Nonlinear Vibrations of Parametrically Excited Complex Mechanical Systems

Ph.D. dissertation

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Abstract

We analyzed the dynamical behaviour of various parametrically excited mechanical systems: fluid conveying articulated rigid pipes attached together with resilient joints, two different models of an elastic pipe containing pulsatile flow, and the experimental rig of a single railway wheelset. The derived mathematical models were investigated from their simplest—linearized, autonomous—from to the complicate—nonlinear, time-periodic—case. The results of (semi-)analytical stability analysis were checked by numerical simulations. The numerical method based on Chebyshev polynomials was applied in the analysis of periodic systems. The results are presented in stability charts and bifurcation diagrams. Flip, fold and secondary Hopf bifurcations were detected in a super- and sub-critical sense.
Hereby, I would like to acknowledge the support of the Hungarian Scientific Research Foundation (OTKA, Grant No. F 030378) and the valuable comments that I received from my colleagues at the Department of Applied Mechanics. Furthermore, special thanks to Prof. Subhash C. Sinha, for the additional consultations, and last but not least I would like to commemorate my friend and colleague, Árpád Meggyes, who left us with tragic unexpectedness. He gave me not only technical ideas and hints about their implementation under Linux but also several practical tips like how to generate nice outlook documents using $\LaTeX$. 

To my wife and children
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Mathematical Notations

\( f \in C^p_H \)  \( f \) is \( p \) times continuously differentiable on the set \( H \)

\( \mathbb{N} \)  non-negative integers (0, 1, 2, \ldots)

\( \mathbb{R}^n \)  \( n \) dimensional space of real numbers

\( \mathbb{C}^n \)  \( n \) dimensional space of complex numbers

\( \Re \)  real part of a complex number

\( \Im \)  imaginary part of a complex number

\( \lambda_i \)  eigenvalue of a matrix or characteristic polynomial

\( a_i \)  coefficient of the characteristic polynomial

\( H_i \)  \( i^{\text{th}} \) leading principal minor of the \( \text{Hurwitz matrix} \)

\( J_s \)  Jordan block of the stable subspace

\( J_c \)  Jordan block of the critical subspace

\( J_u \)  Jordan block of the unstable subspace

\( T \)  the least positive principal period of the parameters of a system

\( \mathbb{C} \)  the \( \text{Floquet Transition Matrix} \) or principal matrix

\( \text{tr}C \)  trace of matrix \( C \)

\( \delta_{ij} \)  the \( \text{Kronecker symbol} \) (equals to 1 if \( i = j \), 0, otherwise)
LIST OF SYMBOLS

Symbols Related to the Mechanical Model

\( \mathcal{T} \)  
kinetic energy of the system

\( \mathcal{U} \)  
strain energy of the system

\( \delta \mathcal{P} \)  
virtual power of non-conservative forces

\( \delta q \)  
virtual displacement

\( \delta \dot{q} \)  
virtual velocity

\( M \)  
mass per unit length of the pipe

\( m \)  
mass per unit length of the fluid in the pipe

\( \mu = \frac{3m}{M+m} \)  
mass ratio

\( l_i \)  
length of the \( i \)th rigid pipe

\( \lambda_p = \frac{l_1}{l_2} \)  
pipe length ratio of two rigid pipes

\( s_i \)  
stiffness of the \( i \)th joint attaching two rigid pipes

\( \sigma = \frac{s_1}{s_2} \)  
stiffness ratio of the joints attaching the rigid pipes

\( U \)  
mean flow velocity

\( \mathring{U} \)  
dimensionless flow velocity

\( \mathring{U}_{cr} \)  
critical dimensionless flow velocity in case of steady flow

\( \nu \)  
amplitude of perturbation of the flow velocity

\( \omega \)  
angular frequency of perturbation of the flow velocity

\( w = \frac{\nu}{\alpha} \)  
dimensionless (or relative) frequency
Chapter 1

Introduction

Periodically changing state of things is a very basic phenomenon in the Universe, see e.g. the planet motions in the solar system caused by the central force field of the gravity (moreover all the known planets rotate around their axis, too). The beginning of life also needed moving and steadiness (predictability), at the same time. So does the evolution of technology which got a big impulse after inventing the wheel, a machine element which moves (rotates) and still remains attached to its axis.

Many (if not every) systems in mechanical engineering design contain element with rotating parts. These rotating parts can have a periodically changing action to the system. The effect of this periodic excitation can be independent from the state of the system. However, it can also depend on the state variables. In the first case the mathematical model will be a (partial) differential equation with (an *additive*) non-homogeneous part containing the excitation. In the second case we have a non-autonomous system which is excited parametrically (*multiplicative* excitation). In this work systems of the latter type are in the focus of our interest.

The first remarkable analysis of a system with periodic coefficients was done by Mathieu [24, 1868] who investigated the vibrations of an elliptical membrane. One can face an analogous problem analyzing a pendulum with excited support. The derived (Mathieu) equation is one of the simplest parametrically excited ordinary differential equations. The general theory of periodic systems (taking into consideration only the linear part) relates to Floquet’s name [9, 1883]. This theory is summarized in Section 2.2 of the next chapter. However, the analytical stability analysis is fairly impossible in most cases contrasted with the analysis of autonomous systems: Poincaré’s method [36, 1899] using small parameters yields only a narrow acceptable domain in the stability charts of Mathieu equation. We have to use some kind of numerical technique (e.g. Runge–Kutta method) to be able to determine the stability of the system in a wide range of parameter values as it will be shown in Section 2.4.

In the case of large systems the commonly used numerical methods can be very
time consuming, especially when we want to determine the stability not only in one point of the space of interesting (bifurcation) parameters but also in many other points to obtain a stability chart. In Section 2.3 we introduce a numerical method based on Chebyshev polynomials which was developed by Sinha and Wu [41, 1991]. This method can be very efficient because we have to solve ‘only’ a linear algebraic system even if it is larger than the original system of ordinary differential equations. Moreover, using symbolic mathematics softwares like MAPLE or MATHEMATICA, one can also obtain the boundary curves of the stability domains in a semi-analytical way, as shown by Sinha and Butcher [42, 1997].

Since the 80’s, thanks to the increasing computer capacity, more and more parametrically excited systems became the subject of researchers. Patkó and Kollányi [35, 1999] analyzed the parameter dependence of the amplitude of nonlinear vibrations in belt drives of machine tools, Schneider and Hiller [38, 1996] investigated the control of hydraulically driven vibrating manipulators pumping of wet concrete. Karsai [18, 1996] gave a brief study on the stability of periodically operating systems. Márialigeti [25, 1995], [26, 1996] investigated various vibration problems in gear trains caused by the tooth contacts between cog-wheels.

The main motivation of the present work comes from the oscillations of pipelines conveying fluid. The first observation must have been made long ago of the peculiar spontaneous motions imparted to the free end of a rubber pipe, such as might be used to water the lawn, by a sufficiently high flow rate. Evidently, this was first recognized as a self-excited oscillation by Marcel Brillouin in 1885, but his work on the subject remained unpublished. Bourrières, one of Brillouin’s students, was the first to undertake a serious study on the dynamics of flexible pipes conveying fluid. He examined [6, 1939] the oscillatory instability of cantilever pipes, both theoretically and experimentally. He derived the correct equation of motion and, although unable to obtain analytically the critical flow velocity for the onset of the oscillation, he determined most of the salient characteristics of the phenomenon.

Interest in the subject came into foreground again in connection with the investigation of vibration of the Trans-Arabian pipeline, presented by Ashley and Haviland [1, 1950]. Later, Feodos’ev [8, 1951] derived the equation of motion in full for a pipe conveying fluid, and analyzed the case of the pipe with simply-supported ends. The same problem was studied independently by Housner [14, 1952] using Hamilton’s action function in the derivation of the governing equation. A subsequent, elegant, and more general study by Niordson [29, 1953] led to the same equation of motion and to essentially the same conclusions as Feodos’ev and Housner, i.e. a pipe with simply-supported ends may buckle like a column subjected to axial loading if the flow velocity is sufficiently high.
CHAPTER 1. INTRODUCTION

On the other hand Gregory and Païdoussis [10, 1966] showed theoretically and experimentally that cantilever pipes are subject to oscillatory instabilities (flutter) rather than buckling. This behaviour was fully anticipated by Benjamin [3, 1961], who was concerned with the dynamics of fluid-conveying articulated (rigid) pipes which are connected by flexible joints. In his paper Benjamin also pointed out some wrong steps in the derivation of Housner’s result in connection with the application of Hamilton’s principle in the case of open systems, which was generally discussed by McIver [27, 1973].

Païdoussis and Issid [30, 1974] gives a rather extensive review of the development of the subject since the Trans-Arabian pipeline was observed to vibrate presumably as a result of internal flow. In that paper they also extended the analysis to other boundary conditions such as clamped-clamped ends and they also investigated the case of harmonically perturbed fluid velocity. This investigation, where they used Bolotin’s [5, 1964] harmonic balance method, was extended to deal with both parametric and combination resonances via a Floquet analysis by Païdoussis and Sundararajan [31, 1975] and was confirmed experimentally by Païdoussis and Issid [32, 1976].

Semler et al. [39, 1994] reviewed the next stage of the topic—the efforts that had been devoted to the study of the nonlinear dynamics—and concluded both the Hamiltonian and Newtonian approaches in the derivation of the equations of motion. Among many notable paper investigating the nonlinear behaviour of pipes containing fluid flow, those by Holmes [13, 1978], Rousselet and Herrmann [37, 1981], Bajaj et al. [2, 1980]. Semler and Païdoussis [40, 1995] studied the nonlinear dynamics of a cantilever tubular beam with Kelvin–Voigt damping that is conveying a sinusoidally perturbed fluid flow, both theoretically and experimentally. Among the several standard numerical techniques two methods can be successfully applied for a system with large nonlinear inertial terms: the Finite Difference Method (FDM) based on Houbolt’s scheme investigated by Jones and Lee [15, 1985], and the Incremental Harmonic Balance (IHB) method developed by Lau et al. [22, 1982] to treat strong nonlinearities, and which can also find unstable branches of the solution.

The major part of this dissertation is dedicated to the research of pipes with pulsatile flow. The first aim of the present work is to understand and verify the application of the generalized Hamilton principle in the case of systems with through-flowing media. Subsequently, we derive a discretized form of the equation of motion of elastic pipes with flowing fluid. This form can further be used in a self-developed computer code—based on Sinha’s numerical method—to obtain stability charts and to be able to determine the different nonlinear behaviours in critical cases. In Chapter 2, after introducing the mathematical tools and theories to be applied, we show the applicability of Sinha’s method on Mathieu’s equation. In Chapter 3, we try to give a complete
stability analysis of three different construction of articulated pipes, from the linear autonomous equations to the nonlinear periodic equations of motion. In Chapter 4, we do the same procedure for two constructions containing an elastic pipe. However, the full analysis will be done only in the case of a cantilever pipe which has non-conservative governing equations.

Finally, in Chapter 5, we show another example where the method of Chebyshev polynomials can successfully be applied: the stability investigations of an experimental roller rig. A nonlinear mathematical model of a single railway wheelset was investigated by Lóránt and Stépán [20, 1996] and an experimental rig was also built to verify the theoretical results. However, the constructed experimental setup had some undesired new properties, like the small periodic perturbation on the track gauge due to the finite precision in the manufacturing and mounting of the driving wheels. The aim of the investigation was to find out how the magnitude of the perturbation can influence the critical velocity at different stiffness parameters.
Chapter 2

Mathematical Background

In this chapter, we review the analytical tools and theorems used in the stability investigations of ordinary differential equations. In Section 2.1, we give the Routh–Hurwitz criterion, the definition of centre manifold and the Hopf bifurcation theorem in the case of linear and nonlinear autonomous systems. In Section 2.2, we summarize the most important results of Floquet theory. In Sections 2.3 and 2.4, we review Sinha’s numerical method and its application for Mathieu’s equation. In the latter case, we also present the result of Poincaré’s method for comparison.

2.1 Stability Analysis of Autonomous Systems

Consider the nonlinear system

\[ \dot{x} = f(x), \] (2.1)

where \( f \in C^1_X \) (continuously differentiable) and \( X \subset \mathbb{R}^n \) is an open and connected subset. Let \( x^* \) an \textit{equilibrium point} (i.e. a point solution or fixed point) of (2.1), i.e. \( f(x^*) = 0 \).

\textbf{Definition 1} Let \( y = x - x^* \). If \( x \) is a solution of (2.1), then \( y \) is said to be the variation of the solution with respect to \( x^* \) and satisfies the variational system

\[ \dot{y} = f(x^* + y), \] (2.2)

which has \( y \equiv 0 \) as a constant (or point) solution.

The zero solution of (2.2), clearly, corresponds to the solution \( x^* \) of (2.1). The absolute value of a solution of (2.2) is equal to the absolute value of the difference of the corresponding solution \( x \) and \( x^* \): \( ||y|| = ||x - x^*|| \). Therefore, the stability of the zero solution of (2.2) takes care of the stability of \( x^* \).
Definition 2 The zero solution of (2.2) is said to be stable in the Liapunov sense if
to every \( \varepsilon > 0 \) and \( t_0 \geq 0 \) there belongs a \( \delta > 0 \) such that for all \( \|y(t_0)\| < \delta \) the solution
\( y(t) \) is defined in \( [t_0, \infty) \), and for all \( t > t_0 \)
\[
\|y(t)\| < \varepsilon.
\]

Definition 3 We say that the zero solution of (2.2) is attractive if to every \( t_0 \geq 0 \) there
belongs an \( \eta > 0 \) such that for all \( \|y(t_0)\| < \eta \) implies
\[
\lim_{t \to \infty} \|y(t)\| = 0.
\]

Definition 4 The zero solution of (2.2) is said to be asymptotically stable if it is
stable in the Liapunov sense and it is attractive.

For local stability analysis of the point solution \( \mathbf{x}^* \) of (2.1), we investigate the
linearized variational system
\[
\dot{y} = \frac{\partial f}{\partial x}
|_{\mathbf{x} = \mathbf{x}^*} y. \quad (2.3)
\]
In the case of nonlinear autonomous systems, many equilibria may exist (\( f(\mathbf{x}) = 0 \) can
have not only one solution), and their stability properties may also differ. On the
contrary, a linear autonomous system like (2.3) can have only one (trivial) solution, the
zero solution, and its stability depends on the eigenvalues of the constant coefficient
matrix.

2.1.1 Linear Systems

Theorem 5 The linear autonomous system
\[
\dot{y} = Ay \quad (2.4)
\]
is asymptotically stable if and only if all the eigenvalues \( \lambda_i \) of coefficient matrix \( A \) have
negative real parts. (\( \forall i : \Re \lambda_i < 0 \), i.e. \( \lambda_i \)'s are in the left-hand side of the complex
plane. In this case, the solutions tend to zero exponentially as \( t \) tends to infinity. Therefore,
the trivial solution is "exponentially asymptotically stable").

Theorem 6 The linear system with constant coefficients (2.4) is stable in the Liapunov
sense (but not asymptotically) if and only if all the eigenvalues of the coefficient matrix
\( A \) have non-positive real parts, there are eigenvalues with zero real parts, but their
multiplicity in the minimal polynomial of \( A \) is one.

Theorem 7 If coefficient matrix \( A \) of system (2.4) either has at least one eigenvalue
with positive real part or at least one eigenvalue has a zero real parts, and its multiplicity in the minimal polynomial is greater than one, then the system is unstable.

Thus, we see that the asymptotic stability of system (2.4) is determined by the stability of the matrix $A$, i.e. by the stability of its characteristic polynomial. In the case of $n$ degree-of-freedoms mechanical systems, we often have the equation of motion as a second order system of ordinary differential equations,

$$M\ddot{q} + K\dot{q} + Sq = 0,$$  \hspace{1cm} (2.5)

which can be rewritten in first order form (or Cauchy form) like (2.4). The characteristic polynomial can be calculated in the following way:

$$\text{det}(\lambda^2 M + \lambda K + S) \equiv a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0.$$  \hspace{1cm} (2.6)

The basic (Routh –) Hurwitz criterion gives the relationship between the location of the characteristic roots and the coefficients of the characteristic polynomial.

**Theorem 8** The roots of the polynomial (2.6) are to the left of the imaginary axis (i.e. they have negative real parts) if and only if

$$a_0 > 0, \quad H_i > 0 \; (i = 1 \cdots 2n),$$

where $H_i$-s are the leading principal minors of the Hurwitz matrix, i.e.

$$H_i = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 & 0 \\ a_5 & a_4 & a_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{i-2} \\ a_{2i-1} & a_{2i-2} & \cdots & a_{i+1} & a_i \end{vmatrix}$$

and $a_k = 0$ if $k > 2n$.

### 2.1.2 Nonlinear Systems

If the equilibrium point of the system (2.1) is critical (i.e. stable only in the Liapunov sense) then its nonlinear part in the critical subspace (in a stable centre manifold) determines the stability. In the following, we investigate the zero solution of the variational system (2.2).
CHAPTER 2. MATHEMATICAL BACKGROUND

Centre Manifold

Consider the Jordan normal form of the third order approximation of (2.2)

\[
\begin{bmatrix}
\dot{y}_c \\
\dot{y}_s
\end{bmatrix} =
\begin{bmatrix}
J_c & 0 \\
0 & J_s
\end{bmatrix}
\begin{bmatrix}
y_c \\
y_s
\end{bmatrix} +
\begin{bmatrix}
f_{c2}(y_c, y_s) + f_{c3}(y_c, y_s) \\
f_{s2}(y_c, y_s) + f_{s3}(y_c, y_s)
\end{bmatrix},
\] (2.7)

where \( J_c \) and \( J_s \) are the critical (\( \Re \lambda_i = 0 \)) and stable (\( \Re \lambda_i < 0 \)) Jordan blocks, respectively. The functions \( f_{c2} \) and \( f_{s3} \) contain only second and third order terms, respectively.

The centre manifold (CM) can be approximated by the equation

\[
y_s = h_2(y_c),
\] (2.8)

which can be determined from the stable part of Eq. (2.7). Let us derivate (2.8):

\[
\dot{y}_s = \frac{\partial h_2}{\partial y_c} \dot{y}_c,
\]

and substitute the corresponding right-hand side of (2.7) into the derivatives \( \dot{y}_s \) and \( \dot{y}_c \):

\[
J_s y_s + f_{c2}(y_c, y_s) + O(|y|^3) = \frac{\partial h_2}{\partial y_c} (J_c y_c + O(|y|^2)),
\]

and finally, neglect the third order terms:

\[
J_s h_2(y_c) - \frac{\partial h_2}{\partial y_c} J_c y_c \approx -f_{c2}(y_c, y_s).
\] (2.9)

Two generic critical cases can occur: when there is a single critical zero eigenvalue and when there are two complex conjugate critical eigenvalues on the imaginary axis.

1. \( J_c \equiv \lambda_c = 0 : f_{c2}(y_t, y_s) = f_{c2}^{(20)} y_1^2 + \cdots \), CM: \( y_s = h_2 y_1^2 \). Substituting these into Eq. (2.9) yields

\[
J_s h_2 = -f_{c2}^{(20)}.
\]

2. \( J_c \equiv \left[
\begin{array}{cc}
0 & \alpha \\
-\alpha & 0
\end{array}\right] : f_{c2}(y_c, y_s) = f_{c2}^{(20)} y_1^2 + f_{c2}^{(11)} y_1 y_2 + f_{c2}^{(02)} y_2^2 + \cdots \) and the CM:

\[
y_s = h_2(y_c) \equiv h_2^{(20)} y_1^2 + h_2^{(11)} y_1 y_2 + h_2^{(02)} y_2^2,
\]

and its gradient:

\[
\frac{\partial h_2}{\partial y_c} = \left(2h_2^{(20)} y_1 + h_2^{(11)} y_2, h_2^{(11)} y_1 + 2h_2^{(02)} y_2\right).
\]
Putting it into Eq. (2.9), we obtain
\[
\begin{bmatrix}
J_s & \alpha I & 0 \\
-2\alpha I & J_s & 2\alpha I \\
0 & -\alpha I & J_s
\end{bmatrix}
\begin{bmatrix}
h_2^{(20)} \\
h_2^{(11)} \\
h_2^{(02)}
\end{bmatrix}
= \begin{bmatrix}
-f_2^{(20)} \\
f_2^{(11)} \\
-f_2^{(02)}
\end{bmatrix}.
\]

Now, we can substitute Eq. (2.8) into the critical part of (2.7):
\[
\dot{y}_c = J_c y_c + f_2(y_c, h_2(y_c)) + f_3(y_c, h_2(y_c)).
\tag{2.10}
\]
One can also notice that if \(f_2(y_c, 0) \equiv 0\) then the CM is \(y_s = h_2(y_c) \equiv 0\).

