(2):233–284, 1992. Twenty-first Symposium on the Theory of Computing (Seattle, WA, 1989).

[22] Tamás Fleiner. A fixed-point approach to stable matchings and some applications. Technical report, EGRES report TR-2001-1, ISSN 1587-4451, March 2001. `http://www.cs.elte.hu/egres`.

[23] Tamás Fleiner. A matroid generalization of the stable matching polytope. In Bert Gerards and Karen Aardal, editors, *Proceedings of the 8th International Conference (IPCO VIII) held in Utrecht, June 13–15, 2001*, pages 105–114, Berlin, 2001. Springer-Verlag.

[24] Tamás Fleiner. Some results on stable matchings and fixed points. Technical report, EGRES report TR-2002-8, ISSN 1587-4451, December 2002. http://www.cs.elte.hu/egres.

[25] Tamás Fleiner. A fixed-point approach to stable matchings and some applications. *Math. Oper. Res.*, 28(1):103–126, 2003.

[26] Tamás Fleiner. On the stable b-matching polytope. *Math. Social Sci.*, 46(2):149–158, 2003.

[27] Tamás Fleiner. On stable matchings and flows. In *Graph-theoretic concepts in computer science*, volume 6410 of *Lecture Notes in Comput. Sci.*, pages 51–62. Springer, Berlin, 2010.

[28] Tamás Fleiner. Stable and crossing structures, August, 2000. PhD dissertation, http://www.renyi.hu/~fleiner.

[29] Tamás Fleiner, Robert W. Irving, and David F. Manlove. Efficient algorithms for generalized stable marriage and roommates problems. *Theoret. Comput. Sci.*, 381(1-3):162–176, 2007.

[30] Tamás Fleiner, Robert W. Irving, and David F. Manlove. An algorithm for a super-stable roommates problem. Technical report, EGRES report TR-2008-06, May 2008. http://www.cs.elte.hu/egres.

[31] D. Gale and L.S. Shapley. College admissions and stability of marriage. *Amer. Math. Monthly*, 69(1):9–15, 1962.

[32] David Gale. The two-sided matching problem. Origin, development and current issues. *Int. Game Theory Rev.*, 3(2-3):237–252, 2001. Special issue on operations research and game theory with economic and industrial applications (Chennai, 2000).

[33] Dan Gusfield and Robert W. Irving. *The stable marriage problem: structure and algorithms*. MIT Press, Cambridge, MA, 1989.

[34] John William Hatfield and Paul Milgrom. Matching with contracts. *American Economic Review*, 95(4):913–935, 2005.

[35] Robert W. Irving. An efficient algorithm for the „stable roommates" problem. *J. Algorithms*, 6(4):577–595, 1985.

[36] Robert W. Irving and David F. Manlove. The stable roommates problem with ties. *J. Algorithms*, 43(1):85–105, 2002.

[37] Zsuzsanna Jankó. Stabil párosítások és egyetemi felvételi ponthatárok (stable matchings and university admissions score limits), May 2009. BSc dissertation, http://www.cs.elte.hu/math/diploma/bsc_mat.html.

[38] Shizuo Kakutani. A generalization of Brouwer's fixed point theorem. *Duke Math. J.*, 8:457–459, 1941.

[39] Jr Kelso, Alexander S. and Vincent P. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50:1483–1504, 1982.

[40] Bettina Klaus and Flip Klijn. Median stable matching for college admissions. *Internat. J. Game Theory*, 34(1):1–11, 2006.

[41] Bronisław Knaster. Un théorème sur les fonctions d'ensembles. *Ann. Soc. Polon. Math.*, 6:133–134, 1928.

[42] Donald E. Knuth. *Stable marriage and its relation to other combinatorial problems*. American Mathematical Society, Providence, RI, 1997. An introduction to the mathematical analysis of algorithms, Translated from the French by Martin Goldstein and revised by the author.

[43] D. Manlove. *Algorithmics of Matching Under Preferences*. Series on Theoretical Computer Science. World Scientific Publishing Company Incorporated, 2013.

[44] Michael Ostrovsky. Stability in supply chain networks. *American Economic Review*, 98(3):897–923, 2006.

[45] Charles R. Plott. Path independence, rationality, and social choice. *Econometrica*, 41:1075–1091, 1973.

[46] Eytan Ronn. NP-complete stable matching problems. *J. Algorithms*, 11(2):285–304, 1990.

[47] Alvin E. Roth. Stability and polarization of intersts in job matching. *Econometrica*, 52(1):47–57, 1984.

[48] Alvin E. Roth. Conflict and coincidence of interest in job matching: some new results and open questions. *Math. Oper. Res.*, 10(3):379–389, 1985.

[49] Alvin E. Roth. On the allocation of residents to rural hospitals: a general property of two-sided matching markets. *Econometrica*, 54(2):425–427, 1986.

[50] Alvin E. Roth, Uriel G. Rothblum, and John H. Vande Vate. Stable matchings, optimal assignments, and linear programming. *Math. Oper. Res.*, 18(4):803–828, 1993.

[51] Alvin E. Roth and Marilda Sotomayor. The college admissions problem revisited. *Econometrica*, 57(3):559–570, 1989.

[52] Alvin E. Roth and Marilda A. Oliveira Sotomayor. *Two-sided matching*. Cambridge University Press, Cambridge, 1990. A study in game-theoretic modeling and analysis, With a foreword by Robert Aumann.

[53] Uriel G. Rothblum. Characterization of stable matchings as extreme points of a polytope. *Math. Programming*, 54(1, Ser. A):57–67, 1992.

[54] Herbert E. Scarf. The core of an N person game. *Econometrica*, 35:50–69, 1967.

[55] Sandy Scott. The study of stable marriage problems with ties, 2005. PhD dissertation, University of Glasgow, Department of Computing Science.

[56] Jay Sethuraman, Chung-Piaw Teo, and Liwen Qian. Many-to-one stable matching: geometry and fairness. *Math. Oper. Res.*, 31(3):581–596, 2006.

[57] L. S. Shapley. On balanced games without side payments. In *Mathematical programming (Proc. Advanced Sem., Univ. Wisconsin, Madison, Wis., 1972)*, pages 261–290. Math. Res. Center Publ., No. 30. Academic Press, New York, 1973.

[58] Ashok Subramanian. A new approach to stable matching problems. *SIAM J. Comput.*, 23(4):671–700, 1994.

[59] Jimmy J. M. Tan. A necessary and sufficient condition for the existence of a complete stable matching. *J. Algorithms*, 12(1):154–178, 1991.

[60] Jimmy J. M. Tan and Yuang Cheh Hsueh. A generalization of the stable matching problem. *Discrete Appl. Math.*, 59(1):87–102, 1995.

[61] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific J. of Math*, 5:285–310, 1955.

[62] Chung-Piaw Teo and Jay Sethuraman. The geometry of fractional stable matchings and its applications. *Math. Oper. Res.*, 23(4):874–891, 1998.

[63] John H. Vande Vate. Linear programming brings marital bliss. *Oper. Res. Lett.*, 8(3):147–153, 1989.