**Poincaré–Andronov–Hopf Bifurcation Theorem**

If the matrix \(J_c\) has a pair of pure imaginary roots then we have to apply a near identity transformation
\[
y = v + g(v) \equiv v + \sum_{i=0}^{2} g^{(2i)} v_1^{2-i} v_2^i + \sum_{i=0}^{3} g^{(3i)} v_1^{3-i} v_2^i
\]
with appropriate values of \(g\)-s which carries the simplified form of Eq. (2.10)
\[
\dot{y} = J_c y + \sum_{i=0}^{2} f^{(2i)} y_1^{2-i} y_2^i + \sum_{i=0}^{3} f^{(3i)} y_1^{3-i} y_2^i
\]
into third order normal form:
\[
\dot{v} = \begin{bmatrix}
0 & \alpha \\
-\alpha & 0
\end{bmatrix} v + v^2 \begin{bmatrix}
\delta & \beta \\
-\beta & \delta
\end{bmatrix} v.
\]
Its polar coordinate form is
\[
\dot{r} = \delta r^3,
\]
\[
\dot{\theta} = \alpha + \beta r^2,
\]
where \(r^2 = v^2\) and \(\tan \theta = v_2/v_1\). That is, the centre manifold is a vague attractor or a vague repeller if the Poincaré–Liapunov constant \(\delta < 0\) or \(\delta > 0\), respectively.

The value of \(\delta\) can be calculated as
\[
\frac{1}{\delta} \left\{ \frac{1}{\alpha} \left( f_2^{(11)} (f_2^{(20)} + f_2^{(02)}) + 2 \left( f_2^{(20)} f_1^{(20)} - f_2^{(02)} f_1^{(02)} \right) - f_1^{(11)} \left( f_1^{(20)} + f_1^{(02)} \right) \right)
+ 3 f_1^{(30)} + f_1^{(12)} + f_2^{(21)} + 3 f_2^{(03)} \right\}.
\]
CHAPTER 2. MATHEMATICAL BACKGROUND

Furthermore, the sign of $\delta$ determines the stability of the limit cycle that bifurcates from the trivial equilibrium in the generic case, i.e.

- $\delta < 0$ refers to super-critical bifurcation where a stable limit cycle exists around the unstable fixed point;
- $\delta > 0$ refers to sub-critical bifurcation with unstable limit cycle.

2.2 Stability Analysis of Periodic Systems

2.2.1 Floquet Theory of Linear Periodic Systems

Let us investigate the periodic homogeneous linear system given in the following form:

$$\dot{x} = A(t)x,$$  \hspace{1cm} (2.11)

where $A \in C_{[0,\infty)}$ and its principal period is $T > 0$: $A(t + T) = A(t)$, $t \in \mathbb{R}^+$.

**Definition 9** $\Phi(t)$ is the State Transition Matrix (or fundamental matrix solution) of (2.11) $\iff$ all of its solutions can be given as $\varphi(t) = \Phi(t)c$, where $c$ is an appropriate vector depending on the initial conditions.

In that case, $\Phi(t + T) = \Phi(t)C$ is a State Transition Matrix, too (for proof, see [7]). In the following, $\Phi(t)$ will denote such a State Transition Matrix where $\Phi(0) = I$, the identity. Hence, $\varphi(t) = \Phi(t)\varphi_0$.

**Definition 10** Let $C = \Phi^{-1}(t)\Phi(t + T)$, and because it is constant, it can be arbitrary, e.g. $t = 0$. Thus, $C = \Phi(T)$ which is the Floquet Transition Matrix (or principal matrix) of (2.11).

**Definition 11** $(\lambda_1, \ldots, \lambda_n)$, the eigenvalues of $C$, are the characteristic multipliers of (2.11).

**Theorem 12** If $\lambda$ is a characteristic multiplier then there exists $\exists \varphi(t)$ non-trivial solution of (2.11) such that $\varphi(t + T) = \lambda \varphi(t)$.

**Theorem 13** If $\varphi(t)$ is a non-trivial solution of (2.11) and $\varphi(T) = \lambda \varphi(0)$ then $\lambda$ is a characteristic multiplier with eigenvector $\varphi(0)$.

**Corollary 14** $\exists T (2T)$ periodic non-trivial solution of (2.11) $\iff \lambda = 1 (\lambda = -1)$

**Theorem 15** If $\exists k$ linearly independent $T$-periodic solution of (2.11) then the multiplicity of $\lambda = 1$ is at least $k$. 
Theorem 16 The State Transition Matrix of (2.11) can be constructed in the following way:
\[ \Phi(t) = P(t) \exp(tB), \]
where \( P(t + T) = P(t) \) and \( \exp(TB) = C \).

Definition 17 \((\nu_1, \ldots, \nu_n)\), the eigenvalues of \( B \), are the characteristic exponents of (2.11): \( \lambda_i = \exp(\nu_iT) \).

The following theorem shows how the linear system (2.11) with non-constant coefficients can be transformed to a linear system with constant coefficients.

Theorem 18 (Liapunov) Let \( P(t) = \Phi(t)e^{-tB} \) \( P \in C^1 \), regular, \( T \)-periodic, \( P(0) = I \), then \( x = P(t)z \) transforms (2.11) to
\[ \dot{z} = Bz, \]
where \( C = \exp(TB) \) is the Floquet Transition Matrix of (2.11).

Proof. Let us substitute \( x = P(t)z \) in (2.11):
\[ \dot{P}z + P\dot{z} = APz \Rightarrow \dot{z} = P^{-1}(AP - \dot{P})z \]
and \( P^{-1}(AP - \dot{P}) \equiv e^{tB}P^{-1}(A\Phi - \Phi \dot{B})e^{-tB} \equiv e^{tB}Be^{-tB} \equiv B \) since \( \Phi = A\Phi \) and \( e^{tB}B \equiv Be^{tB} \).

Stability of Linear Periodic Systems

A homogeneous linear system like (2.11) has only one fixed point which can be stable or unstable (attractor or repeller). Thus, if a solution of the linear system (2.11) is asymptotically stable then every solution is stable. This also corresponds to the stability in the sense of Liapunov and to the instability.

Theorem 19 The stability of the system (2.11) is determined by its characteristic multipliers:

- \( \forall |\lambda_j| < 1 \iff (2.11) \) is asymptotically stable,
- \( \forall |\lambda_j| \leq 1 \) and the multiplicity of \( |\lambda_k| = 1 \) is 1 in the minimal polynomial \( \iff (2.11) \) is stable in the Liapunov sense,
- \( \exists |\lambda_j| > 1 \Rightarrow (2.11) \) is unstable.
2.2.2 Hill’s Equation

As an application of the previous results, let us investigate the following second order differential equation:

\[ \ddot{y} + p(t)y = 0, \quad (2.12) \]

where \( p \in C^0 \), \( p(t + T) = p(t) \) and \( T > 0 \).

Equation (2.12) can be rewritten in the form of (2.11) called Cauchy form. Thus, the coefficient matrix of this form will be

\[
A(t) = \begin{bmatrix}
0 & 1 \\
-p(t) & 0
\end{bmatrix}.
\]

Liouville’s theorem give us the relation between this coefficient matrix and the determinant of the Floquet Transition Matrix.

**Theorem 20** (Liouville) \( \det C = \exp \int_0^T \text{tr} A(\tau) \, d\tau \)

Since the trace of the coefficient matrix \( A \) given above is zero, the determinant of the Floquet Transition Matrix of (2.12) is \( \det C = 1 \). Thus, the characteristic polynomial of \( C \) is

\[ \det(C - \lambda I) = \lambda^2 - \lambda \text{tr} C + \det C = \lambda^2 - b\lambda + 1, \]

and the eigenvalues of \( C \) are

\[ \lambda_{1,2} = \frac{b \pm \sqrt{b^2 - 4}}{2}. \]

Thus, if

- \( |\text{tr } C| > 2 \Rightarrow \forall \lambda_j : |\lambda_j| > 1 \), i.e. (2.12) is unstable,
- \( |\text{tr } C| < 2 \Rightarrow \forall \lambda_j : |\lambda_j| = 1 \), i.e. (2.12) is stable in Liapunov sense.

The same result can be obtained from Vieta’s formula, which gives the relation between the roots of a polynomial of second degree and its coefficients:

\[ (\det C =) 1 = \lambda_1 \lambda_2 \Rightarrow \text{either } |\lambda_1| = |\lambda_2| = 1 \text{ or } \exists \lambda_j : |\lambda_j| > 1, \]

that is, (2.12) cannot be asymptotically stable.

Four cases can be separated with respect to the stability:

- \(-2 < \text{tr } C < 2 : |\Re \lambda_j| < 1\), the system is stable in Liapunov sense with non-\( T \)-periodic solutions (\( \mathcal{S} \));
- \( \text{tr } C = 2 : \lambda_{1,2} = 1 \), the system is stable in Liapunov sense with \( T \)-periodic solutions (\( \mathcal{H}_1 \));
\[ \text{CHAPTER 2. MATHEMATICAL BACKGROUND} \]

- \( \text{tr} \mathbf{C} = -2 : \lambda_{1,2} = -1 \), the system is stable in Liapunov sense with \( 2T \)-periodic solutions \((\mathcal{H}_{-1})\);

- \( |\text{tr} \mathbf{C}| > 2 : \exists |\lambda_j| > 1 \), the system is unstable \((\mathcal{U})\).

If we assume that small changes in \( p(t) \) result small changes in the coefficients (i.e. in \( \text{tr} \mathbf{C} \)) of the characteristic polynomial of \( \mathbf{C} \) then it is obvious that a domain of unstable systems \((\mathcal{U})\) in a so-called stability chart can only be bounded by such curves where the type of stability of (2.12) is either \((\mathcal{H}_1)\) or \((\mathcal{H}_{-1})\), i.e. \( |\text{tr} \mathbf{C}| = 2 \).

### 2.2.3 Nonlinear Periodic Systems

Now, we investigate the following nonlinear periodic system

\[
\dot{x} = \mathbf{A}(t)x + \begin{bmatrix} 0 \\ f(t,x) \end{bmatrix}
\]  

(2.13)

as described in [34], where \( \mathbf{A}(t + T) = \mathbf{A}(t) \) and \( f(t + T, x) = f(t, x) \). Let us define the \( 2T \)-periodic Liapunov–Floquet transformation as

\[
\mathbf{P}(t) = \Phi(t)e^{-t\mathbf{R}},
\]

where \( \Phi(t) \) is the State Transition Matrix and \( \exp(2T\mathbf{R}) = \mathbf{C}^2 \). Substitution of \( x = \mathbf{P}(t)\mathbf{z} \) in Eq. (2.13) results a constant coefficient matrix in the linear term:

\[
\dot{\mathbf{z}} = T^{-1}\mathbf{R}\mathbf{z} + T^{-1}\mathbf{P}^{-1}(t) \begin{bmatrix} 0 \\ f(t, \mathbf{P}(t)\mathbf{z}) \end{bmatrix}.
\]

If \( \mathbf{T} \) is the matrix of eigenvectors of \( \mathbf{R} \) then the coefficient matrix of the linear term is in Jordan form where the stable and unstable manifolds can be separated. As \( \mathbf{P}(t) \) is a periodic function of \( t \) with principal period \( 2T \), it can be expanded in Fourier series. Substituting this expansion in the nonlinear term and doing some trigonometrical simplifications we can easily separate the constants and the time-dependent terms. Taking into account only the time-independent terms we can go on with the analysis as in autonomous case: centre manifold reduction, etc. (see Section 2.1.2). After this procedure, we shall know the nonlinear behaviour of the time-periodic system in the vicinity of critical parameters, namely, whether there can be super- or sub-critical non-trivial solutions.
2.3 Numerical Method of Chebyshev Polynomials

In order to perform the (linear) stability analysis of a periodic system, we need the Floquet Transition Matrix $C$ which can be constructed from the known independent solutions of (2.11). However, these solutions generally cannot be obtained analytically: one has to apply some kind of numerical technique to get an approximated solution. According to the idea given by Sinha and Wu [41], the properties of Chebyshev polynomials described in Appendix A can be applied efficiently to obtain good numerical approximation of the solutions of non-autonomous ordinary differential equations with high degree of freedom.

We consider a second order system of differential equations of the following type:

$$ M_0 \ddot{q} + (K_0 + K(t)) \dot{q} + (S_0 + S(t)) q = 0 $$

or using Einstein’s convention:

$$ m^{ij}_0 \ddot{q}^j + (k^{ij}_0 + k^{ij}(t)) \dot{q}^j + (s^{ij}_0 + s^{ij}(t)) q^j = 0. \quad (2.14) $$

2.3.1 Deriving The Linear Algebraic Equations

Let us integrate Eq. (2.14) with respect to time $t$ in the time-interval $[0, t]$:

$$ m^{ij}_0 (\dot{q}^j - \dot{q}^j(0)) + k^{ij}_0 (q^j - q^j(0)) + \int_0^t k^{ij}(\tau) \dot{q}^j d\tau + s^{ij}_0 \int_0^t q^j d\tau + \int_0^t s^{ij}(\tau) q^j d\tau = 0, $$

where $\int_0^t k^{ij}(\tau) \dot{q}^j d\tau = [k^{ij}q^j]_0^t - \int_0^t \dot{k}^{ij} q^j d\tau$. Substituting it back and integrating again, we get

$$ m^{ij}_0 (q^j - q^j(0)) + k^{ij}_0 \int_0^t \dot{q}^j d\tau + \int_0^t \dot{k}^{ij} q^j d\tau - \int_0^t \ddot{k}^{ij} q^j d\tau + s^{ij}_0 \int_0^t q^j d\tau + \int_0^t s^{ij}(\tau) q^j d\tau = (m^{ij}_0 \dot{q}^j(0) + (k^{ij}_0 + k^{ij}(0)) q^j(0)) t. $$

Applying the operators of Chebyshev polynomials:

$$ T_r(t) M^{ij}_{orp} (q^j_{Q} - q^j_Q(0)) + T_r(t) G_{rs} K^{ij}_{orp} q^j_r - T_r(t) G_{rs} K^{ij}_{orp} \dot{q}^j_r - T_r(t) G_{rs} \tilde{K}^{ij}_{orp} \ddot{q}^j_r $$

$$ + T_r(t) G_{rs} G_{qs} S^{ij}_{orp} q^j_s + T_r(t) G_{rs} G_{qs} \tilde{S}^{ij}_{orp} \dot{q}^j_s = T_r(t) G_{rs} (M^{ij}_{orp} \dot{q}^j_r + (K^{ij}_{orp} + \tilde{K}^{ij}_{orp}(0)) \dot{q}^j_r(Q)), $$

where $M^{ij}_{orp} = m^{ij}_0 \delta_{rp}, K^{ij}_{orp} = k^{ij}_0 \delta_{rp}, S^{ij}_{orp} = s^{ij}_0 \delta_{rp}, q^j(0) = q^j(0) \delta_p$, and $\delta_p$ is the Kronecker symbol. Furthermore, $\dot{q}^j(t) = T_r(t) \ddot{q}^j_r$ and $\dddot{q}^j(t) = T_r(t) \dddot{q}^j_r$, $\tilde{K}^{ij}_{orp} = Q_{orp} \tilde{K}^{ij}_{q}$. 

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This yields the following system of linear algebraic equations for \( q^j(t) \approx T_p(t)q^j_p \):

\[
\begin{aligned}
(m_0^{ij} \delta_{rp} + G_{rs} (k_0^{ij} \delta_{sp} + K_{sp}^{ij}) + G_{qs} G_{qs} \left(-\tilde{K}^{ij}_{ap} + s_{0}^{ij} \delta_{sp} + S_{sp}^{ij}\right)) q^j_p \\
= (m_0^{ij} \delta_{rp} + (k_0^{ij} + k^{ij}(0)) G_{rp}) q^j_p(0) + m_0^{ij} G_{rp} \tilde{q}^j_p(0)
\end{aligned}
\]  

(2.15)

And finally, knowing \( \tilde{q}^j_p \), one can obtain its velocity \( \dot{q}^j_p \), as well:

\[
\begin{aligned}
m_0^{ij} \dot{q}^j + \int_0^t (k_0^{ij} + k^{ij}(\tau)) \dot{q}^j \, d\tau = m_0^{ij} \dot{q}^j(0) - \int_0^t (s_0^{ij} + s^{ij}(\tau)) q^j \, d\tau \\
(m_0^{ij} \delta_{rp} + G_{rs} (k_0^{ij} \delta_{sp} + K_{sp}^{ij})) \tilde{q}^j_p = m_0^{ij} \tilde{q}^j_p(0) - G_{rs} (s_0^{ij} \delta_{sp} + S_{sp}^{ij}) q^j_p.
\end{aligned}
\]  

(2.16)

Applying different initial values for \( q(0) \) and \( \tilde{q}(0) \), the State Transition Matrix \( \Phi(t) \) can be constructed. In case of periodic coefficients, Floquet theory needs the solution at the end of the principal period, where \( T_p(1) = 1 \). This is equivalent with the sum of the Chebyshev coefficients:

\[
C = \begin{bmatrix} q^{ij,k}(T) \\ \tilde{q}^{ij,k}(T) \end{bmatrix} = \sum_{p=0}^{m-1} q_p^{kj,0} \left[ \sum_{p=0}^{m-1} \tilde{q}_p^{kj,0} \right],
\]

where \( j = 1, \ldots, n \) spans the degrees of freedom of the system and \( (q_p^{jk}, q_{\tilde{p}}^{jk}) \) are obtained using the \( k \)th initial condition: \( q_p^{jk}(0) = \delta_p 1 \delta_{jk} \) and \( \tilde{q}_{\tilde{p}}^{jk}(0) = \delta_{\tilde{p}} 1 \delta_{jk} \), where \( \delta_{jk} = 1 \iff j = k \) and \( k = 1, \ldots, 2n \).

2.3.2 Fourier Expansion of Coefficient Matrices

Since we need the solution in the closed interval \([0, T]\), we can introduce the transformation \( t = T\tau \) and create the Fourier expansion of \( K(\tau) \) and \( S(\tau) \) of (2.14):

\[
\begin{aligned}
K(\tau) &= K_1 \cos 2\pi \tau + K_2 \sin 2\pi \tau + K_3 \cos 4\pi \tau + K_4 \sin 4\pi \tau + \cdots \\
\dot{K}(\tau) &= (-K_1 \sin 2\pi \tau + K_2 \cos 2\pi \tau - 2K_3 \sin 4\pi \tau + 2K_4 \cos 4\pi \tau + \cdots) 2\pi \\
S(\tau) &= S_1 \cos 2\pi \tau + S_2 \sin 2\pi \tau + S_3 \cos 4\pi \tau + S_4 \sin 4\pi \tau + \cdots
\end{aligned}
\]

Thus,

\[
\begin{aligned}
K_{sp}^{ij} &= Q_{sr} \left( q_1^{ij} \gamma_1 + q_2^{ij} \sigma_1 + q_3^{ij} \gamma_2 + q_4^{ij} \sigma_2 + \cdots \right) \\
-\tilde{K}_{sp}^{ij} &= Q_{sr} \left(-q_2^{ij} \gamma_1 + q_1^{ij} \sigma_1 - 2q_4^{ij} \gamma_2 + 2q_3^{ij} \sigma_2 + \cdots \right) 2\pi \\
S_{sp}^{ij} &= Q_{sr} \left(q_1^{ij} \gamma_1 + q_2^{ij} \sigma_1 + q_3^{ij} \gamma_2 + q_4^{ij} \sigma_2 + \cdots \right)
\end{aligned}
\]
Table 2.1: Coefficients of Chebyshev expansions of cosine and sine.

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<th>$\cos 4\pi r \gamma_r^k$</th>
<th>$\sin 2\pi r \sigma_r^k$</th>
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</table>

where $\cos 2k\pi r = T_r(\tau) \gamma_r^k$ and $\sin 2k\pi r = T_r(\tau) \sigma_r^k$, i.e.

$$
\begin{align*}
\frac{\gamma_r^k}{\sigma_r^k} &= \frac{1}{\delta_r} \int_0^1 \frac{T_r(\tau) w(\tau)}{\sqrt{\tau - r^2}} \left\{ \frac{\cos 2k\pi r}{\sin 2k\pi r} \right\} \cos 2k\pi k \pi r \left\{ (-1)^{k+r} \right\} \cos 2k\pi r \left\{ (-1)^{k+r} \right\} \sin 2k\pi r \left\{ (-1)^{k+r} \right\} \sin 2k\pi r \left\{ (-1)^{k+r} \right\} \\
&= 2J_r(k\pi) \left\{ (-1)^{k+r} \right\} \cos 2k\pi r \left\{ (-1)^{k+r} \right\} \sin 2k\pi r \left\{ (-1)^{k+r} \right\}
\end{align*}
$$

and $J_r(x)$ is the $r^{th}$ order Bessel function of the first kind. Furthermore, $\gamma_{2i+1}^k = \sigma_{2i}^k = 0 \mid i \in \mathbb{N}.$

The coefficients of Chebyshev expansion of cosine and sine can be seen in Table 2.1

### 2.3.3 Error of The Method

In the case of a numerical method, it is important to estimate the possible error. At the computational results presented in this dissertation, the following formula was used:

$$
\max_{j,k} \left( \frac{\sum_{p=0}^{n-1} |q_p^{j,k}(m) - \tilde{q}_p^{j,k}(m-1)|}{\sum_{p=0}^{n-1} |q_p^{j,k}(m)|}, \frac{\sum_{p=0}^{n-1} |\tilde{q}_p^{j,k}(m) - \tilde{q}_p^{j,k}(m-1)|}{\sum_{p=0}^{n-1} |\tilde{q}_p^{j,k}(m)|} \right),
$$

(2.17)
where \( q_{j}^{p}(m) \) is the coefficient of the \( p \)th shifted Chebyshev polynomial of the \( j \)th component of the solution obtained by using the \( k \)th initial condition and \( m \) Chebyshev polynomials. The expression in (2.17) first determines the relative error in each component (given by \( j, k \)) of the State Transition Matrix, and then takes the greatest of these values.

### 2.4 Mathieu’s Equation

Let us investigate Mathieu’s well-known equation which is the special case of Hill’s equation with \( p(t) = \delta - 2\varepsilon \cos 2t \) and \( T = \pi \) as principal time-period:

\[
\ddot{y} + (\delta - 2\varepsilon \cos 2t)y = 0, \tag{2.18}
\]

where parameters \( \delta \) and \( \varepsilon \) determine the stability of the system.

Representing the stable systems in the \((\delta, \varepsilon)\) plane, we obtain Ince–Strutt’s stability chart. Now, we look for the \( \delta(\varepsilon) \) limit curves where only \( \pi \) and \( 2\pi \) periodic solutions exist, respectively.

#### 2.4.1 Points of Axis \( \delta \)

At first, let us see where the limit curves intersect the axis \( \delta \):

\[
\varepsilon = 0 : \quad \ddot{y} + \delta y = 0.
\]

It is obvious that for any \( \delta < 0 \) the fixed point \( y = 0 \) is unstable (\( U \)):

\[
y(t) = C_{1}e^{\sqrt{-\delta}t} + C_{2}e^{-\sqrt{-\delta}t} \quad t \to \infty \quad \infty.
\]

In case of \( \delta > 0 \), we have periodic solutions (which are stable in Liapunov sense):

\[
y(t) = C_{1}\cos(t\sqrt{\delta}) + C_{2}\sin(t\sqrt{\delta})
\]

\[
\dot{y}(t) = -C_{1}\sqrt{\delta}\sin(t\sqrt{\delta}) + C_{2}\sqrt{\delta}\cos(t\sqrt{\delta})
\]

Let us construct the State Transition Matrix that satisfies the conditions described in Section 2.2:

\[
\Phi(t) = \begin{bmatrix}
\cos(t\sqrt{\delta}) & \frac{1}{\sqrt{\delta}}\sin(t\sqrt{\delta}) \\
\sqrt{\delta}\sin(t\sqrt{\delta}) & \cos(t\sqrt{\delta})
\end{bmatrix}.
\]

Now, \( \Phi(0) = I \), \( C = \Phi(T) \) and the characteristic multipliers can be obtained in the
following way:

\[
\det(C - \lambda I) \equiv \lambda^2 - 2\lambda \cos \pi \sqrt{\delta} + 1 = 0 \implies \lambda_{1,2} = e^{\pm i \sqrt{\delta}} \quad (|\lambda_{1,2}| = 1).
\]

This means that the points on the positive \(\delta\) axis can be classified as:

- \((\mathcal{H}_1)\) \(T\)-periodic solutions: \(\lambda_{1,2} = 1 \iff \delta = \delta_0 = 0, 4, 16, \ldots, (2k)^2\)
- \((\mathcal{H}_2)\) \(2T\)-periodic solutions: \(\lambda_{1,2} = -1 \iff \delta = \delta_0 = 1, 9, 25, \ldots, (2k + 1)^2\)
- \((\mathcal{S})\) non-\(T\)-periodic solutions: \(\lambda_{1,2} \notin \mathbb{R} \iff \delta \neq k^2\)

### 2.4.2 Poincaré’s Idea

We expand the \(\delta(\varepsilon)\) graph of the limit curves and the solutions as the series of \(\varepsilon:\)

\[
\delta(\varepsilon) = \delta_0 + \delta_1 \varepsilon + \delta_2 \varepsilon^2 + \cdots
\]

\[
y(t, \varepsilon) = y_0(t) + y_1(t) \varepsilon + y_2(t) \varepsilon^2 + \cdots
\]

where \(y_j(t + T) = y_j(t)\) are \(T\)-periodic functions \(\forall j\) and

\[
\begin{bmatrix}
y_0(0) \\
y_1(0)
\end{bmatrix} \in \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad
\begin{bmatrix}
y_j(0) \\
y_j(0)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

We obtain the functions \(y_j(t)\) substituting the formula of \(\delta(\varepsilon)\) and \(y(t, \varepsilon)\) above in Mathieu’s equation (2.18) and separating the coefficients of \(\varepsilon^j:\)

\[
\ddot{y}_0(t) + \ddot{y}_1(t) \varepsilon + \ddot{y}_2(t) \varepsilon^2 + \cdots + (\delta_0 + \delta_1 \varepsilon + \delta_2 \varepsilon^2 + \cdots - 2 \varepsilon \cos 2t)(y_0(t) + y_1(t) \varepsilon + y_2(t) \varepsilon^2 + \cdots) = 0.
\]

**First curve** \((k = 0)\): \(\delta_0 = 0\)

\[
\varepsilon^0: \quad \ddot{y}_0 + \delta_0 y_0 = 0 \quad \Rightarrow \quad y_0(t) = 1
\]

\[
\varepsilon^1: \quad \ddot{y}_1 + \delta_0 y_1 + \delta_1 y_0 - 2y_0 \cos 2t = 0 \quad \Rightarrow \quad \delta_1 = 0, \quad y_1(t) = \frac{1}{2}(1 - \cos 2t)
\]

\[
\varepsilon^2: \quad \ddot{y}_2 + \delta_0 y_2 + (\delta_1 - 2 \cos 2t) y_1 + \delta_2 y_0 = 0 \quad \Rightarrow \quad \delta_2 = -\frac{1}{2}, \quad y_2(t) = \frac{1}{32}(7 - 8 \cos 2t + \cos 4t)
\]

\[
\varepsilon^3: \quad \delta_3 = 0
\]

\[
\varepsilon^4: \quad \delta_4 = \frac{7}{128}
\]

The values of \(\delta_j\) are obtained from the assumption that \(y_j(t)\) must be periodic (the homogeneous part cannot be resonant). Thus, the the limit curve is

\[
\delta(\varepsilon) = -\frac{1}{2} \varepsilon^2 + \frac{7}{128} \varepsilon^4 - \frac{29}{2304} \varepsilon^6 + O(\varepsilon^8).
\]
The second (pair of) curve(s) \((k = 1)\) : \(\delta_0 = 1\)

\[ \varepsilon^0: \ 0 + y_0 = 0 \Rightarrow \]

- \(y_0(t) = \cos t\)
  
  \[ \varepsilon^1: \ \dot{y}_1 + y_1 = -\delta_1 \cos t + 2 \cos t \cos 2t \Rightarrow \delta_1 = 1 \]
  
  \[ \varepsilon^2: \ \dot{y}_2 + y_2 = -\delta_2 \cos t - (1 - 2 \cos 2t)(\cos t - \cos 3t) \Rightarrow \delta_2 = -\frac{3}{8} \]
  
  \[ \varepsilon^3: \ \delta_3 = -\frac{4}{64} \]
  
  \[ \varepsilon^4: \ \delta_4 = -\frac{1}{1536} \]

- \(y_0(t) = \sin t\)

  \[ \varepsilon^1: \ \dot{y}_1 + y_1 = -\delta_1 \sin t + 2 \sin t \cos 2t \Rightarrow \delta_1 = -1 \]
  
  \[ \varepsilon^2: \ \dot{y}_2 + y_2 = -\delta_2 \sin t + (1 + 2 \cos 2t)(3 \sin t - \sin 3t) \Rightarrow \delta_2 = -\frac{1}{8} \]
  
  \[ \varepsilon^3: \ \delta_3 = \frac{1}{64} \]
  
  \[ \varepsilon^4: \ \delta_4 = -\frac{1}{1536} \]

The two limit curves starting at \((1,0)\) are

\[ \delta(\varepsilon) = 1 \pm \varepsilon - \frac{1}{8} \varepsilon^2 + \frac{1}{64} \varepsilon^3 - \frac{1}{1536} \varepsilon^4 + \frac{11}{36864} \varepsilon^5 + \frac{49}{589824} \varepsilon^6 + O(\varepsilon^7). \]

The third pair of curves \((k = 2)\) : \(\delta_0 = 4\)  

The third pair of curves can be obtained similar as the previous limit curves:

\[
\delta(\varepsilon) = 4 + \frac{5}{12} \varepsilon^2 - \frac{763}{13824} \varepsilon^4 + \frac{1002401}{79626240} \varepsilon^6 + O(\varepsilon^8), \\
\delta(\varepsilon) = 4 - \frac{1}{12} \varepsilon^2 + \frac{5}{13824} \varepsilon^4 - \frac{289}{79626240} \varepsilon^6 + O(\varepsilon^8).
\]

The graphs of the curves are shown in Figure 2.1. It can be seen that some of the boundary curves (e.g. \(\delta_0 = 0\) and the second branch of \(\delta_0 = 4\)) become unreliable when \(\varepsilon > 1\). Unfortunately, this situation does not change significantly even if we determine more coefficients in the approximation of \(\delta(\varepsilon)\).

2.4.3 Numerically Computed Stability Charts

Figure 2.2(a) shows the results of numerical computations in which Hindmarsh’s [12] ODE solver LSODE was used at different parameter values of \(\delta\) and \(\varepsilon\). The initial grid was \(151 \times 51\) in the range \((\delta, \varepsilon) \in [-5,10] \times [0,5]\) and the final step size was 0.00625 after refining the boundary curves.
Figure 2.1: Ince–Strutt’s stability chart obtained by Poincaré’s method.

<table>
<thead>
<tr>
<th>Numerical method:</th>
<th>LSODE</th>
<th>Chebyshev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of points</td>
<td>39531</td>
<td>39541</td>
</tr>
<tr>
<td>CPU time</td>
<td>2h 27’15”</td>
<td>10’38”</td>
</tr>
</tbody>
</table>

Table 2.2: Computational statistics of the applied numerical methods.

Similar result, shown in Figure 2.2(b), was obtained using the method based on 18 Chebyshev polynomials. Now, equations (2.15) and (2.16) have the following forms $(m_0 = 1, s_0 = \delta \pi^2, s_1 = -2\varepsilon \pi^2)$:

\[
(I_{rp} + G_{rQ}G_{qs} \left( \delta I_{sp} - 2\varepsilon Q_{sP} \gamma_{s}^1 \right) \pi^2) q_p = I_{rp}q_p(0) + G_{rP}q_p(0),
\]

\[
\bar{q}_r = \bar{q}_r(0) - G_{rs} \left( \delta I_{sp} - 2\varepsilon Q_{sP} \gamma_{s}^1 \right) \pi^2 q_p.
\]

The computations were performed on a Celeron 450 MHz processor under Linux 2.2 operating system using the free octave-2.0.16-2 package of Debian distribution. The difference between CPU times can be seen in Table 2.2, which clearly shows the power of Sinhas’ method. Moreover, there cannot be seen any significant differences in the stability charts in the investigated range of $\delta$ and $\varepsilon$. 
Figure 2.2: Boundaries of stability domains of Mathieu’s equation.
Chapter 3

Joint Articulated Pipes

In this chapter we investigate three models of articulated pipes, which are similar to Benjamin’s model (see [3]). However, we do not take into consideration the gravity but do assume that the pipes contain pulsatile flow. We perform an extensive stability analysis to see how the parameters of this pulsatile flow affect the stability of the system.

In the considered ‘pipe-chain’, $i$ pieces of rigid pipes are attached together with resilient joints and the first one is attached to the wall in the same manner. The length of the $i^{th}$ pipes is $l_i$ and the components of the $q = [q_i]$ generalized coordinate vector are the $\varphi_i$ angles between the axis of the $i^{th}$ pipe and the axis $x$. The bending stiffnesses of the joints have the magnitude $s_i$.

The mass per unit length of the pipe and the fluid are $M$ and $m$, respectively. All motion is assumed to occur in a horizontal plane. The fluid with upstream mass-flow $mu(t)$ is modelled as a frictionless incompressible medium and the flow velocity is generally considered to be a harmonic function of time:

$$u(t) = U(1 + \nu \sin \omega t).$$

3.1 One Rigid Pipe

Let us investigate the simplest case consisting only one rigid pipe. The model can be seen in Figure 3.1.

In this case $i = 1$ and $q_1 = \varphi$, which is the only degree of freedom.
3.1.1 Equation of Motion

The equation of motion is derived from the Lagrangian equation of the second kind:

\[ \frac{d}{dt} \frac{\partial (T - U)}{\partial \dot{q}_i} - \frac{\partial (T - U)}{\partial q_i} = Q_i, \quad (i = 1, 2, \ldots, n) \]  

where now the degree of freedoms \( n = 1 \), \( T \) is the kinetic energy of the pipe and the fluid, \( U \) is the potential energy of the elastic joint and \( Q_i \) is a generalized force component defined by the relationship

\[ \delta P = \sum_i Q_i \delta q_i, \]

where \( P \) is the energy flowed in the system during a unit time.

The kinetic energy comes from the motion of the pipe and the fluid particles inside the pipe:

\[ T = \frac{1}{2} M l^3 \dot{\phi}^2 + \int_0^l \frac{m}{2} v^2(X) \, dX, \]  

where \( \frac{1}{2} M l^3 \) is the approximate mass moment of inertia around the origin (\( l \gg D \), the diameter of the pipe) and \( v \) is the absolute velocity vector of the fluid particles depending on \( X \) (the distance from the joint at the wall, measured along the pipe):

\[ v(X) = \begin{bmatrix} -X \dot{\phi} \sin \phi + u(t) \cos \phi \\ X \dot{\phi} \cos \phi + u(t) \sin \phi \end{bmatrix}. \]

The potential energy stored in the joint can be given as

\[ U = \frac{1}{2} s \dot{\phi}^2. \]  

The energy per unit time carried in the system by the fluid is

\[ P = \frac{mu(t)}{2} (v^2(0) - v^2(t)) \equiv -\frac{mu(t)}{2} l^2 \dot{\phi}^2. \]
and hence,
\[ \delta P = -mu(t)l^2 \ddot{\varphi} \delta \varphi. \]

After substituting expressions (3.2)–(3.4) into Eq. (3.1), we obtain the equation of motion of the system, a linear ordinary differential equation:

\[ (M + m) \frac{\ddot{\varphi}}{3} + mu(t)l^2 \dot{\varphi} + s \varphi = 0, \]  
(3.5)

which can be written in simplified form introducing dimensionless quantities:

\[ \varphi'' + \tilde{u}(\tau) \varphi' + \varphi = 0, \]  
(3.6)

where \( \tau = \alpha t, \alpha^2 = s\mu/(ml^3), \mu = 3m/(M + m), \tilde{u}(\tau) = u(\frac{\tau}{\alpha})\mu/(\alpha l) \) and ‘prime’ denotes \( \frac{d}{d\tau} \).

### 3.1.2 Autonomous Case

In time-independent case, \( \tilde{u}(\tau) \equiv \tilde{U} \) is constant. The roots of the characteristic polynomial of (3.6) are

\[ \lambda_{1,2} = -\frac{\tilde{U}}{2} \pm \sqrt{\frac{\tilde{U}^2}{4} - 1}. \]

The real part of both roots is negative if \( \tilde{U} > 0 \), i.e. the fluid flows out from the pipe. In this case the system is always asymptotically stable, i.e. every solution tends to the trivial solution: \( \lim_{t \to \infty} \varphi(t) = 0 \).

### 3.1.3 Non-autonomous Case

Now, we perform similar steps as in Section 2.2.2. However, Eq. (3.6) differs from Hill’s equation (2.12) in a dissipative term, which suggest that the investigated time-periodic model can also be asymptotically stable, not only in Liapunov sense. (Can a small perturbation in the parameter of (3.6) cause stability loss?)

Rewriting Eq. (3.6) into Cauchy form, we get a first order system of ordinary differential equations:

\[ \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & -\tilde{u}(\tau) \end{bmatrix} \mathbf{x} \equiv \mathbf{A}(\tau)\mathbf{x}, \]  
(3.7)

where \( \mathbf{x} = (\varphi, \varphi') \) and \( \tilde{u}(\tau) = \tilde{U}(1 + \nu \sin w\tau), w = \omega/\alpha \) (\( \omega \) is the angular frequency of the perturbation of the flow).

According to Floquet Theory, system (3.7) is asymptotically stable if and only if all of the characteristic multipliers belonging to the coefficient matrix \( \mathbf{A}(\tau) \) are located in the open unit disk of the complex plane (see Section 2.2). The characteristic polynomial
of the Floquet Transition Matrix \( \mathbf{C} \) is

\[
\det(\mathbf{C} - \lambda \mathbf{I}) \equiv \lambda^2 - \lambda \text{tr} \mathbf{C} + \det \mathbf{C}.
\]

Applying Liouville’s Theorem, we get that

\[
\det \mathbf{C} = \exp \int_0^T \text{tr} \mathbf{A}(\tau) \, d\tau \equiv e^{-\bar{U}T} < 1,
\]

if \( \bar{U} > 0 \). Moreover, we know from Viéta’s formula that

\[
\lambda_1 \lambda_2 = \frac{c}{a} \equiv \det \mathbf{C} < 1.
\]

This means that the magnitudes of \( \lambda_1 \) and \( \lambda_2 \) can be smaller than one at the same time:

\[
|\lambda_{1,2}| < 1,
\]

which implies that system (3.7) can be asymptotically stable.

However, if we apply the transformation

\[
\varphi = ye^{-\frac{1}{\bar{U}}\int_0^\tau \hat{u}(\theta) \, d\theta}
\]

(3.8)

to the system (3.6) then we get an equation of Hill’s type:

\[
y'' + p(\tau)y = 0,
\]

(3.9)

where \( \lambda_1 \lambda_2 = 1 \) (as we saw in Section 2.2.2), i.e. this system can be stable only in Liapunov sense \( |\lambda_1| = |\lambda_2| = 1 \). This means that transformation (3.8) shrinks the stability domain because it filters the exponentially decaying coefficient, which can provide stability:

\[
\varphi = ye^{-\frac{\bar{U}}{2} \int_0^\tau \sin \theta \, d\theta}.
\]

Sinha et al. has developed a numerical method based on the behaviour of Chebyshev polynomials. If we write the solution vector as an expansion of these polynomials we obtain a set of linear algebraic equations from the original differential equations. The solution of these systems gives the coefficients of the Chebyshev expansion. This way, we obtain an approximation of the Floquet Transition Matrix \( \mathbf{C} \) and we can analyze the characteristic multipliers at given values of the system parameters. Fixing some of them (e.g. \( \bar{U} \)) and choosing the others (e.g. \( \nu, w \)) as bifurcation parameters we get the
<table>
<thead>
<tr>
<th>Numerical method:</th>
<th>LSODE</th>
<th>Chebyshev-18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of points</td>
<td>22665</td>
<td>24570</td>
</tr>
<tr>
<td>CPU time</td>
<td>4h 10’29”</td>
<td>8’29”</td>
</tr>
<tr>
<td>Relative CPU time</td>
<td>0.6631 s</td>
<td>0.0207 s</td>
</tr>
</tbody>
</table>

Table 3.1: Computational statistics of the applied numerical methods.

stability domains of the system as shown in Figure 3.2(a).

The points below the curves are asymptotically stable, i.e. this system is technically stable if we assume that \( \nu \ll 1 \). To the left of the curve \( err_{0.1} \) the relative error between the results based on 17 Chebyshev polynomials and 18 Chebyshev polynomials are less than 10%. To the right of this ‘error’ boundary the results of the computations are not reliable: the error is too high. For reference we computed the same chart with LSODE shown in Figure 3.2(b).

The difference between CPU times can be seen in Table 3.1: the method of Chebyshev polynomials is roughly 30 times faster than the integrating method LSODE. However, its error becomes greater and greater with decreasing forcing frequency and one has to apply more polynomials in the approximation to reduce the error.

### 3.2 Two Rigid Pipes with Free Ending

Now, we extend our previous model attaching another pipe to the free end of the first pipe as it can be seen in Figure 3.3. Let \( q_1 = \varphi \) and \( q_2 = \psi \) the two independent variables, thus, we have a system with 2 degrees of freedom.

#### 3.2.1 Equations of Motion

We intend to use again the Lagrangian equations (3.1).

*The kinetic energy* of this two-pipe system is

\[
\mathcal{T} = \frac{1}{2} M \frac{1}{3} \dot{\varphi}^2 + \frac{1}{2} M \frac{l_2^3}{12} \dot{\psi}^2 + \frac{1}{2} M l_2 v_{S_2}^2 + \int_0^{l_1} \frac{m}{2} \dot{v}_1^2 dX + \int_0^{l_2} \frac{m}{2} \dot{v}_2^2 dX
\]  

(3.10)

where the integrals correspond to the kinetic energy of the fluid. Moreover,

\[
v_{S_2} = \begin{bmatrix} -l_1 \dot{\varphi} \sin \varphi - \frac{l_2}{2} \dot{\psi} \sin \psi \\ l_1 \dot{\varphi} \cos \varphi + \frac{l_2}{2} \dot{\psi} \cos \psi \end{bmatrix}
\]
CHAPTER 3. JOINT ARTICULATED PIPES

(a) Computed with 18 Chebyshev polynomials

(b) Computed with LSODE

Figure 3.2: Stability charts of one rigid pipe.

Figure 3.3: Sketch of two rigid pipes with elastic joints.
is the velocity of the centre of gravity of the second pipe,

\[
v_1(X) = \begin{bmatrix} -X\dot{\varphi}\sin\varphi + u(t)\cos\varphi \\ X\dot{\varphi}\cos\varphi + u(t)\sin\varphi \end{bmatrix}
\]

is the flow velocity in the first pipe along its length \((X \in [0,l_1])\), and finally

\[
v_2(X) = \begin{bmatrix} -l_1\dot{\varphi}\sin\varphi - X\dot{\psi}\sin\psi + u(t)\cos\psi \\ l_1\dot{\varphi}\cos\varphi + X\dot{\psi}\cos\psi + u(t)\sin\psi \end{bmatrix}
\]

is the flow velocity in the second pipe where \((X \in [0,l_2])\).

The potential energy is

\[
U = \frac{1}{2}s_1\varphi^2 + \frac{1}{2}s_2(\varphi - \psi)^2
\]

and the work per unit time of the upstream and downstream flow is

\[
P = \frac{mu}{2}(v_1^2(0) - v_2^2(l_2)) \equiv \frac{mu}{2} \left( l_1^2\dot{\varphi}^2 + l_2^2\dot{\psi}^2 + 2l_1l_2\dot{\varphi}\dot{\psi}\cos(\varphi - \psi) - 2u_1\dot{\varphi}\sin(\varphi - \psi) \right).
\]

If we substitute Eqs. (3.10)–(3.12) into Lagrangian equation (3.1) then we obtain

\[
\begin{bmatrix} \lambda_p^3 + 3\lambda_p^2 \frac{3}{2}\lambda_p \cos\Delta\varphi \\ \frac{3}{2}\lambda_p \cos\Delta\varphi \end{bmatrix} q'' + \tilde{u}(\tau) \begin{bmatrix} \lambda_p^2 \\ 0 \end{bmatrix} q' = \begin{bmatrix} 1 + \sigma & -1 \\ -1 & 1 \end{bmatrix} q = \begin{bmatrix} \frac{3}{2}\lambda_p \psi^2 \sin\Delta\varphi - f(\tau)\sin\Delta\varphi \\ \frac{3}{2}\lambda_p \psi^2 \sin\Delta\varphi \end{bmatrix}
\]

where \(q = (\varphi, \psi), \lambda_p = l_1/l_2, \sigma = s_1/s_2, f(\tau) = \lambda_p \left( \frac{1}{2}\tilde{u}^2(\tau) + \tilde{u}' \right), \tilde{u}(\tau) = \tilde{U}(1 + \nu \sin\omega\tau), \tilde{U} = U\mu/(\alpha l_2), \alpha^2 = \mu s_2/(ml_2^3) \) and \(\Delta\varphi = \psi - \varphi\). The rest of the parameters \((\mu, \tau, w)\) is defined in a similar way as at Eq. (3.6).

Neglecting the fifth and higher order terms of (3.13) yields

\[
\begin{bmatrix} \lambda_p^3 + 3\lambda_p^2 \frac{3}{2}\lambda_p \cos\Delta\varphi \\ \frac{3}{2}\lambda_p \cos\Delta\varphi \end{bmatrix} q'' + \tilde{u}(\tau) \begin{bmatrix} \lambda_p^2 \\ 0 \end{bmatrix} q' + \begin{bmatrix} 1 + \sigma & -1 \\ -1 & 1 \end{bmatrix} q = \begin{bmatrix} 1 + \sigma - f(\tau) & -1 + f(\tau) \\ -1 & 1 \end{bmatrix} q
\]

\[
= \frac{3}{4} \begin{bmatrix} 0 & \lambda_p \\ \lambda_p & 0 \end{bmatrix} \Delta\varphi^2 q'' + \left[ \frac{3}{2}\lambda_p \psi^2 \Delta\varphi + \tilde{u}(\tau)\lambda_p \psi'\Delta\varphi^2 + \frac{1}{6}f(\tau)\Delta\varphi^3 \right].
\]

### 3.2.2 Autonomous Case

In the following we assume that \(\tilde{u}(\tau) \equiv \tilde{U}\) is constant.
CHAPTER 3. JOINT ARTICULATED PIPES

Linearized System

The linear part (i.e. the left-hand side) of Eq. (3.14) can be asymptotically stable if and only if the real parts of the roots of the characteristic polynomial are less than zero, i.e. the Hurwitz determinants of the polynomial coefficients are greater than zero (see Section 2.1.1). In case of $\lambda_p = 1$ and $\sigma = 1$, the characteristic polynomial is

$$
\frac{7}{4}\xi^4 + 2\tilde{U}\xi^3 + \frac{1}{2} \left(18 - \frac{5 - 2\mu}{\mu} \tilde{U}^2\right) \xi^2 + \frac{5\mu - \tilde{U}^2}{\mu} \tilde{U} \xi + 1.
$$

(3.15)

The conditions of stability are

$$
H_2 = \begin{vmatrix}
\frac{5\mu - \tilde{U}^2}{\mu} & \frac{1}{2} \\
\frac{5\mu - \tilde{U}^2}{\mu} & \tilde{U} - \frac{1}{2}\tilde{U}^2
\end{vmatrix} \equiv \tilde{U} \left(\tilde{U}^4 (5 - 2\mu) + \mu \tilde{U}^2 (10\mu - 43) + 86\mu^2\right)
\begin{vmatrix}
\frac{5\mu - \tilde{U}^2}{\mu} & \frac{1}{2} \\
\frac{5\mu - \tilde{U}^2}{\mu} & \tilde{U} - \frac{1}{2}\tilde{U}^2
\end{vmatrix} \equiv \tilde{U}^2 \left(\tilde{U}^4 (13 - 8\mu) + 2\mu \tilde{U}^2 (20\mu - 51) + 169\mu^2\right)
\frac{4\mu^2}{4\mu^2}
\end{align}

H_3 = a_3 H_2 - a_4

Condition $H_1 > 0$ constrains the solution set of $H_2 > 0$ and $H_3 > 0$ to two curves, from which the latter one will determine the stability:

$$
\mu > \mu_{cr} = \tilde{U}^2 \frac{4\tilde{U}^2 + 51 + 2\sqrt{4\tilde{U}^4 - 28\tilde{U}^2 + 101}}{40\tilde{U}^2 + 169}
$$

(3.16)

for arbitrary $\tilde{U}$.

In the critical case, when $\mu = \mu_{cr}$

$$
\xi_{1,2} = \pm i \sqrt{\frac{7 - 2\tilde{U}^2 + \sqrt{4\tilde{U}^4 - 28\tilde{U}^2 + 101}}{13}},
$$

i.e. Hopf bifurcation occurs.

If $\mu = 1$ then the critical flow velocity $\tilde{U}_{cr} = \sqrt{\frac{1}{5}(31 - 2\sqrt{29})} = 2.01145$ and $\xi_{1,2} = \pm i \sqrt{\frac{2}{5}(\sqrt{29} - 3)} = 0.6907i$. That is, a solution of the linearized Eq. (3.14) exists in the form $q = q_0 e^{i0.6907t} = q_0 e^{i0.691\omega t}$. Thus, the system loses its stability with frequency of $0.691\frac{\omega}{2\pi}$ Hertz.
CHAPTER 3. JOINT ARTICULATED PIPES

Nonlinear System

Before we could apply the Hopf bifurcation theorem, we have to determine the centre manifold where the stability loss occurs. Furthermore, we have to eliminate $q''$ from the nonlinear part of Eq. (3.14).

Let us multiply it with

$$
I + \frac{3}{4} \lambda_p \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} M^{-1} \Delta \varphi^2
$$

and neglect the fifth order terms. We get

$$
Mq'' + Kq' + Sq = b(q, q') - \frac{3}{4} \lambda_p \begin{bmatrix}
0 & \Delta \varphi^2 \\
\Delta \varphi^2 & 0
\end{bmatrix} M^{-1} (Kq' + Sq)
$$

where $M$, $K$ and $S$ are the coefficient matrices of $q''$, $q'$ and $q$ in (3.14), respectively, and $b(q, q')$ is the second term of (3.14):

$$
b(q, q') = \begin{bmatrix}
\frac{3}{2} \lambda_p \psi' \Delta \varphi + \bar{u}(\tau) \lambda_p \psi' \Delta \varphi^2 + \frac{1}{5} f(\tau) \Delta \varphi^3 \\
-\frac{3}{2} \lambda_p \varphi' \Delta \varphi
\end{bmatrix}.
$$

Now, we can write it into first order form with $x = (q, q') \equiv (\varphi, \psi, \varphi', \psi')$:

$$
x' = \begin{bmatrix}
0 & 1 \\
-M^{-1} S & -M^{-1} K
\end{bmatrix} x + \begin{bmatrix}
0 \\
M^{-1} \tilde{b}_3(x)
\end{bmatrix},
$$

(3.17)

where

$$
\tilde{b}_3(q, q') = b(q, q') - \frac{3}{4} \lambda_p \begin{bmatrix}
0 & \Delta \varphi^2 \\
\Delta \varphi^2 & 0
\end{bmatrix} M^{-1} (Kq' + Sq)
$$

and the index 3 denotes that the expression contains only third order nonlinearities.

The eigenvalues of the coefficient matrix of the linear part are

$$
\pm 0.69i, \quad -0.798, \quad -1.500,
$$

if $\lambda_p = 1$, $\sigma = 1$, $\mu = 1$ and $\bar{U} = \bar{U}_c = 2.011$.

After transforming it into Jordan form using the matrix of eigenvectors $T$, we can separate the centre manifold (CM):

$$
y' = \begin{bmatrix}
J_c & 0 \\
0 & J_s
\end{bmatrix} \begin{bmatrix}
y_c \\
y_s
\end{bmatrix} + T^{-1} \begin{bmatrix}
0 \\
M^{-1} \tilde{b}_3(Ty)
\end{bmatrix}.
$$

(3.18)

The nonlinear part consists of only third order terms. Hence, $y_s = O(y_c^3)$ in the
vicinity of the CM and we can simply go on with the analysis using the first two lines of Eq. (3.18) and dropping $y_s$:

$$y'_c = J_c y_c + \begin{bmatrix}
30 y_1^2 + f_{21} y_1 y_2 + f_{12} y_1 y_2 + f_{03} y_2^3 \\
g_{30} y_1^2 + g_{21} y_1 y_2 + g_{12} y_1 y_2 + g_{03} y_2^3
\end{bmatrix},
$$

\hspace{1cm} \text{(3.19)}

$$y'_c = 0.691 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y_c + \begin{bmatrix} 0.3060 y_1^2 - 0.3855 y_1 y_2 + 0.4926 y_1 y_2 + 0.0968 y_2^2 \\
-0.0327 y_1 y_2 - 0.0045 y_1 y_2 + 0.0441 y_1 y_2^3 + 0.0058 y_2^2 \end{bmatrix}.$$

According to the Hopf bifurcation theorem, system (3.19) has a super-critical bifurcation if the Poincaré–Liapunov constant is less than zero:

$$\delta \equiv \frac{1}{S} (3f_{30} + f_{12} + g_{21} + 3g_{03}) < 0.$$

Choosing the critical case $\mu = 1$ and $\tilde{U} = 2.011$ it yields $\delta = -0.107$, which refers to super-critical bifurcation, i.e. there will be stable oscillations if $\tilde{U} > \tilde{U}_{cr}$.

\subsection*{3.2.3 Non-autonomous Case}

\textbf{Linearized System}

Similar computations can be made as in the case of the one-pipe system. The results are shown in Figures 3.4(a) and 3.4(b). The points of the dotted regions are asymptotically stable and only the points of the boundaries are critical. The red curves mean catastrophe-like stability loss (fold), at the blue curves the critical multiplier leaves the complex unit disk at $-1$, i.e. flip bifurcation occurs, and at the green curves a pair of complex eigenvalues leaves the unit disk, which corresponds to Hopf bifurcation.

The $\nu = 0$ axis refers to the autonomous case: the points of this axis are (asymptotically) stable if $\tilde{U} < \tilde{U}_{cr}$ and unstable if $\tilde{U} > \tilde{U}_{cr}$.

The CPU times are listed in Table 3.2.
Figure 3.4: Stability charts of two rigid pipes with free ending.
<table>
<thead>
<tr>
<th>(N)</th>
<th>(\nu)</th>
<th>(w^{-1})</th>
<th>(\lambda_{cr})</th>
<th>type of bif.</th>
<th>P–L const.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.142076</td>
<td>(\alpha \pm i\beta)</td>
<td>Hopf</td>
<td>-0.0937</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.558868</td>
<td>-1</td>
<td>flip</td>
<td>-1.0188</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.952442</td>
<td>-1</td>
<td>flip</td>
<td>-0.2362</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>0.434657</td>
<td>-1</td>
<td>flip</td>
<td>-1.1510</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td>1.18966</td>
<td>-1</td>
<td>flip</td>
<td>-0.9357</td>
</tr>
<tr>
<td>6</td>
<td>0.2</td>
<td>1.88457</td>
<td>-1</td>
<td>flip</td>
<td>-14.4510</td>
</tr>
<tr>
<td>7</td>
<td>0.2</td>
<td>2.21891</td>
<td>-1</td>
<td>flip</td>
<td>0.2788</td>
</tr>
<tr>
<td>8</td>
<td>0.2</td>
<td>2.65855</td>
<td>1</td>
<td>fold</td>
<td>-10.4525</td>
</tr>
<tr>
<td>9</td>
<td>0.2</td>
<td>2.85674</td>
<td>1</td>
<td>fold</td>
<td>13.9665</td>
</tr>
</tbody>
</table>

Table 3.3: Poincaré–Liapunov constants at critical points.

Nonlinear System

The nonlinear analysis described in Section 2.2.3 was performed at some typical points of the boundary curves. The results are summarized in Table 3.3.

One can notice that sub-critical and super-critical flip \((N = 2, \ldots, 7)\), fold \((N = 8, 9)\) and Hopf bifurcations \((N = 1)\) appear.

Figures 3.5–3.8 show four bifurcation diagrams and the corresponding Poincaré maps as increasing \(w^{-1}\) from 0.56 to 0.96 \((\nu = 0.1)\), from 0.44 to 1.19, from 1.89 to 2.18 and from 2.67 to 2.86 \((\nu = 0.2)\), respectively. The magnitude of the derivatives of generalized coordinates can be seen on the vertical axes. In Figures 3.5–3.7 maps of stable 2\(T\)-periodic solutions are shown and Figure 3.8 shows the diagrams of stable \(T\)-periodic solutions. No further bifurcation of the fixed points can be observed in any case. It can be seen in the left-hand side of Figures 3.5 and 3.6 that as the zero solution (the origin) becomes stable again, the 2\(T\)-periodic solutions disappear, which correspond to the super-critical bifurcations forecasted in Table 3.3 \((N = 2, \ldots, 5)\).
Figure 3.5: Nonlinear analysis, $w^{-1} \in [0.56, 0.96]$. 

Figure 3.6: Nonlinear analysis, $w^{-1} \in [0.44, 1.19]$. 
Figure 3.7: Nonlinear analysis, $w^{-1} \in [1.89, 2.18]$.

Figure 3.8: Nonlinear analysis, $w^{-1} \in [2.67, 2.86]$. 
3.3 Two Rigid Pipes with Supported Ending

Now the end of the second pipe can only move in the $x$ direction. Hence,

$$l_1 \sin \varphi + l_2 \sin \psi = 0$$  \hspace{1cm} (3.20)

(see Figure 3.9).

3.3.1 Equation of Motion

The expressions of the kinetic energy $T$, the potential energy $U$ and the energy-flow $P$ are the same as in Eqs. (3.10)–(3.12). However, $\varphi$ and $\psi$ are not independent variables and the Lagrangian equations (3.1) yield a one-dimensional system of equation of motion.

Now, we expand the terms containing $\psi$ into power series of $\varphi$ according to Eq. (3.20):

$$\psi = -\lambda_p \varphi - \frac{1}{6} \lambda_p (\lambda_p^2 - 1) \varphi^3 + O(|\varphi|^5)$$

and we put these expansions in Eqs. (3.10)–(3.12).

After substituting these approximations into Eq. (3.1), doing some algebra and transforming the equation of motion into dimensionless form, we obtain

$$\varphi'' + \left( \frac{\lambda_p + 2}{\lambda_p^3} + \frac{\sigma}{\lambda_p^2(\lambda_p + 1)} - \frac{1}{\lambda_p} \left( \frac{1}{\mu} \ddot{u}(\tau)^2 + \dot{u}' \right) \right) \varphi^3 \hspace{1cm} (3.21)$$

$$= - (\lambda_p + 2) \varphi^2 \varphi'' - (\lambda_p + 2) \varphi^2 \varphi - \ddot{u}(\tau)(\lambda_p + 1) \varphi' \varphi^2$$

$$- \left( 4(\lambda_p^2 - 1) + (3\lambda_p + 1) \left( \frac{1}{\mu} \ddot{u}(\tau)^2 + \dot{u}' \right) \right) \frac{\varphi^3}{6\lambda_p},$$

where the fifth and higher order terms in $\varphi$ and $\varphi'$ were neglected and the dimensionless variables are the same as those defined after Eq. (3.14).
3.3.2 Autonomous Case

Linearized System

It is equivalent to the case $\nu = 0$, i.e. $\bar{u}(\tau) \equiv \bar{U}$ and $\bar{u}'(\tau) \equiv 0$ at any ‘time’ instant $\tau$. Since the linear part (i.e. the left-hand side) of Eq. (3.21) is a 1 degree-of-freedom Hamiltonian system, it can be stable only in Liapunov sense. At the stability boundaries a pair of pure imaginary roots leaves the imaginary axis for the real axis at the origin. The equilibrium $\varphi = 0$ is stable if and only if the coefficient of $\varphi$ is positive:

$$\bar{U} < \bar{U}_{cr} \equiv \sqrt{\frac{\sigma + (\lambda_p + 1)^2}{\lambda_p (\lambda_p + 1)}}.$$  

In case of $s_1 = s_2$ and $l_1 = l_2$, this implies

$$\bar{U}_{cr} = \sqrt{2.5\mu}.$$  

Nonlinear System

In super-critical case ($\bar{U} > \bar{U}_{cr}$) the approximated nonlinear system (3.21) has three equilibria:

$$\varphi_0 = 0, \quad \varphi_{1,2} = \pm \frac{\frac{2\bar{U}^2}{\mu} \lambda_p (\lambda_p + 1) - 6(\lambda_p + 1)^2 + 6\sigma}{\lambda_p (\lambda_p + 1) \left(4(\lambda_p^2 - 1) + (3\lambda_p + 1) \frac{1}{\mu} \bar{U}^2\right)}.$$  

Below $\bar{U}_{cr}$ the stability of the equilibrium point $\varphi = 0$ is determined by the nonlinear terms of Eq. (3.21). In order to investigate the nonlinear part we have to eliminate $\varphi''$ from the nonlinearities. Let us multiply Eq. (3.21) with $1 - (\lambda_p + 2)\varphi^2$ and neglect the fifth order terms:

$$\varphi'' + \left(\frac{\lambda_p + 1}{\lambda_p^2} + \frac{\sigma}{\lambda_p^2 (\lambda_p + 1)} - \frac{\bar{U}^2}{\lambda_p \mu}\right) \varphi$$

$$= -(\lambda_p + 2)\varphi^2 \varphi - \bar{U}(\lambda_p + 1)\varphi' \varphi^2 + (\lambda_p + 2) \frac{(\lambda_p + 1)^2 + \sigma \varphi^3}{\lambda_p (\lambda_p + 1)}$$

$$= -\left(4(\lambda_p^2 - 1) + \frac{9\lambda_p + 13}{\mu} \bar{U}^2\right) \varphi^3.$$  

This equation can be written into Cauchy form as

$$x' = \begin{bmatrix} 0 & \bar{\alpha} \\ -\bar{\alpha} & 0 \end{bmatrix} x + \begin{bmatrix} \ldots \frac{x^3}{\bar{\alpha}^2 x_1} - \frac{(\lambda_p + 1) x^2}{\bar{\alpha} x_1 x_2} - (\cdots) x_1 x_2 \end{bmatrix},$$  

(3.22)
where
\[ \dot{\alpha}^2 = \frac{\lambda_p + 1}{\lambda_p^2} + \frac{\sigma}{\lambda_p^2(\lambda_p + 1)} - \frac{\bar{U}^2}{\lambda_p \mu} \]
and \( x_1 = \sqrt{\alpha} \varphi, \quad x_2 = \sqrt{\frac{\sigma}{\alpha}} \varphi' . \)

Transforming this system into its third order normal form, we find that the amplitude of the oscillation is determined by
\[ \dot{r} = \delta r^3 + \text{h.o.t.}, \]

where \( \delta = -\frac{1}{8} \frac{\lambda_p + 1}{\alpha} \bar{U} \) of which sign is always negative independently from any parameter, that is, the equilibrium point \( \varphi = 0 \) is always locally stable (after small perturbation the amplitude of the oscillation decreases).

### 3.3.3 Non-autonomous Case

**Linearized System**

The linear part of Eq. (3.21) is a *Hill’s equation* (see Eq. (3.9)) where
\[
p(\tau) = \frac{\sigma}{\lambda_p^2(\lambda_p + 1)} + \frac{\lambda_p + 1}{\lambda_p^2} - \frac{\bar{U}^2}{\mu \lambda_p} \left( 1 + \nu^2 \right) \]
\[ -\frac{\bar{U} \nu}{\mu \lambda_p} \left( \mu w \cos w \tau + 2 \bar{U} \sin w \tau - \frac{1}{2} \nu \bar{U} \cos 2w \tau \right). \]

If we choose \( \mu = 1 \) then \( \bar{U}_{cr} \approx 1.5811 \). *Sinha’s* numerical method presents the stability charts in the space of the parameters \( \bar{U}, w \) and \( \nu \), which are shown in Figures 3.10(a) and 3.10(b).

Because of the three bifurcation parameters, the computations were made at certain values of \( \bar{U} \) around \( \bar{U}_{cr} \). The points of the dotted region correspond to such systems that are stable in *Liapunov sense*. At the red and blue curves *fold* \( (\lambda_{cr} = 1, T\text{-per. solutions}) \) and *flip* \( (\lambda_{cr} = -1, 2T\text{-per. solutions}) \) bifurcation occurs, respectively. It can also be seen that there are unstable regions below \( \bar{U}_{cr} \) and these regions are growing fast when approaching to \( \bar{U}_{cr} \). However, some stable regions still remain even if \( \bar{U} \) becomes greater than \( \bar{U}_{cr} \). Since the linear part of (3.21) cannot be asymptotically stable, further analysis is required that takes into account the effect of the nonlinearities in the system.

If we compare the CPU times in Table 3.4 to the ones in Table 3.2 (and in Table 3.1) then we can see that the computations of the 2 DOF model take roughly 2.5 times more time than in the case of 1 DOF system.
### Table 3.4: Computational statistics of two rigid pipes (1 DOF).

<table>
<thead>
<tr>
<th></th>
<th>( \tilde{U} = 1.50 )</th>
<th>( \tilde{U} = 1.60 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of points</td>
<td>36170</td>
<td>28610</td>
</tr>
<tr>
<td>CPU time</td>
<td>11’58”</td>
<td>9’10”</td>
</tr>
<tr>
<td>Relative CPU time</td>
<td>0.0198 s</td>
<td>0.0192 s</td>
</tr>
</tbody>
</table>

Figure 3.10: Stability charts of two rigid pipes with supported ending.
3.4 New Results

**Thesis 1** I determined the stability chart of an elastically supported horizontal rigid pipe containing incompressible and frictionless pulsatile flow. I showed with these calculations that the flow velocity fluctuation do not cause stability problem in typical engineering cases (when the amplitude of the fluctuation is smaller than the mean flow velocity). I carried out the stability analysis of Floquet theory applying a method based on Runge–Kutta type numerical simulation, as well as with a method that transforms the integral form of a differential equation into linear algebraic equation system using Chebyshev polynomials. Comparing the results and estimating the error of the numerical algorithm, I showed that the latter method do not give reliable result, if the frequency of the pulsatile flow is much smaller than the natural frequency of the system.

**Thesis 2** I determined the stability chart of a cantilever construction of two rigid pipes attached together with resilient joints in the vicinity of the critical speed of a steady flow. I classified the stability boundary curves in the parameter plane of the relative pulsation frequency and amplitude with respect to the possible types of stability loss. According to this classification, I found that both period doubling (or period 2) bifurcations and topologically equivalent pitchfork and Hopf bifurcations may appear in case of realistic engineering parameters.

I investigated four typical unstable parameter ranges. Using numerical simulations, I checked the analytical results which determined the stability of the periodic motions in the vicinity of the equilibrium of the pipes. I illustrated the change of the periodic solutions in bifurcation diagrams and the corresponding strolling fixed points by Poincaré maps. No further bifurcations were found in the investigated parameter ranges.

**Thesis 3** I constructed the stability charts of the system of two rigid pipes mentioned in the second thesis but with a pinned end. I showed that it is a 1 degree-of-freedom system with conservative linear part and its stability chart is the transformed Ince–Strutt chart of Mathieu equation. The conservativeness implies that only T- and 2T-periodic solutions can be found at the boundaries of stable domains.
Chapter 4

Elastic Pipes

In this chapter we consider an elastic continuum pipe conveying fluid. From the several different models can be found in the technical literature, we investigate the dynamical behaviour of two fundamental cases, which differ only in the boundary conditions. The derivation of the equation of motion confirm the results found in [39]. After the necessary analysis of the steady (autonomous) case, the main goal is to investigate the effect of the number of applied basis functions on the stability charts of the time-periodic system, as well as, using Sinhas’ powerful numerical method, to compute stability charts for a wide range of parameter values, especially for high pulsatile frequency of the flow. We finish this chapter with the nonlinear analysis in critical (unstable) parameter intervals.

We assume that the points of the pipe, shown in Figure 4.1, can have large displacements but small deformations and it is made of a homogeneous, isotropic material. The bending stiffness of the pipe is $I_E$. The analyzed two cases are as follows: in both cases the upstream end of the pipe is clamped, however, the downstream end can be simply supported or unsupported. All motion is considered in a horizontal plane as in the previous chapter. The mass per unit length of the pipe and that of the frictionless incompressible fluid are $M$ and $m$, respectively. The upstream mass-flow $mu(t)$ is generally a periodic function of time in the same manner as in Chapter 3:

$$u(t) = U(1 + \nu \sin \omega t).$$

\(^1\)apart from the difference in terms related to the gravity, which is neglected in the models to presented in this chapter
4.1 Condition of Inextensibility

The length of the pipe is $L$ and its axis is inextensible (i.e. the cross-sectional area of the pipe remains constant):

$$r^2 = x^2 + y^2 = 1,$$  \hspace{1cm} (4.2)

where the position vector of a point (given by the arc-length $X$) on the pipe axis is $r = (x(X, t), y(X, t))$ and $'$ (‘prime’) denotes the derivation with respect to the spatial coordinate (the arc-length, in this case): $\partial / \partial X$. Hence,

$$x(X, t) = \int_0^X \sqrt{1 - y'^2(\xi, t)} \, d\xi. \hspace{1cm} (4.3)$$

Let $\alpha$ be the angle of the tangent of the pipe axis at a given $X$ relative to the axis $x$. Then

$$\cos \alpha = x' = \frac{1}{\sqrt{1 + y'^2}} = \sqrt{1 - y'^2},$$

where $\tilde{y}(x, t) = y(X(x), t)$ is the graph of the axis in an orthogonal coordinate system,

$$\tilde{y}'_x = \frac{\partial y}{\partial X} \frac{\partial X}{\partial x} = \frac{y'}{x'},$$

where $'$ (‘prime’) denotes the partial derivation with respect to the own spatial coordinate (in case of $y$ and $\tilde{y}$, $X$ and $x$, respectively). Furthermore,

$$\tilde{y}''_{xx} = \frac{\partial y'}{\partial x} x' = \frac{y'' x^2}{x'^3} = \frac{y''(x^2 + y^2)}{x'^3} \frac{1}{x'} = \frac{y''}{x'^4}$$

since derivating the inextensibility condition (4.2) yields

$$x' x'' + y' y'' = 0 \quad \implies \quad -y' x'' = \frac{y'^2 y''}{x'^4}.$$
Thus, the curvature $\kappa$ of the pipe axis is
\[ \kappa = \frac{-y_x''}{(1 + y_x'^2)^{3/2}} = \frac{-y''}{x'} \equiv \frac{-y''}{\sqrt{1 - y'^2}}. \tag{4.4} \]

### 4.2 Equation of Motion

According to the *generalized Hamilton’s principle*
\[ \delta \int_{t_1}^{t_2} (U - T) \, dt = \int_{t_1}^{t_2} \delta W \, dt \tag{4.5} \]

where $U$, $T$ and $\delta W$ are the strain energy, the whole kinetic energy and the virtual work, respectively. The bending moment of a beam-like pipe is a linear function of the curvature: $M_z = \kappa I_z E$. The *strain energy* of a beam is
\[ U = \frac{1}{2} I_z E \int_0^L \kappa^2 \, dX \approx \frac{1}{2} I_z E \int_0^L y''^2 (1 + y'^2) \, dX, \tag{4.6} \]

where the fourth-degree approximation takes into account that we are investigating the stability of the trivial equilibrial shape: $y(X, t) = y'(X, t) = 0$.

Neglecting the terms of rotation (containing $\dot{r}'$), we get a simple expression for the *kinetic energy* of the pipe and the fluid:
\[ T = \int_0^L \left( \frac{1}{2} M \dot{r}^2 + \frac{m}{2} \left( \dot{\mathbf{r}} + u(t) \mathbf{r}' \right)^2 \right) \, dX \tag{4.7} \]

where $\cdot$ (‘dot’) denotes $\partial / \partial t$. The first and second term in the parenthesis are the kinetic energy of an infinitesimal pipe and fluid segment, respectively: $\dot{\mathbf{r}}$ is the absolute velocity of the pipe segment and $u(t) \mathbf{r}'$ is the relative velocity of the fluid.

The external forces changing the momentum of the flow between upstream and downstream at the ends of the pipe are
\[ \mathbf{F} = -\int_0^L mu(t) (\dot{\mathbf{r}} + u(t) \mathbf{r}') \, dX \equiv -mu [\dot{\mathbf{r}} + u \mathbf{r}']_0^L \equiv \mathbf{F}_L + \mathbf{F}_0 \tag{4.8} \]

Thus, the *virtual work* of these forces is
\[ \delta W = \mathbf{F}_L \delta \mathbf{r}_L + \mathbf{F}_0 \delta \mathbf{r}_0. \tag{4.9} \]
Putting the expressions of $\mathcal{U}$, $\mathcal{T}$ and $\delta \mathcal{W}$ in Eq. (4.5), we get

$$\int_{t_1}^{t_2} \int_0^L \left( I_z E \left( y'' \delta y'' + y'^2 y' \delta y' \right) + (M + m) \dot{r} \delta \dot{r} - m \dot{u}(t) (\dot{r} \delta \dot{r} + \ddot{r} \delta r') \right) dX dt$$

$$= - \int_{t_1}^{t_2} [m \dot{u}(t) (\dot{r} + u(t) \dot{r}') \delta r]_0^L dt. \tag{4.10}$$

After integrating by parts (excluding the term of $I_z E$) and eliminating $\delta x$ (see Appendix B), one can obtain

$$\int_{t_1}^{t_2} \int_0^L \left\{ I_z E \left( y'' \delta y'' + y'^2 y' \delta y' \right) + \delta y \left( G \left<y(X, t)\right> - \left(1 + \frac{1}{2} y'^2\right) y G \left<x(X, t)\right> \right) \right.\right.$$  

$$+ \delta y \left(1 + \frac{3}{2} y'^2\right) y'' \int_X^L G \left<x(\xi, t)\right> d\xi \right\} dX dt = 0, \tag{4.11}$$

where

$$G \left<z(X, t)\right> = (M + m) \ddot{z} + 2m \dot{u} \dot{z}' + m \ddot{u} ' + m u^2 z''.$$  

From Eq. (4.3) one can express the derivatives of $x$ as the function of the derivatives of $y$. Thus, we can eliminate all the derivatives of $x$ from Eq. (4.11). After neglecting the fifth and higher order terms, we obtain the equation of motion in dimensionless form which corresponds to the results presented by Semler et al. in [39] on page 586:

$$\int_{\tau_1}^{\tau_2} \int_0^2 \left\{ \left(y'' \delta y'' + y'^2 y' \delta y' \right) + \delta y \left(3y + 2\bar{u}(\tau) y' \left(1 + y'^2\right)\right) \right.\right.$$  

$$+ \delta y \left(\frac{1}{\mu} \bar{u}^2(\tau) y'' \left(1 + y'^2\right) + 3y' \int_0^\xi (y'y' + y'^2) d\eta + \frac{d\bar{u}}{d\tau} \left(2 - \xi\right) y'' \left(1 + \frac{3}{2} y'^2\right)\right)$$  

$$- \delta y \dot{y}' \int_0^2 \left(3 \int_0^\eta (y'y' + y'^2) d\eta + 2\bar{u}(\tau) \ddot{y}' + \frac{d\bar{u}}{d\tau} y'^2 + \frac{1}{\mu} \bar{u}^2(\tau) y'' y' \right) d\eta \right\} d\xi d\tau = 0, \tag{4.12}$$

where

$$\tau = \alpha t, \quad \alpha^2 = I_z E \frac{\mu}{m H}, \quad \mu = \frac{3m}{M + m}, \quad l = \frac{L}{2}, \quad \xi = \frac{X}{l}, \quad \bar{u}(\tau) = \frac{\mu}{\alpha l} \bar{u}(\tau),$$

that is, we transformed the interval $[0, L]$ to $[0, 2]$ with introducing $\xi$. Furthermore, $\bar{y}(\xi, \tau) = \frac{1}{l} y(\xi l, \tau/\alpha)$ but the ‘tilde’-s were dropped in Eq. (4.12) and in the remainder
of this work. Similarly, · (‘dot’) and ′ (‘prime’) denote from this point $\partial/\partial\tau$ and $\partial/\partial\xi$, respectively.

### 4.2.1 Boundary Conditions

The boundary conditions of the two different constructions mentioned in the introduction of this chapter are listed in Table 4.1.

### 4.2.2 Discretizing in the Space Domain

We apply Galerkin’s method to discretize Eq. (4.12) with respect to its spatial coordinate $\xi$. Thus, we assume

$$y(\xi, \tau) = \sum_{i=1}^{n} y_i(\tau) \varphi_i(\xi),$$  \hspace{1cm} (4.13)

where $\varphi_i(\xi)$ are appropriate basis functions which satisfy the boundary conditions and $n$ is the number of modes approximated by basis functions.

Substituting the formula (4.13) into Eq. (4.12) yields

$$\int_{\tau_1}^{\tau_2} \delta y_i \left\{ S_{0ij} y_j + 3M_{ij} \ddot{y}_j + 2\ddot{u}(\tau)K_{ij} \dot{y}_j + \frac{1}{\mu} \ddot{u}^2(\tau)S_{1ij} y_j + \frac{d\ddot{u}}{d\tau}(2S_{1ij} - S_{20ij}) y_j ight\} \, d\tau = 0,$$  \hspace{1cm} (4.14)

where the $\sum$-s were dropped according to Einstein’s convention. Furthermore,

$$M_{ij} = \int_{0}^{2} \varphi_i \varphi_j d\xi, \hspace{0.5cm} K_{ij} = \int_{0}^{2} \varphi_i \varphi_j' d\xi, \hspace{0.5cm} S_{0ij} = \int_{0}^{2} \varphi_i \varphi_j'' d\xi; \hspace{0.5cm} S_{1ij} = \int_{0}^{2} \varphi_i \varphi_j''' d\xi,$$

$$S_{20ij} = \int_{0}^{2} \xi \varphi_i \varphi_j'' d\xi, \hspace{0.5cm} M_{11ijkl} = \int_{0}^{2} \varphi_i \varphi'_j \int_{0}^{\xi} \varphi'_k \varphi'_l d\eta \, d\xi; \hspace{0.5cm} K_{11ijkl} = \int_{0}^{2} \varphi_i \varphi'_j \varphi'_k \varphi'_l d\xi.$$
\[ S_{0ijkl} = \int_0^2 (\varphi_i'\varphi'_j + \varphi_i''\varphi'_j')\varphi_k\varphi'_k d\xi, \quad S_{11ijkl} = \int_0^2 \varphi_i\varphi'_j\varphi'_k\varphi'_l d\xi, \quad S_{2ijkl} = \int_0^2 \xi\varphi_i\varphi'_j\varphi_k\varphi'_l d\xi, \]

\[ M_{12ijkl} = \int_0^2 \varphi_i\varphi'_j d\xi \int_0^\eta \varphi'_k\varphi' k d\eta d\xi, \quad M_{13ijkl} = \int_0^2 \varphi_i\varphi'_j d\xi \int_0^\eta \varphi'_k\varphi' k d\eta d\xi, \]

\[ K_{13ijkl} = \int_0^2 \varphi_i\varphi'_j d\xi \int_0^\eta \varphi'_k\varphi' k d\eta d\xi. \]

In Eq. (4.14) \( \delta y_l \) can be arbitrary. Hence, its coefficient (i.e. the expression in the braces) must be zero:

\[ 3(M + (M_{11kl} - M_{12kl})y_k y_l) \ddot{y} + 2\ddot{u}(\tau)K\dot{y} + \left( S_0 + \frac{1}{\mu} \ddot{u}^2(\tau)S_1 + \frac{d\ddot{u}}{d\tau}S_2 \right) y \]

\[ + 3(M_{11kl} - M_{12kl})y_k y_l + 2\ddot{u}(K_{11kl} - K_{12kl})\dot{y} y_k y_l \]

\[ + \left( S_{0ijkl} + \frac{1}{\mu} \ddot{u}^2(S_{11kl} - S_{12kl}) + \frac{1}{2}\frac{d\ddot{u}}{d\tau}(6S_{11kl} - 3S_{21kl} - K_{12kl}) \right) y_k y_l y_l = 0, \]

(4.15)

where, for sake of brevity, we write the coefficients partially in matrix representation (omitting the indices \( i \) and \( j \)) and keeping Einstein’s convention for the third and fourth indices \( (k \) and \( l \)). It can be seen that \( \ddot{y}_l \) is also in the nonlinear part of (4.15). However, if we multiply Eq. (4.15) with \( I = (M_{11mn} - M_{12mn})M^{-1}y_m y_n \), we obtain

\[ 3M\ddot{y} + 2\ddot{u}(\tau)K\dot{y} + \left( S_0 + \frac{1}{\mu} \ddot{u}^2(\tau)S_1 + \frac{d\ddot{u}}{d\tau}S_2 \right) y \]

\[ + 3(M_{11kl} - M_{12kl})\ddot{y}_k y_l + 2\ddot{u}(K_{11kl} - K_{12kl} - \dddot{I}_{kl}K)\dot{y} y_k y_l \]

\[ + \left( S_{0kl} - \dddot{I}_{kl}S_0 + \frac{1}{\mu} \ddot{u}^2(S_{11kl} - S_{12kl} - \dddot{I}_{kl}S_1) \right) y_k y_l \]

\[ + \frac{1}{2}\frac{d\ddot{u}}{d\tau}(6S_{11kl} - 3S_{21kl} - K_{12kl} - 2\dddot{I}_{kl}S_2) y_k y_l y_l = 0, \]

(4.16)

where the fourth and higher order terms were neglected, and \( y = [y_j], \dddot{I}_{kl} = (M_{11kl} - M_{12kl})M^{-1}, S_2 = 2S_1 - S_2 \).

### 4.3 Pipe with Supported Ending

In the following we consider a continuum elastic pipe shown in Figure 4.2. Its upstream end is attached to the wall while the downstream end can only move in the \( x \) direction.
4.3.1 Basis Functions

Polynomial Functions

The following set of basis functions satisfies the boundary conditions listed in Table 4.1

\[
\varphi_i(\xi) = \xi^{i+1}(c_{i2}\xi^2 + c_{i1}\xi + c_{i0}), \quad (i = 1, 2, \ldots),
\]

(4.17)

where

\[
c_{i0} = \frac{i + 2}{i + 1}c_{i2}, \quad c_{i1} = -\frac{2i + 3}{i + 1}c_{i2},
\]

and \(c_{i2}\) is arbitrary. These functions are normed on the interval \([0, 2]\) if

\[
c_{i2} = \frac{(i + 1)\sqrt{i + 3\sqrt{2i + 3\sqrt{2i + 5\sqrt{2i + 7}}} + 7}}{2^i\sqrt{1792i + 3072}}.
\]

The first six functions are shown in Figure 4.3.
Trigonometric Functions

The following set of basis functions also satisfies the boundary conditions:

\[ \varphi_i(\xi) = \sin \frac{\pi}{4} \xi \sin \frac{i\pi}{2} \xi. \]  \hspace{1cm} (4.18)

Figure 4.4 shows the graphs of \( \varphi_i(\xi) \) for \( i = 1, 2, \ldots, 6 \).

Chebyshev Polynomials

Applying Chebyshev polynomials (see Appendix A), we can construct another set which is also based on polynomials:

\[ \varphi_i(\xi) = \sum_{j=i-1}^{i+3} c_{j-i+2} T_j(\xi - 1). \]  \hspace{1cm} (4.19)

Here \( T_j(\xi) \) denotes the \( j^{\text{th}} \) Chebyshev polynomial of the first kind. The first 6 of these basis functions are shown in Figure 4.5.

One can see the differences between the three type of sets comparing Figures 4.3–4.5.

### 4.3.2 Stability Analysis in Autonomous Case

If the perturbation amplitude of the flow velocity is zero \( (\nu = 0) \), Eq. (4.16) becomes a system of autonomous differential equations, since \( \ddot{u}(\tau) \equiv \ddot{U} \equiv U \frac{d^2}{d\tau^2} \) and \( \ddot{u}'(\tau) \equiv 0. \)
Chebyshev type Base Functions of Supported Pipe

Figure 4.5: Basis functions of Chebyshev polynomials.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Trigonometric</th>
<th>Chebyshev</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_0 : \frac{-5\pi^2}{48\mu} \tilde{U}^2 + \frac{41\pi^4}{768}$</td>
<td>$\frac{-18}{19} \frac{1}{\mu} \tilde{U}^2 + \frac{189}{38}$</td>
</tr>
<tr>
<td></td>
<td>$a_1 : 0$</td>
<td>$0$</td>
</tr>
<tr>
<td>2</td>
<td>$a_0 : \frac{259\pi^4}{6412\mu^2} \tilde{U}^4 + \frac{6095\pi^6}{30200\mu} \tilde{U}^2 + \frac{51331\pi^8}{170092}$</td>
<td>$\frac{165}{112} \frac{1}{\mu^2} \tilde{U}^4 - \frac{801}{14} \frac{1}{\mu} \tilde{U}^2 + \frac{3465}{14}$</td>
</tr>
<tr>
<td></td>
<td>$a_1 : 0$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$a_2 : \left( \frac{-35\pi^2}{12\mu} + \frac{256}{243} \right) \tilde{U}^2 + \frac{707\pi^4}{112}$</td>
<td>( \left( -\frac{173}{39} \frac{1}{\mu} + \frac{142}{144} \right) \tilde{U}^2 + \frac{702}{13} )</td>
</tr>
<tr>
<td></td>
<td>$a_3 : 0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Table 4.2: Coefficients of the characteristic polynomials.

Its characteristic polynomial has the form

$$\lambda^{2n} + \sum_{i=0}^{2n-1} a_i \lambda^i,$$  \hspace{1cm} (4.20)

where $n$ is the number of applied basis functions. The coefficients of the characteristic polynomial are summarized in Table 4.2 for $n \leq 2$.

It is worth noticing that the system with these boundary conditions is a Hamiltonian system: the coefficient matrix of $\mathbf{y}$ is a gyroscopic matrix and there are only $\lambda$-s with even powers ($\lambda^{2k}$) in the characteristic polynomial. Thus, if there is a root $\lambda_1 = \alpha + i\beta$ with non-zero real part ($\alpha \neq 0$) then $\lambda_2 = -\alpha + i\beta$ is a root, too. Moreover, $\lambda_{3,4} = \pm \alpha - i\beta$ are roots, as well. Hence, this system can be stable only in Liapunov sense ($\lambda_{2k-1,2k} = \pm i\beta_k$). To achieve this, the roots of the ‘reduced’ polynomial

$$\beta^n + \sum_{i=0}^{n-1} a_i \beta^i$$  \hspace{1cm} (4.21)
must be negative real numbers, i.e. $\forall i : a_i > 0$ and the discriminant $\Delta$ of this polynomial (see Korn and Korn [19]) and its derivatives must be positive, as well. The stable intervals where $\bar{U}$ satisfies these conditions are listed in Table 4.3 for $n \leq 3$.

### 4.3.3 Non-autonomous Case

The numerical method presented in Section 2.3 was applied to get the stability domains of the system. Figures 4.6(a) and 4.6(b) show slices of the three dimensional ($\bar{U}$, $w$, $\nu$) theoretic stability domain in the vicinity of the critical value $\bar{U}_{cr}$ obtained in the autonomous case. The dotted region represents the stable domain of the analyzed space. The colour of the boundary curves shows the type of stability loss, i.e. the way the characteristic multipliers leave the unit circle while crossing the stability boundary.

Figures 4.7 and 4.8 were obtained by using 2 and 3 modes, respectively. It is worth noticing that these stability charts have more complicated structure for low values of $w^{-1} \equiv \alpha/\omega$. Thus, one should consider applying more basis functions to describe the more exact behaviour of the model in this region.

Table 4.4 summarize the CPU time needed to obtain these stability charts.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Trigonometric basis fun.</th>
<th>Chebyshev basis fun.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 2.249)</td>
<td>(0, 2.2913)</td>
</tr>
<tr>
<td>2</td>
<td>(0, 2.2473) (3.8642, 3.8753)</td>
<td>(0, 2.2554) (4.0636, 4.0661)</td>
</tr>
<tr>
<td>3</td>
<td>(0, 2.2469) (3.8631, 3.8690) (5.450, 5.453)</td>
<td>(0, 2.2469) (3.9253, 4.0112) (5.747, 5.873)</td>
</tr>
</tbody>
</table>

Table 4.3: Stable intervals of $\bar{U}$ in autonomous case.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Number of pts.</th>
<th>CPU time</th>
<th>Rel. CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21795</td>
<td>6’42”</td>
<td>0.0184 s</td>
</tr>
<tr>
<td></td>
<td>30615</td>
<td>9’53”</td>
<td>0.0194 s</td>
</tr>
<tr>
<td>2</td>
<td>57175</td>
<td>48’16”</td>
<td>0.0507 s</td>
</tr>
<tr>
<td></td>
<td>55495</td>
<td>46’48”</td>
<td>0.0506 s</td>
</tr>
<tr>
<td>3</td>
<td>87675</td>
<td>2h 38’37”</td>
<td>0.1085 s</td>
</tr>
<tr>
<td></td>
<td>71805</td>
<td>1h 55’54”</td>
<td>0.0968 s</td>
</tr>
</tbody>
</table>

Table 4.4: Computational statistics of elastic pipe with supported ending.
Figure 4.6: Stability charts of pipe with supported ending (1 trigonometric mode).
Figure 4.7: Stability charts of pipe with supported ending (2 trigonometric modes).
Figure 4.8: Stability charts of pipe with supported ending (3 trigonometric modes).
4.4 Cantilever Pipe

Now, we release the downstream end of the pipe as shown in Figure 4.1, so it can move freely in the \(xy\)-plane.

4.4.1 Basis Functions

Polynomial Functions

We can search the basis functions in the same form as in (4.17). Now,

\[
c_{i0} = 4 \frac{i^2 + 5i + 6}{i(i + 1)} c_{i2}, \quad c_{i1} = -4 \frac{i + 3}{i + 1} c_{i2}
\]

and

\[
c_{i2} = \frac{i(i + 1) \sqrt{2i + 3} \sqrt{2i + 5} \sqrt{2i + 7}}{2^i \sqrt{41728i^2 + 169728i + 161280}}
\]

Figure 4.9 shows the resulting set.

Rayleigh–Krylov Functions

The functions in Figure 4.10 also satisfies the boundary conditions:

\[
\varphi_i(\xi) = U(\beta_i \xi) - \frac{S(2\beta_i)}{T(2\beta_i)} V(\beta_i \xi), \tag{4.22}
\]

where \(S(x), T(x), U(x)\) and \(V(x)\) are the Rayleigh–Krylov functions, and \(\beta_i\) are successive roots of \(\cos 2x \cosh 2x = -1\) (see Appendix C).
Chebyshev Polynomials

We can also apply Chebyshev polynomials in this case. Using formula (4.19) with the appropriate boundary conditions we obtain a new set of functions. Their graphs are shown in Figure 4.11.

4.4.2 Stability Analysis in Autonomous Case

If the perturbation amplitude of the flow velocity is zero ($\nu = 0$), Eq. (4.16) will be a system of autonomous differential equations ($\ddot{u}(\tau) \equiv \ddot{U} \equiv U \frac{d}{dt}$ and $\dddot{u}(\tau) \equiv 0$).

The coefficients of the characteristic polynomial defined in (4.20) are summarized in Table 4.5 for $n \leq 2$.

After constructing the Hurwitz determinant from $a_i$-s we can apply the Routh–Hurwitz criterion to determine the stability of the characteristic polynomial ($\Re \lambda_i < $
Figure 4.11: Basis functions of Chebyshev polynomials.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Rayleigh–Krylov stable intervals</th>
<th>P-L constant</th>
<th>Chebyshev stable intervals</th>
<th>P-L constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[0, \infty)$</td>
<td>-</td>
<td>$[0, \infty)$</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>$[0, 3.3419)$</td>
<td>-0.0805</td>
<td>$[0, 3.3336)$</td>
<td>-0.351</td>
</tr>
<tr>
<td>3</td>
<td>$[0, 4.2904)$</td>
<td>-0.2752</td>
<td></td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>$[0, 4.2716)$</td>
<td>-0.5367</td>
<td>$[0, 4.3031)$</td>
<td>-2.307</td>
</tr>
<tr>
<td>5</td>
<td>$[0, 4.2408)$</td>
<td></td>
<td>$[0, 4.2614)$</td>
<td>-2.256</td>
</tr>
<tr>
<td>6</td>
<td>$[0, 4.2411)$</td>
<td></td>
<td>$[0, 4.2354)$</td>
<td>-2.221</td>
</tr>
</tbody>
</table>

Table 4.6: Stable intervals of $\bar{U}$ in autonomous case.

0, $\forall i$. The stability intervals for $\bar{U}$ listed in the Table 4.6 were obtained at $\mu = 1$.

When we use only one basis function ($\eta = 1$) the system is always stable. However, in case of $n > 1$ a critical value of $\bar{U}$ appears above which the linearized system loses its asymptotically stable behaviour and becomes unstable. At these critical values of $\bar{U}$ a pair of pure imaginary roots crosses the imaginary axis, i.e. Hopf bifurcations occur. Because of the symmetric nonlinearities (there isn’t any second order term) the plane of the critical eigenvectors approximates the centre manifold in second order. Hence, the centre manifold reduction can be done easily. The bifurcation analysis results negative Poincaré–Liapunov constants (for $n > 1$), i.e. super-critical Hopf bifurcations take place in these cases. Figure 4.12 shows the graphs of two solutions obtained by numerical simulations which converge to a stable limit limit cycle.
\[ \tilde{U} = 4.31, \nu = 0 \]

Figure 4.12: Numerical solutions above \( \tilde{U}_{cr} \) (4 Chebyshev modes).

### 4.4.3 Non-autonomous Case

**Linearized System**

Sinha’s numerical method provides the stability domains of the linearized system. Figures 4.13(a) and 4.13(b) show slices of the three dimensional \( (\tilde{U}, w, \nu) \) theoretic stability domain in the vicinity of the critical value \( \tilde{U}_{cr} \) obtained in the autonomous case. The dotted region represents the stable domain of the analyzed space. The colour of the boundary curves shows the type of stability loss, i.e. the way the characteristic multipliers leave the unit circle while crossing the stability boundary.

Figures 4.14 and 4.15 were obtained by using 4 and 5 modes, respectively. It is worth noticing that the more basis functions are used in the discretization, the structure of the stability domains becomes the more complicated for low values of \( w^{-1} \equiv \alpha/\omega \). Thus, one should consider applying more spatial modes to describe the more exact behaviour of the model if \( w^{-1} < 0.09 \).

Stability charts of Figures 4.16–4.18 were computed using Chebyshev type basis functions.

Table 4.7 summarize the CPU time needed to obtain these stability charts.
\[ \tilde{U} = 4.29 \]

\[ \lambda = \pm 1 \]
\[ \lambda_{1,2} = \sqrt{1 + \varphi^2} \]
\[ \lambda = -1 \]
stable
err \[ \tilde{U} = 4.30 \]

(a) Computed below \( \tilde{U}_{\alpha} \)

(b) Computed above \( \tilde{U}_{\alpha} \)

Figure 4.13: Stability charts of a cantilever pipe (3 Krylov modes).
Figure 4.14: Stability charts of a cantilever pipe (4 Krylov modes).
Figure 4.15: Stability charts of a cantilever pipe (5 Krylov modes).
Figure 4.16: Stability charts of a cantilever pipe (4 Chebyshev modes).
Figure 4.17: Stability charts of a cantilever pipe (5 Chebyshev modes).
Figure 4.18: Stability charts of a cantilever pipe (6 Chebyshev modes).
CHAPTER 4. ELASTIC PIPES  

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\tilde{U}$</th>
<th>$\tilde{U}$</th>
<th>Krylov modes</th>
<th>Chebyshev modes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td># of pts.</td>
<td>CPU</td>
<td># of pts.</td>
<td>CPU</td>
</tr>
<tr>
<td>3</td>
<td>4.29</td>
<td>31215</td>
<td>53'6&quot;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.30</td>
<td>36920</td>
<td>1h 03'40&quot;</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4.27</td>
<td>36685</td>
<td>1h 46'36&quot;</td>
<td>4.30</td>
</tr>
<tr>
<td></td>
<td>4.28</td>
<td>42310</td>
<td>2h 03'13&quot;</td>
<td>4.31</td>
</tr>
<tr>
<td>5</td>
<td>4.24</td>
<td>37455</td>
<td>2h 37'29&quot;</td>
<td>4.26</td>
</tr>
<tr>
<td></td>
<td>4.25</td>
<td>43860</td>
<td>3h 03'11&quot;</td>
<td>4.27</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>-</td>
<td></td>
<td>4.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4.24</td>
</tr>
</tbody>
</table>

Table 4.7: Computational statistics of a cantilever elastic pipe.

**Nonlinear System**

In order to be able to perform the stability investigations of the nonlinear system (4.16) we follow the procedure described in Section 2.2.3. Because of the symmetric nonlinearities the centre manifold reduction is trivial.

The bifurcation analysis of the system with 4 Chebyshev modes was performed at some typical points in plane $\tilde{U} = 4.31$ where line $\nu = 0.15$ crosses the curves of the stability boundary (see Figure 4.16(b) for the location of some of these points). The results of the analysis are summarized in Table 4.8. One can notice that flip ($N = 1, 2, 9, 10$) and fold ($N = 5, 6$) bifurcations bounded unstable regions alternate each other with Hopf bifurcation bounded regions ($N = 3, 4, 7, 8$) in between. On the right-most boundary ($N = 11$) and at the bottom ($N = 12$) Hopf bifurcation can be observed. The bifurcation analysis yielded negative Poincaré–Liapunov constants that means super-critical bifurcations occur. However, at points 2, 6 and 10 the centre manifold has positive coefficient in the nonlinear term, i.e. sub-critical flip and fold bifurcations take place at these points, respectively.

We computed bifurcation diagrams in three critical interval starting from the points of $N = 1, 5, 9$ (starting with flip, fold, and flip bifurcations of the trivial solution, respectively):

$N = 1 : w^{-1} = 0.024, \ldots, 0.028$ Figure 4.19(b) shows a bifurcation diagram at $(\tilde{U}, \nu) = (4.31, 0.15)$ changing the value of $w^{-1}$ from 0.024 to 0.028 by step size $10^{-4}$. On the vertical axis are the values of $g_3(t)$ acquired with frequency $2/T$, where $T$ is the principal period. At the beginning, the flip bifurcation can be seen (2$T$-periodic solution), and after $w^{-1} = 0.0251$ it goes through a Hopf bifurcation with a frequency something greater than $1/10T$. This frequency of the quasi-periodic solution decreases slightly, and when it equals $1/10T$ at $w^{-1} = 0.027$ the phase-locked solution (see the Poincaré map in Figure 4.20(a)) has another Hopf bifurcation: small ‘cycles’ appear
at the place of the ten points of the 10T-periodic solution in the Poincaré map. The
frequency of these ‘cycles’ can be seen at the leftmost side of Figure 4.20(b) near 1/30T.
And finally chaotic behaviour appears as we increase $w^{-1}$ further.

All of these can be observed in the waterfall diagram in Figure 4.19(a): the peak at
$f = 0.5$ of the flip bifurcation, the other peak around $f = 0.1$ of the mentioned Hopf
bifurcation, and the ‘noisy’ chaotic solution when $w^{-1} > 0.027.$

$N = 5 : w^{-1} = 0.0566, \ldots, 0.0603$ Figure 4.21 shows a bifurcation diagram at
$(\vec{U}, \nu) = (4.31, 0.15)$ changing the value of $w^{-1}$ from 0.0566 to 0.0603 by $10^{-4}\ldots$. On the
vertical axis are the values of $\dot{y}_2(t)$ acquired with frequency $2/T$. At the beginning,
the fold bifurcation can be seen ($T$-periodic solution), and after $w^{-1} = 0.0587$ it goes
through a Hopf bifurcation (see the Poincaré map in Figure 4.22). This stable quasi-
periodic solution remains when the origin becomes stable again at $w^{-1} = 0.0603$ where
sub-critical bifurcation should be, i.e. there must be an unstable branch between the
two stable sets.

$N = 9 : w^{-1} = 0.097, \ldots, 0.162$ In this interval no further bifurcation was detected.
The Poincaré map of each coordinate of $y_i(t)$ ($i = 1, 2, 3, 4$) consists of two stable points
because of the flip bifurcation, i.e. $2T$-periodic nontrivial solutions appear. Figure 4.23
shows how the locations of these points are changing at $(\vec{U}, \nu) = (4.31, 0.15)$ as we
increase the value of $w^{-1}$ from 0.097 to 0.162 by step size $10^{-3}\ldots$. 

\[
\begin{array}{|c|c|c|c|c|}
\hline
N & \nu & w^{-1} & \lambda_{cr} & \text{type} & \text{P–L} \\
\hline
& & & (\alpha^2 + \beta^2 = 1) & \text{of bif.} & \text{const.} \\
\hline
1 & 0.15 & 0.0238963 & -1 & \text{flip} & -236.90 \\
2 & 0.15 & 0.0331140 & -1 & \text{flip} & 82.14 \\
3 & 0.15 & 0.0428920 & \alpha \pm i\beta & \text{Hopf} & -9.320 \\
4 & 0.15 & 0.0505600 & \alpha \pm i\beta & \text{Hopf} & -0.058 \\
5 & 0.15 & 0.0565510 & 1 & \text{fold} & -44.82 \\
6 & 0.15 & 0.0602135 & 1 & \text{fold} & 31.69 \\
7 & 0.15 & 0.0733880 & \alpha \pm i\beta & \text{Hopf} & -1.981 \\
8 & 0.15 & 0.0757590 & \alpha \pm i\beta & \text{Hopf} & -1.839 \\
9 & 0.15 & 0.0974200 & -1 & \text{flip} & -8.125 \\
10 & 0.15 & 0.1620620 & -1 & \text{flip} & 2.673 \\
11 & 0.15 & 0.1976700 & \alpha \pm i\beta & \text{Hopf} & -2.390 \\
12 & 0.033 & 0.1 & \alpha \pm i\beta & \text{Hopf} & -3.915 \\
\hline
\end{array}
\]

Table 4.8: Poincaré–Liapunov constants at critical points.
$\tilde{U} = 4.31, \nu = 0.15$

(a) Waterfall Diagram

$\tilde{U} = 4.31, \nu = 0.15$

(b) Bifurcation Diagram of $\dot{y}_3$

Figure 4.19: Bifurcation analysis between $w^{-1} = 0.024, \ldots, 0.028$. 
Figure 4.20: Analyzing the numerical solution at $w^{-1} = 0.027$. 
CHAPTER 4. ELASTIC PIPES

\[ \ddot{U} = 4.31, \nu = 0.15 \]

Relative frequency, \( w^{-1} = \alpha/\omega \)

Figure 4.21: Bifurcation diagram of \( \dot{y}_2 \) between \( w^{-1} = 0.0566, \ldots, 0.0603 \).

\[ \ddot{U} = 4.31, \nu = 0.15, w^{-1} = 0.060 \]

Figure 4.22: Poincaré map of \( y_i(t) \) at \( w^{-1} = 0.060 \).
Figure 4.23: Changing fixed points in the Poincaré map \((w^{-1} = 0.097, \ldots, 0.162)\).

4.5 New Results

**Thesis 4** Applying Hamilton's principle, I derived the equation of motion of an elastic pipe conveying incompressible and frictionless fluid, and I performed the critical analysis of the special literature. I showed that the boundary conditions of clamped–pinned end case result a conservative system. The critical flow velocities of the autonomous system showed three-digit agreement with each other even using 2–3 basis functions only. However, increasing the number of basis functions, the stability chart of the periodic system shows more and more complex, fractal-like structure: newer and newer stable regions appear in the high frequency range, while their sizes decrease rapidly above the critical speed. Hence, the implementation of stabilization with relatively high pulsation frequency becomes very difficult and doubtful.

**Thesis 5** I determined the stability charts of a cantilever elastic pipe conveying fluid applying various kind and number of basis functions. This system is non-conservative and at least 4–5 basis functions are needed to obtain results acceptable for the engineering practice. The structure of the stable domains becomes intricate as the number of applied basis functions increases. I showed that above the critical flow velocity the modelled structure can be stabilized by pulsatile flow and the bifurcations that can be observed in the stability chart are similar to those described in Thesis 2.

I also determined the super- or sub-critical nature of each point on the stability
boundary of three typical unstable parameter ranges. I constructed bifurcation diagrams by numerical simulations at the super-critical points (where stable oscillations occur). Decreasing the highest critical relative frequency through the unstable parameter zone, the fixed points originated in $2T$-periodic motions born from a super-critical flip bifurcation go through Hopf bifurcation and the motion becomes quasi-periodic. Decreasing the relative frequency further, a stable $10T$-periodic solution appears which becomes chaotic after further bifurcations.

I showed the appearance of stable quasi-periodic oscillations, i.e. the Hopf bifurcation of the stable fixed point in the unstable interval bounded by $T$-periodic stability loss that is super-critical on one side and sub-critical on the other side.
Chapter 5

A Single Railway Wheelset

As a construction containing rotating parts like wheels, the model of a single railway wheelset can also have parametric excitation. However, the goal of the investigation to describe in this chapter is to analyze how the errors in mounting and manufacturing of the driving wheels of the built experimental rig influence its dynamical behaviour.

We examine the motion of the wheelset running along a straight but harmonically perturbed, smooth and horizontal track when its centre of gravity is towed with a constant velocity $v$ along the track. The model has two degrees of freedom, the lateral displacement and the yaw angle (i.e. the rotation around the vertical axis). We neglect the roll motion and also the self-aligning torque induced by the so-called spin creepage. Linear creep force law is used and the flange contact is not modelled. Analytical results of the nonlinear model without harmonic excitation are given by Kaas–Petersen [16] and by Lóránt and Stépán [20].

The numerical tools used here also for harmonic parametric excitation are based on the approximation of the State Transition Matrix with Chebyshev polynomials. During the experiments we are able to detect the exact values of the towing velocity and the lateral acceleration of the wheelset. Based on the above two signals, a waterfall diagram is generated and the stability loss is determined with it.

5.1 Wheelset Model

The model in question is presented in Figure 5.1. The system has two degrees of freedom, the corresponding generalized coordinates are $q_1$ (lateral displacement of the centre of gravity) and $q_2$ (yaw angle). The parameters are as follows: $m$ and $J$ denote the mass of the wheelset and its mass moment of inertia about the vertical axis, respectively. There are three geometrical parameters: $h$ is the half track gauge, $r$ is the radius of wheels when they are centred on the track, and $c$ is the wheel conicity. The longitudinal velocity component $v$ of the wheelset centre of gravity is constant. This
non-stationary constraint results that the mechanical model in question is rheonomic, so there is energy introduced and/or dissipated into and/or from the system via the constraining force which provides the prescribed constant speed.

There are two types of active forces in the model: the spring forces and the creep forces. The wheelset is connected to the frame of constant speed with linearly elastic springs: $k_1$ is the stiffness of springs normal to the track, and $k_2$ is the stiffness of the springs along the track which are located at a distance $d$ from the centre of gravity. The wheels and the rails are in contact during the motion. The contact force components (also called creep forces) $F_x$ and $F_y$ depend on the creeps $\xi_x$ and $\xi_y$ defined later.

The Newtonian equations of motion lead to a system of two nonlinear second order autonomous ordinary differential equations (see Lóránt and Stépán [20]). After linearization at the stationary motion (trivial solution) it assumes the form:

$$
\begin{bmatrix}
m & 0 \\
0 & J
\end{bmatrix} \ddot{q} + 2 \begin{bmatrix}
k_1 & 0 \\
0 & k_2d^2
\end{bmatrix} q + 2 \begin{bmatrix}
f_2\xi_x \\
h_{11}\xi_y
\end{bmatrix} = 0 ,
$$

(5.1)

where the last term on the left-hand side describes the creep forces. The creeps $\xi_x, \xi_y$ have the following definitions (see Kalker [17]):

$$
\begin{bmatrix}
\xi_x \\
\xi_y
\end{bmatrix} = \begin{bmatrix}
\frac{1}{v} & 0 \\
0 & \frac{1}{v}
\end{bmatrix} \dot{q} + \begin{bmatrix}
0 & -1 \\
\xi & 0
\end{bmatrix} q .
$$

(5.2)
Thus, the mathematical model near the stationary motion reads as

\[
\ddot{q} + \frac{1}{v} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \dot{q} + \begin{bmatrix} \alpha_1^2 & -c_1 \\ Cc_2 & \alpha_2^2 \end{bmatrix} q = 0,
\]

where the natural angular frequencies of the creep-force-free system are denoted by \( \alpha_{1,2} \), while \( c_1 \) and \( c_2 \) are the linear creep force coefficients transformed to the given system, that is

\[
\alpha_1^2 = \frac{2k_1}{m}, \quad \alpha_2^2 = \frac{2k_2d_2^2}{J}, \quad c_1 = \frac{2}{m} f_{22}, \quad c_2 = \frac{2h^2}{J} f_{11}.
\]

There is one parameter left which involves the conicity and all the other geometrical parameters in the form

\[
C = \frac{c}{hr}.
\]

This parameter will be referred to as the conicity parameter and is assumed to be positive.

The Routh–Hurwitz stability investigation shows the following critical value for the towing speed:

\[
v_{\text{crit}} = \sqrt{\frac{(c_1 + c_2) (c_1 \alpha_2^2 + c_2 \alpha_1^2)}{C (c_1 + c_2)^2 - (\alpha_1^2 - \alpha_2^2)^2}}.
\]

Below this critical velocity the stationary motion is exponentially asymptotically stable while it is unstable above \( v_{\text{crit}} \).

### 5.2 Experimental Roller Rig

Figure 5.2 shows the experimental setup. The rails are realized by two big wheels. These wheels provide the constant angular velocity \( \omega \) for the wheelset. The velocity of the wheelset is assumed as

\[
v = \omega r = \Omega R,
\]

where \( \Omega \) is the angular velocity of the driving wheels. Their radius \( R = 90 \text{ mm} \) is nearly five times greater than the radius of the railway wheel \( r = 21.75 \text{ mm} \). This results in a much better approximation of the plain rails than the usual small rollers. The wheelset is connected to the board by four springs. The board offers various connecting positions for the springs. Thus, the equivalent linear stiffnesses denoted by \( k_1 \) and \( k_2 \) (see Figure 5.1) can be set to different values. The conicity is \( c = 1/8.3 \), and the half length of the distance of the contact points is \( h = h_0 = 35 \). Hence, \( C = 158.27 \text{ m}^{-2} \).

The lateral acceleration is measured with an accelerometer connected to the end
of the axle while the driving wheel angular velocity is sampled with an optical sensor. The acquired data are processed by a personal computer.

For suspension configuration case $N = 7$ in Table 5.1, Figure 5.3 shows a typical waterfall diagram of the wheelset motion going through the critical velocity. Each curve was obtained by applying the Fast Fourier Transformation on the acquired lateral acceleration data at a given speed. If the geometry of the experimental rig were ideal, no vibration response would appear at low running speed. However, the deviation in the geometry of the driving wheelset results a sharp peak in the spectrum exactly at the frequency of its speed of rotation. The loss of stability at about $10 \div 10.5$ m/s is recognized from the collapse of this peak and the appearance of a rich frequency spectrum. Lóránt and Stépán [20] proves the existence of chaotic motion right above the critical speed. When the wheel flanges start to hit the driving wheels (imitating the rails), the frequency domain becomes rich also in sub-harmonics.

In order to see how the wheel geometry deviation influences the critical velocity and the dynamics of the rig, we improve the equations of motion (5.3) by introducing time-dependent coefficients. The next section describes the corresponding parametric excitation problem.

### 5.3 Time-Periodic Equations of Motion

The distance of the driving wheels changes periodically due to the finite precision in the manufacturing and mounting of the wheelset of the rig representing the rails. Thus, we assume

$$h(t) = h_0(1 + \eta \sin \Omega t),$$

where $\eta$ is the relative perturbation amplitude of the distance between the rails.
Figure 5.3: Waterfall diagram of the measurements.

Introducing \( \tau = vt/K \) as dimensionless time (\( K = 2\pi R \) is the circumference of the driving wheel) we obtain the following form of the equations of motion

\[
v^2 q'' + K \begin{bmatrix} c_1 & 0 \\ 0 & c_2 (1 + \eta \sin 2\pi \tau)^2 \end{bmatrix} q' + K^2 \begin{bmatrix} 0 & \alpha_2^2 \\ \alpha_2^2 & -c_1 \end{bmatrix} q = 0, \quad (5.6)
\]

where \( ' \) denotes \( d/d\tau \) and

\[
c_2 = \frac{2h_0^2}{J} f_{11}, \quad C = \frac{c}{h_0 r}.
\]

### 5.3.1 Computational Results

The numerical investigation based on Chebyshev polynomials (see Section 2.3) provides us the stability chart of the system. We have chosen \( v \) and \( \eta \) as bifurcation parameters.

The cases \( N = 1, \ldots, 8 \) refer to eight different suspension spring configurations which result different stiffnesses, i.e. different natural frequencies and different creep coefficients. These are summarized in Table 5.1. Figure 5.4 presents the stability boundaries of each case in the \( (v, \eta) \) parameter plane. At a certain suspension configuration \( N \), the stationary motion is asymptotically stable for parameters situated to the left of the corresponding boundary curve.

Because of the large steepness of the curves the critical number of rotations will not much differ from the one of the autonomous case at small value of \( \eta \), since \( \eta = 0.003 \) is
<table>
<thead>
<tr>
<th>N</th>
<th>(c_1) [m/s^2]</th>
<th>(c_2) [m/s^2]</th>
<th>(\alpha_1/2\pi) [Hz]</th>
<th>(\alpha_2/2\pi) [Hz]</th>
<th>Analytically Computed</th>
<th>Numerically Comp. (\eta = 0)</th>
<th>Numerically Comp. (\eta = 0.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12007.1</td>
<td>18007.9</td>
<td>9.034</td>
<td>13.474</td>
<td>5.5076</td>
<td>5.5094</td>
<td>5.4938</td>
</tr>
<tr>
<td>2</td>
<td>14781.6</td>
<td>22168.9</td>
<td>9.008</td>
<td>17.149</td>
<td>6.4422</td>
<td>6.4437</td>
<td>6.4281</td>
</tr>
<tr>
<td>3</td>
<td>16508.0</td>
<td>24758.2</td>
<td>7.638</td>
<td>19.531</td>
<td>6.8428</td>
<td>6.8438</td>
<td>6.8344</td>
</tr>
<tr>
<td>4</td>
<td>17023.0</td>
<td>25530.5</td>
<td>10.077</td>
<td>19.653</td>
<td>7.3322</td>
<td>7.3344</td>
<td>7.3187</td>
</tr>
<tr>
<td>5</td>
<td>18220.7</td>
<td>27326.7</td>
<td>10.948</td>
<td>21.129</td>
<td>7.9068</td>
<td>7.9094</td>
<td>7.8906</td>
</tr>
<tr>
<td>6</td>
<td>19943.5</td>
<td>29910.6</td>
<td>10.687</td>
<td>23.639</td>
<td>8.5386</td>
<td>8.5406</td>
<td>8.5250</td>
</tr>
<tr>
<td>7</td>
<td>21330.9</td>
<td>31991.3</td>
<td>11.183</td>
<td>25.837</td>
<td>9.2420</td>
<td>9.2438</td>
<td>9.2281</td>
</tr>
<tr>
<td>8</td>
<td>22882.7</td>
<td>34318.7</td>
<td>10.970</td>
<td>28.179</td>
<td>9.8680</td>
<td>9.8688</td>
<td>9.8562</td>
</tr>
</tbody>
</table>

Table 5.1: Critical velocities in [m/s].

![Stability boundary](image)

Figure 5.4: Boundaries of stability domains for different parameter values.
the realistic value in the above described experiment. That is, small deviation in the
graphy of the driving wheelset modelling the rails will not influence the boundary of
the stability significantly.

At the stability boundary, a pair of complex conjugate roots leave the unit disk, i.e.
Hopf bifurcation occurs. If nonlinear creep force terms are also included in equation
(5.6), the Liapunov–Floquet transformation is to be carried out to obtain an equivalent
autonomous system for bifurcation analysis (see Section 2.2.3).

5.4 New Results

Thesis 6 In the experimental rig of a single railway wheelset, I investigated the effect of
the parametric excitation on the critical speed. The parametric excitation is originated
in the geometrical inaccuracies occurring during the manufacturing of the roller rigs.
I proved that the critical running speeds for a single railway wheelset are not affected
substantially by the parametric excitation for realistic finite precision in manufacturing
and mounting. The dependence of the critical speed on the perturbation amplitude
qualitatively the same for a wide range of suspension configurations.
Chapter 6

Summary

We analyzed the dynamical behaviour of different mechanical systems, which were parametrically excited. We derived the equations of motions of different pipe models containing fluid flow and investigated the mathematical models from their simplest—linearized, autonomous—form to the complicate—nonlinear, time-periodic—case. The results of analytical and semi-analytical stability analysis was checked by numerical simulations. The analysis of periodic systems was performed by applying the numerical method based on Chebyshev polynomials.

After we introduce the applied analytical tools of the stability theory of differential equations, in Section 2.4 we compared the stability charts obtained by using the method of small-parameters, Runge–Kutta method and Chebyshev polynomials, respectively. We saw that Poincaré’s method gives acceptable result only if the parameter $\varepsilon$ is really small ($\varepsilon < 1$). However, the other two methods gave indistinguishable results in the analyzed interval of $(\delta, \varepsilon)$.

In Chapter 3 we analyzed three different models of articulated rigid pipes with elastic joints. The model consisting of only one pipe has only one degree of freedom, and it is a linear system. It was shown that this system is technically stable even if the flow inside the pipe pulsates but does not change its direction ($\nu < 1$). The method of Chebyshev polynomials can be 30 times faster than the commonly used numerical integration techniques, however, it yields uncertain results if the forcing frequency is relatively small ($w^{-1}$ is large). Thus, one must take care of the estimated error boundary when generating stability charts and repeat the computations with increased number of Chebyshev polynomials if it is required.

The other model consisting of two pipes with free end can be destabilized by increasing the flow velocity. At the point of stability loss, super-critical Hopf bifurcation takes place, i.e. the pipes will flutter. Adding small harmonic perturbation to the flow velocity the system can be stabilized as the computed stability charts showed. Furthermore, the effects of nonlinear terms were also investigated: super- and sub-critical flip,
fold and Hopf bifurcations were detected. These results were confirmed by numerical simulations which did not show any secondary bifurcations.

In the last model of Chapter 3, the end of the second pipe was constrained to move only in the $x$ direction. Since, its linearization yielded a Hamiltonian system, only marginally stable points could be detected in the stability charts.

In Chapter 4 we investigated elastic pipes with flowing fluid and two different boundary conditions. The derived partial differential equation of motion was in integral form, which was simplified by using Galerkin method. In both cases the upstream end of the pipe was clamped. In the first case the downstream end of the pipe was simply supported. As the articulated model, this one has also a Hamiltonian equation of motion. In autonomous case, the critical flow velocity, at which the pipe buckles, can be well approximated even if we use only two or three basis functions. However, the more basis functions were used in non-autonomous case, the more sophisticated structure appeared in the obtained stability charts. The second observation was that the more basis functions were used (i.e. the larger linear algebraic system is to be solved), the error boundary was shifted the more left (i.e. the smaller the area was where the results are satisfactory).

In the case of cantilever elastic pipe, we need four or five basis functions to obtain acceptable approximation of the critical flow velocity at which the pipe will flutter (super-critical Hopf bifurcation). In the stability charts of the non-autonomous system, it could be seen that the more details we get in the high frequency domain (small $w^{-1}$) as the more basis functions are applied. The nonlinear bifurcation analysis at certain critical points showed super- and sub-critical flip, fold and Hopf bifurcations, respectively. The results of numerical simulations confirmed the analytically computed types of bifurcations, and in the generated bifurcation diagram higher order Hopf bifurcations could also be observed which lead to chaotic behaviour.

In Chapter 5 we investigated the parametrically excited linear equations of a roller rig of a single railway wheelset. The excitation came from the deviation of geometry of the wheels driving the rig. The investigation yielded that this error does not influence significantly the critical towing velocity.
Appendix A

Chebyshev Polynomials

A.1 Chebyshev Polynomials of The First Kind

Chebyshev polynomials of the first kind, shown in Figure A.1, are defined as follows:

\[ T_p(t) = \cos p\theta, \quad t = \cos \theta, \quad (-1 \leq t \leq 1, \ p = 0, 1, 2, \ldots). \] (A.1)

Their multiplicative property comes from the trigonometry:

\[ \cos p\theta \cos r\theta = \frac{1}{2} (\cos (p+r)\theta + \cos (p-r)\theta) \implies 2T_p(t)T_r(t) = T_{p+r}(t) + T_{p-r}(t) \]

Thus, they can be defined recursively:

\[ T_{p+1}(t) = 2tT_p(t) - T_{p-1}(t), \] (A.2)

where \( T_0(t) \equiv 1 \) and \( T_1(t) = t \).

Hence, the powers of \( t \) can be given using Chebyshev polynomials:

\[ T_2(t) = \ 2t^2 - 1 \quad \implies \ t^2 = \frac{1}{2}(T_0 + T_2) \]
\[ T_3(t) = \ 4t^3 - 3t \quad \implies \ t^3 = \frac{1}{4}(3T_1 + T_3) \]
\[ T_4(t) = \ 8t^4 - 8t^2 + 1 \quad \implies \ t^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4) \]

It can be seen that the index of a Chebyshev polynomial refers to its order.

The integrating formula below can also be proved:

\[ \int_0^t T_p(\tau) \, d\tau = \frac{1}{2} \left( \frac{T_{p+1}(t)}{p+1} - \frac{T_{p-1}(t)}{p-1} \right), \quad (p \geq 2). \] (A.3)
Chebyshev polynomials are orthogonal in the interval $[-1, 1]$:

$$\int_{-1}^{1} T_p(t) T_r(t) \frac{1}{1 - t^2} \, dt = \delta_{pr}.$$  \hspace{1cm} (A.4)

These properties can also be shown in the case of Chebyshev polynomials of the second kind:

$$U_p(t) = \frac{T_{p+1}}{p+1} = \frac{\sin(p+1)\theta}{\sin \theta}, \quad t = \cos \theta, \quad (-1 \leq t \leq 1, \ p = 0, 1, 2, \ldots).$$ \hspace{1cm} (A.5)

### A.2 Shifted Chebyshev Polynomials

The shifted Chebyshev polynomials of the first kind are defined as follows:

$$\tilde{T}_p(t) = T_p(2t - 1), \quad (0 \leq t \leq 1).$$ \hspace{1cm} (A.6)

However, we drop the ‘tilde’ because we shall use only the shifted polynomials.

They are orthogonal in the following manner:

$$\int_{-1}^{1} T_p(t) T_r(t) w(t) \, dt = \begin{cases} 0, & (p \neq r) \\ \pi, & (p = r = 0) \\ \frac{\pi}{2}, & (p = r > 0) \end{cases}$$ \hspace{1cm} (A.7)

where the weight function is $w(t) = (t - t^2)^{-1/2}$. 
A.2.1 Chebyshev Expansion of Arbitrary Continuous Function

An arbitrary function that is continuous in the interval \([0, 1]\) can be expanded in terms of shifted Chebyshev polynomials:

\[
f(t) = \sum_{p=0}^{\infty} f_p T_p(t), \quad (0 \leq t \leq 1),
\]

where the coefficients of the Chebyshev polynomials can be determined as

\[
f_p = \frac{1}{\delta_p} \int_{-1}^{1} w(t) f(t) T_p(t) dt, \quad (p = 0, 1, 2, \ldots),
\]

with \(\delta_0 = \pi\) and \(\delta_p = \frac{\pi}{2}\) if \(p > 0\).

The expansion of any function that is continuous in any closed interval \([t_1, t_2]\) can be done after the corresponding transformation to the interval \([0, 1]\).

A.2.2 Operator Matrices of Chebyshev Polynomials

Integrate Operator Matrix

The *integrating formula* of the shifted Chebyshev polynomials of the first kind is

\[
\int_{0}^{t} T_p(\tau) d\tau = \frac{1}{4} \left( \frac{T_{p+1}}{p+1} - \frac{T_{p-1}}{p-1} \right) - \frac{(-1)^p}{2(p-1)(p+1)} T_0, \quad (p = 0, 2, 3, 4, \ldots).
\]

In the case of \(p = 1\):

\[
\int_{0}^{t} T_1(\tau) d\tau = \frac{T_2 - T_0}{8}.
\]

Generally, Eqs. (A.9) and (A.10) can be written as

\[
\int_{0}^{t} T_p(\tau) d\tau = T_r(t) G_{rp},
\]
where the integral operator matrix is

\[
[G_{rp}]_{(m \times m)} =
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{16} & \cdots & -\frac{1}{4(m-1)} \\
\frac{1}{2} & 0 & -\frac{1}{4} & 0 & \cdots & 0 \\
0 & \frac{1}{8} & 0 & -\frac{1}{8} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \frac{1}{4(m-2)} & 0 \\
0 & \cdots & \cdots & 0 & 0 & 0 \\
\end{bmatrix}.
\] (A.12)

Thus, the integral of a continuous function \( f \) is

\[
\int_0^t f(\tau) d\tau = T_r(t) G_{rp} f_p,
\] (A.13)

where \( p \) and \( r \) are summing indices according to Einstein’s convention.

**Multiplication Operator Matrix**

According to the multiplicative property

\[
[T_p(t)T_r(t)] = \frac{1}{2}
\begin{bmatrix}
2T_0 & 2T_1 & 2T_2 & \cdots & 2T_{m-1} \\
2T_1 & T_2 + T_0 & T_1 + T_3 & \cdots & T_{m-2} + T_m \\
2T_2 & T_3 + T_1 & T_2 + T_0 & \cdots & T_{m-3} + T_{m+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2T_{m-1} & T_m + T_{m-2} & T_{m-1} + T_{m-3} & \cdots & T_{2m-2} + T_0 \\
\end{bmatrix} = T_q(t) Q_{qpr}.
\]

Hence, the multiplication of two functions \( f(t) = f_p T_p(t) \) and \( g(t) = g_r T_r(t) \) which are continuous on the interval \([0,1]\) can be written as

\[
f(t)g(t) = T_q Q_{qpr} f_p g_r = T_q(t) F_{qrfpr} ,
\] (A.14)

where \( F_{qrf} = Q_{qrf} f_p \) is the multiplication operator matrix of function \( f(t) \):

\[
F_{qr} = \frac{1}{2}
\begin{bmatrix}
2f_0 & f_1 & f_2 & \cdots & f_{m-1} \\
2f_1 & 2f_0 + f_2 & f_1 + f_3 & \cdots & f_{m-2} + f_m \\
2f_2 & f_1 + f_3 & 2f_0 + f_4 & \cdots & f_{m-3} + f_{m+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2f_{m-1} & f_{m-2} + f_m & f_{m-3} + f_{m+1} & \cdots & 2f_0 + f_{2m-2} \\
\end{bmatrix}.
\] (A.15)
A.3 Applying Operator Matrices in Linear Algebra

If all the components of the vector $\mathbf{x}(t) \equiv [x^i(t)]$ are continuous in the interval $[0, 1]$ then its Chebyshev expansion is

$$\mathbf{x}(t) = \left[ T_p(t)x^i_p \right]. \quad (A.16)$$

and similarly, the expansion of $\mathbf{A}(t) \equiv [a^{ij}(t)]$ is

$$\mathbf{A}(t) = \left[ T_p(t)a^{ij}_p \right]. \quad (A.17)$$

Thus,

$$\mathbf{A}(t)\mathbf{x}(t) = \left[ T_q Q_{pq} a^{ij}_p x^j_p \right] \equiv \left[ T_q(t)A^{ij}_q x^j_r \right]. \quad (A.18)$$

The integral of $\mathbf{x}(t)$ is

$$\int_0^t \mathbf{x}(\tau) d\tau = \left[ T_p(t)G_{pr} x^i_r \right]. \quad (A.19)$$
Appendix B

Deriving the Equation of Elastic Pipe

If we integrate Eq. (4.10) by parts we get

\[
\int_{t_1}^{t_2} \int_0^L (I_x E \frac{\partial}{\partial t} (y^{'''} (1 + y^2) + 4y^'' y' + y'^3) + \delta \frac{\partial}{\partial t} (M + m)\mathbf{r} + 2mu\mathbf{r}' + m\mathbf{u}\mathbf{r}')) dX dt
\]

\[
= \int_{t_1}^{t_2} \int_0^L \left[ (y^{'''} (1 + y^2) + y^''')y' + (1 + y'^2) \right]_{t_1}^{t_2} dX dt + \int_{t_1}^{t_2} \int_0^L [(M + m)\mathbf{r} + u\mathbf{r}'] dX dt
\]

\[
+ \int_{t_1}^{t_2} \left[ mu\mathbf{r} + m\mathbf{u}\right]_{t_1}^{t_2} dX dt - \int_{t_1}^{t_2} \left[ mu(\mathbf{r} + u\mathbf{r}')\mathbf{r} + m\mathbf{u}\mathbf{r}') dX dt \equiv - \int_{t_1}^{t_2} \int_0^L mu^2 \mathbf{r}^2 dX dt,
\]

because the integrands in the second row are zeros due to the boundary conditions fix end variation. After rewriting the equation we obtain

\[
\int_{t_1}^{t_2} \int_0^L \delta x \left( (M + m)\ddot{x} + 2mu\dot{x}' + m\dot{x}' + m\mathbf{u}\mathbf{r}'' + mu^2 \mathbf{r}'' \right) dX dt = 0. \quad (B.1)
\]

It can be derived from Eq. (4.2) that

\[
\delta x \approx \int_0^X -\delta y \left( 1 + \frac{1}{2}y'^2 \right) y' d\xi \equiv -\delta y \left( 1 + \frac{1}{2}y'^2 \right) y' + \int_0^X \delta y \left( 1 + \frac{3}{2}y'^2 \right) y'' d\xi,
\]
and the following formula holds, as well:

\[
\int_{0}^{L} f(X) \int_{0}^{X} g(\xi) \mathrm{d}\xi \mathrm{d}X = \left[ (F(X) - F(0)) (G(X) - G(0)) \right]_{0}^{L} - \int_{0}^{L} g(X) \int_{0}^{X} f(\xi) \mathrm{d}\xi \mathrm{d}X
\]

\[
\equiv \int_{0}^{L} g(X) \int_{0}^{L} f(\xi) \mathrm{d}\xi \mathrm{d}X - \int_{0}^{L} g(X) \int_{0}^{X} f(\xi) \mathrm{d}\xi \mathrm{d}X \equiv \int_{0}^{L} g(X) \int_{X}^{L} f(\xi) \mathrm{d}\xi \mathrm{d}X.
\]

Using these expressions in Eq. (B.1), we obtain

\[
\int_{t_1}^{t_2} \int_{0}^{L} \delta y \left( \mathcal{F} \langle y \rangle G \langle y \rangle - \left( 1 + \frac{1}{2} y'^2 \right) y' \mathcal{G} \langle x \rangle + \left( 1 + \frac{3}{2} y'^2 \right) y'' \int_{L}^{X} \mathcal{G} \langle x(\xi, t) \rangle \mathrm{d}\xi \right) \mathrm{d}X \mathrm{d}t,
\]

where

\[
\mathcal{F} \langle z(X, t) \rangle = \ I \ E \left( z''' (1 + z'^2) + 4z'' z' z'' + z''' \right),
\]

\[
\mathcal{G} \langle z(X, t) \rangle = (M + m) \ddot{z} + 2mu \ddot{z} + m \dot{u} z' + mu^2 z'.
\]

The integral is zero for arbitrary \( \delta y \), i.e. its coefficient in Eq. B.2 must be zero.

From Eq. (4.3) one can express the derivatives of \( x \) as the function of the derivatives of \( y \):

\[
x' = 1 - \frac{1}{2} y'^2 + O(\epsilon^4),
\]

\[
x'' = y'y' + O(\epsilon^4),
\]

\[
x''' = y'y' + O(\epsilon^4),
\]

\[
x = -\int_{0}^{X} y' \mathrm{d}\xi + O(\epsilon^4),
\]

\[
x = -\int_{0}^{X} (y'y' + y'^2) \mathrm{d}\xi + O(\epsilon^4).
\]

Substituting these expressions into Eq. (B.2) and neglecting the fifth and higher order terms, we obtain the following partial differential equation as the equation of motion of an elastic pipe conveying fluid flow:

\[
I \ E \left( y''' (1 + y'^2) + 4y'' y'y' + y'^3 \right) + (M + m) \ddot{y} + 2mu \ddot{y} (1 + y'^2) + mu^2 y'' (1 + y'^2)
\]

\[
+ (M + m) \ddot{y} \int_{0}^{X} (y'y' + y'^2) \mathrm{d}\xi + m \dot{u} (L - X) y'' \left( 1 + \frac{3}{2} y'^2 \right)
\]
\[
- y'' \int_{x}^{L} \left( (M + m) (\dot{y}' + \ddot{y}'^{2}) \right) d\xi + 2mu\dot{y}'y' + \frac{1}{2}muy'^{2} + mu^{2}y''y' \right) d\xi = 0.
\]

However, if we want to use a discretization method based on the extreme of the possible integrals of the system energy (e.g. Finite Element Method, Galerkin Method) we don’t need to integrate the term of strain energy by parts \((y''\delta y'(\ldots) \Rightarrow \delta y(y''' + \ldots))\). This explains the difference between Eq. (4.11) and Eq. (B.2) (see \(\mathcal{F}(y(X, t))\)).
Appendix C

Rayleigh–Krylov Functions

Rayleigh–Krylov functions are defined as follows:

\[ S(x) = \frac{1}{2} (\cos x + \cosh x) \]  \hspace{1cm} \text{(C.1)}
\[ T(x) = \frac{1}{2} (\sin x + \sinh x) \]  \hspace{1cm} \text{(C.2)}
\[ U(x) = \frac{1}{2} (-\cos x + \cosh x) \]  \hspace{1cm} \text{(C.3)}
\[ V(x) = \frac{1}{2} (-\sin x + \sinh x) \]  \hspace{1cm} \text{(C.4)}

These functions are successive derivatives of each other (e.g. \( S(x) = T'(x) \), \( T(x) = U''(x) \), etc.). At zero all of them equals to zero, \( T(0) = U(0) = V(0) = 0 \), except \( S \): \( S(0) = 1 \).

Let us assume that the basis functions \( \{ \varphi_i(x) \} \) describing the shape of a beam-like continuum are the linear combination of Rayleigh–Krylov functions:

\[ \varphi_i(x) = c_{i1}S(\beta_i x) + c_{i2}T(\beta_i x) + c_{i3}U(\beta_i x) + c_{i4}V(\beta_i x). \]

However, all of them must satisfy the boundary conditions. In the case of a clamped–free end beam of length \( L \) that means

\[ \varphi_i(0) \equiv c_{i1} \quad = \quad 0, \]

\[ \varphi_i'(0) \equiv c_{i2} \beta_i \quad = \quad 0, \]

\[ \varphi_i''(L) \equiv c_{i3} \beta_i^2 S(\beta_i L) + c_{i4} \beta_i^2 T(\beta_i L) \quad = \quad 0, \]

\[ \varphi_i'''(L) \equiv c_{i3} \beta_i^3 V(\beta_i x) + c_{i4} \beta_i^3 S(\beta_i L) \quad = \quad 0, \]

where we have already applied the obvious result of the first two equations in the last two.

Thus, we have a system of homogeneous linear algebraic equations with two un-
knowns $c_{i3}$ and $c_{i4}$:

$$\begin{bmatrix} S(\beta_i L) & T(\beta_i L) \\ V(\beta_i L) & S(\beta_i L) \end{bmatrix} \begin{bmatrix} c_{i3} \\ c_{i4} \end{bmatrix} = 0$$

In order to get non-trivial solutions the determinant of the coefficient matrix must be zero:

$$S^2(\beta_i L) - T(\beta_i L)V(\beta_i L) = 0. \quad (C.5)$$

Applying the definitions of Rayleigh–Krylov functions given in (C.4) yields

$$\frac{1}{4}(\cos x + \cosh x)^2 - \frac{1}{4}(\sinh x - \sin x)(\sinh x + \sin x) = 0, \quad (x = \beta_i L)$$

and, after expanding it, we obtain the following transcendental equation for possible $\beta_i$-s:

$$2 + 2 \cos \beta_i L \cosh \beta_i L = 0. \quad (C.6)$$

Hence, for a given $\beta_i$ which satisfies Eq. (C.6), we have

$$\frac{c_{i4}}{c_{i3}} = \frac{S(\beta_i L)}{T(\beta_i L)} \left( = \frac{V(\beta_i L)}{S(\beta_i L)} \right),$$

and

$$\varphi_i(x) = U(\beta_i x) - \frac{S(\beta_i L)}{T(\beta_i L)}V(\beta_i x). \quad (C.7)$$
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